VNAV Lecture 18

Welcome!!

Previous Lecture:

1. Non linear least squares problem

$$\min_{x \in \mathbb{R}^n} \|r(x)\|^2 \qquad r : \mathbb{R}^n \to \mathbb{R}^m$$

2. Linear case: $r(x) = Ax - b$

   Normal equations: $A^T A x = A^T b$

   Various solvers: QR, Cholesky

3. Gauss Newton Method.

Today:

1. Levenberg-Marquardt method (LM)
2. Optimization on Riemannian Manifolds

   $SO(3)$, $SE(3)$.

Recall GN:

$$\|r(x)\|^2 \qquad J(x) = \begin{bmatrix} - \nabla r_1(x)^T - \\ - \nabla r_2(x)^T - \\ \vdots \\ - \nabla r_m(x)^T - \end{bmatrix}$$

$$x_{t+1} = x_t - \alpha_t \left(J(x_t)^T J(x_t) \right)^{-1} J(x_t)^T r(x_t).$$

Q: What if $J^T J$ is not invertible?

Side: We didn't really take an inverse.
$J^T J d = -J^T r$.

Easy fix:

$$x_{t+1} = x_t - \alpha_t \left(J(x_t)^T J(x_t) + \underline{\underline{\lambda I_n}} \right)^{-1} J(x_t)^T r(x_t).$$

$\lambda > 0$.

Levenberg-Marquardt Method.

Q: What does it do?

$$(J^T J + \lambda I_n) d = -J^T r \quad \text{— (1)}$$

$$\Updownarrow$$

$$\begin{pmatrix} J \\ \sqrt{\lambda} I_n \end{pmatrix}^T \begin{pmatrix} J \\ \sqrt{\lambda} I_n \end{pmatrix} d = - \begin{pmatrix} J \\ \sqrt{\lambda} I_n \end{pmatrix}^T \begin{pmatrix} r \\ 0 \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} J \\ \sqrt{\lambda} I_n \end{pmatrix}^T \left[\begin{pmatrix} J \\ \sqrt{\lambda} I_n \end{pmatrix} d + \begin{pmatrix} r \\ 0 \end{pmatrix} \right] = 0.$$

$$\frac{\partial}{\partial d}\Big[\underbrace{\Big\| \begin{pmatrix} J \\ \sqrt{\lambda} I_n \end{pmatrix} d + \begin{pmatrix} r \\ 0 \end{pmatrix} \Big\|^2}_{f(d).}\Big] = 0. \quad (*)$$

Note: $f(d)$ is Convex!!

$\therefore$ $(*)$ is necc. suff. for optimality.

(i) Chooses d

$$= \underset{d \in \mathbb{R}^n}{\operatorname{argmin}} \Big\| \begin{pmatrix} J \\ \sqrt{\lambda} I_n \end{pmatrix} d + \begin{pmatrix} r \\ 0 \end{pmatrix} \Big\|^2$$

$$d_{LM} = \underset{d}{\operatorname{argmin}} \; \underbrace{\| Jd + r \|^2}_{\text{Linear Approx.}} + \underbrace{\lambda \|d\|^2}_{\text{Regularizer!!}}.$$

$$d_{GN} = \underset{d}{\operatorname{argmin}} \; \| Jd + r \|^2.$$

$\boxed{\text{Riemannian Optimization}}$:

→ We saw optimization over Euclidean spaces.

NLLS. — GD
GN
LM

→ $SO(3)$, $SE(3)$: Lie Groups = Riemannian Manifolds + Group Structure

R_i T_i

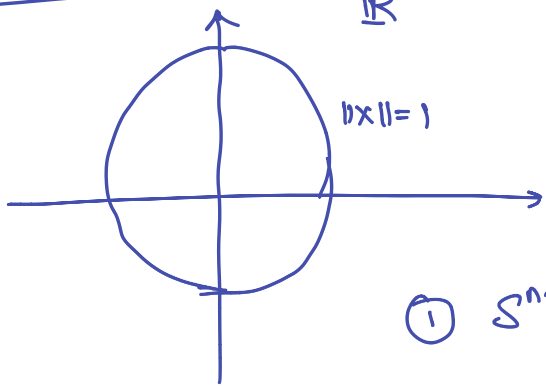
Lie Algebras : $\mathfrak{so}(3)$, $\mathfrak{se}(3)$.
(vector space)

$$\text{Exp} : \mathfrak{so}(3)/\mathfrak{se}(3) \longrightarrow SO(3)/SE(3).$$

$$\mathcal{M} = \text{Manifold} \in \{SO(3), SE(3)\}.$$

$\boxed{\text{SIMPLE EXAMPLE}}$

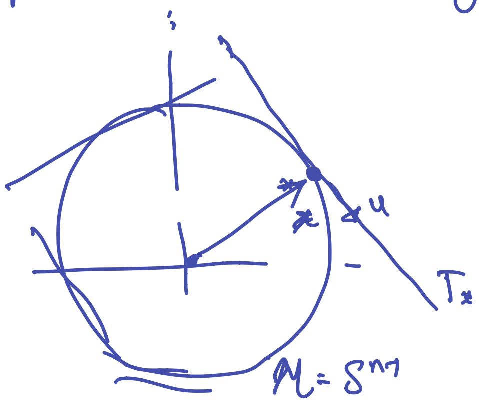
$\mathbb{R}^n$



$S^{n-1} = \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$.

