Internal gravity waves (non-rotating)

Boussinesq

From $\rho = \rho_0(1 - b/g)$, $p = -\rho_0 gz + \rho_0 P$, and $b << g$ (and $c_s >> \sqrt{gH}$)

$$\begin{aligned} \frac{D}{Dt}\mathbf{u} &= -\nabla P + \hat{\mathbf{z}}b \\ \nabla \cdot \mathbf{u} &= 0 \\ \frac{D}{Dt}b &= 0 \end{aligned} \qquad (igw.1)$$

Linearized

Split into a static state $\frac{\partial}{\partial z}\overline{b} = N^2$, $\overline{P} = \int^z \overline{b}$ and deviations. Assume all products of deviations are negligible.

$$\begin{aligned} \frac{\partial}{\partial t}u &= -\frac{\partial}{\partial x}P \\ \frac{\partial}{\partial t}v &= -\frac{\partial}{\partial y}P \\ \frac{\partial}{\partial t}w &= -\frac{\partial}{\partial z}P + b \\ \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w &= 0 \\ \frac{\partial}{\partial t}b + wN^2 &= 0 \end{aligned} \qquad (igw.2)$$

Wave solutions

If all fields are proportional to $\exp(\imath kx + \imath \ell y + \imath mz - \imath \omega t)$ then

$$\begin{pmatrix} -\imath\omega & 0 & 0 & \imath k & 0 \\ 0 & -\imath\omega & 0 & \imath\ell & 0 \\ 0 & 0 & -\imath\omega & \imath m & -1 \\ \imath k & \imath l & \imath m & 0 & 0 \\ 0 & 0 & N^2 & 0 & -\imath\omega \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ P \\ b \end{pmatrix} = 0$$

Nontrivial solutions exist when

$$\imath\omega \left[\omega^2(k^2 + \ell^2 + m^2) - N^2(k^2 + \ell^2)\right] = 0$$

The $\omega = 0$ root corresponds to geostrophic (here, just horizontally nondivergent), hydrostatic balance; we'll return to that later. The other root

$$\omega = \pm N \left[\frac{k^2 + \ell^2}{k^2 + \ell^2 + m^2}\right]^{1/2} = \pm N \cos\phi \qquad (igw.3)$$

Fields

If we pick the structure of one field, we can find the others. For $\theta = \mathbf{k} \cdot \mathbf{x} - \omega t$, we have

$$
\begin{aligned}
b &= B \cos\theta \\
w &= -\frac{\omega}{N^2} B \sin\theta \\
p &= \frac{m}{|\mathbf{k}|^2} B \sin\theta \\
u &= \frac{mk}{\omega |\mathbf{k}|^2} B \sin\theta \\
v &= \frac{m\ell}{\omega |\mathbf{k}|^2} B \sin\theta
\end{aligned}
$$

The dispersion relation follows from the continuity eqn.

$$\frac{mk^2 + m\ell^2}{\omega |\mathbf{k}|^2} - \frac{\omega m}{N^2} = 0$$

Single variable version

We can eliminate variables in favor of a single field, w, to get the analogue to the classical wave equation. We begin with the divergence of the momentum equations which gives a diagnostic eqn. for the pressure

$$\nabla^2 P = \frac{\partial}{\partial z} b$$

which we can use to eliminate P from the vertical momentum equation

$$\frac{\partial}{\partial t} \nabla^2 w = -\frac{\partial^2}{\partial z^2} b + \nabla^2 b = \nabla_h^2 b$$

This gives

$$\frac{\partial^2}{\partial t^2} \nabla^2 w = -N^2 \nabla_h^2 w \qquad (igw.4)$$

after using the buoyancy equation.

Vorticity version

In the simplest 2D case $(x,\, z)$, we can write the equation for the y component of the vorticity, $\xi = \frac{\partial}{\partial z}u - \frac{\partial}{\partial x}w$,

$$\frac{\partial}{\partial t}\xi = -\frac{\partial}{\partial x}b$$

and use the y component of the streamfunction ψ

$$u = \frac{\partial}{\partial z}\psi \quad , \quad w = -\frac{\partial}{\partial x}\psi \quad \Rightarrow \quad \xi = \nabla^2\psi$$

to find

$$\frac{\partial}{\partial t}\nabla^2\psi = -\frac{\partial}{\partial x}b$$
$$\frac{\partial}{\partial t}b = N^2\frac{\partial}{\partial x}\psi$$

Taking another time-derivative of the vorticity equation gives (4) written in terms of ψ.

Fourier solution

If we have an initial condition $w(\mathbf{x}, 0)$, we can solve for later times by finding the transform

$$w(\mathbf{x}, t) = \int d^3\mathbf{k}\, \hat{w}(\mathbf{k}, t)\exp(\imath\mathbf{k}\cdot\mathbf{x})$$

Substituting in the dynamical equation (4) gives

$$\frac{\partial^2}{\partial t^2}\hat{w} = -N^2\frac{k^2+\ell^2}{\mathbf{k}\cdot\mathbf{k}}\hat{w} = -\Omega^2\hat{w}$$

which has solutions like

$$\hat{w}(\mathbf{k}, t) = \hat{w}(\mathbf{k}, 0)\exp(-\imath\Omega(\mathbf{k})t)$$

(There's also the negative sign case – determining how much of each we have will depend on the initial conditions on w and $\frac{d}{dt}w$. We'll begin assuming those are set so that we have energy in only the positive frequency root.) Then

$$w(\mathbf{x}, t) = \int d^3\mathbf{k}\, \hat{w}(\mathbf{k}, 0)\exp(\imath\mathbf{k}\cdot\mathbf{x} - \imath\Omega(\mathbf{k})t)$$

`Demos, Page 3: IGW solutions` *<k=2,m=1,cp=0,4>* *<k=1,m=2,cp=0.2>* *<k=2,m=1,cp=0.4,c 0.18>* *<k=1,m=2,cp=0.2,cg=0.36,-0.18>* *<k=2,m=2,cp=0.25,cg=0.18,-0.18>*

