VNAV Lecture 19: Welcome !!

* Non linear least squares on Lie Groups. (Riem. Manifolds / Manifolds)

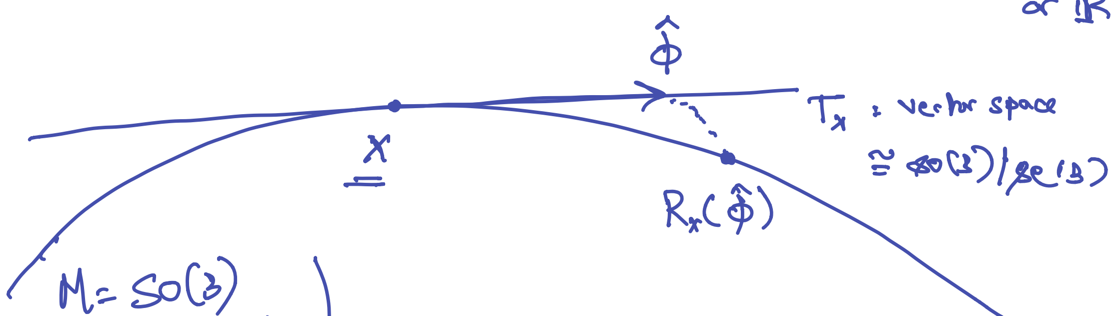
* Product Manifolds

* Examples.

* Step Sizes. & choosing λ_t in LM.

$\boxed{\text{Recall}}$

$\Sigma = \mathbb{R}^{3\times3}$ or $\mathbb{R}^{4\times4}$.



$M = SO(3)$ or $SE(3)$

Retraction $R_x(\hat{\phi})$

$$R_x : T_x \longrightarrow M$$

s.t. $R_x(0) = X$.

$$R_x(\hat{\phi}) = X \cdot \underline{\exp}(\hat{\phi}).$$

$T_x \;\; (= so(3) \text{ or } se(3))$

$$= \left\{ \hat{\phi} = \sum_{i=1}^{K} \phi_i G_i \;\middle|\; \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_K \end{pmatrix} \in \mathbb{R}^K \right\}.$$

$K = 3$ for $so(3)$

$K = 6$ for $se(3)$.

→ Least Sq.

$$\min_{x \in M} \|\underline{r(x)}\|^2 \quad \text{— (i)}$$

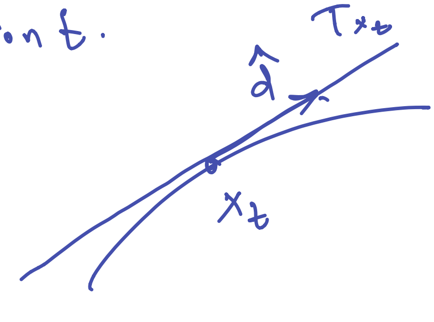
$M \subseteq \mathbb{R}^n$

$r : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

→ Iterative Algo. Gauss-Newton or LM to solve (i).

→ At $x_t \in M$ at iteration t.

$\underline{\hat{d}} \in T_{x_t}$



$x_{t+1} = R_{x_t}(\hat{d})$.

$R_{x_t} = x_t \cdot \exp(\hat{d})$

$$\min_{\underline{\underline{d \in \mathbb{R}^K}}} \| \underbrace{r(R_{x_t}(\hat{d}))}_{\text{Nonlinear fun.}} \|^2$$

$r : \mathbb{R}^n = \mathcal{E} \longrightarrow \mathbb{R}^m$

linearize at $d = 0$

$r(R_{x_t}(\hat{\cdot})) : \mathbb{R}^K \longrightarrow \mathbb{R}^m$.

$$r(R_{x_t}(\hat{d})) = r(x_t \cdot \exp(\hat{d}))$$

$$\approx \underline{\underline{r(x_t) + J(x_t)\, d}}$$

$$\underline{\underline{J(x_t) = \frac{\partial}{\partial d} r(x_t \exp(\hat{d}))\Big|_{d=0}}}. \longleftarrow \circledast$$

At iteration t:

$$\min_{d \in \mathbb{R}^K} \| r(x_t) + J(x_t)\, d \|^2$$

Use: GN, LM.

To Get d_t.

