Single-variable Calculus Problems (and some solutions, too!)

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Abstract

This is a compilation of a lot of quizzes, tests, and exams, and many of their solutions, from some of the calculus courses taught by the author at Trent University. Typos and other errors have been preserved for your enjoyment!

This compilation was was assembled with LaTeX, using the fullpage, hyperref, and graphicx packages. The original quizzes, tests, exams, and solutions collected in this work were included as pdf images, and were originally typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX. Various software, most frequently AppleWorks and Maple, was used to create the diagrams and graphs.

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Part I

Many Questions

Mathematics 110
Calculus I: Calculus of one variable
2001-2002

Mathematics 110 – Calculus of one variable
Trent University 2001-2002

QUIZZES

Quiz #1. Friday, 21 September, 2001. *[15 minutes]*

1. Sketch the graph of a function $f(x)$ with domain $(-1, 2)$ such that $\lim_{x\to 2} f(x) = 1$ but $\lim_{x\to -1} f(x)$ does not exist. *[4]*
2. Use the $\epsilon - \delta$ definition of limits to verify that $\lim_{x\to\pi} 3 = 3$. *[6]*

Quiz #2. Friday, 28 September, 2001. *[15 minutes]*

Evaluate the following limits, if they exist.

$$1. \quad \lim_{x\to -1} \frac{x+1}{x^2-1} \quad [5] \qquad 2. \quad \lim_{x\to 1} \frac{x+1}{x^2-1} \quad [5]$$

Quiz #3. Friday, 5 October, 2001. *[20 minutes]*

1. Is $g(x) = \begin{cases} \dfrac{x^2-6x+9}{x-3} & x \neq 3 \\ 0 & x = 3 \end{cases}$ continuous at $x = 3$? *[5]*
2. For which values of c does $\lim_{x\to\infty} \dfrac{13}{cx^2+41}$ exist? *[5]*

Quiz #3. *(Late version.)* Friday, 5 October, 2001. *[20 minutes]*

1. For which values of the constant c is the function $f(x) = \begin{cases} ce^x & x \geq 0 \\ c - x & x < 3 \end{cases}$ continuous at $x = 0$? *[5]*
2. Compute $\lim_{x\to\infty} \dfrac{(x+13)^2}{2x^2+\frac{1}{x}}$, if it exists. *[5]*

Quiz #4. Friday, 12 October, 2001. *[10 minutes]*

1. Use the definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{5}{7x}$. *[10]*

Quiz #5. Friday, 19 October, 2001. *[17 minutes]*

Compute $\dfrac{dy}{dx}$ for each of the following:

$$1. \quad y = \frac{2x+1}{x^2} \quad [3] \qquad 2. \quad y = \ln(\cos(x)) \quad [3] \qquad 3. \quad y = (x+1)^5 e^{-5x} \quad [4]$$

Quiz #6. Friday, 2 November, 2001. *[20 minutes]*

Find $\dfrac{dy}{dx}$ ...

1. ... at the point that $y = 3$ and $x = 1$ if $y^2 + xy + x = 13$. *[4]*
2. ... in terms of x if $e^{xy} = x$. *[3]*
3. ... in terms of x if $y = x^{3x}$. *[3]*

Quiz #7. Friday, 9 November, 2001. *[13 minutes]*

1. Find all the maxima and minima of $f(x) = x^2 e^{-x}$ on $(-\infty, \infty)$ and determine which are absolute. *[10]*

Quiz #8. Friday, 23 November, 2001. *[15 minutes]*

1. A spherical balloon is being inflated at a rate of 1 m^3/s. How is the diameter of the balloon changing at the instant that the radius of the balloon is 2 m? *[10]*
   [The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.]

Quiz #9. Friday, 30 November, 2001. *[20 minutes]*

1. Use the Right-hand Rule to compute $\int_0^3 \left(2x^2 + 1\right) dx$. *[6]*

   [You may need to know that $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$.]

2. Set up and evaluate the Riemann sum for $\int_0^2 (3x+1)dx$ corresponding to the partition $x_0 = 0$, $x_1 = \frac{2}{3}$, $x_2 = \frac{4}{3}$, $x_3 = 2$, with $x_1^* = \frac{1}{3}$, $x_2^* = 1$, and $x_3^* = \frac{5}{3}$. *[4]*

Quiz #9. *(Late version.)* Friday, 30 November, 2001. *[20 minutes]*

1. Use the Right-hand Rule to compute $\int_1^2 (x+1)dx$. *[6]*

   [You may need to know that $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$.]

2. Set up and evaluate the Riemann sum for $\int_0^4 x^2 dx$ corresponding to the partition $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$, with $x_1^* = 0$, $x_2^* = 2$, $x_3^* = 2$, and $x_4^* = 4$. *[4]*

Quiz #10. Friday, 7 December, 2001. *[20 minutes]*

Given that $\int_1^4 x\,dx = 7.5$ and $\int_1^4 x^2\,dx = 21$, use the properties of definite integrals to:

1. Evaluate $\int_1^4 (x+1)^2\,dx$. *[5]*

2. Find upper and lower bounds for $\int_1^4 x^{3/2}\,dx$. *[5]*

Quiz #10. *(Late version.)* Friday, 7 December, 2001. *[20 minutes]*

1. Without evaluating them, put the following definite integrals in order, from smallest to largest. *[5]*

$$\int_0^2 \sqrt{x^2+1}\,dx \qquad \int_0^1 x\,dx \qquad \int_0^1 \sqrt{x^2+1}\,dx \qquad \int_0^1 x^3\,dx \qquad \int_0^2 (x+3)\,dx$$

2. Write down (you need not evaluate it) a definite integral(s) representing the area of the region bounded by the curves $y = x - x^3$ and $y = x^3 - x$. *[5]*

Quiz #11. Friday, 11 January, 2002. *[15 minutes]*

1. Compute the indefinite integral $\int \left(x^2+x+1\right)^3 (4x+2)\,dx$. *[5]*
2. Find the area under the graph of $f(x)=\sin(x)\cos(x)$ for $0 \le x \le \frac{\pi}{2}$. *[5]*

Quiz #12. Friday, 18 January, 2002. *[15 minutes]*

1. Compute $\int_1^e \frac{\ln(x^2)}{x}\,dx$. *[5]*
2. Find the area of the region between the curves $y=x^3-x$ and $y=x-x^3$. *[5]*

Quiz #12. *(Late version.)* Friday, 18 January, 2002. *[15 minutes]*

1. Compute $\int_1^{\ln(2)} \frac{e^x}{e^{2x}+1}\,dx$. *[5]*
2. Find the area of the region bounded below by the curve $y=x^2-1$ and above by the curve $y=\cos\left(\frac{\pi}{2}x\right)$, where $-1 \le x \le 1$. *[5]*

Quiz #13. Friday, 25 January, 2002. *[19 minutes]*

1. Find the volume of the solid obtained by revolving the region in the first quadrant bounded by $y=\frac{1}{x}$, $y=x$, and $x=2$ about the x-axis. *[10]*

Quiz #14. Friday, 1 February, 2002. *[17 minutes]*

1. Suppose the region bounded above by $y=1$ and below by $y=x^2$ is revolved about the line $x=2$. Sketch the resulting solid and find its volume. *[10]*

Quiz #15. Friday, 15 February, 2002. *[25 minutes]*

Evaluate each of the following integrals.

$$1. \quad \int_0^{\pi/4} \tan^2(x)\,dx \quad [4] \qquad 2. \quad \int \sqrt{x^2+4x+5}\,dx \quad [6]$$

Quiz #16. Friday, 1 March, 2002. *[25 minutes]*

1. Evaluate the following integral:

$$\int \frac{x^2-2x-6}{\left(x^2+2x+5\right)(x-1)}\,dx$$

Quiz #17. Friday, 8 March, 2002. *[25 minutes]*

1. Evaluate the following integral:

$$\int_2^\infty \frac{1}{x(x-1)^2}\,dx$$

Quiz #18. Friday, 15 March, 2002. *[18 minutes]*

Determine whether each of the following series converges or diverges.

$$1. \quad \sum_{n=0}^{\infty}\left[\frac{1}{n+1}+\frac{3^n}{3^n+1}\right] \quad [4] \qquad 2. \quad \sum_{n=0}^{\infty}\frac{253}{3^n+1} \quad [6]$$

Bonus Quiz. Monday, 18 March, 2002. *[15 minutes]*

Compute any *two* of 1–3.

1. $\lim_{t \to \infty} te^{-t}$ *[5]* 2. $\int_0^\infty te^{-t}\,dt$ *[5]* 3. $\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}$ *[5]*

Quiz #19. Friday, 22 March, 2002. *[20 minutes]*

Determine whether each of the following series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{1}{n^n}$ *[5]* 2. $\sum_{n=0}^{\infty} \frac{4n+12}{n^2+6n+13}$ *[5]*

Quiz #19. *(Alternate version.)* Friday, 22 March, 2002. *[20 minutes]*

Consider the series

$$\sum_{n=0}^{\infty} \frac{\arctan(n+1)}{n^2+2n+2}$$

Determine whether this series converges or diverges using:

1. The Comparison Test. *[5]*
2. The Integral Test. *[5]*

Quiz #20. Tuesday, 2 April, 2002. *[10 minutes]*

1. Determine whether the series $\sum_{n=0}^{\infty} \frac{(-1)^n + \cos(n\pi)}{n+1}$ converges absolutely, converges conditionally, or diverges. *[10]*

Quiz #20. *(Alternate version.)* Tuesday, 2 April, 2002. *[10 minutes]*

1. Determine whether the series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} + \sin\left(n\pi + \frac{\pi}{2}\right)}{n+1}$ converges absolutely, converges conditionally, or diverges. *[10]*

Quiz #21. Friday, 5 April, 2002. *[10 minutes]*

1. Find a power series which, when it converges, equals $f(x) = \frac{3x^2}{(1-x^3)^2}$. *[10]*

Quiz #21. *(Alternate version.)* Friday, 5 April, 2002. *[10 minutes]*

1. Find a function which is equal to the power series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!}$ (when the series converges). *[10]*

Mathematics 110 – Calculus of one variable
TRENT UNIVERSITY, 2001-2002

Test #1
Friday, 16 November, 2001
Time: 50 minutes

1. Do any *two* of **a-c**. *[10 = 2 × 5 ea.]*

a. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 2}(5 - x) = 3$.

b. Use the definition of the derivative to verify that $f'(2) = 5$ if $f(x) = 2x^2 - 3x + 4$.

c. Determine whether $f(x) = \begin{cases} \dfrac{x-1}{x^2-1} & x \neq 1 \\ 41 & x = 1 \end{cases}$ is continuous at $a = 1$ or not.

2. Find $\dfrac{dy}{dx}$ in any *three* of **a-d**. *[12 = 3 × 4 ea.]*

a. $y = 4^x$ **b.** $y = e^{-x}\cos(x)$ **c.** $y = \sin(\cos(-x))$ **d.** $\ln(xy) = 0$

3. Do any *two* of **a-c**. *[8 = 2 × 4 ea.]*

a. Find the equation of the tangent line to $y = \arctan(x)$ at the point $\left(1, \frac{\pi}{4}\right)$.

b. How many functions $h(x)$ are there such that $h'(x) = h(x)$?

c. Draw the graph of a function $f(x)$ which is continuous on the domain $(0, \infty)$ such that $\lim_{x \to 0} f(x) = \infty$ and with a local maximum at $x = 5$ as its only critical point, or explain why there cannot be such a function.

4. Find the intercepts, maxima and minima, inflection points, and vertical and horizontal asymptotes of $f(x) = \dfrac{x}{x^2+1}$ and sketch the graph of $f(x)$ based on this information. *[10]*

Mathematics 110 – Calculus of one variable
Trent University, 2001-2002

Test #2
Friday, 8 February, 2002
Time: 50 minutes

1. Compute any *three* of the integrals **a-e**. *[12 = 3 × 4 ea.]*

$$\textbf{a.} \int_{-\pi/2}^{\pi/2} \cos^3(x)\,dx \qquad \textbf{b.} \int x^2 \ln(x)\,dx \qquad \textbf{c.} \int_0^1 (e^x)^2\,dx$$
$$\textbf{d.} \int \frac{e^{2x}\ln\left(e^{2x}+1\right)}{e^{2x}+1}\,dx \qquad \textbf{e.} \int_1^e (\ln(x))^2\,dx$$

2. Do any *two* of **a-c**. *[8 = 2 × 4 ea.]*

a. Compute $\int_0^1 (2x+3)\,dx$ using the Right-hand Rule.

b. Compute $\dfrac{dy}{dx}$ if $y = \int_0^{x^2} \sqrt{t}\,dt$ (where $x \geq 0$) without evaluating the integral.

c. Compute $\int_{-1}^{1} \sqrt{1-x^2}\,dx$ by interpreting it as an area.

3. Water is poured at a rate of 1 m^3/min into a conical tank (set up point down) 2 m high and with radius 1 m at the top. How quickly is the water rising in the tank at the instant that it is 1 m deep over the tip of the cone? *[8]*

(The volume of a cone of height h and radius r is $\frac{1}{3}\pi r^2 h$.)

4. Consider the region in the first quadrant with upper boundary $y = x^2$ and lower boundary $y = x^3$, and also the solid obtained by rotating this region about the y-axis.

a. Sketch the region and find its area. *[4]*

b. Sketch the solid and find its volume. *[7]*

c. What is the average area of either a washer or a shell (your pick!) for the solid? *[1]*

Mathematics 110 – Calculus of one variable
FINAL EXAMINATION
Trent University, 24 April, 2002

Time: 3 hours *Brought to you by* Stefan Bilaniuk.

Instructions: Show all your work and justify all your conclusions. *If in doubt about something,* **ask!**

Aids: Calculator; $8.5'' \times 11''$ aid sheet or the pamphlet *Formula for Success*; one brain.

Part I. Do all four of **1** – **4**.

1. Find $\frac{dy}{dx}$ in any *three* of **a** – **e**. *[9 = 3 × 3 ea.]*

a. $y = x^2 \ln(x) - 1$ **b.** $y = \dfrac{\ln(x)}{x^2} + x$ **c.** $\sin^2(y) = e^{-x^2}$

d. $y = \displaystyle\int_x^0 \cos\left(t^2\right)\, dt$ **e.** $y = \sqrt{\sec^2(\arctan(x)) - 1}$

2. Evaluate any *three* of the integrals **a** – **e**. *[12 = 3 × 4 ea.]*

a. $\displaystyle\int \frac{2x+3}{\sqrt{1-x^2}}\, dx$ **b.** $\displaystyle\int_{-\pi/4}^{\pi/4} \arctan(x)\, dx$ **c.** $\displaystyle\int \cos^2(t)\, dt$

d. $\displaystyle\int_1^\infty \frac{1}{x^3+x}\, dx$ **e.** $\displaystyle\int \frac{1}{x^2+2x+5}\, dx$

3. Evaluate any *three* of the limits **a** – **e**. *[9 = 3 × 3 ea.]*

a. $\displaystyle\lim_{x\to -1} \frac{x^2+2x+1}{x+1}$ **b.** $\displaystyle\lim_{n\to\infty} \frac{2^n}{4^n+\pi}$ **c.** $\displaystyle\lim_{x\to 0} \frac{\arctan(x)}{x}$

d. $\displaystyle\lim_{n\to\infty} \frac{n^2+2n+2}{2n+2}$ **e.** $\displaystyle\lim_{x\to\pi} \frac{\sin(x)}{x-\pi}$

4. Determine whether the given series converges absolutely, converges conditionally, or diverges in any *three* of **a** – **e**. *[12 = 3 × 4 ea.]*

a. $\displaystyle\sum_{n=2}^\infty \frac{2}{n\ln\left(n^2\right)}$ **b.** $\displaystyle\sum_{n=0}^\infty \frac{(-1)^n(n+1)}{n^2+3n+9}$ **c.** $\displaystyle\sum_{n=1}^\infty \frac{(-1)^{4n}}{2n+3}$

d. $\displaystyle\sum_{n=0}^\infty \frac{(-2)^n}{n!}$ **e.** $\displaystyle\sum_{n=0}^\infty \frac{\arctan(-n)}{5^n}$

1

Part II. Do *both* of **5** and **6**.

5. Find the domain, all maximum, minimum, and inflection points, and all vertical and horizontal asymptotes of $f(x) = x\ln(x)$, and sketch its graph. *[14]*

6. Consider the region in the first quadrant bounded by $y = \sin(x)$ and $y = \frac{2}{\pi}x$.

a. Sketch the region. *[2]*

b. Sketch the solid obtained by revolving the region about the y-axis. *[3]*

c. Find the volume of the solid. *[7]*

Part III. Do *one* of **7** or **8**.

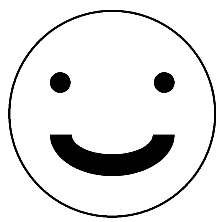
7. Find the MacLaurin series of $\sin(x)$ and determine its radius of convergence. *[12]*

8. Find a function which is equal to $2x + 3x^2 + 8x^3 + 15x^4 + 32x^5 + 63x^6 + 128x^7 + \cdots$, at least when this power series converges. (Note that the coefficient of x^n is 2^n when n is odd and $2^n - 1$ when n is even.) *[12]*

Part IV. Do *one* of **9** or **10**.

9. Use the $\epsilon - \delta$ definition of limits to verify that $\lim_{x \to -3} \frac{x^2 - 9}{x - 3} = 0$. *[10]*

10. A happy face is painted on the surface of a spherical balloon. The face expands as the balloon is inflated at a rate of 10 *litres/sec*. If the distance between the eyes is 10 *cm* at the instant that the diameter of the balloon is 20 *cm*, how is the distance between the eyes changing at the same instant? *[10]*



[Total = 90]

Part V. Bonus!

$\mathbf{42_{13} = 6 \times 9}$. Write a little poem about calculus or mathematics in general. *[2]*

I HOPE YOU'VE HAD FUN IN MATH 110! HAVE A GOOD SUMMER!

2

Mathematics 110, Section A
Calculus I: Calculus of one variable
2002-2003

Mathematics 110 – Calculus of one variable
Trent University 2002-2003

§A Quizzes

Quiz #1. Wednesday, 18 September, 2002. *[10 minutes]*

12:00 Seminar

1. Compute $\lim_{x \to 2} \dfrac{x^2 - x - 2}{x - 2}$ or show that this limit does not exist. *[5]*
2. Sketch the graph of a function $f(x)$ which is defined for all x and for which $\lim_{x \to 0} f(x) = 1$, $\lim_{x \to 2^+} f(x)$ does not exist, and $\lim_{x \to 2^-} f(x) = 4$. *[5]*

13:00 Seminar

1. Compute $\lim_{x \to 2^-} \dfrac{x^2 - x + 2}{x - 2}$ or show that this limit does not exist. *[5]*
2. Sketch the graph of a function $g(x)$ which is defined for all x, and for which $\lim_{x \to 0} g(x) = \infty$, $\lim_{x \to 2} g(x)$ does not exist, and $g(x)$ does not have an asymptote at $x = 2$. *[5]*

Quiz #2. Wednesday, 25 September, 2002. *[10 minutes]*

12:00 Seminar

1. Use the $\epsilon - \delta$ definition of limits to verify that $\lim_{x \to 3} (5x - 7) = 8$. *[10]*

13:00 Seminar

1. Use the $\epsilon - \delta$ definition of limits to verify that $\lim_{x \to 2} (3 - 2x) = -1$. *[10]*

Quiz #3. Wednesday, 2 October, 2002. *[10 minutes]*

12:00 Seminar

1. For which values of the constant c is the function

$$f(x) = \begin{cases} e^{cx} & x \geq 0 \\ cx + 1 & x < 0 \end{cases}$$

continuous at $x = 0$? Why? *[10]*

13:00 Seminar

1. For which values of the constant c is the function

$$f(x) = \begin{cases} e^{cx} & x \geq 0 \\ c(x + 1) & x < 0 \end{cases}$$

continuous at $x = 0$? Why? *[10]*

Quiz #4. Wednesday, 9 October, 2002. *[12 minutes]*

12:00 Seminar

Suppose

$$f(x) = \begin{cases} x & x < 0 \\ 0 & x = 0 \\ 2x^2 + x & x > 0 \end{cases} .$$

1. Use the definition of the derivative to check whether $f'(0)$ exists and compute it if it does. *[7]*
2. Compute $f'(1)$ (any way you like). *[3]*

13:00 Seminar

Suppose $g(x) = \dfrac{1}{x+1}$. Compute $g'(x)$ using

1. the rules for computing derivatives *[3]*, and
2. the definition of the derivative. *[7]*

Quiz #5. Wednesday, 16 October, 2002. *[10 minutes]*

12:00 Seminar

Compute $\dfrac{d}{dx}\sqrt[5]{x}$ using

1. the Power Rule *[2]*, and
2. the fact that $f(x) = \sqrt[5]{x}$ is the inverse function of $g(x) = x^5$. *[8]*

13:00 Seminar

1. Compute $\dfrac{d}{dx}\arccos(x)$ given that $\cos(\arccos(x)) = x$ and $\cos^2(x) + \sin^2(x) = 1$. *[10]*

Quiz #6. Wednesday, 30 October, 2002. *[10 minutes]*

12:00 Seminar

1. Find the absolute and local maxima and minima of $f(x) = x^3 + 2x^2 - x - 2$ on $[-2, 2]$. *[10]*

13:00 Seminar

1. Find the absolute and local maxima and minima of $f(x) = x^3 - 3x^2 - x + 3$ on $[-2, 2]$. *[10]*

Quiz #7. Wednesday, 6 November, 2002. *[15 minutes]*

12:00 Seminar

1. Find the intercepts, critical and inflection points, and horizontal asymptotes of $f(x) = (x-2)e^x$ and sketch its graph. *[10]*

13:00 Seminar

1. Find the intercepts, critical and inflection points, and horizontal asymptotes of $h(x) = (x+1)e^{-x}$ and sketch its graph. *[10]*

Quiz #8. Wednesday, 27 November, 2002. *[15 minutes]*

12:00 Seminar

1. Compute:
$$\int_1^{e^\pi} \frac{1}{x} \sin\left(\ln(x)\right)\, dx \qquad [5]$$

2. What definite integral does the Right-hand Rule limit
$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right) \cdot \frac{1}{n}$$
correspond to? *[5]*

13:00 Seminar

1. Compute:
$$\int_0^{\pi/4} \frac{\tan(x)}{\cos^2(x)}\, dx \qquad [5]$$

2. What definite integral does the Right-hand Rule limit
$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n} - 1\right) \cdot \frac{1}{n}$$
correspond to? *[5]*

Quiz #9. Wednesday, 4 December, 2002. *[15 minutes]*

12:00 Seminar

1. Find the area of the region enclosed by $y = -x^2$ and $y = x^2 - 2x$. *[10]*

13:00 Seminar

1. Find the area of the region enclosed by $y = (x-2)^2 + 1 = x^2 - 4x + 5$ and $y = x + 1$. *[10]*

Quiz #10. Wednesday, 8 January, 2003. *[25 minutes]*

12:00 Seminar

1. Sketch the solid obtained by rotating the region bounded by $y = 0$ and $y = \cos(x)$ for $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$ about the y-axis and find its volume. *[10]*

13:00 Seminar

1. Sketch the solid obtained by rotating the region bounded by $y = -1$ and $y = \cos(x)$ for $0 \leq x \leq \pi$ about the y-axis and find its volume. *[10]*

Quiz #11. Wednesday, 15 January, 2003. *[20 minutes]*

12:00 Seminar

1. Compute $\displaystyle\int \frac{1}{1-x^2}\, dx$. *[10]*

13:00 Seminar

1. Compute $\int \frac{x^2}{\sqrt{1-x^2}}\, dx$. *[10]*

Quiz #12. Wednesday, 22 January, 2003. *[20 minutes]*

12:00 Seminar

1. Compute $\int \frac{3x^2+4x+2}{x^3+2x^2+2x}\, dx$. *[10]*

13:00 Seminar

1. Compute $\int \frac{2x+1}{x^3+2x^2+x}\, dx$. *[10]*

Quiz #13. Wednesday, 29 January, 2003. *[15 minutes]*

12:00 Seminar

1. Compute $\int_{-\infty}^{\infty} e^{-|x|}\, dx$ or show that it does not converge. *[10]*

13:00 Seminar

1. Compute $\int_{-1}^{1} \frac{x+1}{\sqrt[3]{x}}\, dx$ or show that it does not converge. *[10]*

Quiz #14. Wednesday, 5 February, 2003. *[20 minutes]*

12:00 Seminar

1. Sketch the solid obtained by rotating the region bounded by $x = 0$, $y = 4$ and $y = x^2$ for $0 \le x \le 2$ about the y-axis. *[2]*
2. Compute the surface area of this solid. *[8]*

13:00 Seminar

1. Sketch the curve given by the parametric equations $x = 1 + \cos(t)$ and $y = \sin(t)$, where $0 \le t \le 2\pi$. *[3]*
2. Compute the arc-length of this curve using a suitable integral. *[7]*

Quiz #15. Wednesday, 26 February, 2003. *[20 minutes]*

12:00 Seminar

1. Graph the polar curve $r = \sin(2\theta)$, $0 \le \theta \le 2\pi$. *[4]*
2. Find the area of the region enclosed by this curve. *[6]*

13:00 Seminar

1. Graph the polar curve $r = \cos(\theta)$, $0 \le \theta \le 2\pi$. *[4]*
2. Find the arc-length of this curve. *[6]*

Quiz #16. Wednesday, 5 March, 2003. *[15 minutes]*

12:00 Seminar

Let $a_k = \dfrac{1}{(k+1)(k+2)}$ and $s_n = \displaystyle\sum_{k=0}^{n} a_k$.

1. Find a formula for s_n in terms of n. *[5]*
2. Does $\displaystyle\sum_{k=0}^{\infty} a_k$ converge? If so, what does it converge to? *[5]*

13:00 Seminar

Let $a_k = \ln\left(\dfrac{k}{k+1}\right)$ and $s_n = \displaystyle\sum_{k=0}^{n} a_k$.

1. Find a formula for s_n in terms of n. *[5]*
2. Does $\displaystyle\sum_{k=0}^{\infty} a_k$ converge? If so, what does it converge to? *[5]*

Quiz #17. Wednesday, 12 March, 2003. *[15 minutes]*

12:00 Seminar

Determine whether each of the following series converges or diverges:

1. $\displaystyle\sum_{n=0}^{\infty} e^{-n}$ *[5]* 2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\arctan(n)}$ *[5]*

13:00 Seminar

Determine whether each of the following series converges or diverges:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1}$ *[5]* 2. $\displaystyle\sum_{n=1}^{\infty} 2^{1/n^2}$ *[5]*

Quiz #18. Wednesday, 19 March, 2003. *[15 minutes]*

12:00 Seminar

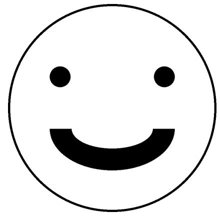
Determine whether each of the following series converges absolutely, converges conditionally, or diverges:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n^2+2}$ *[5]* 2. $\displaystyle\sum_{n=1}^{\infty} \frac{n!(-1)^n}{n^n}$ *[5]*

13:00 Seminar

Determine whether each of the following series converges absolutely, converges conditionally, or diverges:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\left(2n^2+3n+4\right)}{3n^2+4n+5}$ *[5]* 2. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2}$ *[5]*



Bonus Quiz. Friday, 21 March, 2003. *[15 minutes]*

1. A smiley face is drawn on the surface of a balloon which is being inflated at a rate of 10 cm^3/s. At the instant that the radius of the balloon is 10 cm the eyes are 10 cm apart, as measured *inside* the balloon. How is the distance between them changing at this moment? *[10]*

Quiz #19. Wednesday, 26 March, 2003. *[20 minutes]*

12:00 Seminar

Consider the power series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^{2n}}{n!}$.

1. For which values of x does this series converge? *[6]*
2. This series is equal to a (reasonably nice) function. What is it? Why? *[4]*

13:00 Seminar

Consider the power series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n+1}$.

1. For which values of x does this series converge? *[6]*
2. This series is equal to a (reasonably nice) function. What is it? Why? *[4]*

Quiz #20. Wednesday, 2 April, 2003. *[20 minutes]*

12:00 Seminar

Let $f(x) = \sin(\pi - 2x)$.

1. Find the Taylor series at $a = 0$ of $f(x)$. *[6]*
2. Find the radius and interval of convergence of this Taylor series. *[4]*

13:00 Seminar

Let $f(x) = \ln(2 + x)$.

1. Find the Taylor series at $a = 0$ of $f(x)$. *[6]*
2. Find the radius and interval of convergence of this Taylor series. *[4]*

Mathematics 110 – Calculus of one variable
TRENT UNIVERSITY, 2002-2003

Test #1 - Section A
Wednesday, 13 November, 2002
Time: 50 minutes

Instructions

- Please show all your work, make every effort to keep your solution neat, and put the final answer in a box.
- If you have a question, *ask it*!
- You may use a calculator and one of an 8.5" × 11" aid sheet or the pamphlet *Formula for Success*.
- Please **return the question sheets** with your answers and **do not discuss the test with anyone who may write it later**, on pain of violating MATH 110's policy on plagiarism.

1. Do any *one* of **a** – **c**. *[10/100]*

a. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 1} (7x + 3) = 10$.

b. Use the limit definition of the derivative to verify that $\frac{d}{dx}(e^x) = e^x$.

(You may assume that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.)

c. Compute $\lim_{x \to 0} \frac{x}{\tan(2x)}$.

2. Do any *two* of **a** – **c**. *[15/100 each]*

a. Find the equation of the tangent line to $y = \sqrt{x}$ at $x = 4$.

b. Is the function $f(x) = x \cdot |x|$ differentiable at $x = 0$? Why or why not?

c. Use the limit definition of the derivative to verify the Sum Rule, *i.e.* that if $h(x) = f(x) + g(x)$, then $h'(x) = f'(x) + g'(x)$.

Questions **3** and **4** are on page 2.

Mathematics 110 – Calculus of one variable
TRENT UNIVERSITY, 2002-2003

Test #2 - Section A
Wednesday, 12 February, 2003
Time: 50 minutes

Instructions

- Please show all your work, make every effort to keep your solution neat, and put the final answer in a box.
- If you have a question, *ask it*!
- You may use a calculator and one of an 8.5" × 11" aid sheet or the pamphlet *Formula for Success*.
- Please **do not discuss the test with anyone who may write it later nor show them the question sheets**, on pain of violating MATH 110's policy on plagiarism.

1. Compute any *three* of the integrals in parts **a-f**. *[36 = 3 × 12 each]*

a. $\displaystyle\int_0^1 \arctan(x)\,dx$ **b.** $\displaystyle\int_e^{e^e} \frac{\ln(\ln(x))}{x\ln(x)}\,dx$ **c.** $\displaystyle\int \frac{1}{x^2-3x+2}\,dx$

d. $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$ **e.** $\displaystyle\int_0^{\pi/4} \sec^4(x)\,dx$ **f.** $\displaystyle\int_2^{\infty} \frac{1}{x^4}\,dx$

2. Do any *two* of parts **a-d**. *[24 = 2 × 12 each]*

a. Compute $\displaystyle\frac{d}{dx}\left(\int_1^{\cos(x)} \arccos(t)\,dt\right)$.

b. Give both a description and a sketch of the region whose area is computed by the integral $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$.

c. Find the area of the region bounded by $y = x^3 - x$ and $y = 3x$.

d. Compute $\displaystyle\int_0^1 (x+1)\,dx$ using the Right-hand Rule.

3. Consider the solid obtained by rotating the region bounded by $y = x^2$ and $y = 1$ about the line $x = 2$.

a. Sketch the solid. *[5]*

b. Find the volume of the solid. *[20]*

4. Find the arc-length of the curve given by the parametric equations $x = \dfrac{t^2}{\sqrt{2}}$ and $y = \dfrac{t^3}{3}$, where $0 \le t \le 1$. *[15]*

[Total = 100]

Mathematics 110 – Calculus of one variable
§A Final Examination
Trent University, 23 April, 2002

Time: 3 hours *Brought to you by* Стефан Біланюк.

Instructions: Show all your work and justify all your answers. *If in doubt,* **ask!**

Aids: Calculator; an $8.5'' \times 11''$ aid sheet or the pamphlet *Formula for Success*; one brain.

Part I. Do all three of **1** – **3**.

1. Find $\frac{dy}{dx}$ (in terms of x and/or y) in any *three* of **a** – **f**. *[15 = 3 × 5 ea.]*

a. $y = \dfrac{x^2-1}{x^2+1}$ **b.** $y = \displaystyle\int_{-x}^{x} e^{4t}\,dt$ **c.** $\begin{array}{l} y = \cos(t) \\ x = \sin(t) \end{array}$

d. $y = \arcsin(3x)$ **e.** $x^2 + 2xy + y^2 = 1$ **f.** $y = \ln\left(x^2 - 2x + 1\right)$

2. Evaluate any *three* of the integrals **a** – **f**. *[15 = 3 × 5 ea.]*

a. $\displaystyle\int \frac{x+2}{x^2+4x+5}\,dx$ **b.** $\displaystyle\int_0^\infty \frac{1}{t^2+1}\,dt$ **c.** $\displaystyle\int \frac{1}{x^2-5x+6}\,dx$

d. $\displaystyle\int_1^e \ln(y)\,dy$ **e.** $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$ **f.** $\displaystyle\int_0^\pi \cos^2(w)\sin(w)\,dw$

3. Do any *five* of **a** – **j**. *[25 = 5 × 5 ea.]*

a. Does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \arctan(n)}{n^2}$ converge absolutely, converge conditionally, or diverge?

b. Evaluate $\displaystyle\lim_{x\to 0} \frac{x}{e^x - 1}$ or show that the limit does not exist.

c. Find the arc-length of the curve given by $y = 1 - t^2$ and $x = 1 + t^2$ for $0 \le t \le 1$.

d. What is the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n!}$ when it converges?

e. Sketch the region described by $0 \le r \le \csc(\theta)$ and $\pi/4 \le \theta \le \pi/2$ in polar coordinates and find its area.

f. Use an $\varepsilon - \delta$ argument to verify that $\displaystyle\lim_{t\to 3}(2x-4) = 2$.

g. Determine the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}}{2^n} x^n$.

h. For which values of c is $f(x) = \begin{cases} \cos(x) & x \le \pi \\ cx & x > \pi \end{cases}$ continuous at $x = \pi$?

i. Find the absolute maximum and minimum points, if any, of $f(x) = x^3 - 3x$ on the interval $-2 \le x \le 2$.

j. Give an integral corresponding to the Right-hand Rule sum $\displaystyle\sum_{i=1}^{n} \frac{2i}{n} \tan\left(1 + \frac{i}{n}\right)$.

Part II. Do *one* of **4** or **5**.

4. Find the domain, all maximum, minimum, and inflection points, and all vertical and horizontal asymptotes of $g(x) = e^{-x^2}$, and sketch its graph. *[15]*

5. Sand is poured onto a level floor at the rate of 60 l/min. It forms a conical pile whose height is equal to the radius of the base. How fast is the height of the pile increasing when the pile is 2 m high? *[15]*
[The volume of a cone of height h and base radius r is $\frac{1}{3}\pi r^2 h$.]

Part III. Do *one* of **6** or **7**.

6. Consider the curve $y = \sqrt{x}$, $0 \leq x \leq 4$.

- **a.** Sketch the curve. *[1]*
- **b.** Sketch the surface obtained by revolving the curve about the x-axis. *[2]*
- **c.** Find the area of the surface. *[12]*

7. Consider the region in the first quadrant bounded by $y = \sqrt{x}$ and $y = \dfrac{x}{2}$.

- **a.** Sketch the region. *[2]*
- **b.** Sketch the solid obtained by revolving the region about the y-axis. *[2]*
- **c.** Find the volume of the solid. *[11]*

Part IV. Do *one* of **8** or **9**.

8. Consider the power series $\displaystyle\sum_{n=1}^{\infty} (-2)^n n x^{n-1} = -2 + 8x - 24x^2 + 64x^3 - 160x^4 + \cdots$

- **a.** Find the radius and interval of convergence of this power series. *[9]*
- **b.** What function has this power series as its Taylor series at $a = 0$? *[6]*

9. Let $f(x) = e^{2x-2}$.

- **a.** Find the Taylor series at $a = 1$ of $f(x)$. *[10]*
- **b.** Find the radius and interval of convergence of this Taylor series. *[5]*

[Total = 100]

Part MMIII. Bonus!

-1. Write a little poem about calculus or mathematics in general. *[2]*

-2. Find the surface area of a cone with base radius r and height h. For maximum credit, do this without using any calculus. *[2]*

I HOPE YOU'VE HAD A GOOD TIME! HAVE A GOOD SUMMER!

Mathematics 110, Section A
Calculus I: Calculus of one variable
2003-2004

Mathematics 110 – Calculus of one variable
Trent University 2003-2004

§A Quizzes

Quiz #1. Friday, 19 September, 2003. *[10 minutes]*

12:00 Seminar

1. How close does x have to be to 1 in order to guarantee that $\frac{1}{x}$ is within $\frac{1}{10}$ of 1? *[10]*

13:00 Seminar

1. Find a value of $\delta > 0$ that ensures that $-1 < \sqrt{x} - 4 < 1$ whenever $-\delta < x - 16 < \delta$. *[10]*

Leftovers

1. Use the $\varepsilon - \delta$ definition of limits to verify that that $\lim_{x \to 0} 1 = 1$. *[10]*

Hint: Try any $\delta > 0$ you like ...

Quiz #2. Monday, 29 September, 2003. *[10 minutes]*

1. Use the $\varepsilon - \delta$ definition of limits to verify that that $\lim_{x \to 0} \sin^2(x) = 0$. *[10]*

Hint: You may use the fact that $|\sin(x)| \leq |x|$.

