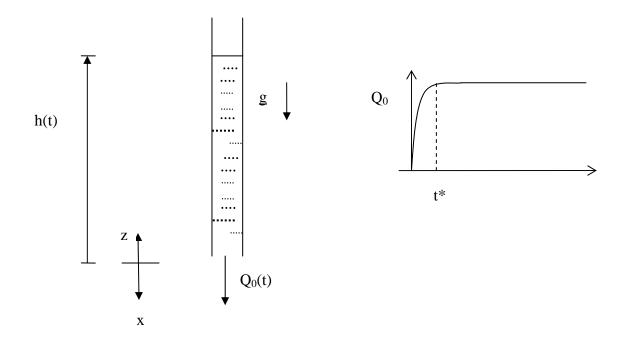# 10.52
Mechanics of Fluids
Spring 2006
Problem Set 5

Problem 1 (Problem adapted from Ain Sonin and Ascher Shapiro. Used with permission.)

Consider a perfect gas ($P=\rho RT$) which flows from one reservoir at pressure P_1 to another at pressure P_2 via a long thin uniform capillary tubing. The gas is Newtonian and the pressure ratio, P_1/P_2, is substantially greater than unity. The entire apparatus exchanges heat readily with the ambient and the system is, therefore, essentially isothermal.

a)   Assume that inertia may be neglected and derive an equation for the mass flow rate in terms of the pressures, the tubing length and radius, the viscosity of the gas, etc.

b)   State the quantitative criteria which must be satisfied for the above assumption to be valid.

c)   If the gas is air at 20°C flowing in a tube with a diameter of 2mm and length of 5m, and if P_1 and P_2 are 10^5Pa and 10^3Pa, respectively, what is the mass flow rate of air? Is the criterion developed in Part (b) satisfied?

Problem 2 (Problem by Ain Sonin and Ascher Shapiro. Used with permission.)

h(t)

g

Q_0

t*

z

$Q_0(t)$

x

A piece of glass tubing is drawn out into a long capillary of constant cross-sectional area as shown in the sketch. The tube is filled with liquid while held vertically, and a finger is placed over the top. When the finger is removed, the liquid begins to flow out the bottom of the tube (x = 0 z = 0; see sketch).

In all that follows, it is agreed to ignore the effects of surface tension.

It is found experimentally that the volume discharge at exit, $Q_0(t)$, varies with time as in the sketch above.

   (a) Explain qualitatively the physical phenomena which account for the shape of the curve $Q_0(t)$.

   (b) Estimate the <u>order of magnitude</u> of the time t*, and explain the basis of your estimate.

   (c) Assume that, for times well in excess of t*, the flow is essentially inertia-free. How long will it take to drain the tube?

   (d) Will your result be applicable if the fluid is glycerin ($\rho = 1260$ kg/m^3, $\nu = 0.6$ x 10^{-3} m^2/s) and if $h_0 = 0.5$ m and R = 1mm? Would the analysis apply if the fluid were water?

Problem 3

Consider a rigid sphere which is fully immersed in an incompressible Newtonian fluid. The sphere is very slowly pushed toward a rigid plane boundary with a constant velocity U. At time zero, the sphere is close to the boundary in the sense that the distance between the sphere and the boundary is very small compared to the radius of the sphere.
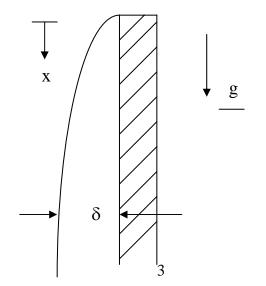
Develop an approximate expression for the force which must be applied to the sphere as a function of time.

Note: Assume that inertia may be neglected and develop a criterion which must be satisfied if this is to be true.

Problem 4

A planar sheet of material is to be coated by dipping it into a bath of Newtonian liquid. The sheet is then hung in a vertical orientation and the film is allowed to drain. This will not lead to a film of uniform thickness. The situation is instead believed to be more or less as shown in the sketch. Because the film is thin ($\delta/x \ll 1$), it is believed that inertial forces may be neglected. Surface tension effects may also be neglected.

A. What is the partial differential equation which describes the evolution of the film thickness as a function of x and t?
B. Use order-of-magnitude techniques to obtain a solution that should be exact at very long times.
C. In the context of the solution obtained in part (B), what criterion must be satisfied if inertial forces are to be truly negligible? Express your answer in terms of variables that can be specified *a priori* such as x, t, g, ρ and μ. Do not express your answer in terms of dependent variables such as u and δ.
D. What criterion must be satisfied if time is to be "long"? Assume that the layer is initially of uniform thickness, δ_i.
E. If, for the fluid in question, $\rho = 1$ gm/cm^3 and $\upsilon = 10$ cm^2/s, will the solution be valid at t = 10s and x = 10 cm? At t = 100s and x = 10 cm? In both cases, $\delta_i = 1$ mm.