Update: $x_{t+1} = R_{x_t}(\hat{d}_t) = x_t \cdot \exp(\hat{d}_t)$. $\boxtimes$

Computing Jacobian:

$$J(x) = \frac{\partial}{\partial d} r(x \cdot \exp(\hat{d})) \Big|_{d=0}.$$

Tips: ① $d \approx 0 \longrightarrow \exp(\hat{d}) \approx I + \hat{d}$

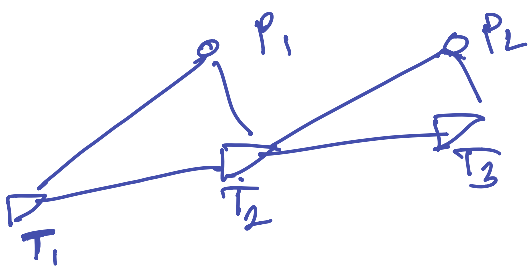
$x + x\hat{d}$.

② $\hat{d}$ · (in $se(3)$ or $so(3)$)

$$\hat{d} = \sum_{i=1}^{K} d_i G_i .$$

$\to \mathcal{M} = SO(3)$ or $SE(3)$.

$\mathbb{R}^2$ or $\mathbb{R}^3$ or $\mathbb{R}^d$.



$P_1, P_2 \in \mathbb{R}^2$ or $\mathbb{R}^3$

$T_1, T_2, T_3 \in SE(3)$

Product Manifolds:

$\{M_i\}_{i=1}^{L}$. $\quad M_i =$ Lie Group.

$$\mathcal{M} = M_1 \times M_2 \times \ldots \times M_L . \Big]$$ Manifold Lie Group?

① $\mathcal{M}$ embedded in some $\mathcal{E}$.

② Tangent space T_x for every $x \in \mathcal{M}$

③ Retraction function $R_x : T_x \to \mathcal{M}$.

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① $\{M_i\}_{i=1}^{L}$. $\quad M_i \subseteq \mathbb{R}^{n_i}$.

$$\mathcal{M} = \bigtimes_{i=1}^{L} M_i \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_L}$$
$$=: \mathcal{E}.$$

② Tangent space:

$$\forall\, x_i \in M_i \quad \exists \quad T_{x_i}$$

$$X \in M \longrightarrow X = (x_1, x_2, \ldots x_L)$$

tangent space for X $\qquad T_X = T_{x_1} \times T_{x_2} \times \cdots \times T_{x_L}$

③ Retraction:

$$R_{x_i} : T_{x_i} \longrightarrow M_i$$

$$R_{(x_1 \ldots x_L)}((\hat{\phi}_1 \ldots \hat{\phi}_L)) = \{ R_{x_1}(\hat{\phi}_1), R_{x_2}(\hat{\phi}_2), \ldots R_{x_L}(\hat{\phi}_L) \}.$$

$X, Y \in M \qquad X \circ Y$ | $SO(3)$ $SE(3)$ $\quad \circ \Leftarrow$ Matrix Mult.

$X, Y \in M$

Product Manifold:

$$\overbrace{(x_1, x_2, \ldots x_L)}^{X} \circ \overbrace{(Y_1, Y_2, \ldots Y_L)}^{Y}$$

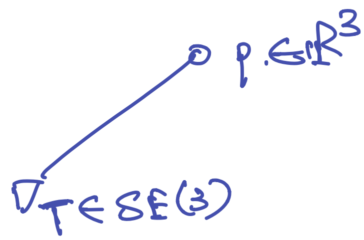
$$= (\underset{\substack{M_1 \\ \| \\ SO(3) \\ X_1 Y_1}}{X_1 \circ Y_1},\ \underset{M_2}{X_2 \circ Y_2}, \ldots \underset{\substack{M_L \\ \| \\ \mathbb{R}^3 \\ X_L + Y_L.}}{X_L \circ Y_L})$$

$\rightarrow$ For any $\{M_i\}_{i=1}^{L}$. M_i are Lie Groups

then so is $M = \overset{L}{\underset{i=1}{\times}} M_i$.