Quiz #3. Friday, 3 October, 2003. *[15 minutes]*

12:00 Seminar

Evaluate

$$1. \quad \lim_{x \to \infty} \frac{x-2}{x^2 - 3x + 2} \quad [5] \qquad 2. \quad \lim_{x \to 0} \frac{e^x - 1}{e^{2x} - 1} \quad [5]$$

13:00 Seminar

Evaluate

$$1. \quad \lim_{x \to 1} \frac{x^2 + x - 2}{x^2 + 2x - 3} \quad [5] \qquad 2. \quad \lim_{x \to \infty} \frac{(x+4)^2}{41x^2 + 43x + 47} \quad [5]$$

Leftovers

1. For what value(s) of the constant c does $\lim_{x \to 2} (cx + 3) = \lim_{t \to \infty} \dfrac{ct^2 + 3 + c}{t^2 + 1}$? *[10]*

Quiz #4. Friday, 10 October, 2003. *[10 minutes]*

12:00 Seminar

1. Use the limit definition of the derivative to find $f'(0)$ if $f(x) = (x+1)^3$. *[10]*

13:00 Seminar

1. Use the limit definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{1}{x}$. *[10]*

Leftovers

1. Use the limit definition of the derivative to find $f'(x)$ if $f(x) = x^2 - 3x$. *[10]*

Quiz #5. Friday, 17 October, 2003. *[10 minutes]*

12:00 Seminar

Find $\frac{dy}{dx}$ in each of the following:

1. $y = \ln\left(\sec(x) + \tan(x)\right)$ *[3]* 2. $e^{xy} = 2$ *[3]* 3. $y = \dfrac{x^2 + 4x + 4}{x + 3}$ *[4]*

13:00 Seminar

Find $\frac{dy}{dx}$ in each of the following:

1. $y = \left(1 + x^2\right) \arctan(x)$ *[3]* 2. $\tan(x + y) = 1$ *[3]* 3. $y = \dfrac{e^x + 1}{e^{2x} - 1}$ *[4]*

Leftovers

Find $\frac{dy}{dx}$ in each of the following:

1. $y = \sqrt{1 - e^{2x}}$ *[3]* 2. $y = \dfrac{\tan(x)}{\cos(x)}$ *[3]* 3. $\ln(x + y) = x$ *[4]*

Quiz #6. Friday, 31 October, 2003. *[15 minutes]*

12:00 Seminar

1. A ladder 5 m long rests against a vertical wall. If the top of the ladder slips down at a rate of 1 m/s, how is angle between the bottom of the ladder and the ground changing when the top of the ladder is $4m$ above the ground? *[10]*

13:00 Seminar

1. A searchlight is on an island 8 km from the nearest point, call it P, on the mainland (which has a straight shore). The searchlight makes one revolution each minute. How swiftly is the light beam moving along the shore when it is 6 km from and moving towards P? *[10]*

Leftovers

1. Kypalo, walking along a straight path at 4 m/s, passes a tree 3 m from the path. How is the distance between Kypalo and the tree changing 1 s after the Kypalo passes the tree? *[10]*

Quiz #7. Friday, 7 November, 2003. *[15 minutes]*

1. Find all the intercepts, the maximum, minimum, and inflection points, and the horizontal and vertical asymptotes of the following function:

12:00: $f(x) = \dfrac{x}{x^2 + 1}$ **13:00:** $g(x) = \dfrac{x - 1}{x + 1}$ **Leftovers:** $h(x) = \dfrac{x^2 + 1}{x}$ *[10]*

Quiz #8. Friday, 21 November, 2003. *[12 minutes]*

1. Use the Right-hand Rule to compute the area under $y = f(x)$ for $0 \leq x \leq 2$. *[10]*

   12:00: $f(x) = (x+1)^2 - 1$ **13:00:** $f(x) = x^2 - 4x + 5$

   Leftovers: $f(x) = 2x^2 + x + 3$

Hint: You may use the facts that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Quiz #9. Friday, 28 November, 2003. *[12 minutes]*

1. Compute the following integrals. *[10]*

12:00: **a.** $\int \frac{\sin(x)}{\cos^2(x)}\,dx$ **b.** $\int \frac{\sqrt{\ln(x)}}{6x}\,dx$

13:00: **a.** $\int \frac{\sin\left(\sqrt{x}\right)}{4\sqrt{x}}\,dx$ **b.** $\int \left(\sin^2(x)\cos(x) + \cos(x)\right)\,dx$

Leftovers: **a.** $\int \frac{4\arctan^3(x)}{1+x^2}\,dx$ **b.** $\int \left(e^x - 1\right)^2 e^x\,dx$

Quiz #10. Friday, 5 December, 2003. *[10 minutes]*

1. Compute the following integral. *[10]*

12:00: $\displaystyle\int_0^{\pi/4} \tan(x)\ln\left(\sec(x)\right)\,dx$ **13:00:** $\displaystyle\int_0^1 \frac{(x-1)^2}{x^2+1}\,dx$

Leftovers: $\displaystyle\int_0^1 \left(x^2+1\right)^{15} x^3\,dx$

Quiz #11. Friday, 9 January, 2004. *[12 minutes]*

1. Compute the following integral. *[10]*

12:00: $\displaystyle\int \sin(2x)\sin(x)\,dx$ **13:00:** $\displaystyle\int \left(\tan^2(x) - 1\right)\,dx$

Leftovers: $\displaystyle\int \sec^{3/2}(x)\tan(x)\,dx$

Quiz #12. Friday, 16 January, 2004. *[15 minutes]*

1. Compute the following integral. *[10]*

12:00: $\displaystyle\int \frac{\sqrt{x^2-4}}{x^2}\,dx$ **13:00:** $\displaystyle\int_0^2 x\sqrt{4-x^2}\,dx$

Leftovers: $\displaystyle\int \frac{x^3}{\sqrt{x^2+9}}\,dx$

3

Quiz #13. Friday, 23 January, 2004. *[15 minutes]*

1. Compute the following integral. *[10]*

$$\textbf{12:00:} \quad \int \frac{5}{(x-1)\,(x^2+2x+2)}\,dx \qquad \textbf{13:00:} \quad \int \frac{4}{(x^2-1)\,(x-1)}\,dx$$

$$\textbf{Leftovers:} \quad \int \frac{x^2-2x+1}{x^2-x-2}\,dx$$

Quiz #14. Monday, 2 February, 2004. *[15 minutes]*

Lecture

1. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 0$, and $x = 1$ about the y-axis. *[10]*

Leftovers

1. Find the volume of the solid obtained by rotating the region bounded by $y = \cos(x)$ and $y = 0$ for $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$ about the y-axis. *[10]*

Quiz #15. Friday, 6 February, 2004. *[15 minutes]*

12:00

1. Find the area of the surface obtained by rotating the curve $y = \frac{x^3}{3}$, $0 \leq x \leq 1$, about the x-axis. *[10]*

13:00

1. Find the arc-length of the curve $y = -\ln(\cos(x))$, $0 \leq x \leq \frac{\pi}{4}$. *[10]*

Leftovers

1. Find the area of the surface obtained by rotating the curve $y = \ln(x)$, $1 \leq x \leq e$, about the y-axis. *[10]*

Quiz #16. Friday, 27 February, 2004. *[15 minutes]*

12:00

1. Sketch the polar curve $r = 1 + \cos(\theta)$, $0 \leq \theta \leq 2\pi$. *[5]*
2. Find the slope of this curve at $\theta = \frac{\pi}{2}$. *[5]*

13:00

1. Sketch the region bounded by the polar curves $r = \theta$, for $0 \leq \theta \leq 2\pi$, and $\theta = 0$. *[5]*
2. Find the area of this region. *[5]*

Leftovers

1. Sketch the region bounded by the polar curves $\theta = 0$, $\theta = \frac{\pi}{4}$, and $r = 4\sec(\theta)$. *[5]*
2. Find the area of this region. *[5]*

Quiz #17. Friday, 5 March, 2004. *[10 minutes]*

12:00

Compute each of the following.

$$1. \quad \lim_{n\to\infty} 2^n 3^{-n} \quad \textit{[5]} \qquad 2. \quad \lim_{n\to\infty} \frac{\cos(n)}{n} \quad \textit{[5]}$$

13:00

Compute each of the following.

$$1. \quad \lim_{n\to\infty} \frac{n}{\ln(n)} \quad \textit{[5]} \qquad 2. \quad \lim_{n\to\infty} (-1)^n \frac{n+1}{n+2} \quad \textit{[5]}$$

Leftovers

Compute each of the following.

$$1. \quad \lim_{n\to\infty} ne^{-n} \quad \textit{[5]} \qquad 2. \quad \lim_{n\to\infty} \frac{n^2}{n+1} \quad \textit{[5]}$$

Quiz #18. Friday, 12 March, 2004. *[20 minutes]*

Determine whether each of the following series converges or diverges.

12:00 1. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{3^n}$ *[5]* 2. $\displaystyle\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{1+\ln(n)}$ *[5]*

13:00 1. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ *[5]* 2. $\displaystyle\sum_{n=0}^{\infty} \frac{\arctan\left(n^2\right)}{2^n+n}$ *[5]*

Leftovers 1. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n\sin(n\pi)}{n^2+1}$ *[5]* 2. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n^2}}{n^3}$ *[5]*

Quiz #19. Friday, 19 March, 2004. *[15 minutes]*

Determine whether the following series is absolutely convergent, conditionally convergent, or divergent.

12:00 1. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$ *[10]*

13:00 1. $\displaystyle\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n-2)\cdot(2n)}{(-1)^n n!}$ *[10]*

Leftovers 1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-5)^{n+2}}{2^n 3^{n+1}}$ *[10]*

5

Quiz #20. Friday, 26 March, 2004. *[15 minutes]*

Find the radius and interval of convergence of the following power series.

12:00 1. $\displaystyle\sum_{n=0}^{\infty}(-4)^n n^2 x^n$ *[10]*

13:00 1. $\displaystyle\sum_{n=0}^{\infty}\frac{(x-2)^n}{n3^n}$ *[10]*

Leftovers 1. $\displaystyle\sum_{n=0}^{\infty}\frac{(3-2x)^n}{n!}$ *[10]*

Quiz #21. Friday, 2 April, 2004. *[20 minutes]*

Find the Taylor series centred at a of $f(x)$ if

12:00 1. $f(x) = \dfrac{1}{\sqrt{x}}$ and $a = 4$ *[10]*

13:00 1. $f(x) = \sin(x)$ and $a = \dfrac{\pi}{2}$ *[10]*

Leftovers 1. $f(x) = (x+1)^3$ and $a = 2$ *[10]*

and determine its radius and interval of convergence.

Mathematics 110 – Calculus of one variable
Trent University, 2003-2004

§A – Test #1
Wednesday, 12 November, 2003
Time: 50 minutes

Instructions

- Show all your work.
- If you have a question, *ask*!
- You may use a calculator and either a two-sided 8.5" × 11" aid sheet or the pamphlet *Formula for Success*.

1. Find $\frac{dy}{dx}$ in any *three* of **a-e**. *[12 = 3 × 4 ea.]*

a. $y = x\ln\left(\frac{1}{x}\right)$ **b.** $x^2 + 2xy + y^2 - x = 1$ **c.** $y = \sin\left(e^{\sqrt{x}}\right)$

d. $y = \frac{2^x}{x+1}$ **e.** $y = \cos(2t)$ where $t = x^3 + 2x$

2. Do any *two* of **a-c**. *[10 = 2 × 5 each]*

a. Determine whether $g(x) = \begin{cases} \frac{x-1}{x^2-1} & x \neq 1 \\ \frac{1}{2} & x = 1 \end{cases}$ is continuous at $x = 1$ or not.

b. Use the definition of the derivative to compute $f'(1)$ for $f(x) = \frac{1}{x}$.

c. Find the equation of the tangent line to $y = \sqrt{x}$ at $x = 9$.

3. Do *one* of **a** or **b**. *[8]*

a. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x\to 2} x^2 = 4$.

b. Use the $\varepsilon - N$ definition of limits to verify that $\lim_{t\to\infty} \frac{1}{t+1} = 0$.

4. Find the intercepts, the maximum, minimum, and inflection points, and the vertical and horizontal asymptotes of $f(x) = xe^{-x^2}$ and sketch the graph of $f(x)$ based on this information. *[10]*

[Total = 40]

Mathematics 110 – Calculus of one variable
TRENT UNIVERSITY, 2003-2004

Test #2 - Section A
Wednesday, 11 February, 2004
Time: 50 minutes

Instructions

- Please show all your work and make every effort to keep your solution neat and legible.
- If you have a question, *ask it*!
- You may use a calculator and one of an 8.5" × 11" aid sheet or the pamphlet *Formula for Success*.

1. Compute any *three* of the integrals in parts **a-f**. *[12 = 3 × 4 each]*

a. $\displaystyle\int_0^{\pi/2} \cos^3(x)\,dx$ **b.** $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$ **c.** $\displaystyle\int_2^{\infty} \frac{1}{\sqrt{x}}\,dx$

d. $\displaystyle\int \frac{\arctan(x)}{x^2+1}\,dx$ **e.** $\displaystyle\int \ln\left(x^2\right)\,dx$ **f.** $\displaystyle\int_1^2 \frac{1}{x^2-2x+2}\,dx$

2. Do any *two* of parts **a-d**. *[8 = 2 × 4 each]*

a. Find a definite integral computed by the Right-hand Rule sum
$$\lim_{n\to\infty}\sum_{i=0}^{n}\left(1+\frac{i^2}{n^2}\right)\cdot\frac{1}{n}.$$

b. Compute $\displaystyle\frac{d}{dx}\left(\int_0^{\tan(x)} e^{\sqrt{t}}\,dt\right)$.

c. Find the area under the parametric curve given by $x = 1+t^2$ and $y = t(1-t)$ for $0 \le t \le 1$.

d. Sketch the region whose area is computed by the integral $\displaystyle\int_0^1 \arctan(x)\,dx$.

3. Find the volume of the solid obtained by rotating the region bounded by $y = \dfrac{1}{x}$, $y = \dfrac{1}{2}$, and $x = 1$ about the line $x = -1$. *[10]*

4. Find the area of the surface obtained by rotating the curve $y = \ln(x)$, $0 < x \le 1$, about the y-axis. *[10]*

[Total = 40]

Mathematics 110 – Calculus of one variable
§A Final Examination
Trent University, 8 April, 2004

Time: 3 hours *Brought to you by* Стефан Біланюк.

Instructions: Show all your work and justify all your answers. *If in doubt,* **ask!**

Aids: Calculator; an $8.5'' \times 11''$ aid sheet or the pamphlet *Formula for Success*; one brain.

Part I. Do all three of **1** – **4**.

1. Find $\dfrac{dy}{dx}$ (in terms of x and/or y) in any *three* of **a** – **f**. *[15 = 3 × 5 ea.]*

a. $y = x^2 \sin(x+3)$ **b.** $e^{xy} = 2$ **c.** $y = \cos^2\left(x^2\right)$

d. $\begin{array}{l} y = t^2 \\ x = t^3 \end{array}$ **e.** $y = \displaystyle\int_0^{x^2} \sqrt{w}\, dw$ **f.** $y = \dfrac{\cos(x)}{1 + \tan(x)}$

2. Evaluate any *three* of the integrals **a** – **f**. *[15 = 3 × 5 ea.]*

a. $\displaystyle\int_e^\infty \frac{1}{x \ln(x)}\, dx$ **b.** $\displaystyle\int x e^{-x}\, dx$ **c.** $\displaystyle\int_{-1}^1 \frac{2s}{1+s^4}\, ds$

d. $\displaystyle\int \frac{1}{\sqrt{x^2+4}}\, dx$ **e.** $\displaystyle\int_0^1 \arctan(t)\, dt$ **f.** $\displaystyle\int \frac{3x-3}{x^2+x-2}\, dx$

3. Determine whether the series converges absolutely, converges conditionally, or diverges in any *two* of **a** – **d**. *[10 = 2 × 5 ea.]*

a. $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n}{n^2}$ **b.** $\displaystyle\sum_{n=0}^\infty (3^n - 2^n)$ **c.** $\displaystyle\sum_{n=0}^\infty \frac{n}{1+n^2}$ **d.** $\displaystyle\sum_{n=0}^\infty 3^{-n} 2^n \cos(n\pi)$

4. Do any *three* of **a** – **f**. *[15 = 3 × 5 ea.]*

a. Use an $\varepsilon - N$ argument to verify that $\displaystyle\lim_{n\to\infty} \frac{1}{n^2} = 0$.

b. Sketch the polar curve $r = \theta$ for $-\pi \le \theta \le \pi$ and find its slope at $\theta = 0$.

c. Evaluate $\displaystyle\lim_{x\to 0} \frac{x^2}{\tan(x)}$ or show that the limit does not exist.

d. Use the Right-hand Rule to compute the definite integral $\int_0^2 (x+1)\, dx$.

e. Find the area of the surface obtained by rotating the curve $y = 2x$, $0 \le x \le 2$, about the y-axis.

f. Determine whether $f(x) = \begin{cases} 1 - e^x & x \le 0 \\ \ln(x+1) & x > 0 \end{cases}$ is continuous at $x = 0$ or not.

Part II. Do *one* of **5** or **6**.

5. A container of volume $54\pi\ cm^3$ is made from sheet metal. Find the dimensions of such a container which require the least amount sheet metal to make if:

a. The container is cylindrical, with a bottom but without a top. *[13]*

b. The container is a sphere. *[2]*

Hint: The volume of a cylinder of radius r and height h is $\pi r^2 h$; the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

6. Find the domain, all maximum, minimum, and inflection points, and all vertical and horizontal asymptotes of $f(x) = \dfrac{x}{1+x^2}$, and sketch its graph. *[15]*

Part III. Do *one* of **7** or **8**.

7. Find the surface area of a cone with height 8 cm and radius 2 cm at the base. *[15]*

8. Consider the region bounded above by $y = \sin(x)$ and below by $y = \dfrac{2}{\pi}x$ for $0 \leq x \leq \dfrac{\pi}{2}$.

a. Sketch this region. *[2]*

b. Sketch the solid obtained by revolving this region about the x-axis. *[3]*

c. Find the volume of this solid. *[10]*

Part IV. Do *one* of **9** or **10**.

9. Let $f(x) = \cos(x)$.

a. Find the Taylor series of $f(x)$ centred at $a = \pi$. *[8]*

b. Determine the radius and interval of convergence of this Taylor series. *[4]*

c. Use your work for **a** to help find the Taylor series of $g(x) = \sin(x)$ at $a = \pi$. *[3]*

10. Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$.

a. Find the radius and interval of convergence of this power series. *[7]*

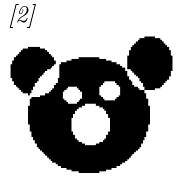
b. What function has this power series as its Taylor series at $a = 0$? *[8]*

[Total = 100]

Part i. Bonus!

$s\pi$**.** Write a little poem about calculus or mathematics in general. *[2]*

$+\iota$**.** Suppose a number of circles are drawn on a piece of paper, dividing it up into regions whose borders are made up of circular arcs. Prove that you can colour these regions with only *two* colours in such a way that no two regions that have a common border have the same colour. *[2]*



I HOPE THAT YOU ENJOYED THE COURSE! ENJOY THE SUMMER TOO!

2

Mathematics 105H
Applied calculus
Summer 2008

Mathematics 105H – Applied calculus
TRENT UNIVERSITY, Summer 2008

Quizzes

Quiz #1. Thursday, 1 May, 2008 *[10 minutes]*

1. Find the equation of the line between $(1,3)$ and $(5,-1)$. *[2]*
2. Find the equation of the line through $(2,2)$ perpendicular to the line in part 1. *[2]*
3. Sketch a graph of the lines in parts 1 and 2. *[1]*

Quiz #2. Tuesday, 6 May, 2008 *[10 minutes]*

1. Find the x- and y-intercepts (if any), and the domain and range, of $f(x) = 1 - \frac{1}{x}$, and give a rough sketch of its graph. *[5]*

Quiz #3. Thursday, 8 May, 2008 *[10 minutes]*

Using the fact that $\log_a(u) = t$ really means that $a^t = u$ and the properties of exponents, explain why each of the following equations works:

1. $\log_2(1) = 0$ *[1]*
2. $\log_2(\log_3(81)) = 2$ *[2]*
3. $\log_2(41) = \log_{1/2}\left(\frac{1}{41}\right)$ *[2]*

Quiz #4. Tuesday, 13 May, 2008 *[10 minutes]*

Do *one* of problems 1 and 2.

1. Use the limit definition of the derivative to compute $f'(2)$ for $f(x) = x^2 + 2x - 3$. *[10]*
2. Use the limit rules to compute $\lim_{x \to -\infty} \frac{x^2 2^x}{(3-x)^2}$. *[10]*

Quiz #5. Thursday, 15 May, 2008 *[10 minutes]*

Do any *two* of problems 1–3.

1. Use the limit definition of the derivative to compute $f'(x)$ for $f(x) = \frac{1}{x}$. *[5]*
2. Compute $f'(1)$ if $f(x) = e^x \ln\left(x^7\right)$. *[5]*
3. Find the point(s) on the graph of $y = x^3 - x$ where the graph has slope 0. *[5]*

Quiz #6. Thursday, 22 May, 2008 *[10 minutes]*

1. Find the coordinates of the point(s), if any, on the graph of $y = x^3 - 9x$ where the graph has slope 3. *[5]*

Quiz #7. Tuesday, 27 May, 2008 *[10 minutes]*

1. Find the domain and all the intercepts, vertical and horizontal asymptotes, and local minimum and maximum points of $y = \frac{x}{1+x^2}$, and sketch the graph of this function. *[5]*

Quiz #8. Thursday, 29 May, 2008 *[10 minutes]*

1. A spherical balloon is being filled at a rate of 1 m^3/s. How is the radius of the balloon changing at the instant that the balloon's radius is 2 m? *[5]*

   The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Quiz #9. Tuesday, 3 June, 2008 *[10 minutes]*

Compute *two* of the following three indefinite integrals. *[10 = 2 × 5 each]*

$$\textbf{1.} \int (w-1)^2\, dw \qquad \textbf{2.} \int \frac{\ln(x)}{x}\, dx \qquad \textbf{3.} \int \frac{1-\sqrt{x}}{\sqrt{x}}\, dx$$

Quiz #10. Thursday, 5 June, 2008 *[10 minutes]*

1. Find the area of the region between the x-axis and the curve $y = x^3 - x = x(x+1)(x-1)$ for $-1 \leq x \leq 1$. *[5]*

Mathematics 105H – Applied calculus
TRENT UNIVERSITY, Summer 2008

MATH 105H Test
20 May, 2008
Time: 50 minutes

Instructions

- *Show all your work.* Legibly!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator, and either (both sides of) one 8.5×11 aid sheet or a copy (annotated as you like) of *Formula for Success*.

1. Do *all* of **a**–**d**.

a. Find the equation of the line through $(0, 3)$ with slope 2. *[2]*

b. Find the tip (*i.e.* vertex) of the parabola $y = -x^2 + 2x + 2$. *[2]*

c. Find the coordinates of the point(s) where the line and the parabola meet. *[3]*

d. Sketch a graph of the line and the parabola. *[3]*

2. Let $f(x) = 1 - e^x$. Do *all* of **a**–**d**.

a. Find the x- and y-intercepts, if any, of $y = f(x)$. *[2]*

b. Find the domain and range of $f(x)$. *[2]*

c. Determine whether $y = f(x)$ has any horizontal asymptotes. *[4]*

d. Sketch the graph of $y = f(x)$. *[2]*

3. Do any *two (2)* of **a**–**c**.

a. Compute $\displaystyle\lim_{x \to -\infty} \frac{1-x}{1+2x} 3^x$. *[5]*

b. Compute $\displaystyle\lim_{t \to 3} \frac{(t-3)(t+2)}{t^2 - 2t - 3}$. *[5]*

c. Use the limit definition of the derivative to compute $f'(1)$ for $f(x) = 2x^2 - 4x + 8$. *[5]*

4. Do any *two (2)* of **a**–**c**.

a. Find $h'(x)$ if $h(x) = e^{\left(x^2+3\right)} - \ln(5x)$. *[5]*

b. Find $g'(w)$ if $g(w) = \dfrac{w\ln(w)}{(w+1)^2}$ *[5]*

c. Find the equation of the tangent line to $y = \frac{1}{2}x^2 + x - 1$ at $x = 3$. *[5]*

[Total = 40]

Mathematics 105H – Applied calculus
Trent University, Summer 2008
Final Examination
Tuesday, 10 June, 2008

Time: 3 hours *Brought to you by* Стефан Біланюк.

Instructions: Show all your work and justify all your answers. *If in doubt,* **ask!**

Aids: Calculator; an $8.5'' \times 11''$ aid sheet or the pamphlet *Formula for Success*; one brain.

Part I. Do all four of **1** – **4**.

1. Do any *three* of **a** – **f**. *[15 = 3 × 5 each]*

a. Find the y-intercepts of the parabola $y = x^2 - 2x - 3$.

b. Evaluate $\lim_{x \to 0} \dfrac{x+1}{x^2+2x+1}$.

c. Find the equation of the line passing through $(1, 0)$ and parallel to $y = 3x + 1024$.

d. Determine whether $f(x) = \dfrac{x}{|x|}$ is continuous at $x = 0$ or not.

e. Use the limit definition of the derivative to compute $f'(x)$ for $f(x) = \frac{1}{2}x^2 + x$.

f. Find the equation of the tangent line to $y = e^x$ at $x = 0$.

2. Find $\dfrac{dy}{dx}$ (in terms of x and/or y) in any *four* of **a** – **f**. *[20 = 4 × 5 each]*

a. $y = (x-1)(x+1)$ **b.** $\ln(x+y) = x$ **c.** $y = \dfrac{x}{x+2}$

d. $y = \displaystyle\int_1^x t^2\,dt$ **e.** $y = e^{x^2 - x}$ **f.** $y = \dfrac{x-1}{x^2+2x-3}$

3. Evaluate any *four* of the integrals **a** – **f**. *[20 = 4 × 5 each]*

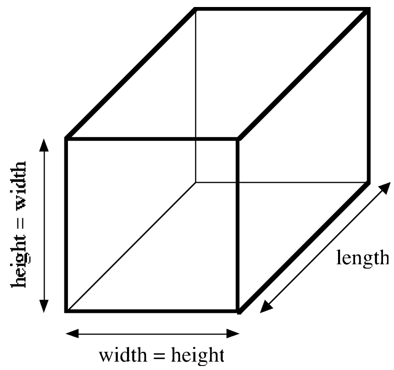
a. $\displaystyle\int_0^2 3x^2\,dx$ **b.** $\displaystyle\int t\ln(t)\,dt$ **c.** $\displaystyle\int_2^3 \frac{2x-2}{x^2-2x+1}\,dx$

d. $\displaystyle\int x\sqrt{1+x^2}\,dx$ **e.** $\displaystyle\int_0^5 \frac{1}{x+5}\,dx$ **f.** $\displaystyle\int_0^\infty 2^{-x}\,dx$

4. Find the intercepts, domain, all maximum, minimum, and inflection points, and all vertical and horizontal asymptotes of $f(x) = \dfrac{x^2}{x^2+1}$, and sketch its graph. *[15]*

Part II. Do any *two* of **5** – **8**.

5. A steel frame is to be made for a box with a square cross-section (so height = width) that is to have a volume of 36 m^3. What is the maximum volume of the box if the total length of steel available for the frame is 16 m? *[15]*



6. A 10 m ladder is placed with its base on the positive x-axis and its tip on the positive y-axis. Its base begins to slip to the right at a constant rate of 1 m/s. Determine how the top of the ladder is moving at the instant that the base of the ladder is 6 m from the y-axis. *[15]*

7. A worker sketches the curves $y = x^3$ and $y = 4x$ on a sheet of metal and cuts out the region between the curves to form a metal plate. Find the area of the plate if distances are measured in cm. *[15]*

8. Consider the region bounded above by $y = \sqrt{x}$ and below by the x-axis for $0 \leq x \leq 4$.

- **a.** Sketch the region. *[2]*
- **b.** Find the area of the region. *[3]*
- **c.** Sketch the solid obtained by revolving the region about the x-axis. *[2]*
- **d.** Find the volume of this solid. *[8]*

[Total = 100]

Part □. Bonus!

△. Write a little poem about calculus or mathematics in general. *[2]*

2

Mathematics 1100Y
Calculus I: Calculus of one variable
Summer 2010

Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2010

Quizzes

Quiz #1 *Wednesday, 12 May, 2010. [10 minutes]*

1. Suppose the graph of $y = x^2$ is stretched vertically by a factor of 3, and then shifted by 2 units to the right and 1 unit down. Find the formula of the parabola with this curve as its graph. *[5]*
2. Use the Limit Laws to evaluate $\lim_{x \to 0} \frac{x^2 - 1}{x^2 + 1}$. *[5]*

Quiz #2 *Monday, 17 May, 2010. [12 minutes]*

Do *one* (1) of the following two questions.

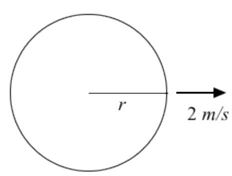
1. Find all the vertical and horizontal asymptotes of $f(x) = \frac{x}{x-1}$ and give a rough sketch of its graph. *[10]*
2. Use the ε–δ definition of limits to verify that $\lim_{x \to 1} (3x - 1) = 2$. *[10]*

Quiz #3 *Wednesday, 19 May, 2010. [10 minutes]*

1. Compute the derivative of $f(x) = \frac{x^2 - 2x}{x - 1}$. *[5]*
2. Compute the derivative of $g(x) = \arctan(e^x)$. *[5]*

Quiz #4 *Wednesday, 26 May, 2010. [12 minutes]*

1. Use logarithmic differentiation to compute the derivative of $g(x) = x^x$. *[5]*
2. A pebble is dropped into a still pond, creating a circular ripple that moves outward from its centre at 2 *m/s*. How is the area enclosed by the ripple changing at the instant that the radius of the ripple is 3 *m*? *[5]*



(*Just in case:* The area of a circle with radius r is πr^2.)

Quiz #5 *Monday, 31 May, 2010. [15 minutes]*

1. Let $f(x) = \frac{x}{x^2 + 1}$. Find the domain and all the intercepts, vertical and horizontal asymptotes, and local maxima and minima of $f(x)$, and sketch its graph using this information. *[10]*

Quiz #6 *Wednesday, 2 June, 2010. [10 minutes]*

1. Use the Left-Hand Rule to compute $\int_0^1 (x+1)\,dx$, the area between the line $y = x+1$ and the x-axis for $0 \leq x \leq 1$. *[10]*

   Hint: You may need the formula $\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$.

Quiz #7 *Monday, 7 June, 2010. [10 minutes]*

1. Compute $\int_0^2 \left(x^2 - 2x + 1\right)\,dx$. *[10]*

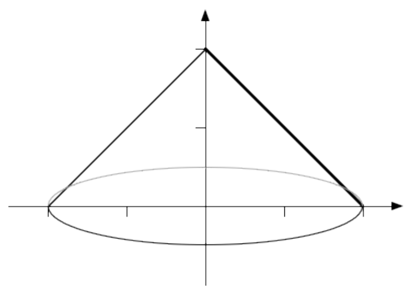
Quiz #8 *Wednesday, 9 June, 2010. [10 minutes]*

1. Find the area between $y = x\cos\left(x^2\right)$ and the x-axis for $-\sqrt{\frac{\pi}{2}} \leq x \leq \sqrt{\frac{\pi}{2}}$. *[10]*

Quiz #9 *Monday, 14 June, 2010. [10 minutes]*

The region between $y = 2 - x$ and the x-axis, for $0 \leq x \leq 2$, is rotated about the y-axis. Find the volume of the resulting solid of revolution using both

1. the disk method *[5]* and
2. the method of cylindrical shells. *[5]*



Quiz #10 *Wednesday, 16 June, 2010. [10 minutes]*

1. Compute $\int_1^e (\ln(x))^2\,dx$. *[10]*

Quiz #11 *Monday, 21 June, 2010. [12 minutes]*

Compute each of the following integrals:

1. $\int_0^{\pi/2} \cos^3(x)\sin^2(x)\,dx$ *[5]* 2. $\int \sec^3(x)\,dx$ *[5]*

Quiz #12 *Wednesday, 23 June, 2010. [15 minutes]*

Compute each of the following integrals:

1. $\int \frac{1}{\sqrt{4-x^2}}\,dx$ *[5]* 2. $\int_1^2 x\sqrt{x^2-1}\,dx$ *[5]*

2

Quiz #13 *Monday, 28 June, 2010. [12 minutes]*

1. Compute $\displaystyle\int \frac{2x^2+3}{(x^2+4)(x-1)}\,dx$. *[10]*

Quiz #14 *Wednesday, 30 June, 2010. [10 minutes]*

1. Compute $\displaystyle\int_0^\infty \frac{1}{x^2+1}\,dx$. *[10]*

Quiz #15 *Monday, 5 July, 2010. [10 minutes]*

1. Compute the arc-length of the curve $y = \frac{2}{3}x^{3/2}$, where $0 \le x \le 1$.

Quiz #16 *Wednesday, 7 July, 2010. [15 minutes]*

1. Find the arc-length of the parametric curve $x = t\cos(t)$ and $y = t\sin(t)$, where $0 \le t \le 1$. *[10]*

Quiz #17 *Monday, 12 July, 2010. [15 minutes]*

1. Sketch the curve given by $r = \sin(\theta)$, $0 \le \theta \le \pi$, in polar coordinates. *[2]*
2. Sketch the curve given by $r = \sin(\theta)$, $\pi \le \theta \le 2\pi$, in polar coordinates. *[2]*
3. Find the area of the region enclosed by the curve given by $r = \sin(\theta)$, $0 \le \theta \le \pi$, in polar coordinates. *[6]*

Bonus: Find an equation in Cartesian coordinates for the curve given by $r = \sin(\theta)$, $0 \le \theta \le \pi$, in polar coordinates. *[2]*

Quiz #18 *Wednesday, 14 July, 2010. [12 minutes]*

1. Use the definition of convergence of a series to compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. *[10]*

Hint: Note that $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$.

Quiz #19 *Monday, 19 July, 2010. [10 minutes]*

1. Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges or diverges. *[10]*

Quiz #20 *Wednesday, 21 July, 2010. [12 minutes]*

1. Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{1+n}$ converges conditionally, converges absolutely, or diverges. *[10]*

Quiz #21 *Monday, 26 July, 2010. [15 minutes]*

1. Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{n3^n}{2^{n+1}}x^n$. *[10]*

Quiz #22 *Wednesday, 28 July, 2010. [15 minutes]*

1. Find the Taylor series of $f(x) = \ln(x)$ at $a = 1$. *[10]*

3

Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2010

Test 1
7 June, 2010

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

a. Find the slope of the tangent line to $y = \tan(x)$ at $x = 0$.

b. Use the limit definition of the derivative to compute $f'(1)$ for $f(x) = x^2$.

c. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x\to 1}(2x-1) = 1$.

2. Find $\frac{dy}{dx}$ in any *three* (3) of **a**–**d**. *[9 = 3 × 3 each]*

a. $y = \dfrac{x}{x+1}$ **b.** $x^2 + y^2 = 4$ **c.** $y = \displaystyle\int_0^x t\cos(3t)\,dt$ **d.** $y = \ln\left(x^3\right)$

3. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

a. Explain why $\lim_{x\to 0} \dfrac{x}{|x|}$ doesn't exist.

b. A spherical balloon is being inflated at a rate of $1\ m^3/s$. How is its radius changing at the instant that it is equal to $2\ m$? [The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.]

c. Use the Left-Hand Rule to find $\displaystyle\int_1^3 x\,dx$. $\left[\ \displaystyle\sum_{i=0}^{n-1} i = 0 + 1 + \cdots + (n-1) = \tfrac{n(n-1)}{2}\ \right]$

4. Let $f(x) = \dfrac{x^2}{x^2+1}$. Find the domain and all the intercepts, vertical and horizontal asymptotes, and maxima and minima of $f(x)$, and sketch its graph using this information. *[11]*

[Total = 40]

Bonus. Find any inflection points of $f(x) = \dfrac{x^2}{x^2+1}$ as well. *[3]*

Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2010

Test 2
5 July, 2010

Instructions
- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Compute any *four* (4) of the integrals in parts **a**-**f**. *[16 = 4 × 4 each]*

a. $\displaystyle\int \frac{1}{4-x^2}\,dx$ **b.** $\displaystyle\int \tan(x)\,dx$ **c.** $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

d. $\displaystyle\int \frac{x^3+x+1}{x^2+1}\,dx$ **e.** $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2(x)\,dx$ **f.** $\displaystyle\int x\ln(x)\,dx$

2. Do any *two* (2) of parts **a**-**e**. *[12 = 2 × 6 each]*

a. Compute $\displaystyle\int_0^2 (x+1)\,dx$ using the Right-hand Rule.

b. Find the area of the region bounded by $y = 2 + x$ and $y = x^2$ for $-1 \le x \le 1$.

c. Without actually computing $\displaystyle\int_0^{10/\pi} \arctan(x)\,dx$, find as small an upper bound as you can on the value of this integral.

d. Compute the arc-length of the curve $y = \ln(\cos(x))$, $0 \le x \le \pi/6$.

e. Give a example of a function $f(x)$ such that $f(x) = 1 + \displaystyle\int_0^x f(t)\,dt$ for all x.

3. Do *one* (1) of parts **a** or **b**. *[12]*

a. Sketch the solid obtained by rotating the region bounded by $y = \sqrt{x}$ and $y = x$, where $0 \le x \le 1$, about the y-axis, and find its volume.

b. Sketch the cone obtained by rotating the line $y = 3x$, where $0 \le x \le 2$, about the x-axis, and find its surface area.

[Total = 40]

Mathematics 1100Y – Calculus I: Calculus of one variable
Trent University, Summer 2010
Final Examination

Time: *09:00–12:00, on Friday, 30 July, 2010.* *Brought to you by* Стефан Біланюк.
Instructions: Show all your work and justify all your answers. *If in doubt,* **ask!**
Aids: Calculator; one aid sheet (all sides!); one brain (no limit on active neurons).

Part I. Do all three (3) of **1**–**3**.

1. Compute $\frac{dy}{dx}$ as best you can in any *three* (3) of **a**–**f**. *[15 = 3 × 5 each]*

a. $x^2 + 3xy + y^2 = 23$ **b.** $y = \ln(\tan(x))$ **c.** $y = \int_x^3 \ln(\tan(t))\, dt$

d. $y = \frac{e^x}{e^x - e^{-x}}$ **e.** $\begin{array}{l} x = \cos(2t) \\ y = \sin(3t) \end{array}$ **f.** $y = (x+2)e^x$

2. Evaluate any *three* (3) of the integrals **a**–**f**. *[15 = 3 × 5 each]*

a. $\int_{-\pi/4}^{\pi/4} \tan(x)\, dx$ **b.** $\int \frac{1}{t^2 - 1}\, dt$ **c.** $\int_0^{\pi} x\cos(x)\, dx$

d. $\int \sqrt{w^2 + 9}\, dw$ **e.** $\int_1^e \ln(x)\, dx$ **f.** $\int \frac{e^x}{e^{2x} + 2e^x + 1}\, dx$

3. Do any *five* (5) of **a**–**i**. *[25 = 5 × 5 ea.]*

a. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $0 \le x \le 4$, the x-axis, and $x = 4$, about the x-axis.

b. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 1} 3x = 3$.

c. Find the Taylor series of $f(x) = \frac{x^2}{1 - x^2}$ at $a = 0$ without taking any derivatives.

d. Sketch the polar curve $r = 1 + \sin(\theta)$ for $0 \le \theta \le 2\pi$.

e. Use the limit definition of the derivative to compute $f'(1)$ for $f(x) = x^2$.

f. Use the Right-hand Rule to compute the definite integral $\int_1^2 \frac{x}{2}\, dx$.

g. Determine whether the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges absolutely, converges conditionally, or diverges.

h. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n^2}{\pi^n} x^n$.

i. Compute the arc-length of the polar curve $r = \theta$, $0 \le \theta \le 1$.