Phase and group velocities

The phase of the wave is $\theta = \mathbf{k}\cdot\mathbf{x} - \omega t$, and the rate of movement in the direction parallel to the wavenumber vector is

$$\delta\theta = \mathbf{k}\cdot\hat{\mathbf{k}}\, c\,\delta t - \omega\delta t = 0 \quad \Rightarrow \quad \delta t\Big[|\mathbf{k}|c - \omega\Big] = 0$$

so that

$$c = \frac{\omega}{|\mathbf{k}|}$$

But the packet doesn't propagate like that at all. To see how it does, let's suppose the initial condition has a sharply-peaked spectrum

$$\hat{w}(\mathbf{k},0) = \epsilon^{-3}\phi(\frac{\mathbf{k}-\mathbf{k}_0}{\epsilon})$$

so that the initial condition represents a large-scale modulation of a small-scale wave

$$\begin{aligned} w(\mathbf{x},t) &= \int d^3\mathbf{k}\,\epsilon^{-3}\phi(\frac{\mathbf{k}-\mathbf{k}_0}{\epsilon})\exp(\imath\mathbf{k}\cdot\mathbf{x}) \\ &= \int d^3\mathbf{K}\,\phi(\mathbf{K})\exp(\imath\mathbf{k}_0\cdot\mathbf{x} + \imath\epsilon\mathbf{K}\cdot\mathbf{x}) \\ &= \exp(\imath\mathbf{k}_0\cdot\mathbf{x})\int d^3\mathbf{K}\,\phi(\mathbf{K})\exp(\imath\mathbf{K}\cdot\epsilon\mathbf{x}) \\ &= A(\epsilon\mathbf{x})\exp(\imath\mathbf{k}_0\cdot\mathbf{x}) \end{aligned}$$

The time-dependent solution is

$$\begin{aligned} w(\mathbf{x},t) &= \int d^3\mathbf{k}\,\epsilon^{-3}\phi(\frac{\mathbf{k}-\mathbf{k}_0}{\epsilon})\exp(\imath\mathbf{k}\cdot\mathbf{x} - \Omega(\mathbf{k})t) \\ &= \int d^3\mathbf{K}\,\phi(\mathbf{K})\exp(\imath\mathbf{k}_0\cdot\mathbf{x} + \imath\epsilon\mathbf{K}\cdot\mathbf{x} - \imath\Omega(\mathbf{k}_0+\epsilon\mathbf{K})t) \\ &= \exp(\imath\mathbf{k}_0\cdot\mathbf{x} - \imath\Omega(\mathbf{k}_0)t)\int d^3\mathbf{K}\,\phi(\mathbf{K})\exp(\imath\mathbf{K}\cdot\epsilon\mathbf{x} - \imath\Omega(\mathbf{k}_0+\epsilon\mathbf{K})t + \imath\Omega(\mathbf{k}_0)t) \\ &\simeq \exp(\imath\mathbf{k}_0\cdot\mathbf{x} - \imath\Omega(\mathbf{k}_0)t)\int d^3\mathbf{K}\,\phi(\mathbf{K})\exp(\imath\mathbf{K}\cdot\epsilon\mathbf{x} - \imath\mathbf{K}\cdot\nabla_{\mathbf{k}}\Omega(\mathbf{k}_0)\epsilon t) \\ &= A(\epsilon[\mathbf{x} - \nabla_{\mathbf{k}}\Omega\, t])\exp(\imath\mathbf{k}_0\cdot\mathbf{x} - \imath\Omega(\mathbf{k}_0)t) \end{aligned}$$

Thus the envelope propagates at the group velocity

$$\mathbf{c}_g = \nabla_k\Omega|_{\mathbf{k}_0} \quad or \quad c_{g_i} = \frac{\partial\Omega}{\partial k_i}$$

For internal waves, the group velocity can be found from

$$\Omega^2 = \frac{N^2(k^2+\ell^2)}{k^2+\ell^2+m^2} = N^2\frac{k^2+\ell^2}{|\mathbf{k}|^2}$$

giving

$$2\Omega\frac{\partial\Omega}{\partial k} = 2\frac{km^2N^2}{|\mathbf{k}|^4} \quad \Rightarrow \quad \frac{\partial\Omega}{\partial k} = N\frac{km^2}{|\mathbf{k}|^3\sqrt{k^2+\ell^2}}$$

and

$$\frac{\partial\Omega}{\partial \ell} = N\frac{\ell m^2}{|\mathbf{k}|^3\sqrt{k^2+\ell^2}} \quad , \quad \frac{\partial\Omega}{\partial m} = -N\frac{m\sqrt{k^2+\ell^2}}{|\mathbf{k}|^3}$$

Thus

$$\mathbf{c}_g = \frac{N}{|\mathbf{k}|^3\sqrt{k^2+\ell^2}}(km^2, \ell m^2, -m(k^2+\ell^2))$$

Note that $\mathbf{k}\cdot\mathbf{c}_g = 0$; the group velocity is parallel to the planes of constant phase.

Vertical modes

If N is constant, and the domain is bounded by a bottom and top at $z = 0$ and $z = H$, respectively, we can write

$$w = W(x,y,t)\sin(M\pi/H)$$

and (for $M = 1$)

$$\frac{\partial^2}{\partial t^2}\left(1 - \frac{H^2}{\pi^2}\nabla_h^2\right)W = +N^2\frac{H^2}{\pi^2}\nabla_h^2 W$$

giving

$$\omega^2 = \frac{N^2K^2}{1+K^2}$$

with $K = kH/\pi$.

Note that the phase speed

$$c = \frac{N}{\left[\frac{M^2\pi^2}{H^2} + k^2\right]^{1/2}} = \frac{NH}{M\pi}\left[1 - \left(\frac{kH}{M\pi}\right)^2\right]^{-1/2}$$

gives only weak dispersion for long waves $\delta c/c \sim k^2H^2/M^2\pi^2$. The long wave speed decreases for modes with more wiggles in the vertical. `Demos, Page 5: Dispersion relation` *<disp rel>*

Another view of group velocity

Consider superimposing two waves,

$$0.5\cos(k_1 x - \omega_1 t) + 0.5\cos(k_2 x - \omega_2 t)$$

with $k_1 < k_2$; the result has a "beat-frequency" modulation. The waves will be back in phase at both their peaks when

$$(N+1)\frac{2\pi}{k_2} = N\frac{2\pi}{k_1} \quad \Rightarrow \quad N = \frac{k_1}{k_2 - k_1} \quad \Rightarrow \quad L = \frac{2\pi}{k_2 - k_1}$$