Example 1:

$$M = SE(3) \times \mathbb{R}^3$$



$$r(T, p) \qquad \begin{array}{l} T \in SE(3) \\ p \in \mathbb{R}^3. \end{array}$$

$$\underset{(T,p) \in M}{\text{MIN}} \quad \| r(T, p) \|^2$$

At (T_t, p_t) at iteration t.

$$\left.\begin{array}{l} d_1 \in \mathbb{R}^6 \Rightarrow \hat{d}_1 \in se(3) \\ d_2 \in \mathbb{R}^3 \Rightarrow \hat{d}_2 \in \mathbb{R}^3 \end{array} \right| \; R_{(T_t, p_t)}(\hat{d}_1, \hat{d}_2)$$

Linearizing $r(\cdot)$:

$$r\left(R_{(T_t,p_t)}(\hat{d}_1, \hat{d}_2)\right)$$

$$= r\left(R_{T_t}(\hat{d}_1), \; R_{p_t}(\hat{d}_2)\right)$$

$$= r\left(T_t \exp(\hat{d}_1), \; p_t + \hat{d}_2\right) \qquad \hat{d}_2 = d_2.$$

$$r(T_t \exp(\hat{d}_1), p_t + \hat{d}_2)$$

$$\approx \underline{\underline{r(T_t, p_t) + J_{1t} d_1 + J_{2t} d_2}}$$

where

$$J_{1t} = \frac{\partial}{\partial d_1} r(T_t \exp(\hat{d}_1), p_t) \Big|_{d_1 = 0}$$

$$J_{2t} = \frac{\partial}{\partial d_2} r(T_t, p_t + d_2) \Big|_{d_2 = 0} = \frac{\partial}{\partial p} r(T_t, p) \Big|_{p = p_t}$$

Solve:

$$\underset{\substack{d_1 \in \mathbb{R}^6 \\ d_2 \in \mathbb{R}^3}}{\mathrm{MIN}} \| r(T_t, p_t) + J_{1t} d_1 + J_{2t} d_2 \|$$

to get $\hat{d}_{1t}$ & $\hat{d}_{2t}$.

Addition / Substraction on Lie Groups:

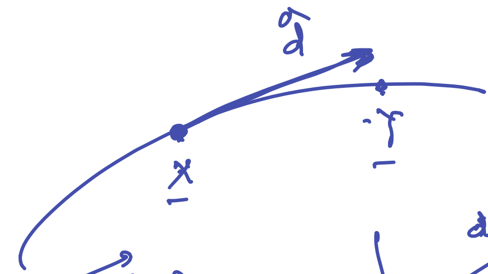
→ $R_x(\hat{d})$ $R_x : T_x \longrightarrow M$.

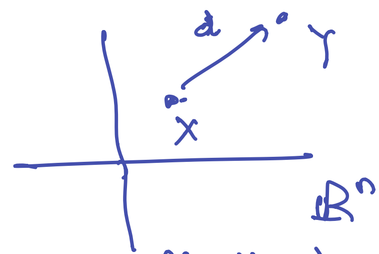
Acts as an $\oplus$

$$X \oplus \hat{d} = R_x(\hat{d})$$

$$= Y.$$

this picture.





$Y = X + d$

$d \notin M$.

→ What should $\hat{d}$ be so that you go from X to Y?

① $\hat{d} \in T_x$.

for $X, Y \in M$

$$Y \ominus X = \log(X^{-1} Y) =$$

$$= \hat{d} \in T_x.$$

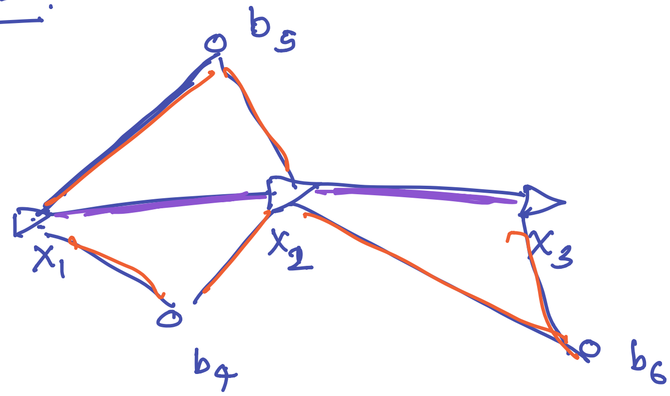
Ex:

- $\hat{d} = Y - X$
- R_x, R_y.

