Shallow-water or long waves

For surface gravity waves, we can simplify the equations for the case of long waves (or shallow-water waves) from either the potential or the original momentum equations.

Potential

Our basic nonlinear equations in the case where the bottom depth varies $H = H_0 + h(\mathbf{x}, t)$ become

$$\nabla^2\phi = 0$$
$$\frac{\partial h}{\partial t} - \nabla\phi\cdot\nabla h = \phi_z \quad at \quad z = -H_0 - h(\mathbf{x}, t)$$
$$\frac{\partial \eta}{\partial t} - \nabla\phi\cdot\nabla \eta = -\phi_z \quad at \quad z = \eta(\mathbf{x}, t)$$
$$\frac{\partial \phi}{\partial t} = g\eta + \frac{1}{2}|\nabla\phi|^2 \quad at \quad z = \eta$$

If we nondimensionalize z by H_0, x, y by L, η by η_0, t by $L/\sqrt{gH_0}$, h by h_0 and ϕ by $g\eta_0 L/\sqrt{gH_0}$, we get

$$\frac{\partial^2\phi}{\partial z^2} + \delta^2\nabla_h^2\phi = 0$$
$$\epsilon_h\delta^2\frac{\partial h}{\partial t} - \epsilon_h\epsilon\delta^2\nabla\phi\cdot\nabla h = \epsilon\phi_z \quad at \quad z = -1 + \epsilon_h h(\mathbf{x}, t)$$
$$\delta^2\frac{\partial \eta}{\partial t} - \delta^2\epsilon\nabla\phi\cdot\nabla \eta = -\phi_z \quad at \quad z = \epsilon\eta(\mathbf{x}, t)$$
$$\frac{\partial \phi}{\partial t} = \eta + \frac{\epsilon}{\delta^2}\frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2 + \epsilon|\nabla_h\phi|^2 \quad at \quad z = \epsilon\eta$$

with $\delta = H_0/L$, $\epsilon = \eta_0/H_0$, and $\epsilon_h = h_0/H_0$. For the long-wave limit, we take $\delta^2 << 1$ and $\epsilon, \epsilon_h \sim 1$ (at least by comparison). Then the lowest order equations tell us that

$$\frac{\partial^2\phi_0}{\partial z^2} = 0 \quad , \quad \frac{\partial\phi}{\partial z} = 0 \quad at \quad z = -1 + \epsilon_h h \quad , \quad \epsilon\eta$$

for which the solution is $\phi_0 = \Phi(x, y, t)$. This is consistent with the synamic equation also. At the next order (δ^2), we find

$$\frac{\partial^2\phi_1}{\partial z^2} = -\nabla_h^2\Phi$$
$$\epsilon_h\frac{\partial h}{\partial t} - \epsilon_h\epsilon\nabla\Phi\cdot\nabla h = \epsilon\frac{\partial}{\partial z}\phi_1 \quad at \quad z = -1 + \epsilon_h h(\mathbf{x}, t)$$
$$\frac{\partial \eta}{\partial t} - \epsilon\nabla\Phi\cdot\nabla \eta = -\frac{\partial}{\partial z}\phi_1 \quad at \quad z = \epsilon\eta(\mathbf{x}, t)$$
$$\frac{\partial \Phi}{\partial t} = \eta + \epsilon|\nabla_h\Phi|^2 \quad at \quad z = \epsilon\eta$$

Integrating Poisson's equation in z and appliying the bounday conditions gives the mass conservation equation

$$\left(\frac{\partial}{\partial t} - \epsilon\nabla\Phi\right)\tilde{H} = \epsilon\tilde{H}\nabla_h^2\Phi$$

with the nondimensional depth of the fluid being $\tilde{H} = 1 + \epsilon\eta + \epsilon_h h$. The dynamic equation is

$$\frac{\partial}{\partial t}\Phi = \eta + \epsilon|\nabla_h\Phi|^2$$

If we look at linear, flat-bottom waves $h = 0$, $\epsilon << 1$ (but now requiring $\delta^2 << \epsilon << 1$), we have

$$\frac{\partial}{\partial t}\eta = \nabla_h^2\Phi$$
$$\frac{\partial}{\partial t}\Phi = \eta$$

giving the nondimensional wave equation

$$\frac{\partial^2}{\partial t^2}\eta = \nabla_h^2\eta$$

From basic equations

For rotating stratified flow, we have

$$\frac{\partial}{\partial t}\mathbf{u} + (\boldsymbol{\zeta} + f\hat{\mathbf{z}}) \times \mathbf{u} = -\nabla(P + \frac{1}{2}|\mathbf{u}|^2) + b\hat{\mathbf{z}}$$
$$\nabla \cdot \mathbf{u}_h + \frac{\partial w}{\partial z} = 0$$
$$\frac{\partial}{\partial t}b + \mathbf{u} \cdot \nabla b = 0$$

From the momentum equations, we can form a vorticity equation (Cartesian form)

$$\frac{\partial}{\partial t}Z_i + \nabla_j(u_j Z_i) - \nabla_j(u_i Z_j) = \nabla \times b\hat{\mathbf{z}} = -\hat{\mathbf{z}} \times \nabla b$$

or

$$\frac{\partial}{\partial t}Z_i + \mathbf{u} \cdot \nabla Z_i - \mathbf{Z} \cdot \nabla u_i = -\hat{\mathbf{z}} \times \nabla b$$

with $\mathbf{Z} = \boldsymbol{\zeta} + f\hat{\mathbf{z}}$. The flow can be irrotational when $f = 0$ and $b = 0$: a non-rotating, constant density fluid.