To calculate the motion consider the time at which the two peaks catch up with each other

$$X_1 = -\frac{2\pi}{k_1} + \frac{\omega_1}{k_1}T = X_2 = -\frac{2\pi}{k_2} + \frac{\omega_2}{k_2}T \quad \Rightarrow \quad T = \frac{2\pi(k_2 - k_1)}{\omega_1 k_2 - \omega_2 k_1}$$

and we see a new maximum constructive interference point at X_1. The speed of motion is therefore

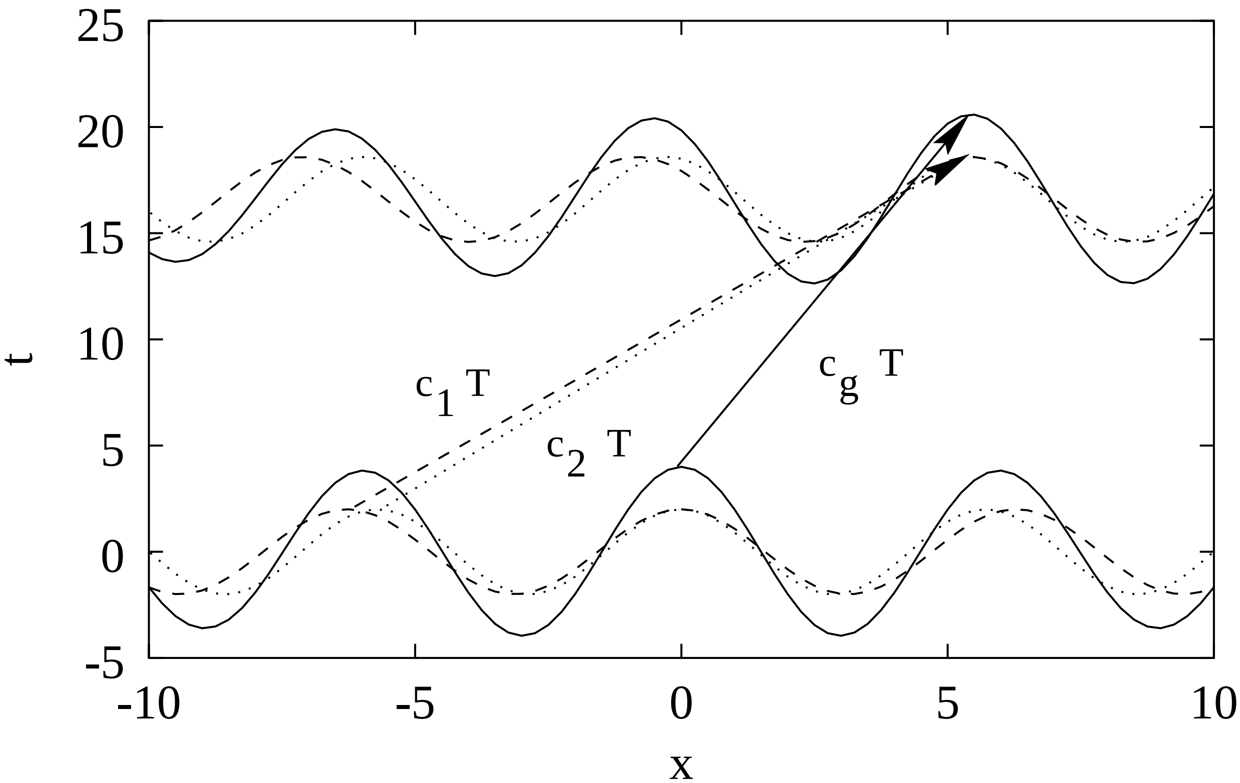
$$\frac{X_1}{T} = \frac{\omega_1}{k_1} - \frac{2\pi}{k_1 T} = \frac{\omega_2 - \omega_1}{k_2 - k_1}$$

`Demos, Page 6: Two waves` *<initial>* *<initial>* `Demos, Page 6: Two waves` *<initial>* *<evolution>* *<NK/(1+K*K)>* *<NK/(1+K*K)**2>*

(Continued on next page.)

Revision of two waves

Assuming that the shorter wave travels more slowly and the two waves start at time $t = 0$ being in phase at $x = 0$, the pattern will repeat exactly when the **previous** crests of each of the two waves match up precisely:



Two wave geometry. Solid=sum; dashed=longer, faster; dotted=shorter, slower

Thus we have

$$c_1 T = \frac{2\pi}{k_1} + c_g T \quad , \quad c_2 T = \frac{2\pi}{k_2} + c_g T$$

We can think of these as simultaneous equations for $1/T$ and c_g:

$$c_g + \frac{2\pi}{k_1}\frac{1}{T} = c_1$$
$$c_g + \frac{2\pi}{k_2}\frac{1}{T} = c_2$$

solving these gives

$$c_g = \frac{c_1 \frac{2\pi}{k_2} - c_2 \frac{2\pi}{k_1}}{\frac{2\pi}{k_2} - \frac{2\pi}{k_1}} = \frac{c_1 k_1 - c_2 k_2}{k_1 - k_2} = \frac{\omega_1 - \omega_2}{k_1 - k_2}$$

Dispersion of group

If we consider the next order in our expansion for sharply peaked spectra

$$w \simeq \exp(\imath \mathbf{k}_0 \cdot \mathbf{x} - \imath\Omega(\mathbf{k}_0)t) \int d^3\mathbf{K}\, \phi(\mathbf{K}) \exp(\imath \mathbf{K} \cdot \epsilon\mathbf{x} - \imath \mathbf{K} \cdot \nabla_{\mathbf{k}}\Omega(\mathbf{k}_0)\epsilon t) \times$$

$$\exp(-\imath \frac{1}{2} \frac{\partial^2 \Omega}{\partial k_i \partial k_j} K_i K_j \epsilon^2 t)$$

In a frame moving with the group $\mathbf{X} = \epsilon\mathbf{x} - \mathbf{c}_g \epsilon t$, the changes on a time scale $\tau = \epsilon^2 t$ are determined by

$$w \simeq \exp(\imath \mathbf{k}_0 \cdot \mathbf{x} - \imath\Omega(\mathbf{k}_0)t) \int d^3\mathbf{K}\, \phi(\mathbf{K}) \exp(\imath \mathbf{K} \cdot \mathbf{X}) \exp(-\imath \frac{1}{2} \frac{\partial^2 \Omega}{\partial k_i \partial k_j} K_i K_j \tau)$$