$\log(R_x^{-1} R_y) \in SO(3)$ $\mathfrak{so}(3)$

$\in T_x$.

Example 2:



$x_i \in SE(2)$

$b_i \in \mathbb{R}^2$

$$u_{12} = (x_2 \ominus x_1) + w_{12}$$

$$u_{ij} = (x_j \ominus x_i) + w_{ij} \quad \text{for } (i,j) = (1,2), (2,3)$$

$$y_{ik} = x_i^{-1} b_k + n_{ik} \quad \text{for } (i,k) = (1,5), (1,4), (2,6), (2,4), (2,5)$$

$w_{ij} \in \mathbb{R}^3$.

$w_{ij} \sim \mathcal{N}(0, \Sigma_{ij})$

$\underline{\Sigma_{ij} \to 3\times3}$.

$n_{14} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} \sim \mathcal{N}(0, \Sigma')$

in $\mathbb{R}^3$

$$\text{MIN} \sum_{(i,j)} \| r_{ij}(X) \|^2 + \sum_{(i,k)} \| r_{ik}(X) \|^2$$

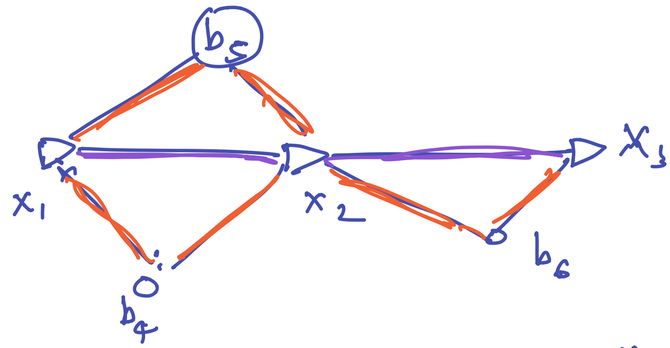
(purple links)

orange links

$X = (x_1, x_2, x_3, b_4, b_5, b_6)$

$r_{12} \leftrightarrow x_1, x_2$

$r_{14} \to x_1, b_4$



$$r = \begin{bmatrix} r_{12} \\ r_{23} \\ r_{14} \\ r_{15} \\ r_{25} \\ r_{26} \\ r_{36} \end{bmatrix}, \qquad J_t = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & b_4 & b_5 & b_6 \end{array} \\ \begin{bmatrix} J^{r_{12}}_{x_1} & J^{r_{12}}_{x_2} & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & 0 & * & 0 \\ 0 & * & 0 & 0 & * & 0 \\ 0 & * & 0 & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 & * \end{bmatrix} \end{array}$$

$\underline{16 \times 15}$

Sparse.

$\boxed{x_t}$

$$d_t = -(J_t^T J_t)^{-1} J_t^T r_t .$$

$$x_{t+1} = \underline{R_{x_t}(d_t)}.$$

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① Choosing Step α_t :

$$x_{t+1} \leftarrow x_t + \alpha_t d_t$$

In practice: $\alpha_t = 1$.

Chap 3 : "Line Search Methods" α_t.
Numerical Opt.

Check out: (a) Armijo's rule

(b) Wolfe's condition.

$\rightarrow$ Convergence theorems in Chap 10, assume such a choice of step sizes α_t.

(2) How to choose λ_t in LM.

- Get $d_t \leftarrow \underset{d}{\operatorname{argmin}} \|J_t d + r_t\|^2 + \lambda_t \|d\|$

(LM method)

- Set $\hat{x} \leftarrow x_t + d_t$.

- If $\|r(\hat{x})\|^2 < \|r(x_t)\|^2$ [If nonlinear least square cost decreases] $\Rightarrow$ We are getting closer to a local optima. Need to discount out the added term $\lambda_t \|d\|^2$.

then $x_{t+1} \leftarrow \hat{x}$

& $\lambda_{t+1} \leftarrow \beta_1 \lambda_t$ for $(0 < \beta_1 < 1)$

- If $\|r(\hat{x})\|^2 \geq \|r(x_t)\|^2$ $\Rightarrow$ linear approximation is bad. We need to increase the regularizer $\lambda_t \|d\|^2$.

then $x_{t+1} \leftarrow x_t$

$\lambda_{t+1} \leftarrow \beta_2 \lambda_t$ for $(\beta_2 > 1)$