Part II. Do any *two* (2) of **4**–**6**.

4. Find the domain, all maximum, minimum, and inflection points, and all vertical and horizontal asymptotes of $f(x) = e^{-x^2}$, and sketch its graph. *[15]*

5. Find the area of the surface obtained by rotating the curve $y = \tan(x)$, $0 \leq x \leq \frac{\pi}{4}$, about the x-axis. *[15]*

6. Find the volume of the solid obtained by rotating the region below $y = 1 - x^2$, $-1 \leq x \leq 1$, and above the x-axis about the line $x = 2$. *[15]*

Part III. Do *one* (1) of **7** or **8**.

7. Do all three (3) of **a**–**c**.

a. Use Taylor's formula to find the Taylor series of e^x centred at $a = -1$. *[7]*

b. Determine the radius and interval of convergence of this Taylor series. *[4]*

c. Find the Taylor series of e^x centred at $a = -1$ using the fact that the Taylor series of e^x centred at 0 is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$. *[4]*

8. Do all three (3) of **a**–**c**. You may assume that the Taylor series of $f(x) = \ln(1+x)$ centred at $a = 0$ is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$.

a. Find the radius and interval of convergence of this Taylor series. *[6]*

b. Use this series to show that $\ln\left(\frac{3}{2}\right) = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}$. *[3]*

c. Find an n such that $T_n\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \cdots + \frac{(-1)^{n+1}}{n2^n}$ is guaranteed to be within $0.01 = \frac{1}{100}$ of $\ln\left(\frac{3}{2}\right)$. *[6]*

[Total = 100]

Part IV - Something different. Bonus!

$\mathbf{e^{i\pi}}$**.** Write a haiku touching on caclulus or mathmatics in general. *[2]*

haiku?

seventeen in three:
five and seven and five of
syllables in lines

I HOPE THAT YOU ENJOYED THE COURSE. ENJOY THE REST OF THE SUMMER!

Mathematics 1101Y
Calculus I: Functions and calculus of one variable
2010-2011

Mathematics 1101Y – Calculus I: functions and calculus of one variable
TRENT UNIVERSITY, 2010–2011

Quizzes

Quiz #1. ~~Friday, 24~~ Monday, 27 September, 2010. *(10 minutes)*

1. Find the location of the tip of the parabola $y = 2x^2 + 2x - 12$, as well as its x- and y-intercepts. *[5]*

Quiz #2. Friday, 1 October, 2010. *(10 minutes)*

1. Solve the equation $e^{2x} - 2e^x + 1 = 0$ for x.

Hint: Solve for e^x first ...

Quiz #3. Friday, 8 October, 2010. *(10 minutes)*

1. Evaluate the limit $\displaystyle\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1}$, if it exists. *[5]*

Quiz #4. Friday, 15 October, 2010. *(10 minutes)*

1. Use the limit definition of the derivative to compute $f'(2)$ if $f(x) = x^2 + 3x + 1$. *[5]*

Quiz #5. ~~Friday, 22 October~~ Monday, 1 November, 2010. *(10 minutes)*

1. Find $f'(x)$ if $f(x) = \dfrac{x^2 + 2x}{x^2 + 2x + 1}$. Simplify $f'(x)$ as much as you reasonably can. *[5]*

Quiz #6. Friday, 5 November, 2010. *(10 minutes)*

1. Find $\dfrac{dy}{dx}$ if $y = \sqrt{x + \arctan(x)}$. *[5]*

Quiz #7. Friday, 12 November, 2010. *(10 minutes)*

1. Find the maximum and minimum of $f(x) = \dfrac{x}{1 + x^2}$ on the interval $[-2, 2]$. *[5]*

Quiz #8. Friday, 26 November, 2010. *(10 minutes)*

1. Find an antiderivative of $f(x) = 4x^3 - 3\cos(x) + \dfrac{1}{x}$. *[5]*

Quiz #9. Friday, 3 December, 2010. *(10 minutes)*

1. Compute the definite integral $\displaystyle\int_0^1 (2x + 1)\, dx$ using the Right-hand Rule. *[5]*

Hint: You may assume that $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Quiz #10. Friday, 10 December, 2010. *(10 minutes)*

1. Find the area between the graphs of $f(x) = \sin(x)$ and $g(x) = \dfrac{2x}{\pi}$ for $0 \le x \le \dfrac{\pi}{2}$. *[5]*

Quiz #11. Friday, 14 January, 2011. *(10 minutes)*

1. Compute $\displaystyle\int_0^{\pi/2} \cos^3(x)\, dx$. *[5]*

1

Quiz #12. Friday, 21 January, 2011. *(10 minutes)*

1. Compute $\int \tan^3(x) \sec(x)\, dx$. *[5]*

Quiz #13. Friday, 28 January, 2011. *(10 minutes)*

1. Compute $\int \frac{1}{\sqrt{4+x^2}}\, dx$. *[5]*

Quiz #14. Friday, 4 February, 2011. *(15 minutes)*

1. Compute $\int \frac{4x^2+3x}{(x+2)(x^2+1)}\, dx$. *[5]*

Quiz #15. Some time or other, 2011. *(15 minutes)*

1. Find the area of the surface obtained by revolving the curve $y = \sqrt{1-x^2}$, where $0 \leq x \leq 1$, about the y-axis. *[5]*

Quiz #16. Some time or other, 2011. *(12 minutes)*

1. Sketch the region bounded by $r = \tan(\theta)$, $\theta = 0$, and $\theta = \frac{\pi}{4}$ in polar coordinates and find its area. *[5]*

Quiz #17. Friday, 11 March, 2011. *(12 minutes)*

1. Find the arc-length of the parametric curve $x = \sec(t)$, $y = \ln(\sec(t) + \tan(t))$, where $0 \leq t \leq \frac{\pi}{4}$.

Quiz #18. Friday, 18 March, 2011. *(10 minutes)*

1. Compute $\lim_{n\to\infty} \frac{n^2}{e^n}$. *[5]*

Quiz #19. Friday, 25 March, 2011. *(10 minutes)*

1. Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{n^2+2^n}$ converges or diverges. *[5]*

Quiz #20. Friday, 1 April, 2011. *(15 minutes)*

1. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ converges absolutely, converges conditionally, or diverges.

Quiz #21. Friday, 8 April, 2011. *(10 minutes)*

1. Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges or diverges.

Mathematics 1101Y – Calculus I: functions and calculus of one variable
Trent University, 2010–2011

Test #1
Friday, 19 November, 2010
Time: 50 minutes

Instructions
- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Find $\frac{dy}{dx}$ in any *three* (3) of **a-e**. *[12 = 3 × 4 each]*

a. $y = x^x$ **b.** $y = \frac{1}{1+x^2}$ **c.** $y = \cos\left(\sqrt{x}\right)$ **d.** $y^2 + x = 1$ **e.** $y = x^2 e^{-x}$

2. Do any *two* (2) of **a-d**. *[10 = 2 × 5 each]*

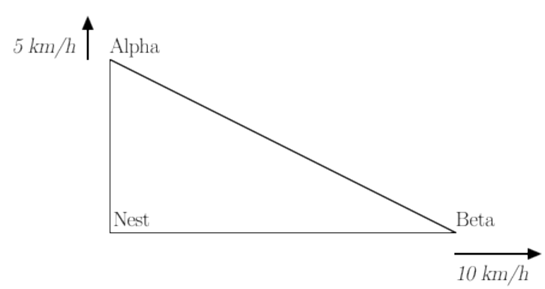
a. Use the limit definition of the derivative to compute $f'(0)$ for $f(x) = x^2 - 3x + \pi$.

b. Suppose $f(x) = \frac{x}{\sin(x)}$ for $x \neq 0$. What would $f(0)$ have to be to make $f(x)$ continuous at $a = 0$?

c. Find the equation of the tangent line to $y = x^2$ at the point $(2, 4)$.

d. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 1}(2x + 3) = 5$.

3. Birds Alpha and Beta leave their nest at the same time, with Alpha flying due north at 5 *km/h* and Beta flying due east at 10 *km/h*. How is the area of the triangle formed by their respective positions and the nest changing 1 *h* after their departure? *[8]*



4. Find the domain and all intercepts, maxima and minima, and vertical and horizontal asymptotes of $f(x) = \frac{x^2 + 2}{x^2 + 1}$ and sketch its graph based on this information. *[10]*

Mathematics 1101Y – Calculus I: functions and calculus of one variable
Trent University, 2010–2011

Test # 2
11 February, 2011
Time: 50 minutes

Instructions
- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Compute any *four* (4) of the integrals in parts **a**-**f**. *[16 = 4 × 4 each]*

a. $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx$ **b.** $\displaystyle\int_0^{\pi/4} \sec(x)\tan(x)\,dx$ **c.** $\displaystyle\int_0^{\infty} e^{-x}\,dx$

d. $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$ **e.** $\displaystyle\int \frac{\cos(x)}{\sin(x)}\,dx$ **f.** $\displaystyle\int_1^{e} \ln(x)\,dx$

2. Do any *two* (2) of parts **a**-**e**. *[12 = 2 × 6 each]*

a. Compute $\displaystyle\int_1^2 \frac{x^3-x^2-x+1}{x+1}\,dx$

b. Find the area between $y=\cos(x)$ and $y=\sin(x)$ for $0 \le x \le \dfrac{\pi}{2}$.

c. Which of $\displaystyle\int_\pi^{41} \arctan\left(\sqrt{x}\right)\,dx$ and $\displaystyle\int_\pi^{41} \arctan\left(x^2\right)\,dx$ is larger? Explain why.

d. Use the Right-hand Rule to compute $\displaystyle\int_1^2 x\,dx$.

e. Find the area of the region bounded by $y=0$ and $y=\ln(x)$ for $0 < x \le 1$.

3. Do *one* (1) of parts **a** or **b**. *[12]*

a. Sketch the solid obtained by rotating the region bounded by $y=x^2$ and $y=0$, where $0 \le x \le 2$, about the y-axis, and find its volume.

b. Sketch the solid obtained by rotating the region bounded by $y=x^2$ and $y=0$, where $0 \le x \le 2$, about the x-axis, and find its volume.

[Total = 40]

Mathematics 1101Y – Calculus I: functions and calculus of one variable
TRENT UNIVERSITY, 2010–2011
Final Examination

Time: *14:00–17:00, on Tuesday, 26 April, 2011.* *Brought to you by* Стефан Біланюк.

Instructions: Do parts **X** and **Y** and, if you wish, part **Z**. Show all your work and justify all your answers. *If in doubt about something,* **ask!**

Aids: Calculator; one aid sheet (all sides!); one brain (10^{100} neuron limit).

Part X. Do all three (3) of **1**–**3**.

1. Compute $\frac{dy}{dx}$ as best you can in any *three* (3) of **a**–**f**. *[15 = 3 × 5 each]*

a. $y = \cos(e^x)$ **b.** $y = \int_1^x e^t \ln(t)\,dt$ **c.** $y = x\ln(x)$

d. $y = \frac{\ln(x)}{x}$ **e.** $\arctan(x+y) = 0$ **f.** $\begin{array}{l} x = e^t \\ y = e^{2t} \end{array}$

2. Evaluate any *three* (3) of the integrals **a**–**f**. *[15 = 3 × 5 each]*

a. $\int \frac{\ln(x)}{x}\,dx$ **b.** $\int \frac{1}{\sqrt{z^2-1}}\,dz$ **c.** $\int_1^4 \sqrt{x}\,dx$

d. $\int_0^1 \frac{x^2+2}{x^2+1}\,dx$ **e.** $\int \tan^2(w)\,dw$ **f.** $\int_0^{\ln(2)} e^{2t}\,dt$

3. Do any *five* (5) of **a**–**i**. *[25 = 5 × 5 each]*

a. Use the limit definition of the derivative to compute $g'(0)$ for $g(x) = 2x+1$.

b. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ converges or diverges.

c. Find the Taylor series of $f(x) = e^{x+1}$ at $a = 0$.

d. Sketch the polar curve $r = 1$, where $0 \le \theta \le 2\pi$, and find the area of the region it encloses.

e. Sketch the surface obtained by rotating $y = \frac{x^2}{2}$, $0 \le x \le 2$, about the y-axis, and find its area.

f. Use the Right-hand Rule to compute the definite integral $\int_0^1 4x\,dx$.

g. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x\to 0}(2x-1) = -1$.

h. Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$.

i. Sketch the solid obtained by rotating the region bounded by $y = x$, $y = 0$, and $x = 2$, about the x-axis, and find its volume.

Part Y. Do any *three* (3) of **4**–**7**. *[45 = 3 × 15 each]*

4. A zombie is dropped into a still pool, creating a circular ripple that moves outward from the point of impact at a constant speed. After 2 s the length of the ripple is increasing at a rate of 2π m/s. How is the area enclosed by the ripple changing at this instant?

Hint: You have (just!) enough information to work out how the radius of the ripple changes with time.

5. Find all the intercepts, maximum, minimum, and inflection points, and all the vertical and horizontal asymptotes of $h(x) = \dfrac{x}{1-x^2}$, and sketch its graph.

6. Show that a cone with base radius 1 and height 2 has volume $\frac{2}{3}\pi$.

Hint: It's a solid of revolution ...

7. Do all *four* (4) of **a**–**d**.

a. Use Taylor's formula to find the Taylor series of $f(x) = \sin(x)$ at $a = 0$. *[7]*

b. Determine the radius and interval of convergence of this Taylor series. *[4]*

c. Find the Taylor series of $g(x) = x\sin(x)$ at $a = 0$ by multiplying the Taylor series for $f(x) = \sin(x)$ by x. *[1]*

d. Use Taylor's formula and your series from **c** to compute $g^{(16)}(0)$. *[3]*

[Total = 100]

Part Z. Bonus problems! Do them (or not), if you feel like it.

$\ln\left(\frac{1}{e}\right)$. Does $\displaystyle\lim_{n\to\infty}\left[\left(\sum_{k=1}^{n}\frac{1}{k}\right)-\ln(n)\right]$ exist? Explain why or why not. *[2]*

$\ln\left(\frac{1}{1}\right)$. Write a haiku touching on calculus or mathematics in general. *[2]*

haiku?

seventeen in three:
five and seven and five of
syllables in lines

I hope that you enjoyed this course,
and even learned a thing or two. :-)
Have a great summer!

Mathematics 1100Y
Calculus I: Calculus of one variable
Summer 2011

Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2011

Quizzes

Quiz #1. Wednesday, 11 May, 2011. *[10 minutes]*

1. Compute $\lim_{x \to -3} \frac{x+3}{x^2-9}$ using the appropriate limit laws and algebra. *[5]*

Quiz #2. Monday, 16 May, 2011. *[10 minutes]*

Do *one* of questions 1 or 2.

1. Use the ε-δ definition of limits to verify that $\lim_{x \to 1}(3x-2) = 1$. *[5]*
2. Find the x- and y-intercepts and all the horizontal asymptotes of $f(x) = \frac{x^2}{x^2+1}$, and sketch its graph. *[5]*

Quiz #3. Wednesday, 18 May, 2011. *[10 minutes]*

1. Use the limit definition of the derivative to compute $f'(a)$ for $f(x) = \frac{1}{x}$. (You may assume that $a \neq 0$.) *[5]*

Quiz #4. Wednesday, 25 May, 2011. *[10 minutes]*

1. Compute $f'(x)$ for $f(x) = \ln\left(\frac{x}{1+x^2}\right)$. *[5]*

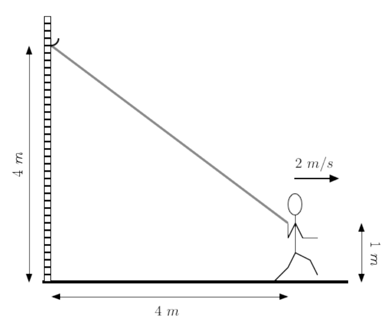
Quiz #5. Monday, 30 May, 2011. *[10 minutes]*

Do *one* of questions 1 or 2.

1. Find $\frac{dy}{dx}$ at the point $(2, 2)$ on the curve defined by $x = \sqrt{x+y}$. *[5]*
2. Find $\frac{dy}{dx}$ at $x = e$ for $y = \ln(x\ln(x))$. *[5]*

Quiz #6. Wednesday, 1 June, 2011. *[15 minutes]*

1. A $3\,m$ long, very stretchy, bungee cord is suspended from a hook $4\,m$ up on a wall. The other end of the cord is grabbed by a child who runs directly away from the wall at $2\,m/s$, holding the end of the cord $1\,m$ off the ground, stretching the cord in the process. How is the length of the cord changing at the instant that the child's end of the cord is $4\,m$ away from the wall? *[5]*



1

Quiz #7. Monday, 6 June, 2011. *[15 minutes]*

1. Find any and all intercepts, intervals of increase and decrease, local maxima and minima, and vertical and horizontal asymptotes, of $y = xe^{-x}$, and sketch this curve based on the information you obtained. *[5]*

Bonus: Find any and all the points of inflection of this curve too. *[1]*

Hint: You may assume that $\lim_{x \to +\infty} xe^{-x} = 0$. For $\lim_{x \to -\infty} xe^{-x}$ you're on your own.

Quiz #8. Monday, 13 June, 2011. *[10 minutes]*

1. Compute $\lim_{x \to \infty} \frac{x^2}{e^x}$. *[5]*

Quiz #9. Monday, 20 June, 2011. *[10 minutes]*

Do *one* of questions 1, 2, or 3.

1. Compute $\int_1^2 (x+1)\,dx$ using the Right-Hand Rule. *[5]*

   Hint: You may assume that $1 + 2 + 3 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

2. Compute $\int_{-1}^{3} (x+1)^2\,dx$. *[5]*
3. Compute $\int \sin(x)\cos(x)\,dx$. *[5]*

Quiz #10. Wednesday, 22 June, 2011. *[10 minutes]*

1. Find the area of the region between the curves $y = \cos(x)$ and $y = \sin(x)$, where $0 \le x \le \pi$. *[5]*

   Hint: Recall that $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

Quiz #11. Monday, 27 June, 2011. *[10 minutes]*

1. Sketch the solid obtained by rotating the region bounded by $y = \sqrt{x}$ and $y = 0$, where $0 \le x \le 4$, about the x-axis and find its volume. *[5]*

Quiz #12. Wednesday, 29 June, 2011. *[10 minutes]*

1. Sketch the solid obtained by rotating the region between $y = e^x$ and $y = 1$, where $0 \le x \le 1$, about the y-axis and find its volume. *[5]*

Quiz #13. Monday, 4 July, 2011. *[10 minutes]*

1. Compute $\displaystyle\int \sec^3(x)\tan^3(x)\,dx$. *[5]*

Quiz #14. Monday, 11 July, 2011. *[15 minutes]*

1. Compute $\displaystyle\int \frac{1}{x^4 + x^2}\,dx$. *[5]*

Quiz #15. Wednesday, 13 July, 2011. *[10 minutes]*

1. Compute $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$. *[5]*

Quiz #16. Monday, 18 July, 2011. *[12 minutes]*

Do *one* of questions 1 or 2.

1. Find the arc-length of the curve $y = \frac{2}{3}x^{3/2}$, where $0 \leq x \leq 3$. *[5]*
2. Find the area of the surface of revolution obtained by rotating the curve $y = 1 - \frac{1}{2}x^2$, where $0 \leq x \leq \sqrt{3}$, about the y-axis. *[5]*

Quiz #17. Wednesday, 19 July, 2011. *[12 minutes]*

Do *one* of questions 1 or 2.

1. Sketch the curve $r = \theta$, $0 \leq \theta \leq \pi$, in polar coordinates and the area of the region between the curve and the origin. *[5]*
2. For which values of x does the series $\sum_{n=0}^{\infty} x^{n+2} = x^2 + x^3 + x^4 + \cdots$ converge? What is the sum when it does converge? *[5]*

Quiz #18. Monday, 25 July, 2011. *[12 minutes]*

1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 2^n}$ converges or diverges. *[5]*

Quiz #19. Wednesday, 27 July, 2011. *[12 minutes]*

Do *one* of questions 1 or 2.

1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{(n+1)!}$ converges absolutely, converges conditionally, or diverges. *[5]*
2. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^n}{n}$ converges absolutely, converges conditionally, or diverges. *[5]*

3

Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, SUMMER 2011

MATH 1100Y Test #1
Wednesday, 8 June, 2011
Time: 50 minutes

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Find $\frac{dy}{dx}$ in any *three* (3) of **a**–**d**. *[9 = 3 × 3 each]*

a. $y = \left(x^2+1\right)^3$ **b.** $\ln(x+y) = 0$ **c.** $y = x^2 e^x$ **d.** $y = \dfrac{\tan(x)}{\sec(x)}$

2. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

a. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 2}(x+1) = 3$.

b. Use the limit definition of the derivative to compute $f'(0)$ for $f(x) = x^3 + x$.

c. Compute $\displaystyle\lim_{x \to 3} \frac{x^2-9}{x-3}$.

3. Do any *two* (2) of **a**–**c**. *[12 = 2 × 6 each]*

a. Each side of a square is increasing at a rate of $3\,cm/s$. At what rate is the area of the square increasing at the instant that the sides are $6\,cm$ long?

b. $f(x) = e^{-1/x^2} = e^{-(x^{-2})}$ has a removable discontinuity at $x = 0$. What should the value of $f(0)$ be to make the function continuous at $x = 0$?

c. What is the smallest possible perimeter of a rectangle with area $36\,cm^2$?

4. Let $f(x) = \sqrt{x^2+1}$. Find any and all intercepts, vertical and horizontal asymptotes, and maxima and minima of $f(x)$, and sketch its graph using this information. *[9]*

[Total = 40]

Bonus. Simplify $\cos(\arcsin(x))$ as much as you can. *[1]*

Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, SUMMER 2011

MATH 1100Y Test 2
6 July, 2011
Time: 50 minutes

Instructions
- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the extra page and the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Compute any *four* (4) of the integrals in parts **a**-**f**. *[16 = 4 × 4 each]*

a. $\displaystyle\int \tan^2(x)\,dx$ **b.** $\displaystyle\int_0^{3/2} 2(2x+1)^{3/2}\,dx$ **c.** $\displaystyle\int xe^x\,dx$

d. $\displaystyle\int_0^{\pi} x\cos(x)\,dx$ **e.** $\displaystyle\int \sec^3(x)\tan(x)\,dx$ **f.** $\displaystyle\int_0^1 \left(x^2+2x+3\right)\,dx$

2. Do any *two* (2) of parts **a**-**e**. *[12 = 2 × 6 each]*

a. Compute $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx$. What does this integral represent?

b. Sketch the solid obtained by rotating the region bounded by $y = x$, $y = 0$, and $x = 2$ about the y-axis, and find its volume.

c. Give an example of a function $f(x)$ with $f'(x) = 1 - \displaystyle\int_0^x f(t)\,dt$ for all x.

d. Sketch the region between $y = \sin(x)$ and $y = -\sin(x)$ for $0 \le x \le 2\pi$, and find its area.

e. Compute $\displaystyle\int_1^2 x\,dx$ using the Right-hand Rule.

3. The region between $y = \sqrt{1-x^2}$ and $y = 2x - 2$, where $0 \le x \le 1$, is rotated about the y-axis to make a solid. Do part **a** and *one* (1) of parts **b** or **c**.

a. Sketch the solid of revolution described above. *[3]*

b. Find the volume of the solid using the disk/washer method. *[9]*

c. Find the volume of the solid using the method of cylindrical shells. *[9]*

[Total = 40]

Mathematics 1100Y – Calculus I: Calculus of one variable
Trent University, Summer 2011
Final Examination

Time: 09:00–12:00, on Wednesday, 3 August, 2011. *Brought to you by* Стефан.
Instructions: Show all your work and justify all your answers. *If in doubt,* **ask!**
Aids: Calculator; two (2) aid sheets; one (1) brain [may be caffeinated].

Part I. Do *all* three (3) of **1**–**3**.

1. Compute $\frac{dy}{dx}$ as best you can in any *three* (3) of **a**–**f**. *[15 = 3 × 5 each]*

a. $x = e^{x+y}$ **b.** $y = \int_0^{-x} te^t\,dt$ **c.** $y = x^2\ln(x)$

d. $y = \frac{x}{\cos(x)}$ **e.** $y = \sec^2(\arctan(x))$ **f.** $y = \sin(e^x)$

2. Evaluate any *three* (3) of the integrals **a**–**f**. *[15 = 3 × 5 each]*

a. $\int \frac{2x}{\sqrt{4-x^2}}\,dx$ **b.** $\int_0^{\pi/2} \sin(z)\cos(z)\,dz$ **c.** $\int x^2\ln(x)\,dx$

d. $\int_{-\infty}^{\ln(3)} e^s\,ds$ **e.** $\int \frac{1}{\sqrt{1+x^2}}\,dx$ **f.** $\int_1^2 \frac{1}{w^2+w}\,dw$

3. Do any *five* (5) of **a**–**i**. *[25 = 5 × 5 ea.]*

a. Determine whether the series $\sum_{n=2}^{\infty} \frac{(-1)^n n^2}{3^n}$ converges absolutely, converges conditionally, or diverges.

b. Why must the arc-length of $y = \arctan(x)$, $0 \le x \le 13$, be less than $13 + \frac{\pi}{2}$?

c. Find a power series equal to $f(x) = \frac{x}{1+x}$ (when the series converges) without using Taylor's formula.

d. Find the area of the region between the origin and the polar curve $r = \frac{\pi}{2} + \theta$, where $0 \le \theta \le \pi$.

e. Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$.

f. Use the limit definition of the derivative to compute $f'(0)$ for $f(x) = 2x - 1$.

g. Compute the area of the surface obtained by rotating the the curve $y = \frac{x^2}{2}$, where $0 \le x \le \sqrt{3}$, about the y-axis.

h. Use the Right-hand Rule to compute the definite integral $\int_0^3 (x+1)\,dx$.

i. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 2}(x+1) = 3$.

1

Part II. Do any *three* (3) of **4**–**8**.

4. Find the domain, all maximum, minimum, and inflection points, and all vertical and horizontal asymptotes of $f(x) = \dfrac{x^2}{x^2+1}$, and sketch its graph. *[15]*

5. Do both of **a** and **b**.

a. Verify that $\displaystyle\int \sqrt{x^2-1}\,dx = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right) + C$. *[7]*

b. Find the arc-length of $y = \dfrac{1}{2}x\sqrt{x^2-1} - \dfrac{1}{2}\ln\left(x+\sqrt{x^2-1}\right)$ for $1 \leq x \leq 3$. *[8]*

6. Sketch the solid obtained by rotating the square with corners at $(1,0)$, $(1,1)$, $(2,0)$, and $(2,1)$ about the y-axis and find its volume and surface area. *[15]*

7. Do all three (3) of **a**–**c**.

a. Use Taylor's formula to find the Taylor series at 0 of $f(x) = \ln(x+1)$. *[7]*

b. Determine the radius and interval of convergence of this Taylor series. *[4]*

c. Use your answer to part **a** to find the Taylor series at 0 of $\dfrac{1}{x+1}$ without using Taylor's formula. *[4]*

8. A spherical balloon is being inflated at a rate of 1 m^3/s. How is its surface area changing at the instant that its volume is 36 m^3? *[15]*
[Recall that a sphere of radius r has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$.]

[Total = 100]

Part MMXI - Bonus problems.

13. Show that $\ln(\sec(x) - \tan(x)) = -\ln(\sec(x) + \tan(x))$. *[2]*

41. Write an original poem touching on calculus or mathematics in general. *[2]*

I HOPE THAT YOU HAD SOME FUN WITH THIS!
GET SOME REST NOW ...

Part II

Some Answers

Mathematics 110
Calculus I: Calculus of one variable
2001-2002
Solutions

Mathematics 110 – Calculus of one variable
Trent University 2001-2002

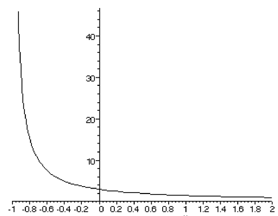
SKELETAL SOLUTIONS TO THE QUIZZES

Quiz #1. Friday, 21 September, 2001. *[15 minutes]*

1. Sketch the graph of a function $f(x)$ with domain $(-1, 2)$ such that $\lim_{x\to 2} f(x) = 1$ but $\lim_{x\to -1} f(x)$ does not exist. *[4]*
2. Use the $\epsilon - \delta$ definition of limits to verify that $\lim_{x\to\pi} 3 = 3$. *[6]*

Solutions.

1. The graph here does the job:



This is the graph of $f(x) = \frac{3}{x+1}$ for $-1 < x < 2$, which does the job since $\lim_{x\to -1^-} \frac{3}{x+1} = \infty$ and $\lim_{x\to 2^+} \frac{3}{x+1} = \frac{3}{3} = 1$. ■

2. We need to check that for any $\varepsilon > 0$ there is some $\delta > 0$ such that if $|x - \pi| < \delta$, then $|3 - 3| < \varepsilon$. Now, given any $\varepsilon > 0$, $|3 - 3| = 0 < \varepsilon$, so *any* $\delta > 0$ does the job ■

Quiz #2. Friday, 28 September, 2001. *[15 minutes]*

Evaluate the following limits, if they exist.

$$1. \quad \lim_{x\to -1} \frac{x+1}{x^2-1} \quad [5] \qquad 2. \quad \lim_{x\to 1} \frac{x+1}{x^2-1} \quad [5]$$

Solutions.

1. $\displaystyle\lim_{x\to -1} \frac{x+1}{x^2-1} = \lim_{x\to -1} \frac{x+1}{(x-1)(x+1)} = \lim_{x\to -1} \frac{1}{x-1} = \frac{1}{-1-1} = -\frac{1}{2}$ ■
2. $\lim_{x\to 1} \frac{x+1}{x^2-1} = \lim_{x\to 1} \frac{x+1}{(x-1)(x+1)} = \lim_{x\to 1} \frac{1}{x-1} = \pm\infty$ (The $\pm$ depends on whether x approaches 1 from the right or the left.) Hence $\lim_{x\to 1} \frac{x+1}{x^2-1}$ does not exist. ■

Quiz #3. Friday, 5 October, 2001. *[20 minutes]*

1. Is $g(x) = \begin{cases} \dfrac{x^2-6x+9}{x-3} & x \neq 3 \\ 0 & x = 3 \end{cases}$ continuous at $x = 3$? *[5]*
2. For which values of c does $\displaystyle\lim_{x\to\infty} \frac{13}{cx^2+41}$ exist? *[5]*

1

Solutions.

1. This boils down to checking whether $\lim_{x\to 3} \frac{x^2-6x+9}{x-3} = 0$ or not.

$$\lim_{x\to 3} \frac{x^2-6x+9}{x-3} = \lim_{x\to 3} \frac{(x-3)^2}{x-3} = \lim_{x\to 3} x-3 = 3-3 = 0$$

   It follows that $f(x)$ is continuous at $x=3$. ■

2. First, if $c=0$, $\displaystyle\lim_{x\to\infty} \frac{13}{cx^2+41} = \lim_{x\to\infty} \frac{13}{41} = \frac{13}{41}$. Second, if $c \neq 0$, then $\displaystyle\lim_{x\to\infty} cx^2+41 = \pm\infty$, depending on whether $c>0$ or $c<0$, but then $\displaystyle\lim_{x\to\infty} \frac{13}{cx^2+41} = 0$.

   Either way, $\displaystyle\lim_{x\to\infty} \frac{13}{cx^2+41}$ exists no matter what the value of the constant c happens to be. ■

Quiz #4. Friday, 12 October, 2001. *[10 minutes]*

1. Use the definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{5}{7x}$. *[10]*

Solutions.

1.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{5}{7(x+h)} - \frac{5}{7x}}{h} = \lim_{h\to 0} \frac{\frac{5\cdot 7x - 5\cdot 7(x+h)}{7(x+h)\cdot 7x}}{h} \\ &= \lim_{h\to 0} \frac{35x-35x-35h}{49xh(x+h)} = \lim_{h\to 0} \frac{-35h}{49xh(x+h)} = \lim_{h\to 0} \frac{-5}{7x(x+h)} = -\frac{5}{7x^2} \qquad ■ \end{aligned}$$

Quiz #5. Friday, 19 October, 2001. *[17 minutes]*

Compute $\dfrac{dy}{dx}$ for each of the following:

1. $y = \dfrac{2x+1}{x^2}$ *[3]* 2. $y = \ln(\cos(x))$ *[3]* 3. $y = (x+1)^5 e^{-5x}$ *[4]*

Solutions.

1. $\dfrac{dy}{dx} = \dfrac{2\cdot x^2 - (2x+1)\cdot 2x}{x^4} = \dfrac{2x^2-4x^2-2x}{x^4} = \dfrac{-2x(x+1)}{x^4} = \dfrac{-2(x+1)}{x^3}$ ■
2. $\dfrac{dy}{dx} = \dfrac{1}{\cos(x)} \cdot \dfrac{d}{dx}\cos(x) = \dfrac{1}{\cos(x)} \cdot (-\sin(x)) = -\dfrac{\sin(x)}{\cos(x)} = -\tan(x)$ ■
3.

$$\begin{aligned} \frac{dy}{dx} &= 5(x+1)^4 \cdot e^{-5x} + (x+1)^5 \cdot (-5)e^{-5x} \\ &= 5(x+1)^4 e^{-5x} - 5(x+1)^5 e^{-5x} \\ &= 5(x+1)^4 e^{-5x}(1-(x+1)) \\ &= -5x(x+1)^4 e^{-5x} \qquad ■ \end{aligned}$$

Quiz #6. Friday, 2 November, 2001. *[20 minutes]*

Find $\dfrac{dy}{dx}$ ...

1. ... at the point that $y = 3$ and $x = 1$ if $y^2 + xy + x = 13$. *[4]*
2. ... in terms of x if $e^{xy} = x$. *[3]*
3. ... in terms of x if $y = x^{3x}$. *[3]*

Solutions.

1. We'll use implicit differentiation. Differentiating both sides of $y^2 + xy + x = 13$ with respect to x and solving for $\frac{dy}{dx}$ gives:

$$\begin{aligned} & 2y\frac{dy}{dx} + 1y + x\frac{dy}{dx} + 1 = 0 \\ \iff & (2y + x)\frac{dy}{dx} + (y + 1) = 0 \\ \iff & \frac{dy}{dx} = \frac{-(y+1)}{2y + x} \end{aligned}$$

When $y = 3$ and $x = 1$, we get:

$$\left.\frac{dy}{dx}\right|_{x=1 \ \& \ y=3} = -\frac{3+1}{2 \cdot 3 + 1} = -\frac{4}{7} \qquad \blacksquare$$

2. In the case of $e^{xy} = x$, it is easiest to solve for y first ...

$$e^{xy} = x \iff xy = \ln(x) \iff y = \frac{\ln(x)}{x}$$

... and then differentiate using the quotient rule:

$$\frac{dy}{dx} = \frac{\frac{1}{x} \cdot x - \ln(x) \cdot 1}{x^2} = \frac{1 - \ln(x)}{x^2} \qquad \blacksquare$$

3. $y = x^{3x}$ is a job for logarithmic differentiation. First,

$$y = x^{3x} \iff \ln(y) = \ln\left(x^{3x}\right) = 3x\ln(x)\,,$$

and differentiating both sides gives

$$\begin{aligned} \frac{d}{dx}\ln(y) = \frac{d}{dx}3x\ln(x) \iff & \left(\frac{d}{dy}\ln(y)\right)\left(\frac{dy}{dx}\right) = 3 \cdot \ln x + 3x \cdot \frac{1}{x} \\ \iff & \frac{1}{y} \cdot \frac{dy}{dx} = 3\ln(x) + 3\,. \end{aligned}$$

3

Solving for $\frac{dy}{dx}$ and substituting back for y now gives:

$$\frac{dy}{dx} = 3y(\ln(x) + 1) = 3x^{3x}(\ln(x) + 1) \qquad \blacksquare$$

Quiz #7. Friday, 9 November, 2001. *[13 minutes]*

1. Find all the maxima and minima of $f(x) = x^2 e^{-x}$ on $(-\infty, \infty)$ and determine which are absolute. *[10]*

Solutions.

1. First,

$$f'(x) = 2x \cdot e^{-x} + x^2 \cdot \left(-e^{-x}\right) = 2xe^{-x} - x^2e^{-x} = x(2-x)e^{-x} \,,$$

which equals 0 exactly when $x = 0$ or $x = 2$. (Note that $e^{-x} > 0$ for all x.) When $x < 0$, $x < 0$ and $2 - x > 0$, so $f'(x) < 0$; when $0 < x < 2$, $2 - x > 0$ and $x > 0$, so $f'(x) > 0$; and when $x > 2$, $2 - x < 0$ and $x > 0$, so $f'(x) < 0$. Thus

x	$x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$2 < x$
$f'(x)$	< 0	0	> 0	0	< 0
$f(x)$	decreasing	local min	increasing	local max	decreasing

so $f(0) = 0$ is a local minimum and $f(2) = 4/e^2$ is a local maximum.

It remains to check whether either local extreme point is an absolute extreme point of the function. This can be done by taking the limit of $f(x)$ as $x \to \infty$ and as $x \to -\infty$, but that is overkill for this problem. It is enough to note that $f(x) = x^2e^{-x} \geq 0$ for all x, so $f(0) = 0$ is an absolute minimum, but that $f(-4) = (-4)^2e^{-(-4)} = 16e^4 > 4/e^2 = f(2)$, so $f(2) = 4/e^2$ is not an absolute maximum. $\blacksquare$

Quiz #8. Friday, 23 November, 2001. *[15 minutes]*

1. A spherical balloon is being inflated at a rate of 1 m^3/s. How is the diameter of the balloon changing at the instant that the radius of the balloon is 2 m? *[10]*
   [The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.]

Solution.