For this we find the amplitude satisfies the Schrödinger equation

$$\frac{\partial}{\partial \tau} w = \frac{\imath}{2} \frac{\partial^2 \Omega}{\partial k_i \partial k_j} \frac{\partial}{\partial X_i} \frac{\partial}{\partial X_j} w$$

This looks like a diffusion equation (with an imaginary diffusivity) and can be solved in much the same way. In particular, we can look for Gaussian solutions

$$w = A(\tau) \exp(-\alpha_{ij}(\tau) X_i X_j)$$

The result can be seen in the 1D case

$$\frac{\partial}{\partial \tau} w = \frac{\imath}{2} \frac{\partial^2 \Omega}{\partial k^2} \frac{\partial^2}{\partial X^2} w$$

Plugging $w = A(\tau)\exp(-\alpha(\tau)X^2)$ into the previous equation and gathering the terms which are proportional to X^2 and to 1 gives

$$\frac{\partial}{\partial t} \alpha = -2\imath\Omega'' \alpha^2$$

$$\frac{\partial}{\partial t} A = -\imath\Omega'' \alpha A$$

This has solutions

$$\alpha = \frac{\alpha_0}{1 + 2\imath\alpha_0\Omega'' t}$$

$$A = A(0)\sqrt{\frac{\alpha(t)}{\alpha_0}}$$

`Demos, Page 8:  1D case` <*packet motion and spread*> <*amplitude decay*>

Energy

We can form an energy equation from the linearized equations

$$\begin{aligned}\frac{\partial}{\partial t}\frac{1}{2}(\mathbf{u}\cdot\mathbf{u}) = \frac{\partial}{\partial t}KE &= -\mathbf{u}\cdot\nabla P + wb \\ &= -\nabla\cdot(\mathbf{u}P) + wb\end{aligned}$$

The last term represents generation of kinetic energy from potential energy (heavy fluid [$b < 0$] moving down [$w < 0$] or light fluid moving up). We can define the available potential energy here as $\frac{1}{2}b^2/N^2$

$$\frac{\partial}{\partial t}\frac{1}{2}\frac{b^2}{N^2} = \frac{\partial}{\partial t}PE = -wb$$

so that the change in total energy in a parcel is given by the flux across the boundary of the parcel $\mathbf{u}P$

$$\frac{\partial}{\partial t}KE + PE = \frac{\partial}{\partial t}E = -\nabla\cdot(\mathbf{u}P)$$

For IGW's, the energy is

$$\begin{aligned}E &= \frac{1}{2}\left(\frac{m^2(k^2+\ell^2)}{\omega^2|\mathbf{k}|^4} + \frac{\omega^2}{N^4}\right)B^2\sin^2\theta + \frac{1}{2}\frac{1}{N^2}B^2\cos^2\theta \\ &= \frac{1}{2}\frac{B^2}{N^2}\sin^2\theta + \frac{1}{2}\frac{B^2}{N^2}\cos^2\theta \\ &= \frac{1}{2}\frac{B^2}{N^2}\end{aligned}$$

and the flux is just

$$\mathbf{F} = \frac{m}{|\mathbf{k}|^2}\left(\frac{mk}{\omega|\mathbf{k}|^2}\,,\ \frac{m\ell}{\omega|\mathbf{k}|^2}\,,\ -\frac{\omega}{N^2}\right)B^2\sin^2\theta$$

which averages to

$$\begin{aligned}\mathbf{F} &= \frac{1}{2}\frac{m}{|\mathbf{k}|^2}\left(\frac{mk}{\omega|\mathbf{k}|^2}\,,\ \frac{m\ell}{\omega|\mathbf{k}|^2}\,,\ -\frac{\omega}{N^2}\right)B^2 \\ &= \frac{1}{2}\frac{B^2}{N^2}\left(\frac{m^2kN}{\sqrt{k^2+\ell^2}|\mathbf{k}|^3}\,,\ \frac{m^2\ell N}{\sqrt{k^2+\ell^2}|\mathbf{k}|^3}\,,\ -\frac{m\sqrt{k^2+\ell^2}N}{|\mathbf{k}|^3}\right)B^2 \\ &= \mathbf{c}_g E\end{aligned}$$

The energy moves at the group velocity.

Reflection from a sloping surface

`Demos, Page 9: Reflection problem` *<geometry>*

Let us consider IGW's incident on a surface sloped at angle β from horizontal. The boundary condition at the surface $z = z\tan\beta$ is that the normal component of the velocity must vanish

$$\mathbf{u} \cdot \hat{\mathbf{n}} = -u \sin\beta + w \cos\beta = 0$$

In terms of the streamfunction, this condition becomes

$$-\sin\beta \frac{\partial}{\partial z}\psi - \cos\beta \frac{\partial}{\partial x}\psi = -\hat{\mathbf{t}} \cdot \nabla\psi = 0$$

so that we can simply assume $\psi(s\cos\beta, s\sin\beta, t) = 0$ with s being the distance along the slope.

We write the solution in terms of an incident wave and a reflected wave

$$\psi = A\cos(kx + mz - \omega t) + A_r \cos(k_r x + m_r z - \omega_r t)$$

and apply the boundary condition

$$A\cos(s\,\mathbf{k} \cdot \hat{\mathbf{t}} - \omega t) + A_r \cos(s\,\mathbf{k}_r \cdot \hat{\mathbf{t}} - \omega_r t) = 0$$

we see that $\omega_r = \omega$, $\mathbf{k} \cdot \hat{\mathbf{t}} = \mathbf{k}_r \cdot \hat{\mathbf{t}}$, and $A_r = -A$ in order that the equation above holds for all s and t.

The geometry is now clear: since the frequencies are the same,

$$\cos(\phi_r) = \cos(\phi)$$

so that the angle of the reflected wavenumber from horizontal is the same as that of the incident wave; secondly, the projection along the slope must match.

`Demos, Page 10: incident/refl` *<geometry>* *<group vel>* *<phi=30,beta=0>* *<wavefronts>* *<phi=30,beta=15>* *<wavefronts>* *<phi=30,beta=25>* *<wavefronts>* *<phi=30,beta=45>* *<wavefronts>*

Note that the reflected wavenumber satisfies

$$K_r \cos(\phi + \beta) = K_i \cos(\phi - \beta)$$

and becomes infinite when $\phi + \beta = 90^{\circ}$ — when the reflected wave group velocity is tangent to the slope. Beyond this critical slope, we satisfy the boundary condition by using $\omega_r = -\omega$ and $\mathbf{k} \cdot \hat{\mathbf{t}} = -\mathbf{k}_r \cdot \hat{\mathbf{t}}$.