① $S^{n-1} \subset \mathbb{R}^n$ — Embedded Manifolds in Euclidean space $\mathbb{R}^n$.

② for every $x \in S^{n-1}$, $\exists$ a tangent space T_x

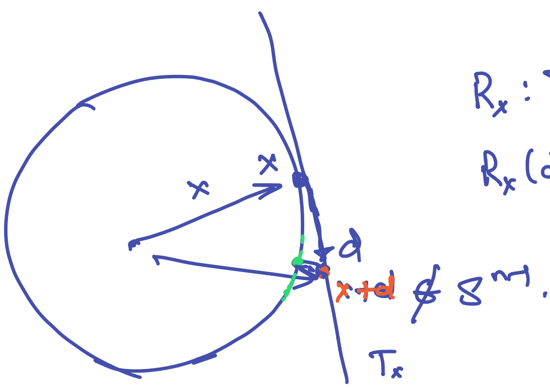


$T_x = \{ u \in \mathbb{R}^n \mid u^T x = 0 \}$.

= vector space structure.

$\dim(T_x) = \ ?$

③



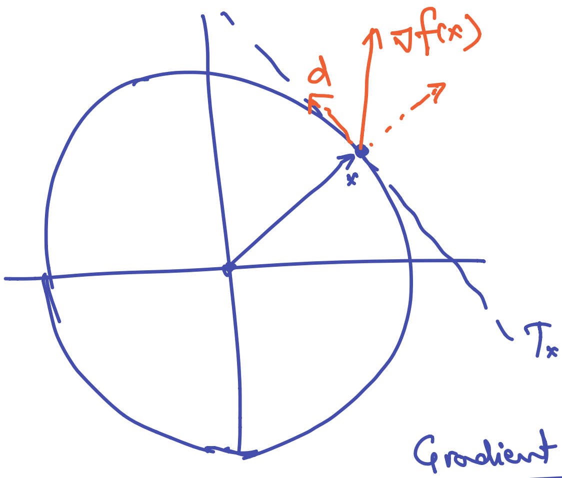
$R_x : T_x \longrightarrow \mathcal{M}$.

$R_x(d) =$ going in the direction d along $\mathcal{M}$

For S^{n-1}: $\quad R_x(d) = \dfrac{x+d}{\|x+d\|}$.

$$\min_{x \in S^{n-1}} f(x)$$

FIRST RIEM-OPT.
GD.



$\text{grad } f(x)$

= unique projection of $\nabla f(x)$ on T_x.

$= \nabla f(x) - (x^T \nabla f(x))\, x$.

Gradient Descent:

- go along $d = -\alpha_t \text{grad } f(x_t)$. when at x_t.

$\rightarrow \quad x_{t+1} = R_{x_t}(d)$

$= R_{x_t}(-\alpha_t \text{ grad } f(x_t))$.

GD-Algo. on RMOpt.

$\underline{\mathcal{M} = SO(3), SE(3):}$

① $SO(3) = \left\{ R \in \mathbb{R}^{3\times 3} \;\middle|\; \begin{matrix} R^T R = I \\ \det(R) = 1 \end{matrix} \right\}$

$\subseteq \mathbb{R}^{3\times 3} = \mathcal{E}.$

$SE(3) = \left\{ \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \;\middle|\; \begin{matrix} R \in SO(3) \\ t \in \mathbb{R}^3 \end{matrix} \right\}.$

$\subseteq \mathbb{R}^{4\times 4} = \mathcal{E}$ Embedded Manifold.

② Tangent space:

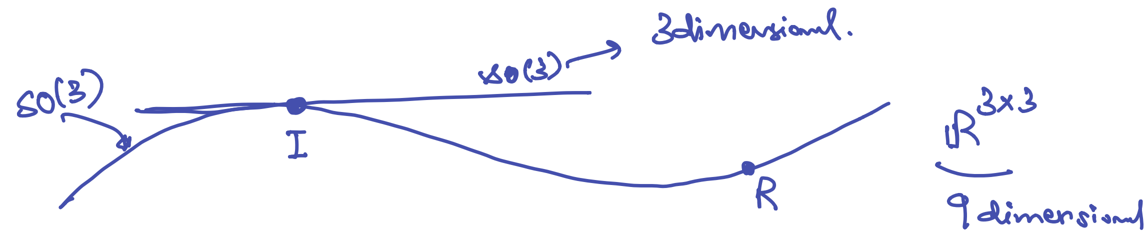
$\mathfrak{so}(3) = \left\{ \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & \phi_1 \\ -\phi_2 & -\phi_1 & 0 \end{bmatrix} \;\middle|\; \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \in \mathbb{R}^3 \right\}.$

= Vectorspace.