1. Since $V = \frac{4}{3}\pi r^3$,

$$1 = \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \cdot \frac{dr}{dt} = \left(\frac{4}{3}\pi \cdot 3r^2\right) \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} \,.$$

It follows that $\frac{dr}{dt} = \frac{1}{4\pi r^2}$. At the instant in question, $r = 2$, so we have $\frac{dr}{dt} = \frac{1}{4\pi 2^2} = \frac{1}{16\pi}$.

Since the diameter, call it s, is twice the radius of the sphere, *i.e.* $s = 2r$, it follows that at the instant that $r = 2$ m, the diameter is changing at a rate of $\frac{ds}{dt} = \frac{d}{dt}(2r) = 2\frac{dr}{dt} = 2 \cdot \frac{1}{16\pi} = \frac{1}{8\pi}$ m/s. $\blacksquare$

Quiz #9. Friday, 30 November, 2001. *[20 minutes]*

1. Use the Right-hand Rule to compute $\int\limits_0^3 \left(2x^2+1\right) dx$. *[6]*

   [You may need to know that $\sum\limits_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$.]

2. Set up and evaluate the Riemann sum for $\int\limits_0^2 (3x+1)dx$ corresponding to the partition $x_0 = 0$, $x_1 = \frac{2}{3}$, $x_2 = \frac{4}{3}$, $x_3 = 2$, with $x_1^* = \frac{1}{3}$, $x_2^* = 1$, and $x_3^* = \frac{5}{3}$. *[4]*

Solutions.

1. The Right-hand Rule comes down to the formula:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) \cdot \frac{b-a}{n}$$

In this problem, $f(x) = 2x^2+1$, $a = 0$, and $b = 3$:

$$\begin{aligned}
\int_0^3 \left(2x^2+1\right) dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2\left(0 + i\frac{3-0}{n}\right)^2 + 1\right) \cdot \frac{3-0}{n} \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(2\frac{9i^2}{n^2} + 1\right) \\
&= \lim_{n\to\infty} \frac{3}{n} \left[\sum_{i=1}^{n} \left(\frac{18i^2}{n^2}\right) + \sum_{i=1}^{n} 1\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \left[\frac{18}{n^2} \sum_{i=1}^{n} i^2 + n\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \left[\frac{18}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} + n\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \left[\frac{3}{n}(n+1)(2n+1) + n\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \left[\frac{3}{n}\left(2n^2+3n+1\right) + n\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \left[6n + 9 + \frac{3}{n} + n\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \left[7n + 9 + \frac{3}{n}\right] \\
&= \lim_{n\to\infty} \left[21 + \frac{27}{n} + \frac{9}{n^2}\right] \\
&= 21 + 0 + 0 = 21
\end{aligned}$$

One could skip the odd step here or there ... ■

2. The Riemann sum for $\int_0^2 (3x+1)dx$ for the given partition and choice of points is (and evaluates to):

$$\sum_{i=1}^{3} f(x_i^*) \cdot (x_i - x_{i-1}) = \sum_{i=1}^{3} (3x_i^* + 1) \cdot (x_i - x_{i-1})$$
$$= \left(3 \cdot \frac{1}{3} + 1\right)\left(\frac{2}{3} - 0\right) + (3 \cdot 1 + 1)\left(\frac{4}{3} - \frac{2}{3}\right) + \left(3 \cdot \frac{5}{3} + 1\right)\left(2 - \frac{4}{3}\right)$$
$$= 2 \cdot \frac{2}{3} + 4 \cdot \frac{2}{3} + 6 \cdot \frac{2}{3} = \frac{4}{3} + \frac{8}{3} + \frac{12}{3} = \frac{4+8+12}{3} = \frac{24}{3} = 8 \qquad \blacksquare$$

Quiz #10. Friday, 7 December, 2001. *[20 minutes]*

Given that $\int_1^4 x\,dx = 7.5$ and $\int_1^4 x^2\,dx = 21$, use the properties of definite integrals to:

1. Evaluate $\int_1^4 (x+1)^2\,dx$. *[5]*
2. Find upper and lower bounds for $\int_1^4 x^{3/2}\,dx$. *[5]*

Solutions.

1. Using the given data and some properties of definite integrals:

$$\begin{aligned}\int_1^4 (x+1)^2\,dx &= \int_1^4 \left(x^2 + 2x + 1\right)\,dx \\ &= \int_1^4 x^2\,dx + \int_1^4 2x\,dx + \int_1^4 1\,dx \\ &= 21 + 2\int_1^4 x\,dx + 1 \cdot (4-1) \\ &= 21 + 2 \cdot 7.5 + 3 \\ &= 21 + 15 + 3 = 39 \qquad \blacksquare\end{aligned}$$

2. First, note that $x \le x^{3/2} \le x^2$ when $x \ge 1$. Using the given data and the order properties of definite integrals gives:

$$7.5 = \int_1^4 x\,dx \le \int_1^4 x^{3/2}\,dx \le \int_1^4 x^2\,dx = 21$$

Thus 7.5 is a lower bound and 21 is an upper bound for $\int_1^4 x^{3/2}\,dx$. ■

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Quiz #11. Friday, 11 January, 2002. *[15 minutes]*

1. Compute the indefinite integral $\int \left(x^2+x+1\right)^3 (4x+2)\,dx$. *[5]*
2. Find the area under the graph of $f(x) = \sin(x)\cos(x)$ for $0 \le x \le \frac{\pi}{2}$. *[5]*

Solutions.

1. We will use the substitution $u = x^2 + x + 1$ to compute the integral; note that $du = (2x+1)dx$.

$$\int \left(x^2+x+1\right)^3 (4x+2)\,dx = \int \left(x^2+x+1\right)^3 2(2x+1)\,dx$$
$$= \int u^3 2du = 2\frac{u^4}{4} + c = \frac{1}{2}u^4 + c = \frac{1}{2}\left(x^2+x+1\right)^4 + c \qquad \blacksquare$$

2. First, note that both $\sin(x)$ and $\cos(x)$, and hence also $\sin(x)\cos(x)$, are non-negative for $0 \le x \le \frac{\pi}{2}$. Hence the area under the graph of $f(x) = \sin(x)\cos(x)$ for $0 \le x \le \frac{\pi}{2}$ is given by the definite integral $\int\limits_0^{\pi/2} \sin(x)\cos(x)\,dx$. We will compute this integral using the substitution $u = \sin(x)$, so $du = \cos(x)dx$. Note also that $u = 0$ when $x = 0$ and $u = 1$ when $x = \frac{\pi}{2}$.

$$\int_0^{\pi/2} \sin(x)\cos(x)\,dx = \int_0^1 u\,du = \left.\frac{u^2}{2}\right|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2} \qquad \blacksquare$$

Quiz #12. Friday, 18 January, 2002. *[15 minutes]*

1. Compute $\int\limits_1^e \frac{\ln\left(x^2\right)}{x}\,dx$. *[5]*
2. Find the area of the region between the curves $y = x^3 - x$ and $y = x - x^3$. *[5]*

Solutions.

1. We'll compute the integral using the substitution $u = \ln(x)$, so $du = \frac{1}{x}dx$. Note that $\ln\left(x^2\right) = 2\ln(x)$, and that $u = 0$ when $x = 1$ and $u = 1$ when $x = e$.

$$\int_1^e \frac{\ln\left(x^2\right)}{x}\,dx = \int_1^e \frac{2\ln(x)}{x}\,dx = \int_0^1 2u\,du = 2\left.\frac{u^2}{2}\right|_0^1 = \left.u^2\right|_0^1 = 1^2 - 0^2 = 1 \qquad \blacksquare$$

2. First, we need to find the points where these curves intersect:

$$x^3 - x = x - x^3 \Leftrightarrow 2x^3 = 2x \Leftrightarrow x^3 = x$$

$x = 0$ is one solution to $x^3 = x$; when $x \ne 0$, we can divide the equation by x to get $x^2 = 1$, so $x = -1$ and $x = 1$ are the other solutions to $x^3 = x$. From -1 to 0, $x^3 - x \ge x - x^3$, and from 0 to 1, $x - x^3 \ge x^3 - x$. (This can be done with some algebra and knowledge of

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inequalities, or you can just test each expression at some points in between, *e.g.* $x = -\frac{1}{2}$ and $x = \frac{1}{2}$.)

The area between the two curves is then given by:

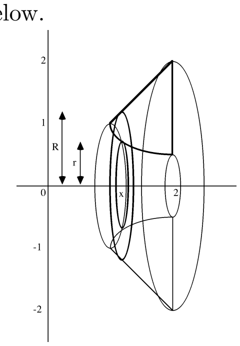
$$
\begin{aligned}
&\int_{-1}^{0} \left[\left(x^3 - x\right) - \left(x - x^3\right)\right]\, dx + \int_{0}^{1} \left[\left(x - x^3\right) - \left(x^3 - x\right)\right]\, dx \\
&= \int_{-1}^{0} \left[2x^3 - 2x\right]\, dx + \int_{0}^{1} \left[2x - 2x^3\right]\, dx \\
&= \left[2\frac{x^4}{4} - 2\frac{x^2}{2}\right]\Bigg|_{-1}^{0} + \left[2\frac{x^2}{2} - 2\frac{x^4}{4}\right]\Bigg|_{0}^{1} \\
&= \left[\frac{x^4}{2} - x^2\right]\Bigg|_{-1}^{0} + \left[x^2 - \frac{x^4}{2}\right]\Bigg|_{0}^{1} \\
&= \left[\left(\frac{0^4}{2} - 0^2\right) - \left(\frac{(-1)^4}{2} - (-1)^2\right)\right] + \left[\left(1^2 - \frac{1^4}{2}\right) - \left(0^2 - \frac{0^4}{2}\right)\right] \\
&= \left[0 - \left(-\frac{1}{2}\right)\right] + \left[\frac{1}{2} - 0\right] \\
&= \frac{1}{2} + \frac{1}{2} = 1 \qquad \blacksquare
\end{aligned}
$$

Quiz #13. Friday, 25 January, 2002. *[19 minutes]*

1. Find the volume of the solid obtained by revolving the region in the first quadrant bounded by $y = \frac{1}{x}$, $y = x$, and $x = 2$ about the x-axis. *[10]*

Solution.

1. First, we find where the curves intersect. $y = \frac{1}{x}$ and $y = x$ intersect when $\frac{1}{x} = x$, *i.e.* when $x^2 = 1$. Since we are looking for a region in the first quadrant, we discard the root $x = -1$ and keep $x = 1$; plugging into either equation for y gives us the point $(1, 1)$. $y = \frac{1}{x}$ and $x = 2$ intersect at the point $\left(2, \frac{1}{2}\right)$, and $y = x$ and $x = 2$ intersect at $(2, 2)$. It is not too hard — if one is careful! — to deduce that the region is bounded above by $y = x$ and below by $y = \frac{1}{x}$ for $1 \leq x \leq 2$. Rotating this region about the x-axis gives the solid sketched below.



We will find the volume of this solid using the washer method. (A typical washer for this solid is also drawn in the sketch.) Since we rotated about a horizontal line, the washers will be stacked along this line (the x-axis) and so we will need to integrate with respect to x. Note that the outer radius of the washer at x is $R = x$ and the inner radius is $r = \frac{1}{x}$, so the volume is given by:

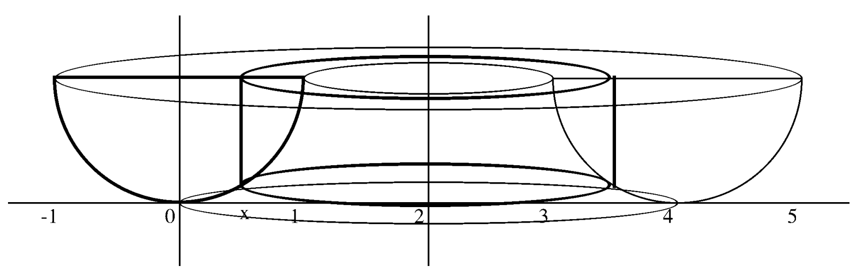
$$\begin{aligned}
\int_1^2 \left(\pi R^2 - \pi r^2\right) \, dx &= \pi \int_1^2 \left(x^2 - \frac{1}{x^2}\right) \, dx \\
&= \pi \left(\frac{x^3}{3} - \frac{-1}{x}\right)\Bigg|_1^2 \\
&= \pi \left(\frac{x^3}{3} + \frac{1}{x}\right)\Bigg|_1^2 \\
&= \pi \left[\left(\frac{8}{3} + \frac{1}{2}\right) - \left(\frac{1}{3} + 1\right)\right] \\
&= \frac{11}{6}\pi \qquad \blacksquare
\end{aligned}$$

Quiz #14. Friday, 1 February, 2002. *[17 minutes]*

1. Suppose the region bounded above by $y = 1$ and below by $y = x^2$ is revolved about the line $x = 2$. Sketch the resulting solid and find its volume. *[10]*

Solution.

1. $y = 1$ intersects $y = x^2$ when $x^2 = 1$, *i.e.* when $x = \pm 1$. The region between $y = 1$ and $y = x^2$, $-1 \leq x \leq 1$, when revolved about $x = 2$, gives the solid sketched below.



We will find the volume of this solid using the shell method. (A typical cylindrical shell for this solid is also drawn in the sketch.) Since we rotated about a vertical line, the shell will be nested around this line ($x = 2$) and so we will need to integrate with respect to x in order to integrate in a direction perpendicular to the shells. Note that the radius of the shell at x is $r = 2 - x$ and the height is $h = 1 - x^2$, so the volume is given by:

$$\begin{aligned}
\int_{-1}^{1} 2\pi rh\,dx &= \int_{-1}^{1} 2\pi(2-x)\left(1-x^2\right)\,dx \\
&= \pi \int_{-1}^{1} \left(4 - 2x - 4x^2 + 2x^3\right)\,dx \\
&= \pi \left(4x - x^2 - \frac{4}{3}x^3 + \frac{1}{2}x^4\right)\Big|_{-1}^{1} \\
&= \pi \left[\left(4 - 1 - \frac{4}{3} + \frac{1}{2}\right) - \left(-4 - 1 + \frac{4}{3} + \frac{1}{2}\right)\right] \\
&= \frac{16}{3}\pi \qquad \blacksquare
\end{aligned}$$

Quiz #15. Friday, 15 February, 2002. *[25 minutes]*

Evaluate each of the following integrals.

1. $\displaystyle\int_0^{\pi/4} \tan^2(x)\,dx$ *[4]* 2. $\displaystyle\int \sqrt{x^2+4x+5}\,dx$ *[6]*

Solution.

1.

$$\begin{aligned}
\int_0^{\pi/4} \tan^2(x)\,dx &= \int_0^{\pi/4} \left(\sec^2(x) - 1\right)\,dx \\
&= \left(\tan(x) - x\right)\Big|_0^{\pi/4} = \left(1 - \frac{\pi}{4}\right) - (0 - 0) = 1 - \frac{\pi}{4} \qquad \blacksquare
\end{aligned}$$

2.

$$\begin{aligned}
\int \sqrt{x^2+4x+5}\,dx &= \int \sqrt{(x+2)^2+1}\,dx \\
&\quad \text{Let } w = x+2, \text{ so } dw = dx. \\
&= \int \sqrt{w^2+1}\,dw \\
&\quad \text{Let } w = \tan(t), \text{ so } dw = \sec^2(t)dt. \\
&= \int \sqrt{\tan^2(t)+1}\,\sec^2(t)\,dt = \int \sqrt{\sec^2(t)}\,\sec^2(t)\,dt \\
&= \int \sec^3(t)\,dt
\end{aligned}$$

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We will use integration by parts to compute this trig integral. Let $u = \sec(t)$ and $dv = \sec^2(t)dt$, so $du = \sec(t)\tan(t)dt$ and $v = \tan(t)$. Then

$$\begin{aligned}
\int \sec^3(t)\,dt &= \sec(t)\tan(t) - \int \sec(t)\tan^2(t)\,dt \\
&= \sec(t)\tan(t) - \int \sec(t)\left(\sec^2(t) - 1\right)\,dt \\
&= \sec(t)\tan(t) - \int \left(sec^3(t) - \sec(t)\right)\,dt \\
&= \sec(t)\tan(t) - \int sec^3(t)\,dt + \int \sec(t)\,dt
\end{aligned}$$

It follows that

$$2\int \sec^3(t)\,dt = \sec(t)\tan(t) + \int \sec(t)\,dt\,,$$

so

$$\begin{aligned}
\int \sec^3(t)\,dt &= \frac{1}{2}\left[\sec(t)\tan(t) + \int \sec(t)\,dt\right] \\
&= \frac{1}{2}\sec(t)\tan(t) + \frac{1}{2}\ln\left(\sec(t) + \tan(t)\right) + C\,.
\end{aligned}$$

(It really helps to have memorized that $\int \sec(t)\,dt = \ln\left(\sec(t) + \tan(t)\right) + C\ \ldots$)
It remains for us to substitute back to put the answer in terms of x:

$$\begin{aligned}
\int \sqrt{x^2 + 4x + 5}\,dx &= \int \sec^3(t)\,dt \\
&= \frac{1}{2}\sec(t)\tan(t) + \frac{1}{2}\ln\left(\sec(t) + \tan(t)\right) + C \\
&= \frac{1}{2}w\sqrt{1 + w^2} + \frac{1}{2}\ln\left(w + \sqrt{1 + w^2}\right) + C \\
&\qquad \ldots \text{ since when } \tan(t) = w,\ \sec(t) = \sqrt{1 + w^2}. \\
&= \frac{1}{2}(x + 2)\sqrt{1 + (x + 2)^2} + \frac{1}{2}\ln\left((x + 2) + \sqrt{1 + (x + 2)^2}\right) + C \quad \blacksquare
\end{aligned}$$

Quiz #16. Friday, 1 March, 2002. *[25 minutes]*

1. Evaluate the following integral:

$$\int \frac{x^2 - 2x - 6}{\left(x^2 + 2x + 5\right)(x - 1)}\,dx$$

Solution. This is a job for partial fractions. First, note that the quadratic factor in the denominator is irreducible since $x^2 + 2x + 5 = x^2 + 2x + 1 + 4 = (x + 1)^2 + 4 > 0$ for all x. Thus

$$\frac{x^2 - 2x - 6}{\left(x^2 + 2x + 5\right)(x - 1)} = \frac{Ax + B}{x^2 + 2x + 5} + \frac{C}{x - 1}$$

for some constants A, B, and C. Putting the right-hand side of the equation above over a common denominator of $x(x-1)^2$ would give a numerator equal to the numerator of the left-hand side:

$$\begin{aligned} x^2-2x-6 &= (Ax+B)(x-1)+C(x^2+2x+5) \\ &= Ax^2-Ax+Bx-B+Cx^2+2Cx+5C \\ &= (A+C)x^2+(-A+B+2C)x+(-B+5C) \end{aligned}$$

Thus $A+C=1$, $-A+B+2C=-2$, and $-B+5C=-6$, which equations we need to solve for A, B, and C. From the first and third of these equations we get that $A=1-C$ and $B=6+5C$. Plugging these into the second equation gives $-2=-(1-C)+(6+5C)+2C=5-8C$, so $C=-7/8$. It follows that $A=15/8$ and $B=13/8$. Hence:

$$\begin{aligned} \int \frac{x^2-2x-6}{(x^2+2x+5)(x-1)}\,dx &= \int \left[\frac{\frac{15}{8}x+\frac{13}{8}}{x^2+2x+5}+\frac{-\frac{7}{8}}{x-1}\right]dx \\ &= \frac{1}{8}\int \frac{15x+13}{x^2+2x+5}\,dx-\frac{7}{8}\int \frac{1}{x-1}\,dx \end{aligned}$$

If $u=x^2+2x+5$, then $du=(2x+2)dx=2(x+1)dx$, so we want to split $15x+13$ into a multiple of $x+1$ plus a constant.

$$\begin{aligned} &= \frac{1}{8}\int \frac{15(x+1)-2}{x^2+2x+5}\,dx-\frac{7}{8}\ln(x-1) \\ &= \frac{1}{8}\int \frac{15(x+1)}{x^2+2x+5}\,dx-\frac{1}{8}\int \frac{-2}{x^2+2x+5}\,dx-\frac{7}{8}\ln(x-1) \end{aligned}$$

Use the substitution $u=x^2+2x+5$ in the first part, and complete the square in the second part.

$$= \frac{15}{8}\int \frac{1}{u}\frac{1}{2}du+\frac{2}{8}\int \frac{1}{(x+1)^2+4}\,dx-\frac{7}{8}\ln(x-1)$$

Use the substitution $w=x+1$, so $dw=dx$, in the second part.

$$= \frac{15}{16}\ln(u)+\frac{1}{4}\int \frac{1}{w^2+4}\,dw-\frac{7}{8}\ln(x-1)$$

Substitute back in the first part, and substitute $w=2s$, so $dw=2ds$ in the second.

$$\begin{aligned} &= \frac{15}{16}\ln\left(x^2+2x+5\right)+\frac{1}{4}\int \frac{1}{4s^2+4}\,2ds-\frac{7}{8}\ln(x-1) \\ &= \frac{15}{16}\ln\left(x^2+2x+5\right)+\frac{1}{4}\cdot\frac{2}{4}\int \frac{1}{s^2+1}\,ds-\frac{7}{8}\ln(x-1) \\ &= \frac{15}{16}\ln\left(x^2+2x+5\right)+\frac{1}{8}\arctan(s)-\frac{7}{8}\ln(x-1)+K \end{aligned}$$

... where K is a constant.

$$
\begin{aligned}
&= \frac{15}{16}\ln\left(x^2+2x+5\right) + \frac{1}{8}\arctan\left(\frac{w}{2}\right) - \frac{7}{8}\ln(x-1) + K \\
&= \frac{15}{16}\ln\left(x^2+2x+5\right) + \frac{1}{8}\arctan\left(\frac{x-1}{2}\right) - \frac{7}{8}\ln(x-1) + K
\end{aligned}
$$

Whew! ■

Quiz #17. Friday, 8 March, 2002. *[25 minutes]*

1. Evaluate the following integral:

$$\int_2^\infty \frac{1}{x(x-1)^2}\,dx$$

Solution. Note that $x(x-1)^2 \neq 0$ for all x with $2 \leq x < \infty$, so we don't have to worry about the integral being improper except by way of the upper limit of ∞. By definition,

$$\int_2^\infty \frac{1}{x(x-1)^2}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x(x-1)^2}\,dx$$

which leaves us with the task of integrating $\int_2^t \frac{1}{x(x-1)^2}\,dx$ and then evaluating the limit as $t \to \infty$.

Computing the integral requires us to decompose $\frac{1}{x(x-1)^2}$ using partial fractions. Note that the numerator has degree less than the degree of the denominator, but that we do have a repeated factor in the denominator. Thus

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

for some constants A, B, and C. Putting the right-hand side of the equation above over a common denominator of $x(x-1)^2$ would give a numerator equal to the numerator of the left-hand side:

$$
\begin{aligned}
1 =& A(x-1)^2 + Bx(x-1) + Cx \\
=& A\left(x^2-2x+1\right) + B\left(x^2-x\right) + Cx \\
=& (A+B)x^2 + (-2A-B+C)x + A\,.
\end{aligned}
$$

Thus $A+B=0$, $-2A-B+C=0$, and $A=1$, from which it quickly follows that $A=1$,

$B = -1$, and $C = 1$. Hence

$$\begin{aligned}
\int_2^\infty \frac{1}{x(x-1)^2}\,dx &= \lim_{t\to\infty} \int_2^t \frac{1}{x(x-1)^2}\,dx \\
&= \lim_{t\to\infty} \int_2^t \left[\frac{1}{x} + \frac{-1}{x-1} + \frac{1}{(x-1)^2}\right] dx \\
&= \lim_{t\to\infty} \left[\int_2^t \frac{1}{x}\,dx - \int_2^t \frac{1}{x-1}\,dx + \int_2^t \frac{1}{(x-1)^2}\,dx\right] \\
&= \lim_{t\to\infty} \left[\ln(x)|_2^t - \ln(x-1)|_2^t + \left.\frac{-1}{(x-1)}\right|_2^t\right] \\
&= \lim_{t\to\infty} \left[(\ln(t) - \ln(2)) - (\ln(t-1) - \ln(2-1)) + \left(\frac{-1}{(t-1)} - \frac{-1}{(2-1)}\right)\right] \\
&= \lim_{t\to\infty} \left[\ln(t) - \ln(t-1) - \frac{1}{(t-1)} - \ln(2) + \ln(1) + \frac{1}{1}\right] \\
&= \lim_{t\to\infty} \left[\ln\left(\frac{t}{t-1}\right) - \frac{1}{(t-1)} - \ln(2) + 0 + 1\right] \\
&= 1 - \ln(2)
\end{aligned}$$

since $\lim_{t\to\infty} \frac{t}{t-1} = \lim_{t\to\infty} \frac{1}{1-1/t} = \frac{1}{1-0} = 1$, so $\lim_{t\to\infty} \ln\left(\frac{t}{t-1}\right) = \ln(1) = 0$, while $\lim_{t\to\infty} \frac{1}{(t-1)} = 0$.
■

Quiz #18. Friday, 15 March, 2002. *[18 minutes]*

Determine whether each of the following series converges or diverges.

$$1.\quad \sum_{n=0}^\infty \left[\frac{1}{n+1} + \frac{3^n}{3^n+1}\right] \quad [4] \qquad 2.\quad \sum_{n=0}^\infty \frac{253}{3^n+1} \quad [6]$$

Solutions.

1. We will apply the Divergence Test.

$$\begin{aligned}
\lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{3^n}{3^n+1}\right] &= \left[\lim_{n\to\infty} \frac{1}{n+1}\right] + \left[\lim_{n\to\infty} \frac{3^n}{3^n+1}\right] = 0 + \lim_{n\to\infty} \frac{3^n}{3^n+1} \cdot \frac{1/3^n}{1/3^n} \\
&= \lim_{n\to\infty} \frac{1}{1+1/3^n} = \frac{1}{1+0} = 1 \neq 0
\end{aligned}$$

It follows that the given series diverges. ■

2. We will apply the Comparison Test. Note that

$$0 < \frac{253}{3^n+1} < \frac{253}{3^n}$$

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for all $n \geq 0$, because reducing the denominator increases the fraction. The series $\sum\limits_{n=0}^{\infty} \frac{253}{3^n}$ is a geometric series with $a = 253$ and $r = \frac{1}{3}$, and since $\left|\frac{1}{3}\right| < 1$, it converges. It follows by the Comparison Test that the given series converges as well. ■

Bonus Quiz. Monday, 18 March, 2002. *[15 minutes]*

Compute any *two* of 1–3.

1. $\lim\limits_{t\to\infty} te^{-t}$ *[5]* 2. $\displaystyle\int_0^\infty te^{-t}\,dt$ *[5]* 3. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}$ *[5]*

Solutions.

1. $\lim\limits_{t\to\infty} te^{-t} = \lim\limits_{t\to\infty} \dfrac{t}{e^t} = \lim\limits_{t\to\infty} \dfrac{\frac{d}{dt}t}{\frac{d}{dt}e^t} = \lim\limits_{t\to\infty} \dfrac{1}{e^t} = 0$, using l'Hôpital's Rule and the fact that $\lim\limits_{t\to\infty} e^t = \infty$. ■
2. We'll need l'Hôpital's Rule and the fact that $\lim\limits_{s\to\infty} e^s = \infty$ again:

$$\int_0^\infty te^{-t}\,dt = \lim_{s\to\infty} \int_0^s te^{-t}\,dt$$

Use integration by parts with $u = t$ and $dv = e^{-t}dt$, so $du = dt$ and $v = -e^{-t}$.

$$\begin{aligned}
&= \lim_{s\to\infty} \left[-te^{-t}\Big|_0^s - \int_0^s \left(-e^{-t}\right)\,dt \right] \\
&= \lim_{s\to\infty} \left[\left(-se^{-s}\right) - \left(-0e^{-0}\right) - e^{-t}\Big|_0^s \right] \\
&= \lim_{s\to\infty} \left[-se^{-s} - \left(e^{-s} - e^{-0}\right) \right] \\
&= \lim_{s\to\infty} \left[-se^{-s} - e^{-s} + 1 \right] \\
&= \lim_{s\to\infty} \left[-\frac{s}{e^s} - \frac{1}{e^s} \right] + 1 \\
&= \lim_{s\to\infty} \left[-\frac{\frac{d}{ds}s}{\frac{d}{ds}e^s} - 0 + 1 \right] \\
&= \lim_{s\to\infty} \left[-\frac{1}{e^s} \right] + 1 \\
&= -0 + 1 \\
&= 1 \qquad ■
\end{aligned}$$

3. Note that $n^2 + 3n + 2 = (n+1)(n+2)$ and that, using the usual partion fraction nonsense, $\dfrac{1}{n^2+3n+2} = \dfrac{1}{n+1} - \dfrac{1}{n+2}$. Thus the kth partial sum of the given series

is

$$\begin{aligned}
S_k &= \sum_{n=0}^{k} \frac{1}{n^2+3n+2} = \sum_{n=0}^{k} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \\
&= \left(\frac{1}{0+1} - \frac{1}{0+2} \right) + \left(\frac{1}{1+1} - \frac{1}{1+2} \right) + \cdots + \left(\frac{1}{k+1} - \frac{1}{k+2} \right) \\
&= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{k+1} - \frac{1}{k+2} \right) \\
&= 1 - \frac{1}{k+2} \,,
\end{aligned}$$

so

$$\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \left(1 - \frac{1}{k+2} \right) = 1 - 0 = 1 \,. \quad \blacksquare$$

Quiz #19. Friday, 22 March, 2002. *[20 minutes]*

Determine whether each of the following series converges or diverges.

$$1. \quad \sum_{n=1}^{\infty} \frac{1}{n^n} \quad [5] \qquad 2. \quad \sum_{n=0}^{\infty} \frac{4n+12}{n^2+6n+13} \quad [5]$$

Solutions.

1. We will use the Comparison Test. Note that for all $n > 2$, $n^n > 2^n$, so

$$0 < \frac{1}{n^n} < \frac{1}{2^n} \,.$$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ is the geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$, and it converges because $|r| = \frac{1}{2} < 1$. It follows by the Comparison Test that $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges as well.

One could also conveniently use the Limit Comparison Test. $\blacksquare$

2. We will use the Limit Comparison Test, comparing the given series to $\sum_{n=0}^{\infty} \frac{1}{n}$. (Why compare the given series to this one? Note that the terms of the given series are rational functions of n in which the top power in the numerator is one less than the top power in the denominator. $\frac{1}{n}$ is the simplest rational function with this pattern.)

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Now:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{4n+12}{n^2+6n+13}} &= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{n^2+6n+13}{4n+12} \\
&= \lim_{n\to\infty} \frac{n^2+6n+13}{4n^2+12n} \\
&= \lim_{n\to\infty} \frac{n^2+6n+13}{4n^2+12n} \cdot \frac{1/n^2}{1/n^2} \\
&= \lim_{n\to\infty} \frac{1+\frac{6}{n}+\frac{13}{n^2}}{4+\frac{12}{n}} \\
&= \frac{1+0+0}{4+0} = \frac{1}{4}
\end{aligned}$$

Since $0 < \frac{1}{4} < \infty$, it follows by the Limit Comparison Test that the series $\sum_{n=0}^{\infty} \frac{1}{n}$ and $\sum_{n=0}^{\infty} \frac{4n+12}{n^2+6n+13}$ both converge or both diverge. Since $\sum_{n=0}^{\infty} \frac{1}{n}$ is known to diverge, it myst be the case that $\sum_{n=0}^{\infty} \frac{4n+12}{n^2+6n+13}$ diverges as well.

This problem could also be done using the Comparison Test or (very conveniently) the Integral Test. ■

Quiz #20. Tuesday, 2 April, 2002. *[10 minutes]*

1. Determine whether the series $\sum_{n=0}^{\infty} \frac{(-1)^n + \cos(n\pi)}{n+1}$ converges absolutely, converges conditionally, or diverges. *[10]*

Solution. The key here is that $\cos(n\pi) = (-1)^n$ since cos is equal to 1 at even multiples of π and -1 at odd multiples of π. Hence

$$\sum_{n=0}^{\infty} \frac{(-1)^n + \cos(n\pi)}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n + (-1)^n}{n+1} = \sum_{n=0}^{\infty} \frac{2 \cdot (-1)^n}{n+1} = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1},$$

so the given series will converge (or not) exactly as $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ does. However, this is a series beaten to death in class and the text in very slight disguise:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = (-1)\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

(Note that the indices are related via $k = n+1$.)

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We know already that $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges – we showed that in class using the Alternating Series Test. It does not converge absolutely because $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge – we showed that, in effect using the Integral Test, on Assignment #6. Thus the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, and hence given series too, converges conditionally. ■

Quiz #21. Friday, 5 April, 2002. *[10 minutes]*

1. Find a power series which, when it converges, equals $f(x) = \frac{3x^2}{(1-x^3)^2}$. *[10]*

Solution. Note that the derivative of part of the denominator $f(x)$, $\frac{d}{dx}\left(1-x^3\right) = -3x^2$, is a constant multiple to the numerator, $3x^2$. This makes it easy to integrate $f(x)$ using the substitution $u = 1 - x^3$ (so $du = -3x^2 dx$ and $(-1)du = 3x^2 dx$):

$$\begin{aligned}\int f(x)\,dx &= \int \frac{3x^2}{(1-x^3)^2}\,dx = \int \frac{1}{u^2}\,(-1)du = -\int u^{-2}\,du \\ &= -\left(-u^{-1}\right) + C = \frac{1}{u} + C = \frac{1}{1-x^3} + C\end{aligned}$$

The point here is that it is easy to find a power series representation of the antiderivative of $f(x)$ because $\frac{1}{1-x^3}$ is the sum of the geometric series with $a = 1$ and $r = x^3$. Thus:

$$\int f(x)\,dx = \frac{1}{1-x^3} + C = \sum_{n=0}^{\infty} \left(x^3\right)^n = \sum_{n=0}^{\infty} x^{3n}$$

The power series of $f(x)$ is the derivative of the power series for $\int f(x)\,dx$ (at least for those x for which this series converges absolutely):

$$f(x) = \frac{3x^2}{(1-x^3)^2} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^{3n}\right) = \sum_{n=0}^{\infty} \frac{d}{dx} x^{3n} = \sum_{n=0}^{\infty} 3nx^{3n-1}$$

Note that the first term, for $n = 0$, has a coefficient of $3 \cdot 0 = 0$, so it doesn't matter that the corresponding power of x is negative. ■

Mathematics 110 – Calculus of one variable
Trent University, 2001-2002

Solutions to Test #1

1. Do any *two* of **a-c**. *[10 = 2 × 5 ea.]*

a. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 2}(5-x) = 3$.

Solution. By definition, $\lim_{x \to 2}(5-x) = 3$ means:

For any $\varepsilon > 0$, there is some $\delta > 0$, such that
if $|x-2| < \delta$, then $|(5-x)-3| < \varepsilon$.

However,

$$\begin{aligned} |(5-x)-3| < \varepsilon &\iff -\varepsilon < (5-x) - 3 < \varepsilon \\ &\iff -\varepsilon < 2 - x < \varepsilon \\ &\iff \varepsilon > x - 2 > -\varepsilon \\ &\iff -\varepsilon < x - 2 < \varepsilon\,, \end{aligned}$$

so, $\delta = \varepsilon$ will do the job. Hence $\lim_{x \to 2}(5-x) = 3$. ■

b. Use the definition of the derivative to verify that $f'(2) = 5$ if $f(x) = 2x^2 - 3x + 4$.

Solution. By definition, $f'(2) = \lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$. Now

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} \frac{\left[2(2+h)^2 - 3(2+h) + 4\right] - \left[2 \cdot 2^2 - 3 \cdot 2 + 4\right]}{h} \\ &= \lim_{h \to 0} \frac{\left[8 + 8h + 2h^2 - 6 - 3h + 4\right] - [8 - 6 + 4]}{h} \\ &= \lim_{h \to 0} \frac{5h + 2h^2}{h} \\ &= \lim_{h \to 0} (5 + 2h) \\ &= 5 + 2 \cdot 0 \\ &= 5\,, \end{aligned}$$

so $f'(2) = 5$. ■

c. Determine whether $f(x) = \begin{cases} \dfrac{x-1}{x^2-1} & x \neq 1 \\ 41 & x = 1 \end{cases}$ is continuous at $a = 1$ or not.

Solution. $f(x)$ is continuous at a if $\lim_{x\to a} f(x) = f(a)$, so we need to check whether or not $\lim_{x\to 1} \frac{x-1}{x^2-1} = 41$. Since

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{x-1}{(x-1)(x+1)} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{1+1} = \frac{1}{2} \neq 41\,,$$

$f(x)$ is *not* continuous at $a = 1$. ■

2. Find $\frac{dy}{dx}$ in any *three* of **a-d**. *[12 = 3 × 4 ea.]*

a. $y = 4^x$ **b.** $y = e^{-x}\cos(x)$ **c.** $y = \sin(\cos(-x))$ **d.** $\ln(xy) = 0$

Solution to a. This is easiest using logarithmic differentiation:

$$y = 4^x \iff \ln(y) = \ln\left(4^x\right) = x\ln(4)$$

Differentiating $\ln(y) = x\ln(4)$ with respect to x gives

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}\ln(y) = \frac{d}{dx}x\ln(4) = \ln(4)\,.$$

Thus

$$\frac{dy}{dx} = y\ln(4) = 4^x\ln(4)\,. \quad ■$$

Solution to b. The main tool needed here is the product rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}e^{-x}\cos(x) \\
&= \left(\frac{d}{dx}e^{-x}\right)\cos(x) + e^{-x}\left(\frac{d}{dx}\cos(x)\right) \\
&= \left(e^{-x}\frac{d}{dx}(-x)\right)\cos(x) + e^{-x}(-\sin(x)) \\
&= -e^{-x}\cos(x) - e^{-x}\sin(x) \\
&= -e^{-x}(\cos(x) + \sin(x)) \quad ■
\end{aligned}$$

Solution to c. This one is chain rule all the way. In rather more detail than is strictly

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desirable:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\sin(\cos(-x)) \\
&= \frac{d}{du}\sin(u)\cdot\frac{du}{dx} \qquad \text{where } u=\cos(-x) \\
&= \cos(u)\frac{du}{dx} \qquad \text{where } u=\cos(-x) \\
&= \cos(\cos(-x))\frac{d}{dx}\cos(-x) \\
&= \cos(\cos(-x))\frac{d}{dv}\cos(v)\cdot\frac{dv}{dx} \qquad \text{where } v=-x \\
&= \cos(\cos(-x))(-\sin(v))\cdot\frac{dv}{dx} \\
&= -\cos(\cos(-x))\sin(-x)\frac{d}{dx}(-x) \\
&= -\cos(\cos(-x))\sin(-x)(-1) \\
&= \cos(\cos(-x))\sin(-x) \\
&= -\cos(\cos(x))\sin(x) \quad \text{since } \cos(-x)=\cos(x) \text{ and } \sin(-x)=-\sin(x) \quad \blacksquare
\end{aligned}$$

Solution to d. This one looks like it ought to be done by implicit differentiation, and it can be done that way, but it is easier to just solve for y in terms of x first:

$$\ln(xy)=0 \iff xy=e^{\ln(xy)}=e^0=1 \iff y=\frac{1}{x}$$

Note that we can't have either y or x equal to 0 in the original equation because $\ln(0)$ is not defined. Now, using the power rule, $\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{x}\right)=-\frac{1}{x^2}$.