`Demos, Page 10: large slope` *<phi=60,beta=15>* *<wavefronts>* *<phi=60,beta=45>* *<wavefronts>* *<phi=60,beta=55>* *<wavefronts>* *<phi=60,beta=85>* *<wavefronts>*

When the slope is shallow, a wave can focus between the surface and the bottom and propagate into the corner, with its scale becoming smaller and smaller. `Demos, Page 10: focus` *<incident>* *<1st refl>* *<2nd refl>* *<3rd refl>* *<4th refl>* *<5th refl>* *<6th refl>* *<7th refl>*

Generation by flow over topography

One mechanism for creating internal gravity wave is flow over topography. We'll consider the simple case with zonal flow at a sinusoidal topography at $z = h_0 \cos(kx)$. The equations of motion will be linearized assuming the mean flow, U, is much larger than the wave flows $\mathbf{u}$.

$$\frac{\partial}{\partial t}\mathbf{u} + U\frac{\partial}{\partial x}\mathbf{u} = -\nabla P + b\hat{\mathbf{z}}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial}{\partial t}b + U\frac{\partial}{\partial x}b + wN^2 = 0$$

The dispersion relation is the same, except we replace ω by $\omega - kU$

$$\omega - kU = \pm\frac{Nk}{|\mathbf{k}|}$$

For the steady response $\omega = 0$, we will need to use the *minus* sign.

$$\omega = kU - \frac{Nk}{|\mathbf{k}|}$$

The condition at the bottom is, again, no normal flow.

$$(U\hat{\mathbf{x}} + \mathbf{u}) \cdot \hat{\mathbf{n}} = (U\hat{\mathbf{x}} + \mathbf{u}) \cdot \frac{\hat{\mathbf{z}} - \nabla h}{\sqrt{1 + |\nabla h|^2}} = 0$$

or

$$(U + u)\frac{\partial}{\partial x}h = w \quad at \quad z = h(x, y)$$

(We can find the normal by thinking about a function $F(x, y, z) = z - h(x, y)$; its three-dimensional gradient is perpendicular to the surfaces of constant F, in particular the one at $F = 0$ which represents the boundary.) This linearizes to

$$w = U\frac{\partial}{\partial x}h \quad at \quad z = 0$$

when the slope and the net height change is small. This can also be written as

$$\psi(x, 0, t) + Uh(x, y) = 0$$

Steady solution [short scales]

For a steady solution, we have

$$\sqrt{k^2+m^2} = \frac{N}{U} \quad \textit{or} \quad m^2 = \frac{N^2}{U^2} - k^2$$

If the topographic scale is short compared to U/N, the m^2 will be negative so that if $\hat{m} = \sqrt{k^2 - N^2/U^2}$ then

$$\psi = -Uh_0 \, \Re(e^{\imath kx \mp \hat{m}z})$$

We must choose the negative sign so that the disturbance decays with height

$$\psi = -Uh_0 \cos(kx) \exp(-\sqrt{k^2 - N^2/U^2}\; z)$$

`Demos, Page 11: short scales` <*Uk/N=1.01*> <*Uk/N=1.1*> gwtop2.png <*Uk/N=1.5*> <*Uk/N=2*>

Long scales

If $k^2 < N^2/U^2$ then m is real and our solution looks like

$$\psi = -Uh_0 \, \Re(e^{\imath kx \pm \imath mz})$$

and we must decide which sign to use (or have some contribution from each). We shall discuss a number of ways of resolving the issue.

`Demos, Page 12: long scales` <*Uk/N=0.99*> <*Uk/N=0.9*> <*Uk/N=0.8*> <*Uk/N=0.7*>

Group Velocity: Since the topography is the source of the waves, we would expect the vertical component of $\mathbf{c}_g$ to be positive. This means that if we suddenly add or eliminate the topography, the disturbance in the wave field would propagate upwards. Therefore

$$\frac{\partial}{\partial m}\left[Uk - \frac{Nk}{\sqrt{k^2+m^2}}\right] = \frac{Nkm}{(k^2+m^2)^{3/2}} > 0$$

The positive sign is the correct one, so that

$$\psi = -Uh_0 \cos(kx + \sqrt{N^2/U^2 - k^2}\; z)$$

Energy Flux: For these 2-D motions, we can write the average (as in zonal average) vertical energy flux as

$$\overline{wP} = -\overline{\frac{\partial \psi}{\partial x} P} = \overline{\psi \frac{\partial P}{\partial x}}$$

and we expect it to be positive. Using the zonal momentum equation gives

$$\overline{wP} = -\overline{\psi \frac{\partial^2 \psi}{\partial t \partial z}} - \overline{U \psi \frac{\partial^2 \psi}{\partial x \partial z}} = -\overline{\psi \frac{\partial^2 \psi}{\partial t \partial z}} + U \overline{\frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial z}}$$

For steady flow with $\psi = -U h_0 \cos(kx \pm mz)$, we have

$$\overline{wP} = \pm \frac{1}{2} U^3 h_0^2 k m$$

again showing the plus sign to be the desired one.

Damping: Another approach is to add damping to the equations so that even the vertically wavy mode decays and reject any growing solution. We take

$$\frac{\partial}{\partial t}\mathbf{u} + U \frac{\partial}{\partial x}\mathbf{u} = -\nabla P + b\hat{\mathbf{z}} - \epsilon \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial}{\partial t} b + U \frac{\partial}{\partial x} b + w N^2 = -\epsilon b$$

We now have

$$(\imath k U + \epsilon)^2 = -\frac{N^2 k^2}{k^2 + m^2} \quad \Rightarrow \quad m^2 = \frac{N^2}{U^2 (1 - \imath \epsilon / kU)^2} - k^2$$

The imaginary part of m is

$$\Im(m) \simeq \frac{1}{\Re(m)} \frac{\epsilon N^2}{k U^3}$$

so that vertically decaying solutions $\Im(m) > 0$ require $\Re(m) > 0$ as before.