$= \left\{ \phi_1 G_1 + \phi_2 G_2 + \phi_3 G_3 \;\middle|\; \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \in \mathbb{R}^3 \right\}.$

$G_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dots$

$\mathfrak{so}(3)$ is a tangent space of $SO(3)$ at I_3



$$\mathfrak{se}(3) = \left\{ \sum_{i=1}^{6} \phi_i G_i \;\middle|\; \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_6 \end{pmatrix} \in \mathbb{R}^6 \right\}$$

$\rightarrow$ $\mathfrak{se}(3)$ is a tangent space for $SE(3)$ at $I_4 \in SE(3)$.

③ <u>Retraction</u>:

$$R_x(\hat{\phi})$$

$$= x \, \mathrm{Exp}(\hat{\phi})$$

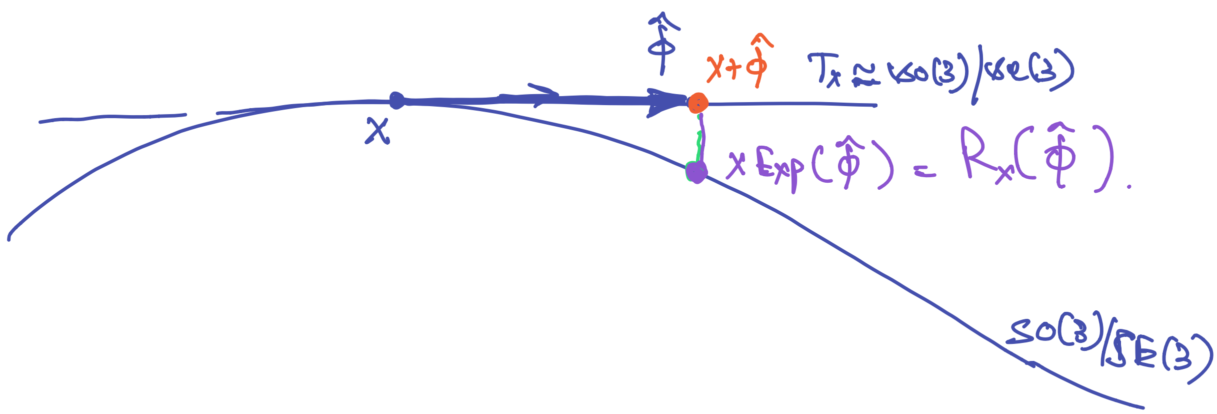
$\mathcal{M} = SO(3) / SE(3)$

$\phi \in \mathbb{R}^3 / \mathbb{R}^6$

$\hat{\phi} \in \mathfrak{so}(3) / \mathfrak{se}(3)$

$x \in SO(3) / SE(3)$

$\mathcal{E}$



Problem:

$$\min_{x \in \mathcal{M}} \|r(x)\|^2$$

$\mathcal{M} \subseteq \mathbb{R}^n$. $r : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

① "Linear case": $r(x) = Ax - b$.

$$\min_{x \in \mathcal{M}} \|Ax - b\|^2$$

$A^T A x = A^T b$?

$$x = x_0 \operatorname{Exp}(\hat{d})$$

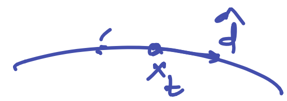
$$\min_{d \in \mathbb{R}^4 / \mathbb{R}^6} \|A x_0 \operatorname{Exp}(\hat{d}) - b\|^2.$$

unconstrained.

No longer linear

② General case:

$$\min_{x \in M} \|r(x)\|^2$$



At x_t. $x = x_t \operatorname{Exp}(\hat{d})$

$$\min_{d \in \mathbb{R}^3 / \mathbb{R}^6} \|\underline{r(x_t \operatorname{Exp}(\hat{d}))}\|^2$$

$\boxed{\text{Gauss Newton \& LM for NLLS on Manifolds}}$

linearize this near $d=0$.

$r(x_t) + J(x_t)\, d$

$d_t \xleftarrow{\text{get}}$

update

$x_{t+1} = x_t \operatorname{Exp}(\hat{d}_t)$

On linearizing $r(x_t \operatorname{Exp}(\hat{d}))$ at $d=0$.

$$r(x_t \operatorname{Exp}(\hat{d})) = r(x_t) + \underbrace{\left[\frac{\partial}{\partial d} r(x_t \operatorname{Exp}(\hat{d}))\right]\bigg|_{d=0}}_{J(x_t)} d$$

Tips to compute $J(x_t)$

① Near $d=0$: $\operatorname{Exp}(\hat{d}) \approx I + \hat{d}$.

② $\hat{d} = \sum_{i=1}^{k} d_i G_i$.

16.485 Visual Navigation for Autonomous Vehicles (VNAV)
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