If one were to do this by implicit differentiation, one would get $\frac{dy}{dx}=-\frac{y}{x}$ and then still have to solve for y in terms of x to have the best possible solution ... $\blacksquare$

3. Do any *two* of **a**-**c**. *[8 = 2 × 4 ea.]*

a. Find the equation of the tangent line to $y=\arctan(x)$ at the point $\left(1,\frac{\pi}{4}\right)$.

Solution. Since $\frac{d}{dx}\arctan(x)=\frac{1}{1+x^2}$, it follows that the slope of the tangent line to $y=\arctan(x)$ at the point $\left(1,\frac{\pi}{4}\right)$ is given by

$$\left.\frac{d}{dx}\arctan(x)\right|_{x=1}=\left.\frac{1}{1+x^2}\right|_{x=1}=\frac{1}{1+1^2}=\frac{1}{2}\,.$$

It follows that the tangent line has an equation of the form $y=\frac{1}{2}x+b$, and since the tangent line passes through the point $\left(1,\frac{\pi}{4}\right)$, we can solve for b by plugging in $x=1$ and

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$y = \dfrac{\pi}{4}$:

$$\frac{\pi}{4} = \frac{1}{2} \cdot 1 + b \iff b = \frac{\pi}{4} - \frac{1}{2},$$

Thus the equation of the tangent line $y = \arctan(x)$ at the point $\left(1, \frac{\pi}{4}\right)$ is $y = \dfrac{1}{2}x + \dfrac{\pi}{4} - \dfrac{1}{2}$. ■

b. How many functions $h(x)$ are there such that $h'(x) = h(x)$?

Solution. Infinitely many, a $h(x) = ce^x$ is such a function for any constant c. ■

c. Draw the graph of a function $f(x)$ which is continuous on the domain $(0, \infty)$ such that $\lim_{x \to 0} f(x) = \infty$ and with a local maximum at $x = 5$ as its only critical point, or explain why there cannot be such a function.

Solution. There can be no such function. Suppose there were. Such an $f(x)$ must have a local maximum at $x = 5$, it would also have values which are less than $f(5)$ for some points to the left of $x = 5$. Since $f(x)$ must also have a vertical asymptote heading to infinity at $x = 0$, which is to the left of $x = 5$, and supposed to be continuous between $x = 0$ and $x = 5$, it must have a local minimum somewhere in between. However, such a local minimum would be a another critical point, but $f(x)$ is only supposed to have the one at $x = 5$ ... ■

4. Find the intercepts, maxima and minima, inflection points, and vertical and horizontal asymptotes of $f(x) = \dfrac{x}{x^2 + 1}$ and sketch the graph of $f(x)$ based on this information. *[10]*

Solution.

i. (*Domain*) Since x and $x^2 + 1$ are defined and continuous for all x, and $x^2 + 1 > 0$ for all x, $f(x) = \dfrac{x}{x^2+1}$ is defined and continuous for all x.

ii. (*Intercepts*) Since $f(x) = \dfrac{x}{x^2+1} = 0 \iff x = 0$, $(0, 0)$ is the only x-intercept and the only y-intercept of $f(x)$.

iii. (*Local maxima and minima*)

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{x}{x^2+1}\right) \\ &= \frac{1 \cdot \left(x^2+1\right) - x \cdot (2x)}{\left(x^2+1\right)^2} \\ &= \frac{1 - x^2}{\left(x^2+1\right)^2} \end{aligned}$$

which is defined everywhere since $x^2 + 1 > 0$ for all x.

Thus $f'(x) = 0 \iff 1 - x^2 = 0 \iff x = \pm 1$. We determine which of these give local maxima or minima by considering the intervals of increase and decrease:

x	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x$
$f'(x)$	< 0	0	> 0	0	< 0
$f(x)$	decreasing	local min	increasing	local max	decreasing

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Thus $f(-1) = \frac{1}{2}$ and $f(1) = \frac{1}{2}$ are, respectively, local minimum and local maximum points of $f(x)$.

iv. (*Points of inflection and curvature*)

$$\begin{aligned}
f''(x) &= \frac{d}{dx} f'(x) \\
&= \frac{d}{dx} \frac{1-x^2}{\left(x^2+1\right)^2} \\
&= \frac{(-2x)\left(x^2+1\right)^2 - \left(1-x^2\right)\cdot 2\left(x^2+1\right)\cdot 2x}{\left(x^2+1\right)^4} \\
&= \frac{(-2x)\left(x^2+1\right) - \left(1-x^2\right)\cdot 2\cdot 2x}{\left(x^2+1\right)^3} \\
&= \frac{2x^3-6x}{\left(x^2+1\right)^3} \\
&= \frac{2x\left(x^2-3\right)}{\left(x^2+1\right)^3}
\end{aligned}$$

which is defined everywhere since $x^2+1>0$ for all x.

Thus $f''(x) = 0$ if $x = 0$ or $x = \pm\sqrt{3}$. To sort out the inflection points and intervals of curvature, we check where $f''(x)$ is positive and where it is negative.

x	$x < -\sqrt{3}$	$x = -\sqrt{3}$	$-\sqrt{3} < x < 0$	$x = 0$	$0 < x < \sqrt{3}$	$x = \sqrt{3}$	$x > \sqrt{3}$
$f''(x)$	< 0	0	> 0	0	< 0	0	> 0
$f(x)$	conc. down	infl. pt.	conc. up	infl. pt.	conc. down	infl. pt.	conc. up

Thus $f\left(-\sqrt{3}\right) = -\frac{\sqrt{3}}{4}$, $f(0) = 0$, and $f\left(\sqrt{3}\right) = \frac{\sqrt{3}}{4}$ are the inflection points of $f(x)$.

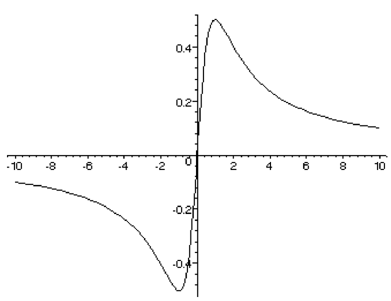
v. (*Vertical asymptotes*) $f(x)$ has no vertical asymptotes because it is defined and continuous for all x.

vi. (*Horizontal asymptotes*) Since $x \to \pm\infty$ and $x^2+1 \to infty$ as $x \to \pm\infty$, we can use l'Hôpital's Rule in the relevant limits. (Not that we really need to, but what the heck!)

$$\lim_{x\to+\infty} \frac{x}{x^2+1} = \lim_{x\to+\infty} \frac{1}{2x} = 0$$
$$\lim_{x\to-\infty} \frac{x}{x^2+1} = \lim_{x\to-\infty} \frac{1}{2x} = 0$$

Thus $f(x)$ has a horizontal asymptote at $y = 0$ in both directions.

vii. (*The graph!*)



Whew! ■

Mathematics 110 – Calculus of one variable
TRENT UNIVERSITY, 2001-2002

Test #2
Friday, 8 February, 2002
Time: 50 minutes

1. Compute any *three* of the integrals **a-e**. *[12 = 3 × 4 ea.]*

$$\textbf{a.} \int_{-\pi/2}^{\pi/2} \cos^3(x)\,dx \qquad \textbf{b.} \int x^2 \ln(x)\,dx \qquad \textbf{c.} \int_0^1 \left(e^x\right)^2\,dx$$

$$\textbf{d.} \int \frac{e^{2x}\ln\left(e^{2x}+1\right)}{e^{2x}+1}\,dx \qquad \textbf{e.} \int_1^e \left(\ln(x)\right)^2\,dx$$

Solutions.

a.

$$\int_{-\pi/2}^{\pi/2} \cos^3(x)\,dx = \int_{-\pi/2}^{\pi/2} \cos^2(x)\cos(x)\,dx = \int_{-\pi/2}^{\pi/2} \left(1-\sin^2(x)\right)\cos(x)\,dx$$

We'll substitute $u = \sin(x)$, so $du = \cos(x)\,dx$, $-1 = \sin(-\pi/2)$, and $1 = \sin(\pi/2)$.

$$= \int_{-1}^{1} \left(1-u^2\right)\,du = \left(u - \frac{u^3}{3}\right)\Bigg|_{-1}^{1} = \left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right) = \frac{4}{3} \quad \blacksquare$$

b. We'll use integration by parts, with $u = \ln(x)$ and $dv = x^2\,dx$, so $du = \frac{1}{x}\,dx$ and $v = \frac{x^3}{3}$.

$$\int x^2\ln(x)\,dx = \frac{x^3}{3}\ln(x) - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx = \frac{x^3}{3}\ln(x) - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C \quad \blacksquare$$

c. After bit of algebra, we'll use the substitution $u = 2x$, so $du = 2\,dx$ (and $\frac{1}{2}\,du = dx$), $0 = 2\cdot 0$, and $2 = 2\cdot 1$.

$$\int_0^1 \left(e^x\right)^2\,dx = \int_0^1 e^{2x}\,dx = \int_0^2 e^u\cdot\frac{1}{2}\,du = \frac{1}{2}e^u\Bigg|_0^2 = \frac{1}{2}\left(e^2-1\right) \quad \blacksquare$$

d. We'll substitute whole hog: let $w = \ln\left(e^{2x}+1\right)$, so $dw = \frac{2e^{2x}}{e^{2x}+1}\,dx$ (and $\frac{1}{2}\,dw = \frac{e^{2x}}{e^{2x}+1}$).

$$\int \frac{e^{2x}\ln\left(e^{2x}+1\right)}{e^{2x}+1}\,dx = \int w\cdot\frac{1}{2}\,dw = \frac{w^2}{4} + C = \frac{1}{4}\left(\ln\left(e^{2x}+1\right)\right)^2 + C \quad \blacksquare$$

e. We'll use integration by parts, with $u = (\ln(x))^2$ and $dv = dx$, so $du = 2\ln(x)\cdot\frac{1}{x}\,dx$ and $v = x$.

$$\int_1^e \left(\ln(x)\right)^2\,dx = x\left(\ln(x)\right)^2\Big|_1^e - \int_1^e x\cdot 2\ln(x)\cdot\frac{1}{x}\,dx = \left(e\cdot 1^2 - 1\cdot 0^2\right) - 2\int_1^e \ln(x)\,dx$$

We use integration by parts again, with $u = \ln(x)$ and $dv = dx$, so $du = \dfrac{1}{x}\,dx$ and $v = x$.

$$= e - 2\left(x\ln(x)|_1^e - \int_1^e x\cdot\frac{1}{x}\,dx\right) = e - 2\left((e\cdot 1 - 1\cdot 0) - \int_1^e 1\,dx\right)$$

$$= e - 2\left(e - x|_1^e\right) = e - 2\left(e - (e-1)\right) = e - 2 \quad \blacksquare$$

2. Do any *two* of **a-c**. *[8 = 2 × 4 ea.]*

a. Compute $\int_0^1 (2x+3)\,dx$ using the Right-hand Rule.

b. Compute $\frac{dy}{dx}$ if $y = \int_0^{x^2} \sqrt{t}\,dt$ (where $x \geq 0$) without evaluating the integral.

c. Compute $\int_{-1}^{1} \sqrt{1-x^2}\,dx$ by interpreting it as an area.

Solutions.

a. If we partition $[0,1]$ into n equal subintervals, then the ith subinterval is $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, which has width $\frac{1}{n}$ and right endpoint $\frac{i}{n}$. Thus the area of the ith rectangle in the Right-hand Rule Riemann sum is $\left(2\frac{i}{n}+3\right)\frac{1}{n}$. Hence

$$\begin{aligned}
\int_0^1 (2x+3)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2\frac{i}{n}+3\right)\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(2\frac{i}{n}+3\right) \\
&= \lim_{n\to\infty} \frac{1}{n}\left[2\left(\sum_{i=1}^{n} \frac{i}{n}\right) + \left(\sum_{i=1}^{n} 3\right)\right] = \lim_{n\to\infty} \frac{1}{n}\left[\frac{2}{n}\left(\sum_{i=1}^{n} i\right) + 3n\right] \\
&= \lim_{n\to\infty} \frac{1}{n}\left[\frac{2}{n}\cdot\frac{n(n+1)}{2} + 3n\right] = \lim_{n\to\infty} \frac{1}{n}\left[(n+1)+3n\right] = \lim_{n\to\infty} \frac{1}{n}\left[4n+1\right] \\
&= \lim_{n\to\infty} \left[\frac{4n}{n} + \frac{1}{n}\right] = \lim_{n\to\infty} \left[4+\frac{1}{n}\right] = 4+0 = 4 \quad \blacksquare
\end{aligned}$$

b. Let $u = x^2$; since $x \geq 0$, $x = \sqrt{u}$. Then, using the Chain Rule and the Fundamental Theorem of Calculus,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \left(\frac{d}{du}\int_0^u \sqrt{t}\,dt\right)\cdot\frac{du}{dx} = \sqrt{u}\cdot\frac{du}{dx} = \sqrt{x^2}\cdot\frac{d}{dx}x^2 = x\cdot 2x = 2x^2 \quad \blacksquare$$

c. Note that $y = \sqrt{1-x^2}$, $-1 \leq x \leq 1$, is the upper half of the unit circle $x^2+y^2=1$. This circle has area $\pi 1^2 = \pi$, so $\int_{-1}^{1} \sqrt{1-x^2}\,dx$, which represents the area of the upper half of the circle, is equal to $\frac{\pi}{2}$. $\blacksquare$

3. Water is poured at a rate of 1 m^3/min into a conical tank (set up point down) 2 m high and with radius 1 m at the top. How quickly is the water rising in the tank at the instant that it is 1 m deep over the tip of the cone? *[8]*

(The volume of a cone of height h and radius r is $\frac{1}{3}\pi r^2 h$.)

Solution. At any given instant, the water in the tank occupies a conical volume, with height – that is, depth in the tank – h and radius r in the same proportions as the tank as a whole.

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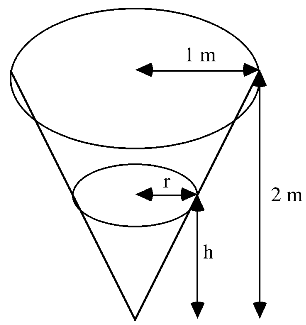


Hence $\frac{r}{h} = \frac{1}{2}$, so $r = \frac{h}{2}$, and it follows that the volume of the water at the given instant is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3 .$$

Note that the rate at which the water is rising in the tank is $\frac{dh}{dt}$.

Since, on the one hand $\frac{dV}{dt} = 1\ m^3/min$, and on the other hand

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{12}\pi h^3\right) = \frac{d}{dh}\left(\frac{1}{12}\pi h^3\right)\cdot\frac{dh}{dt} = \frac{1}{12}\pi\cdot 3h^2\cdot\frac{dh}{dt} = \frac{1}{4}\pi h^2\cdot\frac{dh}{dt} ,$$

we know that any given instant, $\frac{dh}{dt} = \frac{dV}{dt}/\frac{1}{4}\pi h^2 = 1/\frac{1}{4}\pi h^2 = 4/\pi h^2$. At the particular instant that $h = 1\ m$, it follows that $\frac{dh}{dt} = 4/\pi 1^2 = 4/\pi\ m/min$. ■

4. Consider the region in the first quadrant with upper boundary $y = x^2$ and lower boundary $y = x^3$, and also the solid obtained by rotating this region about the y-axis.

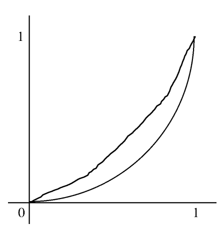
a. Sketch the region and find its area. *[4]*

b. Sketch the solid and find its volume. *[7]*

c. What is the average area of either a washer or a shell (your pick!) for the solid? *[1]*

Solution.

a. First, we find the points of intersection of the two curves: if $x^2 = x^3$, then $x = 0$ or $x = x^3/x^2 = 1$. Note that when $0 \le x \le 1$, then $x^3 = x^2 \cdot x \le x^2 \cdot 1 = x^2$. It's not too hard to see that the region between the curves looks more or less like:



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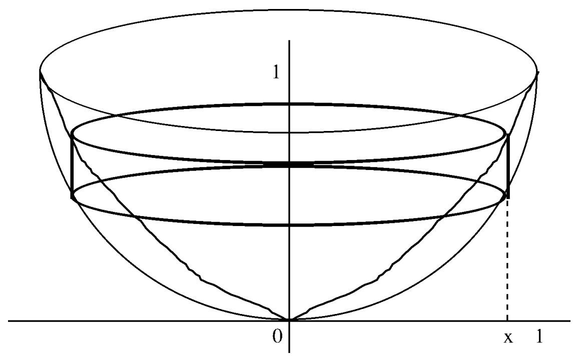
The area of the region is then

$$\int_0^1 \left(x^2 - x^3\right)\, dx = \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \left(\frac{1}{3} - \frac{1}{4}\right) - (0-0) = \frac{1}{12} \quad \blacksquare$$

b. Rotating (revolving, whatever ...) the region about the y-axis produces the following solid.



The volume of this solid is a little easier to compute using shells than using washers. Since we rotated the region about a vertical line, we will use x as the variable of integration; note that $0 \leq x \leq 1$ over the region in question. With respect to x, a generic cylindrical shell has radius $r = x - 0 = x$ and height $h = x^2 - x^3$. Thus the volume of the solid is

$$\begin{aligned}\int_0^1 2\pi r h\, dx &= \int_0^1 2\pi x \left(x^2 - x^3\right)\, dx = 2\pi \int_0^1 \left(x^3 - x^4\right)\, dx \\ &= 2\pi \left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_0^1 = 2\pi \left[\left(\frac{1}{4} - \frac{1}{5}\right) - (0-0)\right] = 2\pi \frac{1}{20} = \frac{\pi}{10}. \quad \blacksquare\end{aligned}$$

c. From **b** we know that the area of the cylindrical shell for x, where $0 \leq x \leq 1$, is $2\pi x \left(x^2 - x^3\right)$. Thus the average area of a cylindrical shell for this solid is

$$\frac{1}{1-0} \int_0^1 2\pi x \left(x^2 - x^3\right)\, dx = 1 \cdot \frac{\pi}{10} = \frac{\pi}{10}. \quad \blacksquare$$

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Mathematics 110, Section A
Calculus I: Calculus of one variable
2002-2003
Solutions

Mathematics 110 – Calculus of one variable
Trent University 2002-2003

QUIZ SOLUTIONS

Quiz #1. (§A) Wednesday, 18 September, 2001. *[10 minutes]*

12:00 Seminar

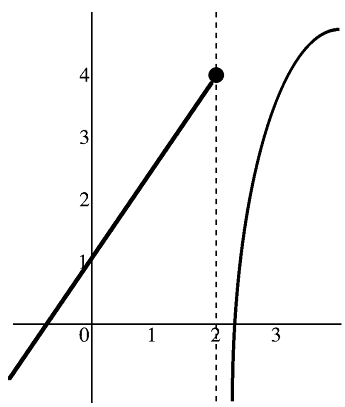
1. Compute $\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2}$ or show that this limit does not exist. *[5]*

Solution.

$$\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+1)}{x - 2} = \lim_{x \to 2} (x+1) = 2 + 1 = 3 \qquad \blacksquare$$

2. Sketch the graph of a function $f(x)$ which is defined for all x and for which $\lim_{x \to 0} f(x) = 1$, $\lim_{x \to 2^+} f(x)$ does not exist, and $\lim_{x \to 2^-} f(x) = 4$. *[5]*

Solution.



$\blacksquare$

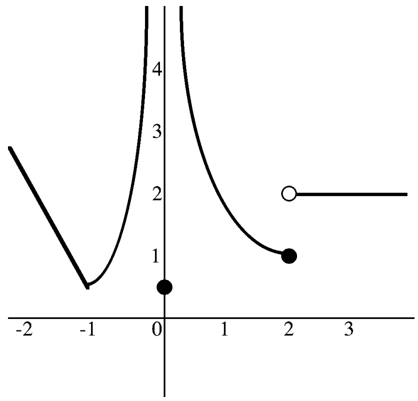
13:00 Seminar

1. Compute $\lim_{x \to 2^-} \frac{x^2 - x + 2}{x - 2}$ or show that this limit does not exist. *[5]*

Solution. Note that $\lim_{x \to 2^-} x^2 - x + 2 = 2^2 - 2 + 2 = 4$ and $\lim_{x \to 2^-} x - 2 = 0$. It follows that $\lim_{x \to 2^-} \frac{x^2 - x + 2}{x - 2}$ fails to exist. (Since when x is a bit less than 2, $x^2 - x + 2$ is about 4 and $x - 2$ is a bit less than 0, it $\lim_{x \to 2^-} \frac{x^2 - x + 2}{x - 2} = -\infty$.) $\blacksquare$

2. Sketch the graph of a function $g(x)$ which is defined for all x, and for which $\lim_{x \to 0} g(x) = \infty$, $\lim_{x \to 2} g(x)$ does not exist, and $g(x)$ does not have an asymptote at $x = 2$. *[5]*

Solution.



■

Quiz #2. (§A) Wednesday, 25 September, 2001. *[10 minutes]*

12:00 Seminar

1. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 3}(5x - 7) = 8$. *[10]*

Solution. We need to show that given any $\varepsilon > 0$, one can find a $\delta > 0$ such that if $|x - 3| < \delta$, then $|(5x - 7) - 8| < \varepsilon$. Suppose we are given an $\varepsilon > 0$. Then

$$\begin{aligned} & |(5x - 7) - 8| < \varepsilon \\ \iff & |5x - 15| < \varepsilon \\ \iff & 5|x - 3| < \varepsilon \\ \iff & |x - 3| < \frac{\varepsilon}{5}, \end{aligned}$$

so $\delta = \frac{\varepsilon}{5}$ will do the job. ■

13:00 Seminar

1. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 2}(3 - 2x) = -1$. *[10]*

Solution. We need to show that given any $\varepsilon > 0$, one can find a $\delta > 0$ such that if $|x - 2| < \delta$, then $|(3 - 2x) - (-1)| < \varepsilon$. Suppose we are given an $\varepsilon > 0$. Then

$$\begin{aligned} & |(3 - 2x) - (-1)| < \varepsilon \\ \iff & |(3 - 2x) + 1| < \varepsilon \\ \iff & |4 - 2x| < \varepsilon \\ \iff & 2|x - 2| = 2|2 - x| < \varepsilon \\ \iff & |x - 2| < \frac{\varepsilon}{2}, \end{aligned}$$

2

so $\delta = \frac{\varepsilon}{2}$ will do the job. ■

Quiz #3. Wednesday, 2 October, 2001. *[10 minutes]*

12:00 Seminar

1. For which values of the constant c is the function

$$f(x) = \begin{cases} e^{cx} & x \geq 0 \\ cx + 1 & x < 0 \end{cases}$$

continuous at $x = 0$? Why? *[10]*

Solution. For $f(x)$ to be continuous at $x = 0$ we need to have $f(0)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0^+} f(x)$ all be defined and equal to each other:

$f(0) = e^{c0} = e^0 = 1$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} cx + 1 = c0 + 1 = 1$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{cx} = e^{c0} = e^0 = 1$

Since all three are defined and equal to 1, no matter what the value of c, $f(x)$ is continuous at $x = 0$. ■

13:00 Seminar

1. For which values of the constant c is the function

$$f(x) = \begin{cases} e^{cx} & x \geq 0 \\ c(x + 1) & x < 0 \end{cases}$$

continuous at $x = 0$? Why? *[10]*

Solution. For $f(x)$ to be continuous at $x = 0$ we need to have $f(0)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0^+} f(x)$ all be defined and equal to each other:

$f(0) = e^{c0} = e^0 = 1$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} c(x + 1) = c(0 + 1) = c$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{cx} = e^{c0} = e^0 = 1$

All three are defined for all values of c, but are equal to each other only for $c = 1$. hence $f(x)$ is continuous at $x = 0$ exactly when $c = 1$. ■

Quiz #4. Wednesday, 9 October, 2002. *[12 minutes]*

12:00 Seminar

Suppose

$$f(x) = \begin{cases} x & x < 0 \\ 0 & x = 0 \\ 2x^2 + x & x > 0 \end{cases} \quad .$$

3

1. Use the definition of the derivative to check whether $f'(0)$ exists and compute it if it does. *[7]*

Solution. By definition, $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$. For this limit to exist, both of $\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$ and $\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$ must be defined and be equal:

i. By the definition of $f(x)$, $f(0) = 0$ and $f(x) = x$ when $x < 0$. It follows that:

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{(0+h) - 0}{h} = \lim_{h \to 0^-} \frac{h}{h} = \lim_{h \to 0^-} 1 = 1$$

ii. By the definition of $f(x)$, $f(0) = 0$ and $f(x) = 2x^2 + x$ when $x > 0$. It follows that:

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0^+} \frac{\left[2(0+h)^2 + (0+h)\right] - 0}{h} \\ &= \lim_{h \to 0^+} \frac{2h^2 + h}{h} = \lim_{h \to 0^+} (2h+1) = 1 \end{aligned}$$

Thus $f'(0)$ exists and equals 1. ■

2. Compute $f'(1)$ (any way you like). *[3]*

Solution. By the definition of $f(x)$, $f(x) = 2x^2 + x$ when $x > 0$. Thus, when $x > 0$, $f'(x) = 2 \cdot 2x + 1 = 4x + 1$. Since $1 > 0$ …), it follows that $f'(1) = 4 \cdot 1 + 1 = 5$. ■

13:00 Seminar

Suppose $g(x) = \frac{1}{x+1}$. Compute $g'(x)$ using

1. the rules for computing derivatives *[3]*, and

Solution. Using the Quotient Rule (and some other bits and pieces …):

$$\begin{aligned} g'(x) = \frac{d}{dx}\left(\frac{1}{x+1}\right) &= \frac{\left(\frac{d}{dx}1\right) \cdot (x+1) - 1 \cdot \left(\frac{d}{dx}(x+1)\right)}{(x+1)^2} \\ &= \frac{0 \cdot (x+1) - 1 \cdot (1+0)}{(x+1)^2} = \frac{-1}{(x+1)^2} \qquad \blacksquare \end{aligned}$$

2. the definition of the derivative. *[7]*

Solution. Here goes!

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{\frac{1}{(x+h)+1} - \frac{1}{x+1}}{h} = \lim_{h \to 0} \frac{\frac{(x+1)-(x+h+1)}{(x+h)+1)(x+1)}}{h} = \lim_{h \to 0} \frac{-h}{h(x+h)+1)(x+1)} \\ &= \lim_{h \to 0} \frac{-1}{(x+h)+1)(x+1)} = \frac{-1}{(x+1)(x+1)} = \frac{-1}{(x+1)^2} \qquad \blacksquare \end{aligned}$$

Quiz #5. Wednesday, 16 October, 2002. *[10 minutes]*

12:00 Seminar

Compute $\frac{d}{dx}\sqrt[5]{x}$ using

1. the Power Rule *[2]*, and

Solution. $\frac{d}{dx}\sqrt[5]{x} = \frac{d}{dx}x^{1/5} = \frac{1}{5}x^{(1/5)-1} = \frac{1}{5}x^{-4/5} = \frac{1}{5x^{4/5}}$ ■

2. the fact that $f(x) = \sqrt[5]{x}$ is the inverse function of $g(x) = x^5$. *[8]*

Solution. Since $f(x) = \sqrt[5]{x}$ is the inverse function of $g(x) = x^5$, $x = g(f(x)) = \left(\sqrt[5]{x}\right)^5$. Differentiating both sides gives:

$$\begin{aligned}
1 = \frac{dx}{dx} &= \frac{d}{dx}\left(\sqrt[5]{x}\right)^5 \\
&= \frac{d}{dx}u^5 \qquad \text{(Where } u = \sqrt[5]{x}.\text{)} \\
&= \left(\frac{d}{du}u^5\right)\cdot\frac{du}{dx} \qquad \text{(Using the Chain Rule.)} \\
&= 5u^4\cdot\frac{du}{dx} \\
&= 5\left(\sqrt[5]{x}\right)^4\cdot\frac{d}{dx}\sqrt[5]{x} \\
&= 5x^{4/5}\cdot\frac{d}{dx}\sqrt[5]{x}
\end{aligned}$$

Solving this equation for $\frac{d}{dx}\left(\sqrt[5]{x}\right)$ gives $\frac{d}{dx}\sqrt[5]{x} = \frac{1}{5x^{4/5}}$. ■

13:00 Seminar

1. Compute $\frac{d}{dx}\arccos(x)$ given that $x = \cos(\arccos(x))$ and $\cos^2(x) + \sin^2(x) = 1$. *[10]*

Solution. Differentiating both sides of $x = \cos(\arccos(x))$ gives:

$$\begin{aligned}
1 = \frac{dx}{dx} &= \frac{d}{dx}\cos(\arccos(x)) \\
&= \frac{d}{dx}\cos(u) \qquad \text{(Where } u = \arccos(x).\text{)} \\
&= \left(\frac{d}{du}\cos(u)\right)\cdot\frac{du}{dx} \qquad \text{(Using the Chain Rule.)} \\
&= (-\sin(u))\cdot\frac{du}{dx} \\
&= (-\sin(\arccos(x)))\cdot\frac{d}{dx}\arccos(x)
\end{aligned}$$

Solving this equation for $\frac{d}{dx}\arccos(x)$ gives

$$\frac{d}{dx}\arccos(x) = \frac{1}{-\sin(\arccos(x))} = \frac{-1}{\sin(\arccos(x))},$$

which answer can be simplified considerably. Since $\cos^2(x) + \sin^2(x) = 1$, it follows that $\sin(x) = \sqrt{1 - \cos^2(x)}$, so

$$\begin{aligned}
\frac{d}{dx}\arccos(x) &= \frac{-1}{\sin\left(\arccos(x)\right)} = \frac{-1}{\sqrt{1 - \cos^2\left(\arccos(x)\right)}} \\
&= \frac{-1}{\sqrt{1 - \left(\cos\left(\arccos(x)\right)\right)^2}} = \frac{-1}{\sqrt{1 - x^2}}
\end{aligned}$$

since, once again, $x = \cos\left(\arccos(x)\right)$. ■

Quiz #6. Wednesday, 30 October, 2002. *[10 minutes]*

12:00 Seminar

1. Find the absolute and local maxima and minima of $f(x) = x^3 + 2x^2 - x - 2$ on $[-2, 2]$. *[10]*

Solution. First, note that $f(x)$ is defined and continuous throughout $[-2, 2]$. At the endpoints we get $f(-2) = (-2)^3 + 2 \cdot (-2)^2 - (-2) - 2 = -8 + 8 + 2 - 2 = 0$ and $f(2) = 2^3 + 2 \cdot 2^2 - 2 - 2 = 8 + 8 - 2 - 2 = 12$.

Second, $f'(x) = \dfrac{d}{dx}\left(x^3 + 2x^2 - x - 2\right) = 3x^2 + 2 \cdot 2x - 1 = 3x^2 + 4x - 1$, which is also defined throughout $[-2, 2]$. To find the critical points we use the quadratic formula:

$$\begin{aligned}
f'(x) = 0 &\iff 3x^2 + 4x - 1 = 0 \\
&\iff x = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 3 \cdot (-1)}}{2 \cdot 3} \\
&\iff x = \frac{-4 \pm \sqrt{16 + 12)}}{6} \\
&\iff x = \frac{-4 \pm \sqrt{28}}{6} \\
&\iff x = \frac{-4 \pm 2\sqrt{7}}{6} \\
&\iff x = \frac{-2 \pm \sqrt{7}}{3}
\end{aligned}$$

The problem here is that $\frac{-2 \pm \sqrt{7}}{3}$ is not a terribly nice pair of numbers to play with. One could use a calculator to get results which are close enough for our purposes, or one can use the fact that $2 < \sqrt{7} < 3$ to observe that $-\frac{5}{3} < \frac{-2-\sqrt{7}}{3} < -\frac{4}{3}$ and $0 < \frac{-2+\sqrt{7}}{3} < \frac{1}{3}$. Either way, both $\frac{-2-\sqrt{7}}{3}$ and $\frac{-2+\sqrt{7}}{3}$ are in the interval $[-2, 2]$.

It remains to determine the values of $f(x)$ at the critical points and compare these to each other and to the values at the endpoints. We leave this to the reader; one can use a calculator or approximations to figure out what is going on ... ■

13:00 Seminar

1. Find the absolute and local maxima and minima of $f(x) = x^3 - 3x^2 - x + 3$ on $[-2, 2]$. *[10]*

Solution. First, note that $f(x)$ is defined and continuous throughout $[-2, 2]$. At the endpoints we get $f(-2) = (-2)^3 - 3 \cdot (-2)^2 - (-2) + 3 = -8 - 12 + 2 + 3 = -15$ and $f(2) = 2^3 - 3 \cdot 2^2 - 2 + 3 = 8 - 12 - 2 + 3 = -3$.

Second, $f'(x) = \dfrac{d}{dx}\left(x^3 - 3x^2 - x + 3\right) = 3x^2 - 3 \cdot 2x - 1 = 3x^2 - 6x - 1$, which is also defined throughout $[-2, 2]$. To find the critical points we use the quadratic formula:

$$\begin{aligned} f'(x) = 0 &\iff 3x^2 - 6x - 1 = 0 \\ &\iff x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 3 \cdot (-1)}}{2 \cdot 3} \\ &\iff x = \frac{6 \pm \sqrt{36 + 12)}}{6} \\ &\iff x = \frac{6 \pm \sqrt{48}}{6} \\ &\iff x = \frac{6 \pm 4\sqrt{3}}{6} \\ &\iff x = \frac{3 \pm 2\sqrt{3}}{3} \end{aligned}$$

The problem here is that $\frac{3 \pm 2\sqrt{3}}{3}$ is not a terribly nice pair of numbers to play with. One could use a calculator to get results which are close enough for our purposes. Using the fact that $1 < \sqrt{3} < 2$ to observe that $-\frac{1}{3} < \frac{3 - 2\sqrt{3}}{3} < \frac{1}{3}$ and $\frac{5}{3} < \frac{3 + 2\sqrt{3}}{3} < 3$ does not even tell us whether the second root falls in the interval $[-2, 2]$ ...