Initial Value Problem: Finally, we can look at what happens if we suddenly turn the flow or the topography on. Using

$$\left(\frac{\partial}{\partial t} + \imath kU\right)^2 \left(\frac{\partial^2}{\partial z^2} - k^2\right)\psi = k^2 N^2 \psi$$

with the initial and boundary conditions

$$\psi(z,0) = 0 \quad , \quad \psi(0,t) = -Uh_0 \quad , \quad \psi(\infty,t) = 0$$

The Laplace transformed problem gives the same z structure equation as in the damped system

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)\psi^T = -\frac{N^2}{(U^2 - 2\imath sU/k - s^2/k^2)}\psi^T$$

with

$$\psi^T(0,s) = -Uh_0/s \quad , \quad \psi^T(\infty,s) = 0$$

Again the positive root is the proper one

$$\psi^T = -Uh_0 \frac{1}{s}\exp(\imath\,[\frac{N^2}{(U^2 - 2\imath sU/k - s^2/k^2)} - k^2]^{1/2} z)$$

The inverse transform

$$\psi = -Uh_0 \int_{-\imath\infty}^{\imath\infty} \frac{1}{s}\exp(\imath\,[\frac{N^2}{(U^2 - 2\imath sU/k - s^2/k^2)} - k^2]^{1/2} z) e^{st}$$

is dominated by the singularity at $s = 0$; for large time, we recover the standing wave solution. `Demos, Page 14: IGW data` <*Cloud patterns*> <*Breaking waves*> <*Appalacian*> <*Surface slicks*> <*Internal tides*> <*Georges Bank*> <*Northern Oregon*>

The Nonlinear Problem

We can also look at the nonlinear problem in simple 2-D cases. The steady equations

$$\mathbf{u}\cdot\nabla q = -\frac{\partial}{\partial x}b$$
$$\mathbf{u}\cdot\nabla(b+N^2 z) = 0$$

can be solved by noting that $\mathbf{u}\cdot\nabla\phi = 0$ implies $\phi = \Phi(\psi)$ – the advected property is constant along streamlines, since the parcels of fluid move along the streamlines in steady flows. The streamfunction here includes both the mean flow and the fluctuations $\psi = Uz + \psi'(\mathbf{x})$. Therefore

$$N^2 z + b'(\mathbf{x}) = B(Uz+\psi'(\mathbf{x})) = \frac{N^2}{U}(Uz+\psi')$$

Uniqueness could be a problem, of course. In any case, we'll take

$$b' = \frac{N^2}{U}\psi'$$

The vorticity equation then tells us that

$$\mathbf{u}\cdot\nabla q = w\frac{N^2}{U} \quad\Rightarrow\quad \mathbf{u}\cdot\nabla(q-\frac{N^2}{U}z) = 0$$

so that

$$\nabla^2\psi' - \frac{N^2}{U}z = Q(Uz+\psi') = -\frac{N^2}{U^2}(Uz+\psi')$$

or

$$\nabla^2\psi' = -\frac{N^2}{U^2}\psi'$$

with the boundary conditions

$$\psi'(x,h)+Uh = 0 \quad , \quad \psi'\to 0 \quad \textit{or} \quad \text{radiation condition}$$

Note that the linear solution is a perfectly good one – we just have to find the topography that matches it!

$$Uh_0\cos(kx)\exp(-\hat{m}h) = Uh \quad \textit{or} \quad h_0\cos(kx) = h\exp(\hat{m}h)$$

$$h_0\cos(kx+mh(x)) = h(x)$$

`Demos, Page 15: topographies` *<Uk/N=1.001>* *<Uk/N=1.01>* *<Uk/N=1.05>* *<Uk/N=1.07>* *<Uk/N=1.08>* *<Uk/N=0.99>* *<Uk/N=0.9>* *<Uk/N=0.5>* *<Uk/N=0.2>*

WKB and modes

We will now consider propagation and vertical modes in the case where the Brunt-Väisälä frequency varies with z. The normal mode problem

$$\frac{\partial^2}{\partial t^2}\nabla^2 w + N^2\nabla_h^2 w = 0 \quad , \quad w(0) = w(H) = 0$$

can be separated as $w = W(z)\exp(\imath[kx + \ell y - \omega t])$, and the eigenvalue problem becomes

$$\frac{\partial^2}{\partial z^2}W - \mathbf{k}_h^2 W + \frac{N^2}{\omega^2}\mathbf{k}_h^2 = 0 \quad , \quad W(0) = W(H) = 0$$

with $\mathbf{k}_h^2 = k^2 + \ell^2$. This is a Sturm-Liouville problem with eigenvalue $\lambda = 1/\omega^2$. For a given $N^2(z)$ and $\mathbf{k}_h^2$, we will find an infinite set of eigenfunctions with increasing λ's and therefore decreasing frequencies. In the case of constant N, we have

$$W = \sin(M\pi z/H) \quad , \quad \omega = N\sqrt{\frac{\mathbf{k}_h^2 H^2}{\mathbf{k}_h^2 H^2 + M^2\pi^2}}$$

When N is not constant, we can look for approximate solutions

$$W \simeq A(\epsilon z)\sin(\epsilon^{-1}\theta(\epsilon z))$$

so that

$$\frac{\partial^2}{\partial z^2}W = \epsilon^2 A''\sin\frac{\theta}{\epsilon} + \epsilon[2A'\theta' + A\theta'']\cos\frac{\theta}{\epsilon} - A\theta'^2\sin\frac{\theta}{\epsilon}$$

The lowest order problem gives

$$\theta'^2 = \mathbf{k}_h^2\left[\frac{N^2}{\omega^2} - 1\right]$$

or

$$\theta = |\mathbf{k}_h|\int_0^z \sqrt{\frac{N^2}{\omega^2} - 1}$$

and an eigenvalue relationship

$$M\pi = |\mathbf{k}_h| H\,\frac{1}{H}\int_0^H \sqrt{\frac{N^2}{\omega^2} - 1}$$

The amplitude satisfies

$$2A'\theta' + A\theta'' = 0 \quad \Rightarrow \quad 2\frac{1}{A}\frac{d}{dz'}A = -\frac{1}{\theta'}\frac{d}{dz'}\theta'$$

so that

$$A = const\frac{\theta_0'}{\theta'}^{1/2} = const|\mathbf{k}_h|^{-1/2}\left[\frac{N^2}{\omega^2} - 1\right]^{-1/4}$$

— the amplitude is largest in the regions of small N and slowly changing phase. This makes sense since the energy flux is proportional to the vertical wavenumber times the amplitude squared.

The results above assume that ω is smaller than the minimum value of N so that the solution remains sinusoidal. When that is not the case, the waves will have a turning point and will be exponentially damped in the region where $\omega > N$.