Hydrostatic

If $L >> H_0$, then the continuity equation implies $w \sim \frac{H_0}{L}\mathbf{u}_h$ and the w terms in the x and y components of $\boldsymbol{\zeta}$ are order δ^2 compared to the others:

$$\zeta_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \simeq -\frac{\partial v}{\partial z}$$

so that the vorticity in the momentum equations is replaced by $\boldsymbol{\zeta}_h = \nabla \times \mathbf{u}_h$. Likewise the w^2 term in the Bernouilli function is order δ^2 compared to the others. Finally, if $P \sim UL/T$ then

$$\frac{[\frac{\partial}{\partial t}w]}{[\frac{\partial}{\partial z}P]} \sim \frac{UH_0/LT}{UL/H_0T} = \delta^2$$

Dropping all the δ^2 terms gives

$$\frac{\partial}{\partial t}\mathbf{u}_h + (\boldsymbol{\zeta}_h + f\hat{\mathbf{z}}) \times \mathbf{u} = -\nabla(P + \frac{1}{2}|\mathbf{u}_h|^2) + b\hat{\mathbf{z}}$$

Note that vertical advection is still significant:

$$[w\frac{\partial}{\partial z}] = U\frac{H_0}{L}\frac{1}{H_0} \sim [\mathbf{u}_h\frac{\partial}{\partial x}]$$

In conventional form, we have

$$\frac{D}{Dt}\mathbf{u}_h + f\hat{\mathbf{z}} \times \mathbf{u}_h = -\nabla_h P \quad , \quad \frac{\partial}{\partial z}P = b$$

Homogeneous fluid

In this case, if the horizontal vorticities are zero initially, they will remain so; i.e. at time 0

$$\frac{\partial}{\partial t}\zeta_1 + \mathbf{u}\cdot\nabla\zeta_1 - (\zeta_3 + f)\frac{\partial}{\partial z}u_1 = 0 = \frac{\partial}{\partial t}\zeta_1 + \mathbf{u}\cdot\nabla\zeta_1 - (\zeta_3 + f)\zeta_2$$

implying $\frac{\partial}{\partial t}\zeta_1 = 0$. Thus we have

$$\frac{\partial}{\partial z}\mathbf{u}_h = 0$$

The vertical momentum equation implies $\frac{\partial}{\partial z}P = 0$ and the continuity equation tells us that $\frac{\partial}{\partial z}w$ is independent of depth so that

$$\frac{\partial}{\partial z}w = \frac{w(\eta(\mathbf{x},t)) - w(-H(\mathbf{x},t))}{H(\mathbf{x},t) + \eta(\mathbf{x},t)} = \frac{1}{H+\eta}(\frac{\partial}{\partial t} + \mathbf{u}_h\cdot\nabla)(H+\eta)$$

Finally, we note that the pressure at the surface is

$$-\rho_0 g\eta(x,y,t) + \rho_0 P(x,y,t) = p_a(x,y,t)$$

where p_a is the atmospheric pressure. Thus

$$P = g\eta + \frac{1}{\rho_0} p_a$$

and our equations become

$$\frac{\partial}{\partial t}\mathbf{u}_h + (\zeta_3 + f)\hat{\mathbf{z}} \times \mathbf{u}_h + \nabla(\frac{1}{2}|\mathbf{u}_h|^2) = \frac{D}{Dt}\mathbf{u}_h + f\hat{\mathbf{z}} \times \mathbf{u}_h = -\nabla g\eta - \nabla\frac{p_a}{\rho_0}$$

$$\frac{\partial}{\partial t}(H + \eta) + \nabla \cdot [\mathbf{u}_h(H + \eta)] = 0$$

These are the "shallow water equations"

Irrotational case

When $f = 0$, ζ_3 will also stay zero, and we can use

$$\mathbf{u}_h = -\nabla\Phi$$

and the momentum equations give

$$\frac{\partial}{\partial t}\nabla\Phi = \nabla(g\eta + \frac{p_a}{\rho_0} + \frac{1}{2}|\nabla\Phi|^2) \quad or \quad \frac{\partial}{\partial t}\Phi = g\eta + \frac{1}{2}|\nabla\Phi|^2 + \frac{p_a}{\rho_0}$$

and

$$\frac{\partial}{\partial t}(H + \eta) - \nabla \cdot [(H + \eta)\nabla\Phi = 0$$

as before.