It remains to determine the values of $f(x)$ at the critical points and compare these to each other and to the values at the endpoints. We leave this to the reader; one can use a calculator or (better) approximations to figure out what is going on ... ■

Quiz #7. Wednesday, 6 November, 2002. *[15 minutes]*

12:00 Seminar

1. Find the intercepts, critical and inflection points, and horizontal asymptotes of $f(x) = (x - 2)e^x$ and sketch its graph. *[10]*

Solution. Note that $f(x) = (x - 2)e^x$ is defined and continuous everywhere; in particular, it has no vertical asymptotes.

i. Intercepts: For the x-intercept, since $e^x \neq 0$ for all x, $f(x) = 0$ exactly when $x - 2 = 0$, *i.e.* when $x = 2$. For the y-intercept, note that $f(0) = (0 - 2)e^0 = -2 \cdot 1 = -2$.

ii. Critical points: First,

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x - 2)e^x = \left(\frac{d}{dx}(x - 2)\right) \cdot e^x + (x - 2)\left(\frac{d}{dx}e^x\right) \\ &= e^x + (x - 2)e^x = (x - 1)e^x , \end{aligned}$$

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which is defined and continuous for all x. Since $e^x > 0$ for all x, $f'(x) = 0$ precisely when $x = 1$. Note that because $f'(x) = (x-1)e^x < 0$ for all $x < 1$ and $f'(x) > 0$ for all $x > 1$, $f(x)$ has a local minimum at the critical point.

iii. Inflection points: First,

$$\begin{aligned} f''(x) &= \frac{d}{dx}(x-1)e^x = \left(\frac{d}{dx}(x-1)\right) \cdot e^x + (x-1) \cdot \left(\frac{d}{dx}e^x\right) \\ &= e^x + (x-1)e^x = xe^x \,, \end{aligned}$$

which is defined and continuous for all x. Since $e^x > 0$ for all x, $f''(x) = 0$ precisely when $x = 0$. Because $f''(x) = xe^x < 0$ for all $x < 0$ and $f''(x) > 0$ for all $x > 0$, $f(x)$ has an inflection point at $x = 0$.

iv. Horizontal asymptotes: First,

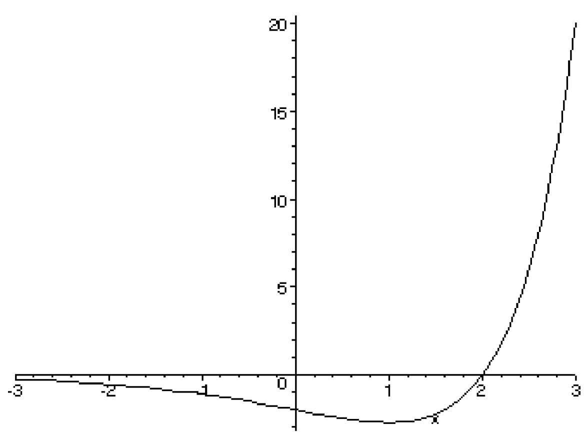
$$\lim_{t \to -\infty} (x-2)e^x = \lim_{t \to \infty} \frac{x-2}{e^{-x}} = \lim_{t \to -\infty} \frac{\frac{d}{dx}(x-2)}{\frac{d}{dx}e^{-x}} = \lim_{t \to -\infty} \frac{1}{e^{-x}} = 0$$

using l'Hôpital's Rule because $\lim_{t \to -\infty} (x-2) = \infty$ and $\lim_{t \to -\infty} e^{-x} = \infty$. It follows that $h(x)$ has a horizontal asymptote of $y = 0$ in the negative x direction. Second,

$$\lim_{t \to \infty} (x-2)e^x = \infty$$

because $\lim_{t \to \infty} (x-2) = \infty$ and $\lim_{t \to \infty} e^x = \infty$. It follows that $h(x)$ has no horizontal asymptote in the positive x direction.

v. The graph:



■

8

13:00 Seminar

1. Find the intercepts, critical and inflection points, and horizontal asymptotes of $h(x) = (x+1)e^{-x}$ and sketch its graph. *[10]*

Solution. Note that $h(x) = (x+1)e^{-x}$ is defined and continuous everywhere; in particular, it has no vertical asymptotes.

i. *Intercepts:* For the x-intercept, since $e^x \neq 0$ for all x, $h(x) = 0$ exactly when $x+1 = 0$, *i.e.* when $x = -1$. For the y-intercept, note that $h(0) = (0-1)e^{-0} = -1 \cdot 1 = -1$.

ii. *Critical points:* First,

$$\begin{aligned} h'(x) &= \frac{d}{dx}(x+1)e^{-x} = \left(\frac{d}{dx}(x+1)\right) \cdot e^{-x} + (x+1)\left(\frac{d}{dx}e^{-x}\right) \\ &= e^{-x} + (x+1)\left(-e^{-x}\right) = -xe^{-x}, \end{aligned}$$

which is defined and continuous for all x. Since $e^x > 0$ for all x, $h'(x) = 0$ precisely when $x = 0$. Note that because $h'(x) = -xe^{-x} > 0$ for all $x < 0$ and $h'(x) < 0$ for all $x > 0$, $h(x)$ has a local maximum at the critical point.

iii. *Inflection points:* First,

$$\begin{aligned} h''(x) &= \frac{d}{dx}\left(-xe^{-x}\right) = \left(\frac{d}{dx}(-x)\right) \cdot e^{-x} + (-x) \cdot \left(\frac{d}{dx}e^{-x}\right) \\ &= -e^{-x} - x \cdot \left(-e^{-x}\right) = (x-1)e^{-x}, \end{aligned}$$

which is defined and continuous for all x. Since $e^x > 0$ for all x, $h''(x) = 0$ precisely when $x = 1$. Because $h''(x) = (x-1)e^{-x} < 0$ for all $x < 1$ and $h''(x) > 0$ for all $x > 1$, $h(x)$ has an inflection point at $x = 1$.

iv. *Horizontal asymptotes:* First,

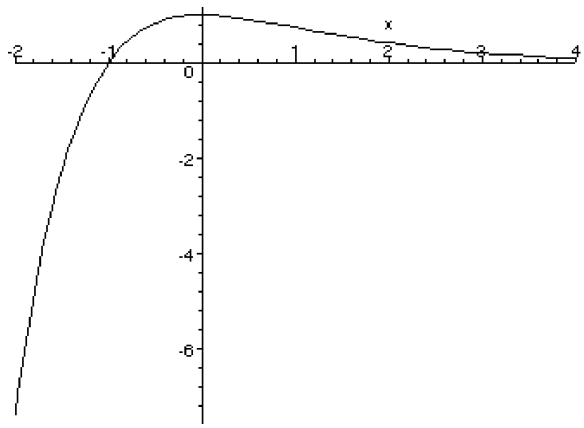
$$\lim_{t \to -\infty} (x+1)e^{-x} = -\infty$$

since $\lim_{t \to -\infty}(x+1) = -\infty$ and $\lim_{t \to -\infty} e^{-x} = \infty$. It follows that $h(x)$ has no horizontal asymptote in the negative x direction. Second,

$$\lim_{t \to \infty} (x+1)e^{-x} = \lim_{t \to \infty} \frac{x+1}{e^x} = \lim_{t \to \infty} \frac{\frac{d}{dx}(x+1)}{\frac{d}{dx}e^x} = \lim_{t \to \infty} \frac{1}{e^x} = 0$$

using l'Hôpital's Rule because $\lim_{t \to \infty}(x+1) = \infty$ and $\lim_{t \to \infty} e^x = \infty$. It follows that $h(x)$ has a horizontal asymptote of $y = 0$ in the positive x direction.

v. The graph:



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Quiz #8. Wednesday, 27 November, 2002. *[15 minutes]*

12:00 Seminar

1. Compute:

$$\int_1^{e^\pi} \frac{1}{x} \sin(\ln(x))\ dx \qquad [5]$$

Solution. We'll use the substitution $u = \ln(x)$ and change the limits when we do. Note that $du = \frac{1}{x}dx$ and that when $x = 1$, $u = \ln(1) = 0$, and that when $x = e^\pi$, $u = \ln(e^\pi) = \pi\ln(e) = \pi \cdot 1 = \pi$. Then

$$\int_1^{e^\pi} \frac{1}{x} \sin(\ln(x))\ dx = \int_0^\pi \sin(u)\, du = \cos(u)|_0^\pi = \cos(\pi) - \cos(0) = (-1) - 1 = -2$$

does the job. ■

2. What definite integral does the Right-hand Rule limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right) \cdot \frac{1}{n}$$

correspond to? *[5]*

Solution. The general formula for the Right-hand Rule is:

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) \cdot \frac{b-a}{n}$$

10

Here $\frac{b-a}{n}$ is the width and $f\left(a+i\frac{b-a}{n}\right)$ is the height of the ith rectangle in the Riemann sum using a partition of $[a,b]$ into n equal subintervals and evaluating $f(x)$ at the right endpoint of each subinterval to find the height of the corresponding rectangle.

We compare the right hand side of the formula to the given limit and try to indentify a, b, and f. Comparing the given common width $\frac{1}{n}$ of the rectangles with $\frac{b-a}{n}$ tells us that $b-a=1$; comparing $1+\frac{i}{n}$ with $f\left(a+i\frac{b-a}{n}\right)$, it is a reasonable guess that $a=1$ (so $b=1+1=2$). If so, then $f(x)$ does nothing to x, *i.e.* $f(x)=x$. Hence the desired definite integral could be:

$$\int_1^2 x\,dx$$

This is *not* the only solution. For example, if one had instead guessed that $a=0$ (so $b=1$), we would get that $f(x)=1+x$ and that the integral being sought was

$$\int_0^1 (1+x)\,dx$$

instead. In general, one could let a be any constant; then $b=a+1$ and $f(x)=1-a+x$ do the job. ■

13:00 Seminar

1. Compute:

$$\int_0^{\pi/4} \frac{\tan(x)}{\cos^2(x)}\,dx \qquad \textit{[5]}$$

Solution. We'll use the fact that $\tan(x)=\frac{\sin(x)}{\cos(x)}$ to rewrite the function and then use the substitution $u=\cos(x)$ and change the limits when we do. Note that $du=-\sin(x)dx$, so $\sin(x)dx=(-1)du$. Also, when $x=0$, $u=\cos(0)=1$, and when $x=\frac{\pi}{4}$, $u=\cos(\pi/4)=\frac{1}{\sqrt{2}}$. Then

$$\begin{aligned}\int_0^{\pi/4} \frac{\tan(x)}{\cos^2(x)}\,dx &= \int_0^{\pi/4} \frac{\sin(x)}{\cos(x)}\cdot\frac{1}{\cos^2(x)}\,dx = \int_0^{\pi/4}\frac{\sin(x)}{\cos^3(x)}\,dx \\ &= \int_1^{1/\sqrt{2}} \frac{-1}{u^3}\,du = -\int_1^{1/\sqrt{2}} u^{-3}\,du = -\left.\frac{u^{-2}}{-2}\right|_1^{1/\sqrt{2}} = \left.\frac{1}{2u^2}\right|_1^{1/\sqrt{2}} \\ &= \frac{1}{2\left(\frac{1}{\sqrt{2}}\right)^2} - \frac{1}{2\,(1)^2} = \frac{1}{2\frac{1}{2}} - \frac{1}{2} = \frac{1}{1} - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}\end{aligned}$$

does the job. ■

2. What definite integral does the Right-hand Rule limit

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}-1\right)\cdot\frac{1}{n}$$

correspond to? *[5]*

Solution. The general formula for the Right-hand Rule is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) \cdot \frac{b-a}{n}$$

Here $\frac{b-a}{n}$ is the width and $f\left(a + i\frac{b-a}{n}\right)$ is the height of the ith rectangle in the Riemann sum using a partition of $[a, b]$ into n equal subintervals and evaluating $f(x)$ at the right endpoint of each subinterval to find the height of the corresponding rectangle.

We compare the right hand side of the formula to the given limit and try to indentify a, b, and f. Comparing the given common width $\frac{1}{n}$ of the rectangles with $\frac{b-a}{n}$ tells us that $b-a = 1$; comparing $\frac{2i}{n} - 1$ with $f\left(a + i\frac{b-a}{n}\right)$, one could guess that $a = 0$ (so $b = 0+1 = 1$). If so, then $f(x)$ takes x to $2x - 1$, *i.e.* $f(x) = 2x - 1$. Hence the desired definite integral could be:

$$\int_0^1 (2x-1)\,dx$$

This is *not* the only solution. For example, if one had instead guessed that $a = 1$ (so $b = 2$), we would get that $f(x) = 2x - 3$ and that the integral being sought was

$$\int_1^2 (2x-3)\,dx$$

instead. In general, one could let a be any constant; then $b = a+1$ and $f(x) = 2x-2a-1$ do the job. ■

Quiz #9. Wednesday, 4 December, 2002. *[15 minutes]*

12:00 Seminar

1. Find the area of the region enclosed by $y = -x^2$ and $y = x^2 - 2x$. *[10]*

Solution. First, we need to find the points of intersection of the two curves. We set $-x^2 = x^2 - 2$, rearrange this to give $2x^2 = 2$, and observe that the solutions are $x = -1$ and $x = 1$.

Second, note that if x is between -1 and 1, $x^2 < 1$, so $-x^2 > -1$ and $x^2 - 2 < 1 - 2 = -1$. This tells us that for x between -1 and 1, the curve $y = -x^2$ is above the curve $y = x^2 - 2$.

Third, the area of the region is thus given by:

$$\begin{aligned}\int_{-1}^{1} \left(-x^2 - \left(x^2 - 2\right)\right)\,dx &= \int_{-1}^{1} \left(-2x^2 + 2\right)\,dx = -\frac{2}{3}x^3 + 2x\Big|_{-1}^{1} \\ &= \left(-\frac{2}{3}1^3 + 2(1)\right) - \left(-\frac{2}{3}(-1)^3 + 2(-1)\right) \\ &= \frac{4}{3} - \left(\frac{8}{3}\right) = \frac{4}{3} + \frac{8}{3} = \frac{12}{3} = 4 \qquad ■\end{aligned}$$

13:00 Seminar

1. Find the area of the region enclosed by $y = (x-2)^2 + 1 = x^2 - 4x + 5$ and $y = x + 1$. *[10]*

Solution. First, we need to find the points of intersection of the two curves. We set $x^2 - 4x + 5 = x + 1$, rearrange this to give $x^2 - 5x + 4 = 0$, and observe that $x^2 - 5x + 4 = (x-4)(x-1)$. (Worst coming to worst, one could get that by using the quadratic formula.) It follows that the solutions are $x = 1$ and $x = 4$.

Second, note that if x is between 1 and 4, $x^2 - 5x + 4 = \left(x^2 - 4x + 5\right) - (x+1) < 0$, because the parabola $y = x^2 - 5x + 4$ opens upwards and its tip must be between its zeros. This tells us that for x between 1 and 4, the line $y = x + 1$ is above the curve $y = x^2 - 4x + 5$.

Third, the area of the region is thus given by:

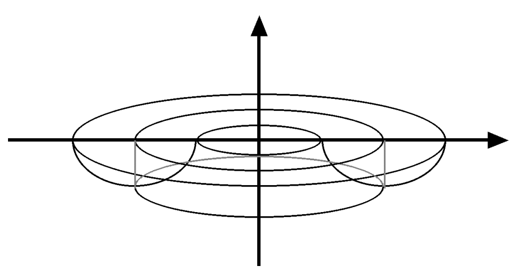
$$\begin{aligned}
\int_1^4 \left((x+1) - \left(x^2 - 4x + 5\right)\right)\, dx &= \int_1^4 \left(-x^2 + 5x - 4\right)\, dx = -\frac{1}{3}x^3 + \frac{5}{2}x^2 - 4x\Big|_1^4 \\
&= \left(-\frac{1}{3}4^3 + \frac{5}{2}4^2 - 4\cdot 4\right) - \left(-\frac{1}{3}1^3 + \frac{5}{2}1^2 - 4\cdot 1\right) \\
&= \frac{44}{3} - \frac{11}{6} = \frac{77}{6} \qquad \blacksquare
\end{aligned}$$

Quiz #10. Wednesday, 8 January, 2003. *[25 minutes]*

12:00 Seminar

1. Sketch the solid obtained by rotating the region bounded by $y = 0$ and $y = \cos(x)$ for $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$ about the y-axis and find its volume. *[10]*

Solution. Observe that $\cos(x) \leq 0$ for $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$. The solid in question looks like this:



We will find the volume of this solid using the method of cylindrical shells. Since we rotated about a vertical line, we will use x as the variable. Note that the cylinder whose edge passes through x has height $h = 0 - \cos(x) = -\cos(x)$ and radius $r = x - 0 = x$. The

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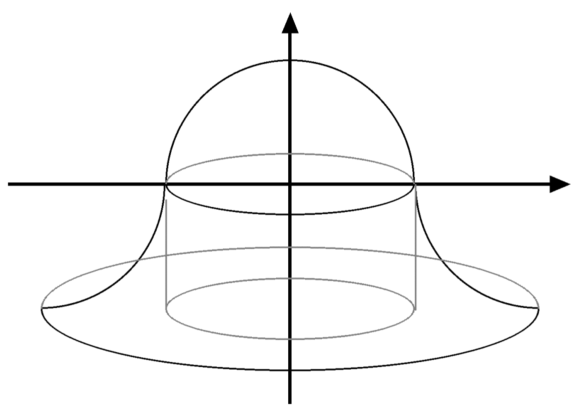
volume of the solid is:

$$\begin{aligned}
V &= \int_{\pi/2}^{3\pi/2} 2\pi r h \, dx \\
&= \int_{\pi/2}^{3\pi/2} 2\pi x \left(-\cos(x)\right) \, dx \\
&= -2\pi \int_{\pi/2}^{3\pi/2} x \cos(x) \, dx \\
&\quad \text{We use integration by parts with } u = x \text{ and } dv = \cos(x)\,dx, \\
&\quad \text{so } du = dx \text{ and } v = \sin(x). \\
&= -2\pi \left[x\sin(x)\Big|_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} \sin(x)\,dx \right] \\
&= -2\pi \left[\left(\frac{3\pi}{2} \sin\left(\frac{3\pi}{2}\right) - \frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) \right) - \left(-\cos(x)\Big|_{\pi/2}^{3\pi/2} \right) \right] \\
&= -2\pi \left[\left(\frac{3\pi}{2}(-1) - \frac{\pi}{2} 1 \right) + \left(\cos\left(\frac{3\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right) \right) \right] \\
&= -2\pi \left[-\frac{4}{2}\pi - (0-0) \right] \\
&= 4\pi^2 \qquad \blacksquare
\end{aligned}$$

13:00 Seminar

1. Sketch the solid obtained by rotating the region bounded by $y = -1$ and $y = \cos(x)$ for $0 \leq x \leq \pi$ about the y-axis and find its volume. *[10]*

Solution. Observe that $\cos(x) \geq -1$ for $0 \leq x \leq \pi$. The solid in question looks like this:



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We will find the volume of this solid using the method of cylindrical shells. Since we rotated about a vertical line, we will use x as the variable. Note that the cylinder whose edge passes through x has height $h = \cos(x) - (-1) = \cos(x) + 1$ and radius $r = x - 0 = x$. The volume of the solid is:

$$\begin{aligned}
V &= \int_0^\pi 2\pi r h \, dx \\
&= \int_0^\pi 2\pi x \left(\cos(x) + 1\right) \, dx \\
&= 2\pi \int_0^\pi x \left(\cos(x) + 1\right) \, dx \\
&\quad \text{We use integration by parts with } u = x \text{ and } dv = \left(\cos(x) + 1\right) \, dx, \\
&\quad \text{so } du = dx \text{ and } v = \sin(x) + x. \\
&= 2\pi \left[x \left(\sin(x) + x\right)\Big|_0^\pi - \int_0^\pi \left(\sin(x) + x\right) \, dx \right] \\
&= 2\pi \left[\left(\pi \left(\sin(\pi) + \pi\right) - 0 \left(\sin(0) + 0\right)\right) - \left(-\cos(x) + \frac{1}{2}x^2 \Big|_0^\pi \right) \right] \\
&= 2\pi \left[\left(\pi(0 + \pi) - 0\right) - \left(\left(-\cos(\pi) + \frac{1}{2}\pi^2 \right) - \left(-\cos(0) + \frac{1}{2}0^2 \right) \right) \right] \\
&= 2\pi \left[\pi^2 - \left(\left(-(-1) + \frac{1}{2}\pi^2 \right) - (-1 - 0) \right) \right] \\
&= 2\pi \left[\pi^2 - \left(2 + \frac{1}{2}\pi^2 \right) \right] \\
&= 2\pi \left[\frac{1}{2}\pi^2 - 2 \right] \\
&= \pi^3 - 4\pi \qquad \blacksquare
\end{aligned}$$

Quiz #11. Wednesday, 15 January, 2002. *[20 minutes]*

12:00 Seminar

1. Compute $\displaystyle\int \frac{1}{1 - x^2} \, dx$.

Solution. We'll use the trigonometric substitution $x = \sin(t)$; note that then $dx = \cos(t)\,dt$ and $\cos(t) = \sqrt{1 - x^2}$.

$$\begin{aligned}
\int \frac{1}{1 - x^2} \, dx &= \int \frac{1}{1 - \sin^2(t)} \cos(t) \, dt = \int \frac{1}{\cos^2(t)} \cos(t) \, dt \\
&= \int \frac{1}{\cos(t)} \, dt = \int \sec(t) \, dt = \ln\left(\sec(t) + \tan(t)\right) + C \\
&= \ln\left(\frac{1}{\cos(t)} + \frac{\sin(t)}{\cos(t)} \right) + C = \ln\left(\frac{1}{\sqrt{1 - x^2}} + \frac{x}{\sqrt{1 - x^2}} \right) + C \qquad \blacksquare
\end{aligned}$$

13:00 Seminar

1. Compute $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$.

Solution. We'll use the trigonometric substitution $x = \sin(t)$; note that then $dx = \cos(t)\,dt$, $t = \arcsin(x)$, and $\cos(t) = \sqrt{1-x^2}$.

$$\begin{aligned}
\int \frac{x^2}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin^2(t)}{\sqrt{1-\sin^2(t)}}\cos(t)\,dt = \int \frac{\sin^2(t)}{\cos(t)}\cos(t)\,dt \\
&= \int \sin^2(t)\,dt = \int \frac{1}{2}\left(1-\cos(2t)\right)\,dt = \frac{1}{2}\left(t + \frac{1}{2}\sin(2t)\right) + C \\
&= \frac{1}{2}t + \frac{1}{4}2\sin(t)\cos(t) + C = \frac{1}{2}\arcsin(x) + \frac{1}{2}x\sqrt{1-x^2} + C \qquad \blacksquare
\end{aligned}$$

Quiz #12. Wednesday, 22 January, 2002. *[20 minutes]*

12:00 Seminar

1. Compute $\displaystyle\int \frac{3x^2+4x+2}{x^3+2x^2+2x}\,dx$.

Solution. This can be done by partial fractions, but since $\frac{d}{dx}\left(x^3+2x^2+2x\right) = 3x^2 + 4x + 2$, there is a quicker alternative, namely the substitution $u = x^3+2x^2+2x$:

$$\int \frac{3x^2+4x+2}{x^3+2x^2+2x}\,dx = \int \frac{1}{u}\,du = \ln(u) + C = \ln\left(x^3+2x^2+2x\right) + C$$

Not many people spotted this shortcut … $\blacksquare$

13:00 Seminar

1. Compute $\displaystyle\int \frac{2x+1}{x^3+2x^2+x}\,dx$.

Solution. Since we are trying to integrate a rational function and there are no readily apparent shortcuts, we use partial fractions.

First, note that the degree of the numerator is already less than the degree of the denominator.

Second, we factor the numerator as far as possible:

$$x^3+2x^2+x = x\left(x^2+2x+1\right) = x\,(x+1)^2$$

Third, it follows that

$$\frac{2x+1}{x^3+2x^2+x} = \frac{2x+1}{x\,(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

for some constants A, B, and C. Putting the right-hand side over a common denominator of $x\,(x+1)^2$ and comparing numerators, we see that we must have:

$$\begin{aligned}
2x+1 &= A(x+1)^2 + Bx(x+1) + Cx \\
&= A\left(x^2+2x+1\right) + B\left(x^2+x\right) + Cx \\
&= (A+B)x^2 + (2A+B+C)x + A
\end{aligned}$$

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Hence $A+B=0$, $2A+B+C=2$, and $A=1$, from which it follows pretty quickly that $B=-1$ and $C=1$.

Fourth, we compute the integral:

$$\begin{aligned}
\int \frac{2x+1}{x^3+2x^2+x}\,dx &= \int \left(\frac{1}{x} + \frac{-1}{x+1} + \frac{1}{(x+1)^2}\right) dx \\
&= \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx + \int \frac{1}{(x+1)^2}\,dx \\
&= \ln(x) - \ln x + 1 + \frac{-1}{x+1} + K \\
&= \ln\left(\frac{x}{x+1}\right) - \frac{1}{x+1} + K
\end{aligned}$$

We're using K for the generic constant because C has already been used ... ■

Quiz #13. Wednesday, 29 January, 2002. *[15 minutes]*

12:00 Seminar

1. Compute $\displaystyle\int_{-\infty}^{\infty} e^{-|x|}\,dx$ or show that it does not converge. *[10]*

Solution. This is obviously an improper integral since there is an infinity in each limit of integration.

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-|x|}\,dx &= \int_{-\infty}^{0} e^{-|x|}\,dx + \int_{0}^{\infty} e^{-|x|}\,dx \\
&= \int_{-\infty}^{0} e^{-(-x)}\,dx + \int_{0}^{\infty} e^{-x}\,dx \\
&\qquad \ldots \text{ since } |a| = -a \text{ when } a \le 0 \text{ and } |a| = a \text{ when } a \ge 0. \\
&= \lim_{t\to-\infty} \int_{t}^{0} e^{x}\,dx + \lim_{t\to\infty} \int_{0}^{t} e^{-x}\,dx \\
&= \lim_{t\to-\infty} e^{x}\Big|_{t}^{0} + \lim_{t\to\infty} -e^{-x}\Big|_{0}^{t} \\
&= \lim_{t\to-\infty} \left(e^0 - e^t\right) + \lim_{t\to\infty} \left(\left(-e^{-t}\right) - \left(-e^{-0}\right)\right) \\
&= \lim_{t\to-\infty} \left(1 - e^t\right) + \lim_{t\to\infty} \left(1 - e^{-t}\right) \\
&= 1 + 1 = 2
\end{aligned}$$

Note that $\displaystyle\lim_{t\to-\infty} e^t = \lim_{t\to\infty} e^{-t} = 0$. ■

13:00 Seminar

1. Compute $\displaystyle\int_{-1}^{1} \frac{x+1}{\sqrt[3]{x}}\,dx$ or show that it does not converge. *[10]*

Solution. This is an improper integral since $f(x) = \dfrac{x+1}{\sqrt[3]{x}}$ has an asymptote at $x=0$, which is in the interval over which the integral is taken. We will do a little bit of algebra first to simplify our task.

$$\begin{aligned}
\int_{-1}^{1} \frac{x+1}{\sqrt[3]{x}}\,dx &= \int_{-1}^{1} \left(\frac{x}{\sqrt[3]{x}} + \frac{1}{\sqrt[3]{x}}\right)\,dx \\
&= \int_{-1}^{1} \left(\frac{x}{x^{1/3}} + \frac{1}{x^{1/3}}\right)\,dx \\
&= \int_{-1}^{1} \left(x^{2/3} + x^{-1/3}\right)\,dx \\
&= \int_{-1}^{1} x^{2/3}\,dx + \int_{-1}^{1} x^{-1/3}\,dx \\
&\qquad \text{Note that the left integral is not improper.} \\
&= \frac{3}{5}x^{5/3}\Big|_{-1}^{1} + \int_{-1}^{0} x^{-1/3}\,dx + \int_{0}^{1} x^{-1/3}\,dx \\
&= \left(\frac{3}{5}1^{5/3} - \frac{3}{5}(-1)^{5/3}\right) + \lim_{t\to 0^-}\int_{-1}^{t} x^{-1/3}\,dx + \lim_{t\to 0^+}\int_{t}^{1} x^{-1/3}\,dx \\
&= \left(\frac{3}{5} - \left(-\frac{3}{5}\right)\right) + \lim_{t\to 0^-}\frac{3}{2}x^{2/3}\Big|_{-1}^{t} + \lim_{t\to 0^+}\frac{3}{2}x^{2/3}\Big|_{t}^{1} \\
&= \left(\frac{3}{5} + \frac{3}{5}\right) + \lim_{t\to 0^-}\left(\frac{3}{2}t^{2/3} - \frac{3}{2}(-1)^{2/3}\right) + \lim_{t\to 0^+}\left(\frac{3}{2}1^{2/3} - \frac{3}{2}t^{2/3}\right) \\
&= \frac{6}{5} + \lim_{t\to 0^-}\left(\frac{3}{2}t^{2/3} - \frac{3}{2}\right) + \lim_{t\to 0^+}\left(\frac{3}{2} - \frac{3}{2}t^{2/3}\right) \\
&= \frac{6}{5} + \left(0 - \frac{3}{2}\right) + \left(\frac{3}{2} - 0\right) \\
&= \frac{6}{5} - \frac{3}{2} + \frac{3}{2} \\
&= \frac{6}{5}
\end{aligned}$$

Note that $\lim_{t\to 0^-} t^{2/3} = \lim_{t\to 0^+} t^{2/3} = 0$. ■

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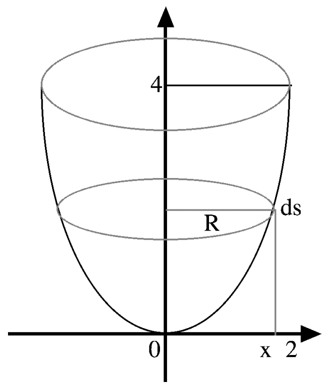
Quiz #14. Wednesday, 5 February, 2002. *[20 minutes]*

12:00 Seminar

1. Sketch the solid obtained by rotating the region bounded by $x = 0$, $y = 4$ and $y = x^2$ for $0 \le x \le 2$ about the y-axis. *[2]*

Solution.



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2. Compute the surface area of this solid. *[8]*

Solution. The surface of the solid has two parts: the disk at the top and the parabolic surface below it. The disk has radius 2 and hence has area $\pi 2^2 = 4\pi$. We compute the area of the parabolic surface using the formula for the area of a surface of revolution. Note that the radius R for the infinitesimal arc length at x is simply $R = x - 0 = x$. Then the area of the parabolic surface is given by:

$$\int_0^2 2\pi R\,ds = \int_0^2 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \pi\int_0^2 2x\sqrt{1+\left(\frac{d}{dx}x^2\right)^2}\,dx$$

$$= \pi\int_0^2 2x\sqrt{1+(2x)^2}\,dx = \pi\int_0^2 2x\sqrt{1+4x^2}\,dx$$

Using the substitution $u = 1 + 4x^2$ we get $du = 8xdx$, so $\frac{1}{4}du = 2xdx$. When $x = 0$, $u = 1$, and when $x = 2$, $u = 17$.

$$= \pi\int_1^{17}\sqrt{u}\,\frac{1}{4}du = \frac{\pi}{4}\;\frac{1}{2\sqrt{u}}\bigg|_1^{17}$$

$$= \frac{\pi}{4}\left(\frac{1}{2\sqrt{17}} - \frac{1}{2\sqrt{1}}\right) = \frac{\pi}{8}\left(\frac{1}{\sqrt{17}} - 1\right)$$

Thus the total surface area of the solid is $4\pi + \frac{\pi}{8}\left(\frac{1}{\sqrt{17}} - 1\right) = \frac{\pi}{8}\left(31 + \sqrt{17}\right)$. ■

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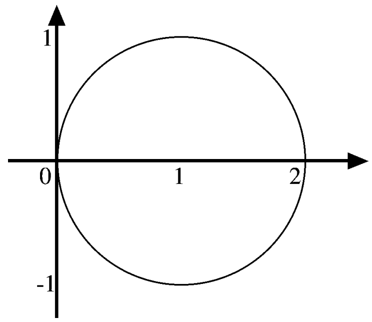
13:00 Seminar

1. Sketch the curve given by the parametric equations $x = 1 + \cos(t)$ and $y = \sin(t)$, where $0 \leq t \leq 2\pi$. *[3]*

Solution.



■

2. Compute the arc-length of this curve using a suitable integral. *[7]*

Solution. We'll use the parametric version of the arc-length formula:

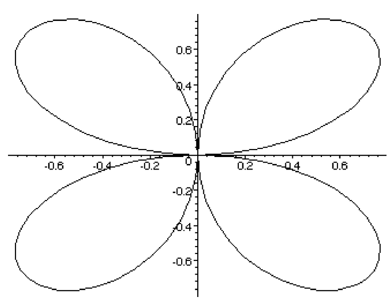
$$\int_0^{2\pi} ds = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{\left(\frac{d}{dt}\left((1+\cos(t)\right)\right)^2 + \left(\frac{d}{dt}\sin(t)\right)^2}\, dt$$
$$= \int_0^{2\pi} \sqrt{(0-\sin(t))^2 + (\cos(t))^2}\, dt = \int_0^{2\pi} \sqrt{\sin^2(t) + \cos^2(t)}\, dt$$
$$= \int_0^{2\pi} \sqrt{1}\, dt = \int_0^{2\pi} 1\, dt = t\Big|_0^{2\pi} = 2\pi - 0 = 2\pi \qquad ■$$

Quiz #15. Wednesday, 26 February, 2002. *[20 minutes]*

12:00 Seminar

1. Graph the polar curve $r = \sin(2\theta)$, $0 \leq \theta \leq 2\pi$. *[4]*

Solution.



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2. Find the area of the region enclosed by this curve. *[6]*

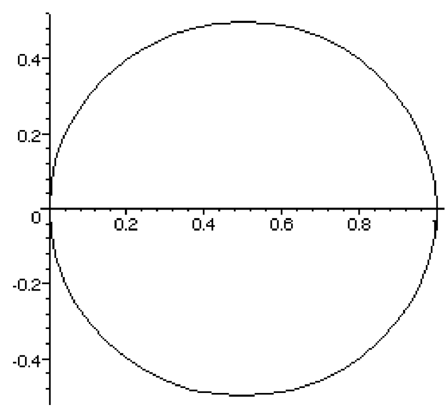
Solution. The area can be computed as follows:

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{2} r^2 \, d\theta &= \int_0^{2\pi} \frac{1}{2} \sin^2(2\theta) \, d\theta = \frac{1}{2} \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2} \cos(2 \cdot 2\theta) \right) \, d\theta \\
&= \frac{1}{4} \int_0^{2\pi} (1 - \cos(4\theta)) \, d\theta = \frac{1}{4} \left(\theta - \frac{1}{4} \sin(4\theta) \right) \Big|_0^{2\pi} \\
&= \frac{1}{4} \left(2\pi - \frac{1}{4} \sin(4 \cdot 2\pi) \right) - \frac{1}{4} \left(0 - \frac{1}{4} \sin(0 \cdot 2\pi) \right) \\
&= \frac{1}{4} \left(2\pi - \frac{1}{4} \sin(8\pi) \right) - \frac{1}{4} \left(0 - \frac{1}{4} \sin(0) \right) \\
&= \frac{1}{4} (2\pi - 0) - (0 - 0) = \frac{pi}{2} \qquad \blacksquare
\end{aligned}$$

13:00 Seminar

1. Graph the polar curve $r = \cos(\theta)$, $0 \le \theta \le 2\pi$. *[4]*

Solution.



$\blacksquare$

2. Find the arc-length of this curve. *[6]*

Solution. Note that to find the arc-length of the curve we only need to trace it once, so we only need $0 \le \theta \le \pi$. The arc-length can then be computed as follows:

$$\begin{aligned}
\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} \, d\theta &= \int_0^{\pi} \sqrt{\cos^2(\theta) + (-\sin(\theta))^2} \, d\theta \\
&= \int_0^{\pi} \sqrt{\cos^2(\theta) + \sin^2(\theta)} \, d\theta = \int_0^{\pi} \sqrt{1} \, d\theta \\
&= \int_0^{\pi} 1 \, d\theta = \theta|_0^{\pi} = \pi - 0 = \pi \qquad \blacksquare
\end{aligned}$$

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Quiz #16. Wednesday, 5 March, 2003. *[15 minutes]*

12:00 Seminar

Let $a_k = \dfrac{1}{(k+1)(k+2)}$ and $s_n = \displaystyle\sum_{k=0}^{n} a_k$.

1. Find a formula for s_n in terms of n. *[5]*

Solution. Note that $\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2}$. (Partial fractions!) Hence:

$$\begin{aligned} s_n &= \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} = \sum_{k=0}^{n} \left[\frac{1}{k+1} - \frac{1}{k+2}\right] \\ &= \left[\frac{1}{1} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \cdots + \left[\frac{1}{n} - \frac{1}{n+1}\right] + \left[\frac{1}{n+1} - \frac{1}{n+2}\right] \\ &= \frac{1}{1} - \frac{1}{n+2} = 1 - \frac{1}{n+2} \qquad \blacksquare \end{aligned}$$

2. Does $\displaystyle\sum_{k=0}^{\infty} a_k$ converge? If so, what does it converge to? *[5]*

Solution. It converges to 1:

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+2}\right) = 1 - 0 = 1 \qquad \blacksquare$$

13:00 Seminar

Let $a_k = \ln\left(\dfrac{k}{k+1}\right)$ and $s_n = \displaystyle\sum_{k=1}^{n} a_k$.

1. Find a formula for s_n in terms of n. *[5]*

Solution. The key here is that $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$.

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \ln\left(\frac{k}{k+1}\right) = \sum_{k=1}^{n} [\ln(k) - \ln(k+1)] \\ &= [\ln(1) - \ln(2)] + [\ln(2) - \ln(3)] + [\ln(3) - \ln(4)] + \cdots \\ &\qquad + [\ln(n-1) - \ln(n)] + [\ln(n) - \ln(n+1)] \\ &= \ln(1) - \ln(n+1) = 0 - \ln(n+1) = -\ln(n+1) \qquad \blacksquare \end{aligned}$$

2. Does $\displaystyle\sum_{k=0}^{\infty} a_k$ converge? If so, what does it converge to? *[5]*

Solution. It does not converge. First, recall that $\ln(x) \to \infty$ as $x \to \infty$. Now:

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} [-\ln(n+1)] = -\lim_{n\to\infty} \ln(n+1) = -\infty \quad \blacksquare$$

Quiz #17. Wednesday, 12 March, 2003. *[15 minutes]*

12:00 Seminar

Determine whether each of the following series converges or diverges:

$$1. \quad \sum_{n=0}^{\infty} e^{-n} \quad \textit{[5]} \qquad 2. \quad \sum_{n=1}^{\infty} \frac{1}{\arctan(n)} \quad \textit{[5]}$$

Solution for 1. $\displaystyle\sum_{n=0}^{\infty} e^{-n} = \sum_{n=0}^{\infty} \frac{1}{e^n} = \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$ converges because this is a geometric series with $|r| = \left|\dfrac{1}{e}\right| < 1$. ■

Solution for 2. Since

$$\lim_{n\to\infty} \frac{1}{\arctan(n)} = \frac{1}{\lim_{n\to\infty} \arctan(n)} = \frac{1}{\pi/2} = \frac{2}{\pi} \neq 0\,,$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\arctan(n)}$ diverges by the Divergence Test. ■

13:00 Seminar

Determine whether each of the following series converges or diverges:

$$1. \quad \sum_{n=0}^{\infty} \frac{1}{n+1} \quad \textit{[5]} \qquad 2. \quad \sum_{n=1}^{\infty} 2^{1/n^2} \quad \textit{[5]}$$

Solution for 1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges by the (general) p-test since the degree of the denominator is 1, the degree of the numerator is 0, and their difference, $1 - 0 = 1$, is not greater than 1. (The divergence of this series can also be verified pretty quickly using the Integral Test.) ■

Solution for 2. Since

$$\lim_{n\to\infty} 2^{1/n^2} = 2^{\lim_{n\to\infty} 1/n^2} = 2^0 = 1 \neq 0\,,$$

$\sum_{n=1}^{\infty} 2^{1/n^2}$ diverges by the Divergence Test. ■

Quiz #18. Wednesday, 17 March, 2003. *[15 minutes]*

12:00 Seminar

Determine whether each of the following series converges absolutely, converges conditionally, or diverges:

$$1. \quad \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n^2+2} \quad \textit{[5]} \qquad 2. \quad \sum_{n=1}^{\infty} \frac{n!(-1)^n}{n^n} \quad \textit{[5]}$$

Solution for 1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n^2+2}$ diverges by the Divergence Test: since

$$\lim_{n\to\infty} \left| \frac{(-1)^n 2^n}{n^2+2} \right| = \lim_{n\to\infty} \frac{2^n}{n^2+2} = \lim_{x\to\infty} \frac{2^x}{x^2+2} = \lim_{x\to\infty} \frac{\ln(2)2^x}{2x+0} = \lim_{x\to\infty} \frac{(\ln(2))^2\, 2^x}{2} = \infty$$

(using l'Hôpital's Rule twice), it follows that $\displaystyle\lim_{n\to\infty} \frac{(-1)^n 2^n}{n^2+2}$ does not exist. ■

Solution for 2. $\displaystyle\sum_{n=1}^{\infty} \frac{n!(-1)^n}{n^n}$ converges absolutely since $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by the Comparison Test:

$$\frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdot \ldots \cdot 3 \cdot 2 \cdot 1}{n \cdot n \cdot n \cdot \ldots \cdot n \cdot n \cdot n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \ldots \cdot \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n} \leq 1 \cdot \frac{2}{n} \cdot \frac{1}{n} = \frac{2}{n^2}$$

and $\displaystyle\sum_{n=1} \frac{2}{n^2} = 2 \sum_{n=1} \frac{1}{n^2}$ converges by the p-test because $2-0>1$. ■

13:00 Seminar

Determine whether each of the following series converges absolutely, converges conditionally, or diverges:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \left(2n^2+3n+4\right)}{3n^2+4n+5}$ *[5]*
2. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2}$ *[5]*

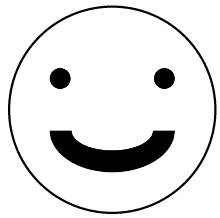
Solution for 1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \left(2n^2+3n+4\right)}{3n^2+4n+5}$ diverges by the Divergence Test: since

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{(-1)^n \left(2n^2+3n+4\right)}{3n^2+4n+5} \right| &= \lim_{n\to\infty} \frac{2n^2+3n+4}{3n^2+4n+5} = \lim_{n\to\infty} \frac{2n^2+3n+4}{3n^2+4n+5} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\ &= \lim_{n\to\infty} \frac{2+\frac{3}{n}+\frac{4}{n^2}}{3+\frac{4}{n}+\frac{5}{n^2}} = \frac{2+0+0}{3+0+0} = \frac{2}{3}\end{aligned}$$

it follows that $\displaystyle\lim_{n\to\infty} \frac{(-1)^n \left(2n^2+3n+4\right)}{3n^2+4n+5} \neq 0$. ■

Solution for 2. Note that $\cos(\pi) = -1$, $\cos(2\pi) = 1 = (-1)^2$, $\cos(3\pi) = -1 = (-1)^3$, $\cos(4\pi) = 1 = (-1)^4$, and so on. It follows that the given series is the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. This converges absolutely because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test since $2-0>1$. ■

Bonus Quiz. Friday, 19 March, 2003. *[15 minutes]*



1. A smiley face is drawn on the surface of a balloon which is being inflated at a rate of 10 cm^3/s. At the instant that the radius of the balloon is 10 cm the eyes are 10 cm apart, as measured *inside* the balloon. How is the distance between them changing at this moment? *[10]*

Solution. ■

Quiz #19. Wednesday, 24 March, 2003. *[20 minutes]*

12:00 Seminar

Consider the power series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^{2n}}{n!}$.