`Demos, Page 16: modes` <*N squared*> <*kh=0.01*> <*kh=0.1*> <*kh=1*> <*kh=10*> <*disp rel*> <*wkb disp rel*>

More general WKB form

We now consider propagation in an inhomogeneous medium. The waves will be locally sinusoidal

$$w = A(\mathbf{x}, t)\exp(\imath\theta(\mathbf{x}, t))$$

with the implicit assumption that the gradients of θ are large (we could put in the ϵ factors as before). Locally, we can identify the effective wavenumber and frequency in terms of the derivatives of θ:

$$\mathbf{k} = \nabla\theta \quad , \quad \omega = -\frac{\partial}{\partial t}\theta$$

To calculate the evolution of the phase, we note that the largest terms in the dynamical equation will give the dispersion relation

$$\left(\frac{\partial\theta}{\partial t}\right)^2 |\nabla\theta|^2 - N^2|\nabla_h\theta|^2 = 0$$

where ∇_h is the horizontal gradient $(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ 0)$ (in Cartesian coordinates). From this, we obtain an evolution equation for θ

$$\frac{\partial}{\partial t}\theta = -N\frac{|\nabla_h\theta|}{|\nabla\theta|} \equiv -\Omega(\nabla\theta, \mathbf{x}, t)$$

This equation implies that the wavenumbers and frequency evolve as the wave packet moves through the medium. To see this, consider the case $N = N(z)$ and suppose that the horizontal and vertical wavenumbers are initially constant

$$\theta(x, z, 0) = k_0 x + m_0 z$$

At time zero, we have

$$\frac{\partial}{\partial t}\theta = -N(z)\frac{k_0}{\sqrt{k_0^2 + m_0^2}}$$

so that

$$\theta(x,z,\delta t) = k_0 x + m_0 z - N(z)\frac{k_0}{\sqrt{k_0^2+m_0^2}}\delta t$$

and the vertical wavenumber is now

$$\frac{\partial}{\partial z}\theta = m_0 - N'(z)\frac{k_0}{\sqrt{k_0^2+m_0^2}}\delta t$$

giving a new frequency as well. Likewise, even with constant N, if $\frac{\partial}{\partial z}\theta$ is not constant, the value of $\frac{\partial}{\partial t}\theta$ will also be non-uniform. Therefore the vertical wavenumber changes. To see this more precisely, we can take a z derivative of the evolution equation to find

$$\frac{\partial}{\partial t}\theta_z = -\frac{\partial\Omega}{\partial\theta_z}\frac{\partial}{\partial z}\theta_z - \frac{\partial\Omega}{\partial z} = -c_{g\,z}\frac{\partial}{\partial z}\theta_z - \frac{\partial\Omega}{\partial z}$$

Using $\frac{\partial}{\partial z}\theta = \theta_z$. In the internal gravity wave case

$$\frac{\partial}{\partial t}\theta_z = -N'(z)\frac{k_0}{\sqrt{k_0^2+\theta_z^2}} + N\frac{k_0\theta_z}{[k_0^2+\theta_z^2]^{3/2}}$$

In general, we find

$$\frac{\partial}{\partial t}\nabla_i\theta + (\mathbf{c}_g\cdot\nabla)\nabla_i\theta \equiv D_g\theta_i = -\nabla_i\Omega$$

Likewise, we note that $\frac{\partial\theta}{\partial t}$ at time δt will be different from $\frac{\partial\theta}{\partial t}$ at time 0, since it will be evaluated using the new vertical wavenumber.

$$\frac{\partial}{\partial t}\frac{\partial\theta}{\partial t} + \mathbf{c}_g\cdot\nabla\frac{\partial\theta}{\partial t} = D_g\theta_t = -\frac{\partial}{\partial t}\Omega$$

Finally, we look at the amplitude equation, which comes from the first order terms in

$$(\imath\theta_t + \frac{\partial}{\partial t})^2(\imath\nabla\theta + \nabla)^2 A + (\imath\nabla_h\theta + \nabla_h)^2 A N^2 = 0$$

This gives

$$\theta_t\frac{\partial}{\partial t}(|\nabla\theta|^2 A) + \frac{\partial}{\partial t}(\theta_t|\nabla\theta|^2 A) + \theta_t^2\nabla\theta\cdot\nabla A + \theta_t^2\nabla\cdot(A\nabla\theta) - \nabla_h\theta\cdot\nabla_h(N^2 A) - \nabla_h\cdot(N^2 A\nabla_h\theta)$$

the terms involving derivatives of A are

$$2\theta_t|\nabla\theta|^2\frac{\partial A}{\partial t} + 2\theta_t|\nabla\theta|^2\frac{\theta_t\nabla\theta}{|\nabla\theta|^2}\cdot\nabla A - 2\theta_t|\nabla\theta|^2\frac{\theta_t\nabla_h\theta}{|\nabla_h\theta|^2}\cdot\nabla A = 2\theta_t|\nabla\theta|^2 D_g A$$

using the dispersion relation

$$\theta_t^2 = N^2\frac{|\nabla_h\theta|^2}{|\nabla\theta|^2}$$

and the definition of the group velocity, which gives

$$\mathbf{c}_g = \theta_t \left[\frac{\nabla\theta}{|\nabla\theta|^2} - \frac{\nabla_h\theta}{|\nabla_h\theta|^2} \right]$$

The amplitude equation becomes

$$2\theta_t|\nabla\theta|^2 D_g A + A\left[2\theta_t \frac{\partial}{\partial t}|\nabla\theta|^2 + |\nabla\theta|^2\frac{\partial\theta_t}{\partial t} + \theta_t^2\nabla^2\theta - \nabla_h\theta\cdot\nabla_h N^2 - \nabla_h\cdot(N^2\nabla_h\theta)\right] = 0$$

Multiplying by $\frac{1}{2}A^*$ and adding the conjugate gives

$$\theta_t|\nabla\theta|^2 D_g|A|^2 + |A|^2\left[2\theta_t \frac{\partial}{\partial t}|\nabla\theta|^2 + |\nabla\theta|^2\frac{\partial\theta_t}{\partial t} + \theta_t^2\nabla^2\theta - \nabla_h\theta\cdot\nabla_h N^2 - \nabla_h\cdot(N^2\nabla_h\theta)\right] = 0$$