Problem 5

1. A thin "thread" of fluid issues from a tube as shown in the sketch. The fluid is
    Newtonian and incompressible. The flow is believed to be viscous dominated. In
    addition, $dR/dz \ll 1$. The roles of surface tension, of gravity, of air drag, of inertia,
    etc., may all be neglected. However, please note that the filament is "pulled" with a
    force F. Except in Part E, v_z may be assumed to be independent of r.

    A. It is believed that τ_{rz} must be zero for $z > 0$ and σ_{rr} must be continuous across the
        interface. Show that:

        $$F = \pi\sigma_{zz}R^2 = \text{constant} \tag{1}$$
        and
        $$q = \pi v_z R^2 = \text{constant} \tag{2}$$

    B. Note that

        $$\sigma_{zz} = -p + 2\mu\frac{\partial v_z}{\partial z} \tag{3}$$

        and

        $$\sigma_{rr} = -p + 2\mu\frac{\partial v_r}{\partial r} = -p_{ambient} = 0 \tag{4}$$

        Furthermore, continuity (see eq. 9) requires $v_r = -\dfrac{r}{2}\dfrac{\partial v_z}{\partial z}$, or

        $$\frac{\partial v_r}{\partial r} = \frac{v_r}{r} = -\frac{1}{2}\frac{\partial v_z}{\partial z} \tag{5}$$

        Thus

        $$\sigma_{zz} = +3\mu\frac{\partial v_z}{\partial z} \tag{6}$$

        Please obtain v_z and R as functions of F, q, z, etc. Set $R = R_0$ at $z = z_0$.
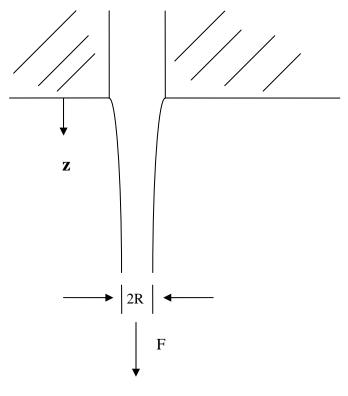
    C. Please provide a criterion which must be satisfied if the assumption of viscous
        dominated flow is to be valid.
    D. How does the rate of momentum flow behave as $z \to \infty$? What are the physical
        consequences of this observation?
    E. Equations (4) and (5) may be combined to give

        $$-p - \mu\frac{\partial v_z}{\partial z} = 0 \tag{7}$$

        which is not consistent with an obvious simplification of eq. (11), i.e.

$$0 = -\frac{\partial p}{\partial z} + \mu \frac{\partial^2 v_z}{\partial z^2} \qquad (8)$$

Is it possible that eq. (7) and eq. (11) can be reconciled if v_z is a function of radial position, r? Would this negate all of Parts A to D? If you believe that v_z is actually a function of r, please give an estimate for the magnitude of the variation in terms of dR/dz, etc.
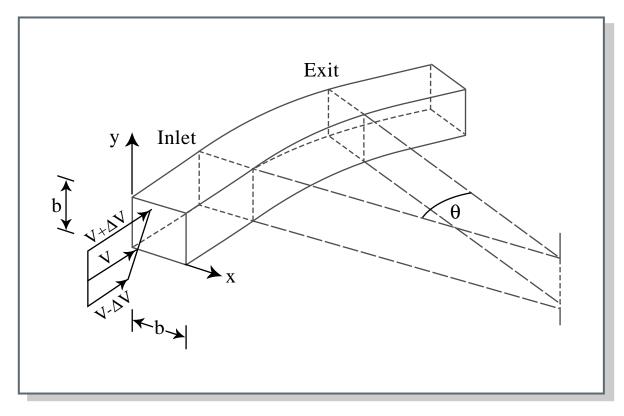
z

|2R|

F

Notes:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z} = 0 \qquad (9)$$

$$v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \upsilon \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{\partial^2 v_r}{\partial z^2} \right] \qquad (10)$$

$$v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \upsilon \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{\partial^2 v_z}{\partial z^2} \right] \qquad (11)$$

Problem 6 (Problem by Ain Sonin and Ascher Shapiro. Used with permission.)

A flow in a square duct of width b passes through a very small angle of turn θ, such that sin $\theta = \theta$; the bend's radius of curvature is large compared with b. At the inlet to the turn, the flow velocity is independent of x, but has a linear velocity gradient in the y-direction, such that there is a very small excess of velocity ΔV at the top of the duct, and a corresponding defect ΔV at the bottom, the mean velocity being V.

The flow is incompressible, steady, and non-viscous, and gravitational effects are negligible.

(a) Show that a streamwise component of vorticity develops in the bend, and that the ratio of the streamwise vorticity at exit (the "secondary vorticity") to the original vorticity is approximately equal to 2θ. To an observer looking downstream, is this secondary vorticity clockwise or counter-clockwise?

(b) Show that the average transverse velocity at the bend exit is of the order of $\theta\Delta V$.