1. For which values of x does this series converge? *[6]*

Solution. ■

2. This series is equal to a (reasonably nice) function. What is it? Why? *[4]*

Solution. ■

13:00 Seminar

Consider the power series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n+1}$.

1. For which values of x does this series converge? *[6]*

Solution. ■

2. This series is equal to a (reasonably nice) function. What is it? Why? *[4]*

Solution. ■

Quiz #20. Wednesday, 2 April, 2003. *[20 minutes]*

12:00 Seminar

Let $f(x) = \sin(\pi - 2x)$.

1. Find the Taylor series at $a = 0$ of $f(x)$. *[6]*

Solution. ■

2. Find the radius and interval of convergence of this Taylor series. *[4]*

Solution. ■

13:00 Seminar

Let $f(x) = \ln(2 + x)$.

1. Find the Taylor series at $a = 0$ of $f(x)$. *[6]*

Solution. ■

2. Find the radius and interval of convergence of this Taylor series. *[4]*

Solution. ■

Mathematics 110 – Calculus of one variable
TRENT UNIVERSITY, 2002-2003

Test #1 - Section A
Wednesday, 13 November, 2002
Time: 50 minutes

Solutions to page 1

1. Do any *one* of **a** – **c**. *[10/100]*

a. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 1} (7x + 3) = 10$.

Solution. We need to check that for any $\varepsilon > 0$ there is a $\delta > 0$ so that if $|x - 1| < \delta$, then $|(7x + 3) - 10| < \varepsilon$. As usual, we try to reverse engineer the δ we need from the desired outcome:

$$\begin{aligned} & |(7x+3) - 10| < \varepsilon \\ \iff & |7x - 7| < \varepsilon \\ \iff & |7(x-1)| < \varepsilon \\ \iff & 7|x-1| < \varepsilon \\ \iff & |x-1| < \frac{\varepsilon}{7} \end{aligned}$$

Since every step above is reversible, $\delta = \frac{\varepsilon}{7}$ does the job. It follows by the $\varepsilon - \delta$ definition of limits that $\lim_{x \to 1} (7x + 3) = 10$. ■

b. Use the limit definition of the derivative to verify that $\frac{d}{dx}\left(e^x\right) = e^x$.

(You may assume that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.)

Solution. Apply the definition $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to $f(x) = e^x$:

$$\begin{aligned} \frac{d}{dx}\left(e^x\right) &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x e^h - e^x}{h} \\ &= \lim_{h \to 0} \frac{e^x\left(e^h - 1\right)}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x \end{aligned}$$

Hence, by the limit definition of the derivative, $\frac{d}{dx}\left(e^x\right) = e^x$. ■

1

c. Compute ompute $\lim_{x \to 0} \dfrac{x}{\tan(2x)}$.

Solution 1. Note that $\lim_{x \to 0} x = 0$ and $\lim_{x \to 0} \tan(2x) = \tan(2 \cdot 0) = \tan(0) = 0$, so the limit in question is an indeterminate form to which we can apply l'Hôpital's Rule:

$$\lim_{x \to 0} \frac{x}{\tan(2x)} = \lim_{x \to 0} \frac{\frac{d}{dx} x}{\frac{d}{dx} \tan(2x)} = \lim_{x \to 0} \frac{1}{\sec^2(2x) \cdot \frac{d}{dx}(2x)}$$
$$= \lim_{x \to 0} \frac{1}{2\sec^2(2x)}$$
$$= \frac{1}{2\sec^2(0)} = \frac{1}{2 \cdot 1^2} = \frac{1}{2}$$

(Note that $\sec(0) = \dfrac{1}{\cos(0)} = \dfrac{1}{1} = 1$.) Hence $\lim_{x \to 0} \dfrac{x}{\tan(2x)} = \dfrac{1}{2}$. ■

Solution 2. Note that $\tan(2x) = \dfrac{\sin(2x)}{\cos(2x)}$ and recall that $\lim_{t \to 0} \frac{\sin(t)}{t} = \lim_{t \to 0} \frac{t}{\sin(t)} = 1$. It follows that:

$$\lim_{x \to 0} \frac{x}{\tan(2x)} = \lim_{x \to 0} \frac{x}{\frac{\sin(2x)}{\cos(2x)}} = \lim_{x \to 0} \frac{x\cos(2x)}{\sin(2x)}$$
$$= \left(\lim_{x \to 0} \cos(2x)\right)\left(\lim_{x \to 0} \frac{2x}{2\sin(2x)}\right)$$
$$= \cos(2 \cdot 0)\left(\frac{1}{2} \lim_{x \to 0} \frac{2x}{\sin(2x)}\right) = \cos(0) \cdot \frac{1}{2} \cdot 1 = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

Hence $\lim_{x \to 0} \dfrac{x}{\tan(2x)} = \dfrac{1}{2}$. ■

2. Do any *two* of **a** – **c**. *[15/100 each]*

a. Find the equation of the tangent line to $y = \sqrt{x}$ at $x = 4$.

Solution. The tangent line in question passes through the point $\left(4, \sqrt{4}\right) = (4, 2)$ and its slope is given by:

$$\left.\frac{d}{dx}\sqrt{x}\right|_4 = \left.\frac{1}{2\sqrt{x}}\right|_4 = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

The equation of the tangent line is therefore given by $y - 2 = \frac{1}{4}(x - 4)$, which works out to $y = \frac{1}{4}x + 1$. ■

b. Is the function $f(x) = x \cdot |x|$ differentiable at $x = 0$? Why or why not?

Solution. We need to check that $f'(0) = \lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$ is defined; this boils down to checking that $\lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h}$ and $\lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h}$ both exist and are equal.

First, note that $f(0) = 0 \cdot |0| = 0$. Second, if $h < 0$, then $|h| = -h$. Hence:

$$\begin{aligned}
&\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{f(h) - 0}{h} \\
&= \lim_{h \to 0^-} \frac{h \cdot |h|}{h} = \lim_{h \to 0^-} \frac{h \cdot (-h)}{h} = \lim_{h \to 0^-} (-h) = -0 = 0
\end{aligned}$$

Third, if $h > 0$, then $|h| = h$. Hence:

$$\begin{aligned}
&\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{f(h) - 0}{h} \\
&= \lim_{h \to 0^+} \frac{h \cdot |h|}{h} = \lim_{h \to 0^+} \frac{h \cdot h}{h} = \lim_{h \to 0^+} h = 0
\end{aligned}$$

It follows that that $f'(0) = \lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$ is defined (and equals 0). ■

c. Use the limit definition of the derivative to verify the Sum Rule, *i.e.* that if $h(x) = f(x) + g(x)$, then $h'(x) = f'(x) + g'(x)$.

Solution. Here goes:

$$\begin{aligned}
h'(x) &= \lim_{k \to 0} \frac{h(x+k) - h(x)}{k} \\
&= \lim_{k \to 0} \frac{(f(x+k) + g(x+k)) - (f(x) + g(x))}{k} \\
&= \lim_{k \to 0} \frac{f(x+k) - f(x) + g(x+k) - g(x)}{k} \\
&= \lim_{k \to 0} \left(\frac{f(x+k) - f(x)}{k} + \frac{g(x+k) - g(x)}{k} \right) \\
&= \left(\lim_{k \to 0} \frac{f(x+k) - f(x)}{k} \right) + \left(\lim_{k \to 0} \frac{g(x+k) - g(x)}{k} \right) \\
&= f'(x) + g'(x)
\end{aligned}$$

... and that's all! ■

Mathematics 110 – Calculus of one variable
Trent University, 2002-2003

Sollutions for Test #2 - Section A

1. Compute any *three* of the integrals in parts **a-f**. *[36 = 3 × 12 each]*

a. $\displaystyle\int_0^1 \arctan(x)\,dx$ **b.** $\displaystyle\int_e^{e^e} \frac{\ln(\ln(x))}{x\ln(x)}\,dx$ **c.** $\displaystyle\int \frac{1}{x^2-3x+2}\,dx$

d. $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$ **e.** $\displaystyle\int_0^{\pi/4} \sec^4(x)\,dx$ **f.** $\displaystyle\int_2^{\infty} \frac{1}{x^4}\,dx$

Solutions:
a. We'll use integration by parts, with $u = \arctan(x)$ and $dv = dx$, so $du = \frac{1}{1+x^2}\,dx$ and $v = x$.

$$\int_0^1 \arctan(x)\,dx = x\arctan(x)\big|_0^1 = 1\cdot\arctan(1) - 0\cdot\arctan(0) = 1\cdot\frac{\pi}{4} - 0\cdot 0 = \frac{\pi}{4} \quad \blacksquare$$

b. Here we'll substitute whole hog, namely $u = \ln(\ln(x))$. Note that then

$$\frac{du}{dx} = \frac{1}{\ln(x)}\cdot\frac{d}{dx}\ln(x) = \frac{1}{\ln(x)}\cdot\frac{1}{x}\,.$$

Also, when $x = e$,

$$u = \ln(\ln(e)) = \ln(1) = 0\,,$$

and when $x = e^e$,

$$u = \ln(\ln(e^e)) = \ln(e\ln(e)) = \ln(e\cdot 1) = \ln(e) = 1\,.$$

Hence

$$\int_e^{e^e} \frac{\ln(\ln(x))}{x\ln(x)}\,dx = \int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}1^2 - \frac{1}{2}0^2 = \frac{1}{2}\,. \quad \blacksquare$$

c. This one is a job for integration by partial fractions. Since $x^2-3x+2 = (x-2)(x-1)$,

$$\frac{1}{x^2-3x+2} = \frac{1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}\,,$$

where $1 = A(x-1) + B(x-2) = (A+B)x + (-A-2B)$. It follows that $0 = A+B$, so $A = -B$, and $1 = -A-2B = B-2B = -B$, so $B = -1$ and $A = 1$. Then:

$$\begin{aligned}\int \frac{1}{x^2-3x+2}\,dx &= \int\left(\frac{1}{x-2} - \frac{1}{x-1}\right)dx = \int \frac{1}{x-2}\,dx - \int\frac{1}{x-1}\,dx \\ &= \ln(x-2) - \ln(x-1) + C = \ln\left(\frac{x-2}{x-1}\right) + C \quad \blacksquare\end{aligned}$$

1

d. Note that $x^2 + 2x + 2 = x^2 + 2x + 1 + 1 = (x + 1)^2 + 1$. We will use the substitution $u = x + 1$, so $du = dx$, and the fact that $\frac{d}{du} \arctan(u) = \frac{1}{u^2 + 1}$:

$$\int \frac{1}{x^2 + 2x + 1}\, dx = \int \frac{1}{(x + 1)^2 + 1}\, dx = \int \frac{1}{u^2 + 1}\, du = \arctan(u) + C \quad \blacksquare$$

e. Here we will use the fact that $\sec^2(x) = 1 + \tan^2(x)$ and the substitution $u = \tan(x)$, so $du = \sec^2(x)\, dx$. Note that when $x = 0$, $u = \tan(0) = 0$, and when $x = \frac{\pi}{4}$, $u = \tan\left(\frac{\pi}{4}\right) = 1$.

$$\begin{aligned}
\int_0^{\pi/4} \sec^4(x)\, dx &= \int_0^{\pi/4} \sec^2(x) \sec^2(x)\, dx \\
&= \int_0^{\pi/4} \left(1 + \tan^2(x)\right) \sec^2(x)\, dx = \int_0^1 \left(1 + u^2\right)\, du \\
&= \left(u + \frac{1}{3}u^3\right)\Big|_0^1 = \left(1 + \frac{1}{3} \cdot 1^3\right) - \left(0 + \frac{1}{3} \cdot 0^3\right) = \frac{4}{3} \quad \blacksquare
\end{aligned}$$

f. Since it has an infinity in one of the limits of integration, this is obviously an improper integral.

$$\begin{aligned}
\int_2^\infty \frac{1}{x^4}\, dx = \int_2^\infty x^{-4}\, dx &= \lim_{t \to \infty} \int_2^t x^{-4}\, dx \\
&= \lim_{t \to \infty} \frac{x^{-3}}{-3}\Big|_2^t = \lim_{t \to \infty} \left(\frac{-1}{3t^3} - \frac{-1}{3 \cdot 2^3}\right) \\
&= \lim_{t \to \infty} \left(\frac{1}{24} - \frac{1}{3t^3}\right) = \frac{1}{24} - 0 = \frac{1}{24}
\end{aligned}$$

Note that as $t \to \infty$, $t^3 \to \infty$ as well, so $\frac{1}{3t^3} \to 0$. $\blacksquare$

2. Do any *two* of parts **a**-**d**. *[24 = 2 × 12 each]*

a. Compute $\frac{d}{dx}\left(\int_1^{\cos(x)} \arccos(t)\, dt\right)$.

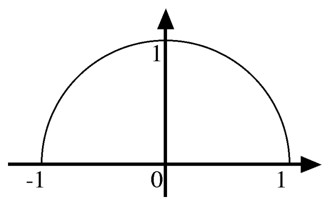
Solution. Let $y = \int_1^{\cos(x)} \arccos(t)\, dt$ and $u = \cos(x)$. Then, using the Chain Rule, the Fundamental Theorem of Calculus, and the fact that $\arccos(\cos(x)) = x$:

$$\begin{aligned}
\frac{d}{dx}\left(\int_1^{\cos(x)} \arccos(t)\, dt\right) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} &= \frac{d}{du}\left(\int_1^u \arccos(t)\, dt\right) \cdot \frac{du}{dx} \\
&= \arccos(u) \cdot \frac{du}{dx} = \arccos(\cos(x)) \cdot \frac{d}{dx} \cos(x) \\
&= x\,(-\sin(x)) = -x \sin(x) \quad \blacksquare
\end{aligned}$$

b. Give both a description and a sketch of the region whose area is computed by the integral $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$.

Solution. Note that if $y = \sqrt{1-x^2}$, then $y^2 = 1 - x^2$, so $x^2 + y^2 = 1$. Since $y = \sqrt{1-x^2} \geq 0$ for $-1 \leq x \leq 1$, it follows that the region in question is the one bounded above by the upper unit semicircle, *i.e.* $y = \sqrt{1-x^2}$, and below by the x-axis, *i.e.* $y = 0$, for $-1 \leq x \leq 1$.

Here's a sketch:



■

c. Find the area of the region bounded by $y = x^3 - x$ and $y = 3x$.

Solution. We first need to determine where $y = x^3 - x$ and $y = 3x$ intersect. Setting $x^3 - x = 3x$, which amounts to $x^3 = 4x$, we see that $x = 0$ must be one solution. The solutions with $x \neq 0$ are given by $x^2 = 4$, *i.e.* $x = -2$ and $x = 2$.

We now need to determine when one curve is above the other, which we do by testing points between the points where the two are equal. Note that $-2 < -1 < 0$ and $0 < 1 < 2$. Plugging in $x = -1$ into both equations gives $y = (-1)^3 - (-1) = 0$ and $y = 3(-1) = -3$, so between $x = -2$ and $x = 0$, $y = x^3 - x$ is above $y = 3x$. Plugging in $x = 1$ into both equations gives $y = 1^3 - 1 = 0$ and $y = 3 \cdot 1 = 3$, so between $x = 0$ and $x = 2$, $y = 3x$ is above $y = x^3 - x$.

It follows that the area in question is given by:

$$
\begin{aligned}
&\int_{-2}^{0} \left(\left(x^3 - x\right) - 3x\right)\,dx + \int_{0}^{2} \left(3x - \left(x^3 - x\right)\right)\,dx \\
=&\int_{-2}^{0} \left(x^3 - 4x\right)\,dx + \int_{0}^{2} \left(4x - x^3\right)\,dx \\
=&\left(\frac{1}{4}x^4 - \frac{4}{2}x^2\right)\Bigg|_{-2}^{0} + \left(\frac{4}{2}x^2 - \frac{1}{4}x^4\right)\Bigg|_{0}^{2} \\
=&\left(\frac{1}{4}\cdot 0^4 - 2\cdot 0^2\right) - \left(\frac{1}{4}\cdot(-2)^4 - 2\cdot 2^2\right) + \left(2\cdot 2^2 - \frac{1}{4}\cdot 2^4\right) - \left(2\cdot 0^2 - \frac{1}{4}0^4\right) \\
=&(0-0) - (4-8) + (8-4) - (0-0) = 8 \quad \blacksquare
\end{aligned}
$$

d. Compute $\int_0^1 (x+1)\,dx$ using the Right-hand Rule.

Solution. The general Right-hand Rule formula is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) \cdot \frac{b-a}{n}$$

Plugging in $a = 0$, $b = 1$, and $f(x) = x + 1$ gives:

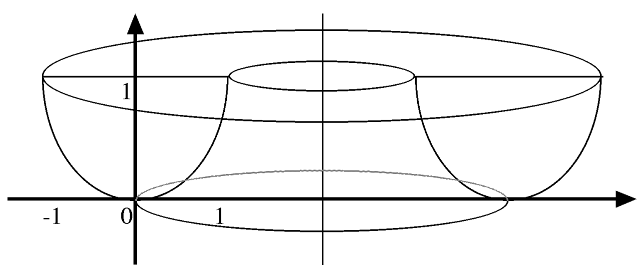
$$\begin{aligned}
\int_0^1 (x+1)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(0 + i\frac{1-0}{n}\right) + 1\right] \cdot \frac{1-0}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[i\frac{1}{n} + 1\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\left(\sum_{i=1}^{n} \frac{i}{n}\right) + \left(\sum_{i=1}^{n} 1\right)\right] = \lim_{n\to\infty} \frac{1}{n} \left[\left(\frac{1}{n} \sum_{i=1}^{n} i\right) + n\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\frac{1}{n} \cdot \frac{n(n+1)}{2} + n\right] = \lim_{n\to\infty} \frac{1}{n} \left[\frac{n+1}{2} + n\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\frac{3}{2}n + \frac{1}{2}\right] = \lim_{n\to\infty} \left[\frac{3}{2} + \frac{1}{2n}\right] = \frac{3}{2} + 0 = \frac{3}{2}
\end{aligned}$$

This computation uses the facts that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} 1 = n$ along the way. ∎

3. Consider the solid obtained by rotating the region bounded by $y = x^2$ and $y = 1$ about the line $x = 2$.

a. Sketch the solid. *[5]*

Solution.

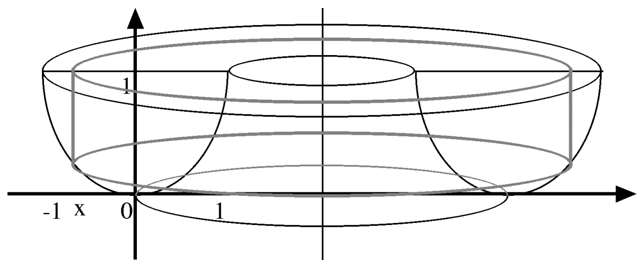


∎

b. Find the volume of the solid. *[20]*

Solution. We will use the method of cylindrical shells to compute the volume of the given solid of revolution. Since we rotated about a vertical line, we need to use x as the variable.

Here is the sketch for part **a** with a cylindrical shell drawn in:



It is easy to see from this diagram that for the cylindrical shell at x, the radius is $r = 2 - x$ and the height is $h = 1 - x^2$.

The volume of the solid is now computed as follows:

$$\begin{aligned}
\int_{-1}^{1} 2\pi r h\, dx &= \int_{-1}^{1} 2\pi(2-x)\left(1-x^2\right)\, dx = 2\pi \int_{-1}^{1} \left(2 - x - 2x^2 + x^3\right)\, dx \\
&= 2\pi \left(2x - \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4\right)\Bigg|_{-1}^{1} \\
&= 2\pi \left(2 \cdot 1 - \frac{1}{2} \cdot 1^2 - \frac{2}{3} \cdot 1^3 + \frac{1}{4} \cdot 1^4\right) \\
&\qquad - 2\pi \left(2(-1) - \frac{1}{2}(-1)^2 - \frac{2}{3}(-1)^3 + \frac{1}{4}(-1)^4\right) \\
&= 2\pi \left(2 - \frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) - 2\pi \left(-2 - \frac{1}{2} + \frac{2}{3} + \frac{1}{4}\right) \\
&= \cdots = \frac{16\pi}{3} \quad \blacksquare
\end{aligned}$$

4. Find the arc-length of the curve given by the parametric equations $x = \dfrac{t^2}{\sqrt{2}}$ and $y = \dfrac{t^3}{3}$, where $0 \le t \le 1$. *[15]*

Solution. Note that $\dfrac{dx}{dt} = \dfrac{2t}{\sqrt{2}} = \sqrt{2}t$ and $\dfrac{dy}{dt} = \dfrac{3t^2}{3} = t^2$. The arc-length is then:

$$\begin{aligned}
\int_0^1 ds &= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{\left(\sqrt{2}t\right)^2 + \left(t^2\right)^2}\, dt \\
&= \int_0^1 \sqrt{2t^2 + t^4}\, dt = \int_0^1 \sqrt{t^2\left(2+t^2\right)}\, dt = \int_0^1 t\sqrt{2+t^2}\, dt
\end{aligned}$$

5

We use the substitution $u = 2 + t^2$, so $du = 2t\,dt$ and $\frac{1}{2}\,du = t\,dt$. Also, $u = 2$ when $t = 0$ and $u = 3$ when $t = 1$.

$$\begin{aligned} &= \int_2^3 \frac{1}{2}\sqrt{u}\,du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2}\Big|_2^3 = \frac{1}{3} \cdot 3^{3/2} - \frac{1}{3} \cdot 2^{3/2} \\ &= \frac{1}{3} \cdot 3\sqrt{3} - \frac{1}{3} \cdot 2\sqrt{2} = \sqrt{3} - \frac{2}{3}\sqrt{2} \quad \blacksquare \end{aligned}$$

[Total = 100]

Mathematics 110, Section A
Calculus I: Calculus of one variable
2003-2004
Solutions

Mathematics 110 – Calculus of one variable
Trent University 2003-2004

SOLUTIONS TO §A QUIZZES

Quiz #1. Friday, 19 September, 2002. *[10 minutes]*

12:00 Seminar

1. How close does x have to be to 1 in order to guarantee that $\frac{1}{x}$ is within $\frac{1}{10}$ of 1? *[10]*

SOLUTION. We'll reverse-engineer the "how close" (*i.e.* the δ) from the tolerance (*i.e.* the ε) we want to ensure.

$$\begin{aligned}
&\frac{1}{x} \text{ is within } \frac{1}{10} \text{ of } 1 \\
\iff & -\frac{1}{10} < \frac{1}{x} - 1 < \frac{1}{10} \\
\iff & 1 - \frac{1}{10} < \frac{1}{x} < 1 + \frac{1}{10} \\
\iff & \frac{9}{10} < \frac{1}{x} < \frac{11}{10} \\
\iff & \frac{10}{9} > x > \frac{10}{11} \\
\iff & \frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1 \\
\iff & \frac{1}{9} > x - 1 > -\frac{1}{11} \\
\Longleftarrow & \frac{1}{11} > x - 1 > -\frac{1}{11} \quad \text{since } \frac{1}{11} < \frac{1}{9}
\end{aligned}$$

Thus x ought to be within $\frac{1}{11}$ of 1 in order to guarantee that $\frac{1}{x}$ is within $\frac{1}{10}$ of 1. ■

13:00 Seminar

1. Find a value of $\delta > 0$ that ensures that $-1 < \sqrt{x} - 4 < 1$ whenever $-\delta < x - 16 < \delta$. *[10]*

SOLUTION. As usual, we reverse-engineer the δ from the ε, which in this case has the value 1.

$$\begin{aligned}
& -1 < \sqrt{x} - 4 < 1 \\
\iff & 4 - 1 < \sqrt{x} < 4 + 1 \\
\iff & 3 < sqrtx < 5 \\
\iff & 9 < x < 25 \\
\iff & 9 - 16 < x - 16 < 25 - 16 \\
\iff & -7 < x - 16 < 9 \\
\Longleftarrow & -7 < x - 16 < 9 \quad \text{since } 7 < 9
\end{aligned}$$

1

Thus $\delta = 7$ will ensure that $-1 < \sqrt{x} - 4 < 1$ whenever $-\delta < x - 16 < \delta$. ■

Leftovers

1. Use the $\varepsilon - \delta$ definition of limits to verify that that $\lim_{x \to 0} 1 = 1$.

   Hint: Try any $\delta > 0$ you like ...

SOLUTION. There's not much to reverse-enginner here. Observe that for any $\varepsilon > 0$ whatsoever we have

$$-\varepsilon < 1 - 1 < \varepsilon$$

for the very simple reason that $1 - 1 = 0$. This is entirely indepependent of the value of x, so we get $-\varepsilon < 1 - 1 < \varepsilon$ whenever $-\delta < x - 0 < \delta$ no matter what value of $\delta > 0$ we try ... Hence, by the $\varepsilon - \delta$ definition of limit, $\lim_{x \to 0} 1 = 1$. ■

Quiz #2. Monday, 29 September, 2002. *[10 minutes]*

1. Use the $\varepsilon - \delta$ definition of limits to verify that that $\lim_{x \to 0} \sin^2(x) = 0$.

   Hint: You may use the fact that $|\sin(x)| \leq |x|$.

SOLUTION. As usual, we reverse-engineer the δ from the ε.

$$\begin{aligned}
& -\varepsilon < \sin^2(x) - 0 < \varepsilon \\
\iff & -\varepsilon < [\sin(x)]^2 < \varepsilon \\
\iff & -\sqrt{\varepsilon} < \sin(x) < \sqrt{\varepsilon} \\
\iff & |\sin(x)| < \sqrt{\varepsilon} \\
\Longleftarrow & |\sin(x)| \leq |x| < \sqrt{\varepsilon} \\
\Longleftarrow & |x - 0| < \sqrt{\varepsilon}
\end{aligned}$$

It follows that $\delta = \sqrt{\varepsilon}$ will do the job, and hence that $\lim_{x \to 0} \sin^2(x) = 0$. ■

Quiz #3. Friday, 3 October, 2003. *[15 minutes]*

12:00 Seminar

Evaluate

$$1. \quad \lim_{x \to \infty} \frac{x-2}{x^2 - 3x + 2} \quad [5] \qquad 2. \quad \lim_{x \to 0} \frac{e^x - 1}{e^{2x} - 1} \quad [5]$$

SOLUTIONS.

1.

$$\lim_{x \to \infty} \frac{x-2}{x^2 - 3x + 2} = \lim_{x \to \infty} \frac{x-2}{(x-2)(x-1)} = \lim_{x \to \infty} \frac{1}{x-1} = 0$$

since $\lim_{x \to \infty} (x - 1) = \infty$. ■

2.

$$\lim_{x \to 0} \frac{e^x - 1}{e^{2x} - 1} = \lim_{x \to 0} \frac{e^x - 1}{(e^x)^2 - 1} = \lim_{x \to 0} \frac{e^x - 1}{(e^x - 1)(e^x + 1)} = \lim_{x \to 0} \frac{1}{e^x + 1} = \frac{1}{1+1} = \frac{1}{2}$$

since $\lim_{x \to 0} e^x = e^0 = 1$. ■

13:00 Seminar

Evaluate

1. $\displaystyle\lim_{x\to 1}\frac{x^2+x-2}{x^2+2x-3}$ *[5]* 2. $\displaystyle\lim_{x\to\infty}\frac{(x+4)^2}{41x^2+43x+47}$ *[5]*

SOLUTIONS.

1.

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2+2x-3}=\lim_{x\to 1}\frac{(x-1)(x+2)}{(x-1)(x+3)}=\lim_{x\to 1}\frac{x+2}{x+3}=\frac{1+2}{1+3}=\frac{3}{4} \quad \blacksquare$$

2.

$$\begin{aligned}\lim_{x\to\infty}\frac{(x+4)^2}{41x^2+43x+47}&=\lim_{x\to\infty}\frac{x^2+8x+16}{41x^2+43x+47}=\lim_{x\to\infty}\frac{x^2+8x+16}{41x^2+43x+47}\cdot\frac{1/x^2}{1/x^2}\\&=\lim_{x\to\infty}\frac{1+\frac{8}{x}+\frac{16}{x^2}}{41+\frac{43}{x}+\frac{47}{x^2}}=\frac{1+0+0}{1+0+0}=1\end{aligned}$$

since $\lim_{x\to\infty}\frac{1}{x}=\lim_{x\to\infty}\frac{1}{x^2}=0$. $\blacksquare$

Leftovers

1. For what value(s) of the constant c does $\displaystyle\lim_{x\to 2}(cx+3)=\lim_{t\to\infty}\frac{ct^2+3+c}{t^2+1}$? *[10]* SOLUTION. First, we'll evaluate both limits:

$$\lim_{x\to 2}(cx+3)=c\cdot 2+3=2c+3$$

and

$$\lim_{t\to\infty}\frac{ct^2+3+c}{t^2+1}=\lim_{t\to\infty}\frac{ct^2+3+c}{t^2+1}\cdot\frac{1/t^2}{1/t^2}=\lim_{t\to\infty}\frac{c+\frac{3}{t}+\frac{c}{t^2}}{1+\frac{1}{t^2}}=\frac{c+0+0}{1+0}=c$$

Second, setting them equal gives the linear equation $2c+3=c$; solving this equation for c gives $c=-3$. $\blacksquare$

Quiz #4. Friday, 10 October, 2003. *[10 minutes]*

12:00 Seminar

1. Use the limit definition of the derivative to find $f'(0)$ if $f(x)=(x+1)^3$. *[10]*

SOLUTION.

$$\begin{aligned}f'(0)&=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{(0+h+1)^2-(0+1)^2}{h}=\lim_{h\to 0}\frac{(h+1)^2-1}{h}\\&=\lim_{h\to 0}\frac{h^2+2h+1-1}{h}=\lim_{h\to 0}\frac{h^2+2h}{h}=\lim_{h\to 0}(h+2)=0+2=2 \quad \blacksquare\end{aligned}$$

13:00 Seminar

1. Use the limit definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{1}{x}$. *[10]*

SOLUTION.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x(x+0)} = \frac{-1}{x^2} \quad \blacksquare
\end{aligned}
$$

Leftovers

1. Use the limit definition of the derivative to find $f'(x)$ if $f(x) = x^2 - 3x$. *[10]*

SOLUTION.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left[(x+h)^2 - 3(x+h)\right] - \left[x^2-3x\right]}{h} \\
&= \lim_{h\to 0} \frac{\left[x^2+2xh+h^2-3x-3h\right] - \left[x^2-3x\right]}{h} \\
&= \lim_{h\to 0} \frac{x^2+2xh+h^2-3x-3h-x^2+3x}{h} \\
&= \lim_{h\to 0} \frac{2xh+h^2-3h}{h} = \lim_{h\to 0} (2x+h-3) = 2x+0-3 = 2x-3 \quad \blacksquare
\end{aligned}
$$

Quiz #5. Friday, 17 October, 2003. *[10 minutes]*

12:00 Seminar

Find $\frac{dy}{dx}$ in each of the following:

1. $y = \ln\left(\sec(x)+\tan(x)\right)$ *[3]* 2. $e^{xy} = 2$ *[3]* 3. $y = \dfrac{x^2+4x+4}{x+3}$ *[4]*

SOLUTIONS.

1.

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\ln\left(\sec(x)+\tan(x)\right) \\
&= \frac{1}{\sec(x)+\tan(x)} \cdot \frac{d}{dx}\left(\sec(x)+\tan(x)\right) \\
&= \frac{1}{\sec(x)+\tan(x)} \cdot \left(\frac{d}{dx}\sec(x) + \frac{d}{dx}\tan(x)\right) \\
&= \frac{1}{\sec(x)+\tan(x)} \cdot \left(\sec(x)\tan(x)+\sec^2(x)\right) \\
&= \frac{\sec(x)\left(\tan(x)+\sec(x)\right)}{\sec(x)+\tan(x)} = \sec(x) \quad \blacksquare
\end{aligned}
$$

4

2.

$$e^{xy} = 2 \iff xy = \ln(2) \iff y = \frac{\ln(2)}{x}$$

so

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln(2)}{x}\right) = \ln(2)\frac{d}{dx}\left(\frac{1}{x}\right) = \ln(2)\cdot\frac{-1}{x^2} = -\frac{\ln(2)}{x^2} \quad \blacksquare$$

3.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x^2+4x+4}{x+3}\right) \\
&= \frac{\frac{d}{dx}\left(x^2+4x+4\right)\cdot(x+3) - \left(x^2+4x+4\right)\cdot\frac{d}{dx}(x+3)}{(x+3)^2} \\
&= \frac{(2x+4)(x+3) - \left(x^2+4x+4\right)\cdot 1}{(x+3)^2} \\
&= \frac{2x^2+10x+12-x^2-4x-4}{(x+3)^2} \\
&= \frac{x^2+6x+8}{(x+3)^2} = \frac{(x+2)(x+4)}{(x+3)^2} \quad \blacksquare
\end{aligned}$$

13:00 Seminar

Find $\frac{dy}{dx}$ in each of the following:

1. $y = \left(1+x^2\right)\arctan(x)$ *[3]* 2. $\tan(x+y) = 1$ *[3]* 3. $y = \frac{e^x+1}{e^{2x}-1}$ *[4]*

Solutions.

1.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[\left(1+x^2\right)\arctan(x)\right] = \frac{d}{dx}\left(1+x^2\right)\cdot\arctan(x) + \left(1+x^2\right)\cdot\frac{d}{dx}\arctan(x) \\
&= 2x\arctan(x) + \left(1+x^2\right)\cdot\frac{1}{1+x^2} = 2x\arctan(x) + 1 \quad \blacksquare
\end{aligned}$$

2. The hard way:

$$\begin{aligned}
&\tan(x+y) = 1 \\
\implies &\frac{d}{dx}\tan(x+y) = \frac{d}{dx}1 \\
\implies &\sec^2(x+y)\cdot\frac{d}{dx}(x+y) = 0 \\
\implies &\sec^2(x+y)\cdot\left(\frac{d}{dx}x + \frac{d}{dx}y\right) = 0 \\
\implies &\sec^2(x+y)\cdot\left(1+\frac{dy}{dx}\right) = 0 \\
\implies &1+\frac{dy}{dx} = 0 \qquad \text{(since } \sec(t) \neq 0 \text{ when defined)} \\
\implies &\frac{dy}{dx} = -1
\end{aligned}$$

. . . and the easy way:

$$\begin{aligned}\tan(x+y)=1 &\iff x+y=n\pi+\frac{\pi}{4} \quad \text{for some integer } n\\ &\iff y=-x+n\pi+\frac{\pi}{4} \quad \text{for some integer } n\end{aligned}$$

so

$$\frac{dy}{dx}=\frac{d}{dx}\left(-x+n\pi+\frac{\pi}{4}\right)=-1+0=-1 \quad \blacksquare$$

3.
$$y=\frac{e^x+1}{e^{2x}-1}=\frac{e^x+1}{(e^x)^2-1}=\frac{e^x+1}{(e^x-1)(e^x+1)}=\frac{1}{e^x-1}$$

so

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{e^x-1}\right)=\frac{-1}{(e^x-1)^2}\cdot\frac{d}{dx}(e^x-1)=\frac{-1}{(e^x-1)^2}\cdot e^x=\frac{-e^x}{(e^x-1)^2} \quad \blacksquare$$

Leftovers

Find $\frac{dy}{dx}$ in each of the following:

1. $y=\sqrt{1-e^{2x}}$ *[3]* 2. $y=\dfrac{\tan(x)}{\cos(x)}$ *[3]* 3. $\ln(x+y)=x$ *[4]*

Solutions.