Next we manipulate the divergence terms

$$\begin{aligned}\theta_t^2\nabla^2\theta - \nabla(N^2\nabla_h\theta) &= \nabla\cdot\left(\theta_t|\nabla\theta|^2\frac{\theta_t\nabla\theta}{|\nabla\theta|^2} - \theta_t|\nabla\theta|^2\frac{\theta_t\nabla_h\theta}{|\nabla_h\theta|^2}\right) - \nabla\theta\cdot\nabla\theta_t^2 \\ &= \nabla\cdot(\theta_t|\nabla\theta|^2\mathbf{c}_g) - \nabla\theta\cdot\nabla\theta_t^2\end{aligned}$$

Combining these with the first two terms gives

$$\begin{aligned}2\theta_t\frac{\partial}{\partial t}|\nabla\theta|^2 + |\nabla\theta|^2\frac{\partial\theta_t}{\partial t} + \theta_t^2\nabla^2\theta - \nabla_h\cdot(N^2\nabla_h\theta &= \theta_t\frac{\partial}{\partial t}|\nabla\theta|^2 + D_g(\theta_t|\nabla\theta|^2) + \theta_t|\nabla\theta|^2\nabla\cdot\mathbf{c}_g - \nabla\theta\cdot\nabla\theta_t^2 \\ &= D_g(\theta_t|\nabla\theta|^2) + \theta_t|\nabla\theta|^2\nabla\cdot\mathbf{c}_g\end{aligned}$$

Finally, we use

$$-2N\nabla_h\theta\cdot\nabla_h N = 2N\nabla_h\theta\cdot\frac{|\nabla\theta|}{\nabla_h\theta|}D_g\nabla_h\theta = -\theta_t|\nabla\theta|^2\frac{1}{|\nabla_h\theta|^2}D_g|\nabla_h\theta|^2$$

Putting these all together gives

$$\theta_t|\nabla\theta|^2 D_g|A|^2 + |A|^2\left[D_g\theta_t|\nabla\theta|^2 - \theta_t|\nabla\theta|^2\frac{1}{|\nabla_h\theta|^2}D_g|\nabla_h\theta|^2 + \theta_t|\nabla\theta|^2\nabla\cdot\mathbf{c}_g\right] = 0$$

Multiply everything by $1/\theta_t|\nabla\theta|^2|A|^2$, put in terms of logs, combine the terms, and undo the logs; we find

$$D_g\theta_t\frac{|\nabla\theta|^2|A|^2}{|\nabla_h\theta|^2} + \theta_t\frac{|\nabla\theta|^2|A|^2}{|\nabla_h\theta|^2}\nabla\cdot\mathbf{c}_g = 0$$

If we substitute the definition of energy from the lowest order relationships

$$\begin{aligned}\imath\theta_t\mathbf{u} &= -\imath\nabla\theta p + b\hat{\mathbf{z}} \\ \nabla\theta\cdot\mathbf{u} &= 0 \\ \imath\theta_t b + wN^2 &= 0\end{aligned}$$

giving

$$E = \frac{|\nabla\theta|^2}{|\nabla_h\theta|^2}|A|^2$$

Thus we arrive at the form

$$\frac{\partial}{\partial t}\theta_t E + \nabla\cdot(\mathbf{c}_g\theta_t E) = 0$$

Using $D_g\theta_t = 0$ (N^2 independent of time) tells us that the energy in the wave changes by divervences of the energy flux

$$\frac{\partial}{\partial t}E + \nabla\cdot(\mathbf{c}_g E) = 0$$

Obviously, this is a non-trivial process; for the internal gravity wave problem, we can take a simpler approach, which is to work directly from the equations of motion assuming all the variables have a WKB form:

$$b = b(\mathbf{x},t)e^{\imath\theta(\mathbf{x},t)} + b^*(\mathbf{x},t)e^{-\imath\theta(\mathbf{x},t)}$$

Then the equations become

$$\imath\theta_t\mathbf{u} + \frac{\partial}{\partial t}\mathbf{u} = -\imath P\nabla\theta - \nabla P + b\hat{\mathbf{z}}$$

$$\imath\nabla\theta\cdot\mathbf{u} + \nabla\cdot\mathbf{u} = 0$$

$$\imath\theta_t b + \frac{\partial}{\partial t}b + wN^2 = 0$$

The kinetic energy, averaged over the rapidly varying phase is

$$\frac{1}{2}\langle(\mathbf{u}e^{\imath\theta(\mathbf{x},t)} + \mathbf{u}^*e^{-\imath\theta(\mathbf{x},t)})^2\rangle = \frac{1}{2}\langle\mathbf{u}\cdot\mathbf{u}e^{2\imath\theta(\mathbf{x},t)} + 2\mathbf{u}\cdot\mathbf{u}^* + \mathbf{u}^*\cdot\mathbf{u}^*e^{-2\imath\theta(\mathbf{x},t)})^2\rangle = \mathbf{u}\cdot\mathbf{u}^*$$

so that we can form the KE eqn. by dotting the first equation with $\mathbf{u}^*$ and adding the conjugate

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{u}\cdot\mathbf{u}^* &= -\imath(P\mathbf{u}^*\cdot\nabla\theta - P^*\mathbf{u}\cdot\nabla\theta) - \mathbf{u}^*\cdot\nabla P - \mathbf{u}\cdot\nabla P^* + w^*b + wb^* \\ &= -P\cdot\nabla\mathbf{u}^* - P^*\nabla\cdot\mathbf{u} - \mathbf{u}^*\cdot\nabla P - \mathbf{u}\cdot\nabla P^* + w^*b + wb^* \\ &= -\nabla\cdot(\mathbf{u}P^* + \mathbf{u}^*P) + w^*b + wb^*\end{aligned}$$

(using the continuity equation). Multiplying the buoyancy equation by b^* and adding the conjugate gives

$$\frac{\partial}{\partial t}bb^* + (wb^* + w^*b)N^2 = 0$$

or, in terms of the available potential energy bb^*/N^2 (assuming N^2 is time-independent),

$$\frac{\partial}{\partial t}\frac{bb^*}{N^2} + wb^* + w^*b = 0$$

From these, we find

$$\frac{\partial}{\partial t}E = -\nabla\cdot(\mathbf{u}P^* + \mathbf{u}^*P)$$

or, using the equipartition of energy at the lowest order (all that's now needed)

$$\frac{\partial}{\partial t}E = -\nabla\cdot(\mathbf{u}P^* + \mathbf{u}^*P) = -\nabla\cdot(\mathbf{c}_g E)$$