1.
$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\sqrt{1-e^{2x}}=\frac{d}{dx}\left(1-e^{2x}\right)^{1/2}=\frac{1}{2}\left(1-e^{2x}\right)^{-1/2}\cdot\frac{d}{dx}\left(1-e^{2x}\right)\\ &=\frac{1}{2}\left(1-e^{2x}\right)^{-1/2}\cdot\left(0-e^{2x}\cdot\frac{d}{dx}(2x)\right)=\frac{1}{2}\left(1-e^{2x}\right)^{-1/2}\cdot\left(-2e^{2x}\right)\\ &=-e^{2x}\left(1-e^{2x}\right)^{-1/2}=\frac{-e^{2x}}{\sqrt{1-e^{2x}}} \quad \blacksquare\end{aligned}$$

2.
$$y=\frac{\tan(x)}{\cos(x)}=\tan(x)\frac{1}{\cos(x)}=\tan(x)\sec(x)$$

so

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left(\tan(x)\sec(x)\right)=\left(\frac{d}{dx}\tan(x)\right)\cdot\sec(x)+\tan(x)\cdot\left(\frac{d}{dx}\sec(x)\right)\\ &=\sec^2(x)\sec(x)+\tan(x)\sec(x)\tan(x)=\sec(x)\left(\sec^2(x)+\tan^2(x)\right) \quad \blacksquare\end{aligned}$$

6

3. The hard way:

$$
\begin{aligned}
\ln(x+y) = x &\Longrightarrow \frac{d}{dx}\ln(x+y) = \frac{d}{dx}x \\
&\Longrightarrow \frac{1}{x+y} \cdot \frac{d}{dx}(x+y) = 1 \\
&\Longrightarrow \frac{1}{x+y} \cdot \left(1 + \frac{dy}{dx}\right) = 1 \\
&\Longrightarrow 1 + \frac{dy}{dx} = x + y \\
&\Longrightarrow \frac{dy}{dx} = x + y - 1
\end{aligned}
$$

... and the easy way:

$$\ln(x+y) = x \iff x + y = e^x \iff y = e^x - x$$

so

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x - x\right) = e^x - 1 \quad \blacksquare$$

Mathematics 110 – Calculus of one variable
TRENT UNIVERSITY, 2003-2004

§A Test #1 Solutions

1. Find $\frac{dy}{dx}$ in any *three* of **a-e**. *[12 = 3 × 4 ea.]*

a. $y = x\ln\left(\frac{1}{x}\right)$ **b.** $x^2 + 2xy + y^2 - x = 1$ **c.** $y = \sin\left(e^{\sqrt{x}}\right)$

d. $y = \frac{2^x}{x+1}$ **e.** $y = \cos(2t)$ where $t = x^3 + 2x$

Solutions.

a. Product rule:

First, the direct approach.

$$\frac{dy}{dx} = \frac{d}{dx}x\ln\left(\frac{1}{x}\right) = \frac{1}{1/x} \cdot \frac{d}{dx}\frac{1}{x} = x \cdot \frac{-1}{x^2} = -\frac{1}{x}$$

Second, an alternative, slightly indirect, approach.

$$y = x\ln\left(\frac{1}{x}\right) = x\ln\left(x^{-1}\right) = -x\ln(x)$$

so

$$\frac{dy}{dx} = \frac{d}{dx}\left(-x\ln(x)\right) = -\frac{d}{dx}\ln(x) = -\frac{1}{x} \qquad \blacksquare$$

b. The direct approach is to use implicit differentiation, plus the product and chain rules along the way:

$$\begin{aligned}
&x^2 + 2xy + y^2 - x = 1 \Longrightarrow \frac{d}{dx}\left(x^2 + 2xy + y^2 - x\right) = \frac{d}{dx}1 \\
\Longrightarrow &\frac{d}{dx}x^2 + \frac{d}{dx}2xy + \frac{d}{dx}y^2 - \frac{d}{dx}x = 0 \Longrightarrow 2x + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} - 1 = 0 \\
\Longrightarrow &(2x+2y)\frac{dy}{dx} + (2x + 2y - 1) = 0 \Longrightarrow (2x+2y)\frac{dy}{dx} = 1 - 2x - 2y \\
\Longrightarrow &\frac{dy}{dx} = \frac{1 - 2x - 2y}{2x + 2y} = \frac{1}{2x+2y} - 1
\end{aligned}$$

An alternate approach is to solve for y first ...

$$\begin{aligned}
x^2 + 2xy + y^2 - x = 1 &\iff (x+y)^2 - x = 1 \iff (x+y)^2 = 1 + x \\
&\iff x + y = \pm\sqrt{1+x} \iff y = -x \pm \sqrt{1+x}
\end{aligned}$$

1

... and then take the derivative:

$$\frac{dy}{dx} = \frac{d}{dx}\left(-x \pm \sqrt{1+x}\right) = -1 \pm \frac{1}{2\sqrt{1+x}} \qquad \blacksquare$$

c. Chain rule, twice:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\sin\left(e^{\sqrt{x}}\right) = \cos\left(e^{\sqrt{x}}\right)\cdot\frac{d}{dx}e^{\sqrt{x}} \\ &= \cos\left(e^{\sqrt{x}}\right)e^{\sqrt{x}}\cdot\frac{d}{dx}\sqrt{x} = \cos\left(e^{\sqrt{x}}\right)e^{\sqrt{x}}\frac{1}{2\sqrt{x}} \qquad \blacksquare\end{aligned}$$

d. Quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{2^x}{x+1}\right) = \frac{\frac{d}{dx}2^x\cdot(x+1) - 2^x\cdot\frac{d}{dx}(x+1)}{(x+1)^2} \\ &= \frac{\ln(2)2^x\cdot(x+1)-2^x\cdot 1}{(x+1)^2} = \frac{2^x\left(\ln(2)(x+1)-1\right)}{(x+1)^2} \qquad \blacksquare\end{aligned}$$

e. Chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{d}{dt}\cos(2t)\cdot\frac{d}{dx}\left(x^3+2x\right) = -\sin(2t)\cdot\frac{d}{dt}2t\cdot\left(3x^2+2\right) \\ &= -2\sin(2t)\cdot\left(3x^2+2\right) = -2\left(3x^2+2\right)\sin\left(2\left(x^3+2x\right)\right) \\ &= -\left(6x^2+4\right)\sin\left(2x^3+4x\right) \qquad \blacksquare\end{aligned}$$

2. Do any *two* of **a**-**c**. *[10 = 2 × 5 each]*

a. Determine whether $g(x) = \begin{cases} \frac{x-1}{x^2-1} & x \neq 1 \\ \frac{1}{2} & x = 1 \end{cases}$ is continuous at $x = 1$ or not.

Solution. $g(x)$ is continuous at $x = 1$ if and only if $\lim_{x\to 1} g(x)$ exists and equals $g(1)$. Since

$$\lim_{x\to 1} g(x) = \lim_{x\to 1}\frac{x-1}{x^2-1} = \lim_{x\to 1}\frac{x-1}{(x-1)(x+1)} = \lim_{x\to 1}\frac{1}{x+1} = \frac{1}{1+1} = \frac{1}{2} = g(1),$$

$g(x)$ is continuous at $x = 1$. $\blacksquare$

b. Use the definition of the derivative to compute $f'(1)$ for $f(x) = \frac{1}{x}$.

Solution. Plug in and run ...

$$\begin{aligned}f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\frac{1}{1+h}-\frac{1}{1}}{h} = \lim_{h\to 0}\frac{\frac{1-(1+h)}{1+h}}{h} \\ &= \lim_{h\to 0}\frac{\frac{-h}{1+h}}{h} = \lim_{h\to 0}\frac{-h}{h(1+h)} = \lim_{h\to 0}\frac{-1}{1+h} = \frac{-1}{1+0} = -1 \qquad \blacksquare\end{aligned}$$

2

c. Find the equation of the tangent line to $y = \sqrt{x}$ at $x = 9$.

Solution. Note that at $x = 9$, $y = \sqrt{9} = 3$. The slope m of the tangent line is equal to the derivative of y at $x = 9$.

$$m = \left.\frac{dy}{dx}\right|_{x=9} = \left.\frac{d}{dx}\sqrt{x}\right|_{x=9} = \left.\frac{1}{2\sqrt{x}}\right|_{x=9} = \frac{1}{2\sqrt{9}} = \frac{1}{2 \cdot 3} = \frac{1}{6}$$

We want the equation $y = mx + b$ of the line with slope $m = \frac{1}{6}$ passing through the point $(9, 3)$, and it remains to compute the y-intercept, b. We do this by plugging in the slope and the coordinates of the point into the equation of the line and solving for b:

$$3 = \frac{1}{6} \cdot 9 + b \iff 3 = \frac{3}{2} + b \iff b = 3 - \frac{3}{2} = \frac{3}{2}$$

Thus the equation of the tangent line to $y = \sqrt{x}$ at $x = 9$ is $y = \frac{1}{6}x + \frac{3}{2}$. ■

3. Do *one* of **a** or **b**. *[8]*

a. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 2} x^2 = 4$.

Solution. We need to show that for every $\varepsilon > 0$, there is a $\delta > 0$ such that if x is within δ of 2, then x^2 is within ε of 4. As usual, given ε, we try to reverse-engineer the necessary δ:

$$-\varepsilon < x^2 - 4 < \varepsilon \iff -\varepsilon < (x-2)(x+2) < \varepsilon \iff -\frac{\varepsilon}{x+2} < x - 2 < \frac{\varepsilon}{x+2}$$

Unfortunately, δ cannot depend on x, so we need to find a suitable bound for $\frac{\varepsilon}{x+2}$. If we arbitrarily decide to ensure that $\delta \leq 1$, then:

$$\begin{aligned} -\delta < x - 2 < \delta &\Longrightarrow -1 < x - 2 < 1 \Longrightarrow 1 < x < 3 \Longrightarrow 3 < x + 2 < 5 \\ &\Longrightarrow \frac{1}{3} > \frac{1}{x+2} > \frac{1}{5} \Longrightarrow \frac{\varepsilon}{3} > \frac{\varepsilon}{x+2} > \frac{\varepsilon}{5} \end{aligned}$$

If we now let $\delta = \min\left(1, \frac{\varepsilon}{5}\right)$, this will do the job:

$$\begin{aligned} -\delta < x - 2 < \delta &\Longrightarrow -\frac{\varepsilon}{5} < x - 2 < \frac{\varepsilon}{5} \qquad \text{because } \delta \leq \frac{\varepsilon}{5} \\ &\Longrightarrow -\frac{\varepsilon}{x+2} < x - 2 < \frac{\varepsilon}{x+2} \\ &\qquad \text{because } -1 \leq \delta < x - 2 < \delta \leq 1 \text{ implies that } \frac{\varepsilon}{5} < \frac{\varepsilon}{x+2} \\ &\Longrightarrow -\varepsilon < (x-2)(x+2) < \varepsilon \\ &\Longrightarrow -\varepsilon < x^2 - 4 < \varepsilon \qquad \ldots \text{ as desired!} \end{aligned}$$

Hence $\lim_{x \to 2} x^2 = 4$. ■

b. Use the $\varepsilon - N$ definition of limits to verify that $\lim_{t\to\infty} \dfrac{1}{t+1} = 0$.

Solution. We need to show that for every $\varepsilon > 0$, there is an $N > 0$ such that if $x > N$, then $\frac{1}{t+1}$ is within ε of 0. Note that as $t \to \infty$, we can assume that $t > -1$, from which it follows that $\frac{1}{t+1} - 0 > 0 > -\varepsilon$. This means we only have to worry about making $\frac{1}{t+1} - 0 < \varepsilon$. As usual, given ε, we try to reverse-engineer the necessary N:

$$\frac{1}{t+1} - 0 < \varepsilon \iff \frac{1}{t+1} < \varepsilon \iff t+1 > \frac{1}{\varepsilon} \iff t > \frac{1}{\varepsilon} - 1$$

Since every step was reversible here, it follows that $N = \dfrac{1}{\varepsilon} - 1$ will do the job. Hence $\lim_{t\to\infty} \dfrac{1}{t+1} = 0$. ■

4. Find the intercepts, the maximum, minimum, and inflection points, and the vertical and horizontal asymptotes of $f(x) = xe^{-x^2}$ and sketch the graph of $f(x)$ based on this information. *[10]*

Solution.

i. (*Domain*) Since x and e^{-x^2} are defined and continuous for all x, it follows that $f(x) = xe^{-x^2}$ is defined and continuous for all x.

ii. (*Intercepts*) $f(x) = xe^{-x^2} = 0 \iff x = 0$, because $e^{-x^2} > 0$ for all x. Thus $(0, 0)$ is the only x-intercept and the only y-intercept of $f(x)$.

iii. (*Local maxima and minima*)

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(xe^{-x^2}\right) = \frac{d}{dx}x \cdot e^{-x^2} + x \cdot \frac{d}{dx}e^{-x^2} = 1 \cdot e^{-x^2} + x \cdot e^{-x^2} \cdot \frac{d}{dx}\left(-x^2\right) \\ &= e^{-x^2} + x \cdot e^{-x^2} \cdot (-2x) = \left(1 - 2x^2\right)e^{-x^2} \end{aligned}$$

which is also defined for all x.

Since $e^{-x^2} > 0$ for all x,

$$f'(x) = 0 \iff 1 - 2x^2 = 0 \iff x^2 = \frac{1}{2} \iff x = \pm\frac{1}{\sqrt{2}}.$$

We determine which of these give local maxima or minima by considering the intervals of increase and decrease. Note that since $e^{-x^2} > 0$ for all x, $f'(x)$ is positive or negative depending on whether $1 - 2x^2$ is positive or negative.

x	$x < -\frac{1}{\sqrt{2}}$	$x = -\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$	$x = \frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}} < x$
$f'(x)$	< 0	0	> 0	0	< 0
$f(x)$	decreasing	local min	increasing	local max	decreasing

Thus $f\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}e^{-1/2}$ and $f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e^{-1/2}$ are, respectively, local minimum and local maximum points of $f(x)$.

iv. (*Points of inflection and curvature*)

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\left[\left(1-2x^2\right)e^{-x^2}\right] = \frac{d}{dx}\left(1-2x^2\right)\cdot e^{-x^2} + \left(1-2x^2\right)\cdot\frac{d}{dx}e^{-x^2}$$
$$= -4xe^{-x^2} + \left(1-2x^2\right)e^{-x^2}\frac{d}{dx}\left(-x^2\right) = -4xe^{-x^2} + \left(1-2x^2\right)e^{-x^2}(-2x)$$
$$= \left(-4x-2x+4x^3\right)e^{-x^2} = \left(4x^3-6x\right)e^{-x^2} = 2x\left(2x^2-3\right)e^{-x^2}$$

which is also defined for all x.

Since $e^{-x^2} > 0$ for all x, $f''(x) = 0$ if $x = 0$ or $x = \pm\sqrt{\frac{3}{2}}$. To sort out the inflection points and intervals of curvature, we check where $f''(x)$ is positive and where it is negative. Again, $e^{-x^2} > 0$ for all x, so $f''(x)$ is positive or negative depending on whether $2x\left(2x^2-3\right)$ is positive or negative

x	$x < -\sqrt{\frac{3}{2}}$	$x = -\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}} < x < 0$	$x = 0$	$0 < x < \sqrt{\frac{3}{2}}$	$x = \sqrt{\frac{3}{2}}$	$x > \sqrt{\frac{3}{2}}$
$f''(x)$	< 0	0	> 0	0	< 0	0	> 0
$f(x)$	conc. down	infl. pt.	conc. up	infl. pt.	conc. down	infl. pt.	conc. up

Thus $f\left(-\sqrt{\frac{3}{2}}\right) = -\sqrt{\frac{3}{2}}e^{-3/2}$, $f(0) = 0$, and $f\left(\sqrt{\frac{3}{2}}\right) = \sqrt{\frac{3}{2}}e^{-3/2}$ are the inflection points of $f(x)$.

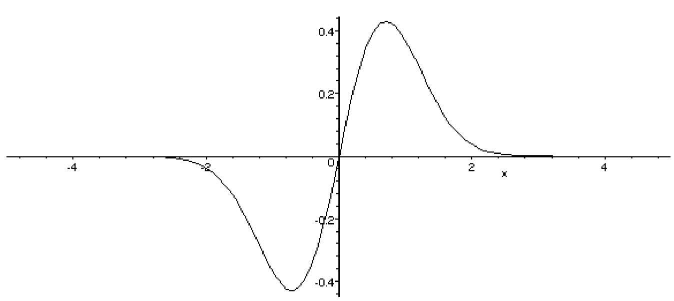
v. (*Vertical asymptotes*) $f(x)$ has no vertical asymptotes because it is defined and continuous for all x.

vi. (*Horizontal asymptotes*) $f(x) = xe^{-x^2} = \frac{x}{e^{x^2}}$ and, since $x \to \pm\infty$ and $e^{x^2} \to infty$ as $x \to \pm\infty$, we can use l'Hôpital's Rule in the relevant limits.

$$\lim_{x\to+\infty}\frac{x}{e^{x^2}} = \lim_{x\to+\infty}\frac{1}{2xe^{x^2}} = 0$$
$$\lim_{x\to-\infty}\frac{x}{e^{x^2}} = \lim_{x\to-\infty}\frac{1}{2xe^{x^2}} = 0$$

Thus $f(x)$ has a horizontal asymptote at $y = 0$ in both directions.

vii. (*The graph!*) Typing `plot(x*exp(-x*x),x=-5..5);` into Maple gives:



Whew! ■

Mathematics 110 – Calculus of one variable
TRENT UNIVERSITY, 2003-2004

§A Test #2 Solutions

1. Compute any *three* of the integrals in parts **a-f**. *[12 = 3 × 4 each]*

a. $\displaystyle\int_0^{\pi/2} \cos^3(x)\,dx$ **b.** $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$ **c.** $\displaystyle\int_2^{\infty} \frac{1}{\sqrt{x}}\,dx$

d. $\displaystyle\int \frac{\arctan(x)}{x^2+1}\,dx$ **e.** $\displaystyle\int \ln\left(x^2\right)\,dx$ **f.** $\displaystyle\int_1^2 \frac{1}{x^2-2x+2}\,dx$

Solutions.

a. Trig identity followed by a substitution:

$$\int_0^{\pi/2} \cos^3(x)\,dx = \int_0^{\pi/2} \cos^2(x)\cos(x)\,dx = \int_0^{\pi/2} \left(1-\sin^2(x)\right)\cos(x)\,dx$$

Letting $u = \sin(x)$, we get $du = \cos(x)\,dx$; note that $u = 0$ when $x = 0$ and $u = 1$ when $x = \pi/2$.

$$= \int_0^1 \left(1-u^2\right)\,du = \left(u - \frac{1}{3}u^3\right)\Big|_0^1$$

$$= \left(1 - \frac{1}{3}1^3\right) - \left(0 - \frac{1}{3}0^3\right) = \frac{2}{3} \quad \blacksquare$$

b. Partial fractions:

$$\int \frac{1}{x^2+3x+2}\,dx = \int \frac{1}{(x+1)(x+2)}\,dx = \int \left(\frac{A}{x+1} + \frac{B}{x+2}\right)\,dx$$

We need to determine A and B:

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)} = \frac{(A+B)x+(2A+B)}{(x+1)(x+2)}$$

Comparing coefficients in the numerators, it follows that $A + B = 0$ and $2A + B = 1$. Subtracting the first equation from the second gives $A = (2A+B)-(A+B) = 1-0 = 1$; substituting this back into the first equation gives $1 + B = 0$, so $B = -1$. We can now return to our integral:

$$\int \frac{1}{x^2+3x+2}\,dx = \int \frac{1}{(x+1)(x+2)}\,dx = \int \left(\frac{1}{x+1} + \frac{-1}{x+2}\right)\,dx$$

$$= \int \frac{1}{x+1}\,dx - \int \frac{1}{x+2}\,dx = \ln(x+1) - \ln(x+2) + C$$

$$= \ln\left(\frac{x+1}{x+2}\right) + C \quad \blacksquare$$

c. Improper integral:

$$\int_2^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}\int_2^t x^{1/2}\,dx$$
$$= \lim_{t\to\infty} \left.\frac{x^{3/2}}{3/2}\right|_2^t = \lim_{t\to\infty}\left(\frac{2}{3}t^{3/2} - \frac{2}{3}2^{3/2}\right) = \infty$$

... because $t^{3/2} > t$ and $t \to \infty$. Hence this improper integral does not converge. ■

d. Substitution:

$$\int \frac{\arctan(x)}{x^2+1}\,dx = \int u\,du \qquad \text{where } u = \arctan(x) \text{ and } du = \frac{1}{x^2+1}\,dx$$
$$= \frac{1}{2}u^2 + C = \frac{1}{2}\arctan^2(x) + C \quad ■$$

e. Integration by parts:

$$\int \ln\left(x^2\right)\,dx = \int 2\ln(x)\,dx \qquad \text{Let } u = \ln(x) \text{ and } v' = 2, \text{ so } u' = \frac{1}{x} \text{ and } v = 2x.$$
$$= \ln(x)\cdot 2x - \int \frac{1}{x}\cdot 2x\,dx = 2x\ln(x) - \int 2\,dx = 2x\ln(x) - 2x + C \quad ■$$

f. Completing the square and substitution:

$$\int_1^2 \frac{1}{x^2-2x+2}\,dx = \int_1^2 \frac{1}{(x^2-2x+1)+1}\,dx = \int_1^2 \frac{1}{(x-1)^2+1}\,dx$$

Let $u = x-1$, then $du = dx$; note that $u = 0$ when $x = 1$ and $u = 1$ when $x = 2$.

$$= \int_0^1 \frac{1}{u^2+1}\,du = \arctan(u)|_0^1$$
$$= \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Note that $\arctan(1) = \frac{\pi}{4}$ and $\arctan(0) = 0$ because $\tan\left(\frac{\pi}{4}\right) = 1$ and $\tan(0) = 0$. ■

2. Do any *two* of parts **a**-**d**. *[8 = 2 × 4 each]*

a. Find a definite integral computed by the Right-hand Rule sum

$$\lim_{n\to\infty}\sum_{i=0}^{n}\left(1+\frac{i^2}{n^2}\right)\cdot\frac{1}{n}.$$

[The sum should have been $\sum_{i=1}^{n}\cdots$ *instead of* $\sum_{i=0}^{n}\cdots$*. Darn typo!]*

2

Solution. The general Right-hand Rule formula is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f\left(a + i\frac{b-a}{n}\right) \cdot \frac{b-a}{n}$$

Comparing the general sum above to the given one reveals that $f\left(a+\frac{b-a}{n}\right) = 1 + \frac{i^2}{n^2}$ and $\frac{b-a}{n} = \frac{1}{n}$. It follows from the latter that $b-a=1$. If we arbitrarily choose $a=0$, it will follow that $b=1$ and $f\left(a+i\frac{b-a}{n}\right) = f\left(0+\frac{i}{n}\right) = f\left(\frac{i}{n}\right)$. It follows that $f\left(\frac{i}{n}\right) = 1 + \frac{i^2}{n^2} = 1 + \left(\frac{i}{n}\right)^2$, that is, $f(x) = 1 + x^2$.

Plugging all this into the integral side of the Right-hand Rule formula, we see that:

$$\lim_{n\to\infty} \sum_{i=1}^n \left(1 + \frac{i^2}{n^2}\right) \cdot \frac{1}{n} = \int_0^1 \left(1+x^2\right)\,dx$$

It is worth noting that we could have chosen a to be any real number. This would, of course, result in a different value of b (since $b-a=1$) and a different function $f(x)$. ■

b. Compute $\dfrac{d}{dx}\left(\displaystyle\int_0^{\tan(x)} e^{\sqrt{t}}\,dt\right)$.

Solution. This is a job for the Fundamental Theorem of Calculus and the Chain Rule:

$$\begin{aligned}
\frac{d}{dx}\left(\int_0^{\tan(x)} e^{\sqrt{t}}\,dt\right) &= \frac{d}{dx}\left(\int_0^u e^{\sqrt{t}}\,dt\right) \qquad \text{where } u = \tan(x) \\
&= \frac{d}{du}\left(\int_0^u e^{\sqrt{t}}\,dt\right) \cdot \frac{du}{dx} \qquad \text{by the Chain Rule} \\
&= e^{\sqrt{u}} \cdot \frac{du}{dx} \qquad \text{by the Fundamental Theorem} \\
&= e^{\sqrt{\tan(x)}} \cdot \frac{d}{dx}\tan(x) \\
&= e^{\sqrt{\tan(x)}} \cdot \sec^2(x)
\end{aligned}$$

If you can simplify this one significantly, you're doing better than I! ■

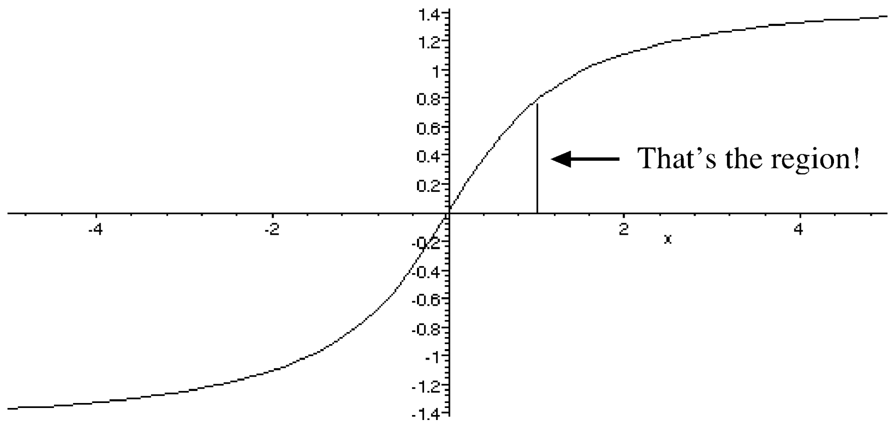
c. Find the area under the parametric curve given by $x = 1 + t^2$ and $y = t(1-t)$ for $0 \le t \le 1$.

Solution. Note that $dx = 2t\,dt$ and that $y = t(1-t) \ge 0$ for $0 \le t \le 1$.

$$\begin{aligned}
\text{Area} &= \int_{t=0}^{t=1} y\,dx = \int_0^1 t(1-t)2t\,dt = 2\int_0^t \left(t^2 - t^3\right)\,dt \\
&= 2\left(\frac{1}{3}t^3 - \frac{1}{4}t^4\right)\Big|_0^1 = 2\left(\frac{1}{3}1^3 - \frac{1}{4}1^4\right) - 2\left(\frac{1}{3}0^3 - \frac{1}{4}0^4\right) = 2\frac{1}{12} = \frac{1}{6} \quad ■
\end{aligned}$$

d. Sketch the region whose area is computed by the integral $\displaystyle\int_0^1 \arctan(x)\,dx$.

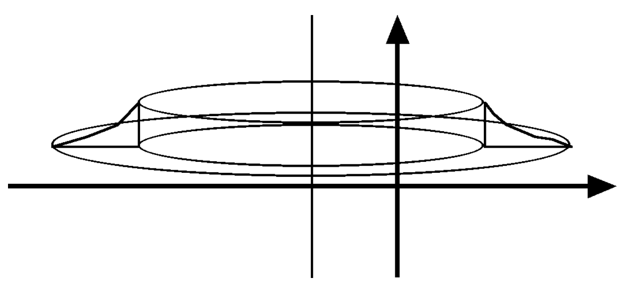
Solution. Note that $\arctan(x) \geq 0$ for $x \geq 0$, $\arctan(0) = 0$, and $\arctan(1) = \dfrac{\pi}{4}$.



One does need to know what the graph of $\arctan(x)$ looks like; the one above was generated using the MAPLE command `plot(arctan(x),x=-5..5);` (with some additions made in a drawing program). ■

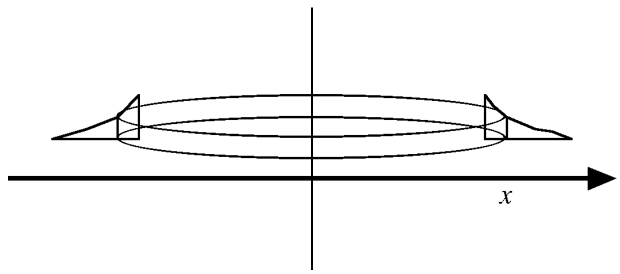
3. Find the volume of the solid obtained by rotating the region bounded by $y = \dfrac{1}{x}$, $y = \dfrac{1}{2}$, and $x = 1$ about the line $x = -1$. *[10]*

Solution. Here's a crude sketch of the solid in question:



Note the region that was rotated includes x values from 1 to 2.

We will tackle this problem using shells rather than washers, not that there is much difference in difficulty between the two methods. Since the axis of revolution is a vertical line, the shells are upright and we will need to integrate with respect to the horizontal coordinate axis, namely x. Here is a sketch of the cylindrical shell at x:



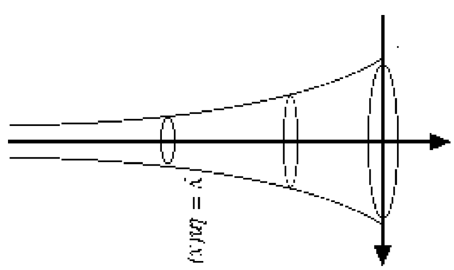
It is not hard to see that this shell has radius $r = x - (-1) = x + 1$ and height $h = \frac{1}{x} - \frac{1}{2}$, and hence area $2\pi rh = 2\pi(x+1)\left(\frac{1}{x} - \frac{1}{2}\right)$.

Thus

$$\begin{aligned}\text{Volume} &= \int_1^2 2\pi rh\,dx = \int_1^2 2\pi(x+1)\left(\frac{1}{x} - \frac{1}{2}\right)\,dx = 2\pi\int_1^2 \left(1 - \frac{x}{2} + \frac{1}{x} - \frac{1}{2}\right)\,dx \\ &= 2\pi\int_1^2 \left(\frac{1}{2} - \frac{x}{2} + \frac{1}{x}\right)\,dx = 2\pi\left(\frac{x}{2} - \frac{x^2}{4} + \ln(x)\right)\Big|_1^2 \\ &= 2\pi\left(\frac{2}{2} - \frac{2^2}{4} + \ln(2)\right) - 2\pi\left(\frac{1}{2} - \frac{1^2}{4} + \ln(1)\right) = 2\pi\left(\ln(2) - \frac{1}{4}\right) \qquad \blacksquare\end{aligned}$$

4. Find the area of the surface obtained by rotating the curve $y = \ln(x)$, $0 < x \leq 1$, about the y-axis. *[10]*

Solution. Here's a crude sketch of the surface:



A slightly nasty feature of this problem is that one must use an improper integral to compute the surface area because $\ln(x)$ has an asymptote at $x = 0$. (Even nastier is the fact that if one does not notice that this requires an improper integral and proceeds blindly using x as the independent variable, one is likely to get the right answer but still lose some marks ...) It should not be too hard to see that the radius of the surface corresponding to the point (x, y) on the curve is just $r = x - 0 = x$. Note that $\frac{dy}{dx} = \frac{d}{dx}\ln(x) = \frac{1}{x}$.

$$\mathrm{A} = \int_0^1 2\pi r\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^1 x\sqrt{1+\left(\frac{1}{x}\right)^2}\,dx = 2\pi\int_0^1 x\sqrt{1+\frac{1}{x^2}}\,dx$$

Note that this last *is* an improper integral.

$$= \lim_{t\to 0^+} 2\pi\int_t^1 x\sqrt{1+\frac{1}{x^2}}\,dx = \lim_{t\to 0^+} 2\pi\int_t^1 \sqrt{x^2\left(1+\frac{1}{x^2}\right)}\,dx = \lim_{t\to 0^+} 2\pi\int_t^1 \sqrt{x^2+1}\,dx$$

This is a job for a trig substitution, namely $x = \tan(\theta)$. Then $dx = \sec^2(\theta)\,d\theta$; we'll keep the old limits and substitute back eventually.

$$= \lim_{t\to 0^+} 2\pi\int_{x=t}^{x=1} \sqrt{\tan^2(\theta)+1}\cdot\sec^2(\theta)\,d\theta = \lim_{t\to 0^+} 2\pi\int_{x=t}^{x=1} \sqrt{\sec^2(\theta)}\cdot\sec^2(\theta)\,d\theta$$

$$= \lim_{t\to 0^+} 2\pi\int_{x=t}^{x=1} \sec(\theta)\cdot\sec^2(\theta)\,d\theta = \lim_{t\to 0^+} 2\pi\int_{x=t}^{x=1} \sec^3(\theta)\,d\theta$$

This is an integral we've seen several times over, so we'll just cut to the chase:

$$= \lim_{t\to 0^+} 2\pi\cdot\frac{1}{2}\left(\tan(\theta)\sec(\theta)+\ln|\tan(\theta)+\sec(\theta)|\right)\Big|_{x=t}^{x=1}$$

$$= \lim_{t\to 0^+} \pi\left(x\sqrt{x^2+1}+\ln\left|x+\sqrt{x^2+1}\right|\right)\Big|_t^1$$

$$= \lim_{t\to 0^+} \left[\pi\left(1\sqrt{1^2+1}+\ln\left|1+\sqrt{1^2+1}\right|\right) - \pi\left(t\sqrt{t^2+1}+\ln\left|t+\sqrt{t^2+1}\right|\right)\right]$$

$$= \lim_{t\to 0^+} \left[\pi\left(\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right) - \pi\left(t\sqrt{t^2+1}+\ln\left|t+\sqrt{t^2+1}\right|\right)\right]$$

$$= \pi\left(\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right)$$

... because $t\sqrt{t^2+1} \to 0$ as $t \to 0$ and $t+\sqrt{t^2+1} \to 1$, so $\ln\left|t+\sqrt{t^2+1}\right| \to \ln(1) = 0$ as $t \to 0$. ■

6

Mathematics 105H
Applied calculus
Summer 2008
Solutions

Mathematics 105H – Applied calculus
TRENT UNIVERSITY, Summer 2008

Quiz Solutions

Quiz #1. Thursday, 1 May, 2008 *[10 minutes]*

1. Find the equation of the line between $(1,3)$ and $(5,-1)$. *[2]*
2. Find the equation of the line through $(2,2)$ perpendicular to the line in part 1. *[2]*
3. Sketch a graph of the lines in parts 1 and 2. *[1]*

Solutions:

1. The line between $(1,3)$ and $(5,-1)$ has slope

$$m = \frac{\text{rise}}{\text{run}} = \frac{-1-3}{5-1} = \frac{-4}{4} = -1\,.$$

To find the y-intercept, we plug the coordinates of one of the points, say $(1,3)$, into $y = mx + b = (-1)x + b$ and solve for b:

$$3 = (-1)\cdot 1 + b \quad\Longrightarrow\quad -1 + b = 3 \quad\Longrightarrow\quad b = 4$$

Thus the equation of the line between $(1,3)$ and $(5,-1)$ is $y = -x + 4$. □

2. The line through $(2,2)$ perpendicular to the line $y = -x + 4$ must have as its slope the negative reciprocal of the slope of $y = -x + 4$:

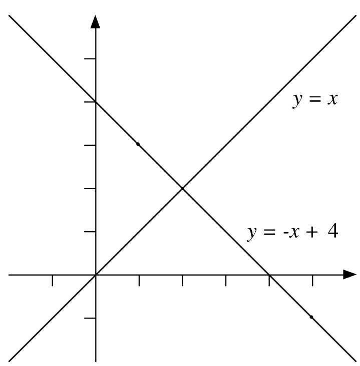
$$m = -\frac{1}{-1} - -(-1) = 1$$

To find the y-intercept, we plug $(2,2)$ into $y = mx + b = x + b$ and solve for b:

$$2 = 1 \cdot 2 + b \quad\Longrightarrow\quad b = 2 - 2 = 0$$

Thus the equation of the line is $y = x$. □

3. A sketch:



□

Quiz #2. Tuesday, 6 May, 2008 *[10 minutes]*

1. Find the x- and y-intercepts (if any), and the domain and range, of $f(x) = 1 - \frac{1}{x}$, and give a rough sketch of its graph. *[5]*

Solution: *(The overkill edition!)* The y-intercept of a function occurs when $x = 0$, but $f(x) = 1 - \frac{1}{x}$ is not defined for $x = 0$. (Since we'd be dividing by 0 in $\frac{1}{x}$ ...) Hence $f(x)$ has no y-intercept.

The x-intercept of a function occurs when $y = 0$, so we need to solve the equation $y = f(x) = 1 - \frac{1}{x} = 0$:

$$1 - \frac{1}{x} = 0 \quad \implies \quad 1 = \frac{1}{x} \quad \implies \quad x = 1$$

(Check: $f(1) = 1 - \frac{1}{1} = 1 - 1 = 0$.) Hence $f(x)$ has one x-intercept, at $x = 1$.

The expression $1 - \frac{1}{x}$ makes sense for all x except, as previously noted, for $x = 0$. Hence the domain of $f(x)$ is the set of all real numbers other than 0, *i.e.* $\{ x \in \mathbb{R} \mid x \neq 0 \}$.

Since $\frac{1}{x} \neq 0$ for all $x \neq 0$ (because if $\frac{1}{x} = 0$, then multiplying both sides by x would give $1 = x \cdot 0 = 0$, which ain't so), $y = f(x) = 1 - \frac{1}{x} \neq 1 - 0 = 1$ for all $x \neq 0$. Hence $y = 1$ is not in the range of $f(x)$. On the other hand, every $y \neq 1$ is in the range of $f(x)$, since we can then solve the equation $y = f(x) = 1 - \frac{1}{x}$ for x:

$$y = 1 - \frac{1}{x} \quad \implies \quad \frac{1}{x} = 1 - y \quad \implies \quad 1 = x(1 - y) \quad \implies \quad x = \frac{1}{1 - y}$$

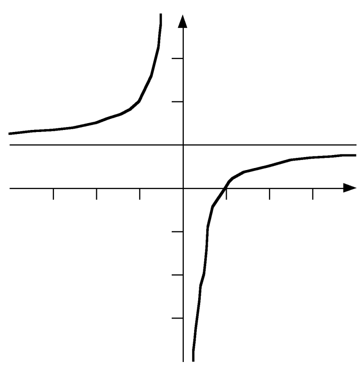
(Note that the last makes sense whenever $y \neq 1$.) Hence the range of $f(x)$ is the set of all real numbers other than 1, *i.e.* $\{ y \in \mathbb{R} \mid y \neq 1 \}$.

To sketch the graph, note first that $\frac{1}{x} < 0$ when $x < 0$ and that $\frac{1}{x} > 0$ when $x > 0$. It follows that $1 - \frac{1}{x} > 1$ when $x < 0$ and that $1 - \frac{1}{x} < 1$ when $x > 0$.

Note also that as x gets very large (in absolute value), $\frac{1}{x}$ gets very close to 0. Combining this with the previous observation, we can see that as x gets very large in the negative direction, $f(x) = 1 - \frac{1}{x}$ gets close to 1 from above, while as x gets very large in the positive direction, $y = f(x) = 1 - \frac{1}{x}$ gets close to 1 from below.

Finally, note that as x gets very close to 0 from the negative direction, $\frac{1}{x}$ shoots off to $-\infty$, and so $y = f(x) = 1 - \frac{1}{x}$ shoots up to $+\infty$. Similalrly, as x gets very close to 0 from the positive direction, $\frac{1}{x}$ shoots off to $+\infty$, and so $y = f(x) = 1 - \frac{1}{x}$ shoots up to $-\infty$.

Here's the (rough!) sketch:



$\square$

2

Quiz #3. Thursday, 8 May, 2008 *[10 minutes]*

Using the fact that $\log_a(u) = t$ really means that $a^t = u$ and the properties of exponents, explain why each of the following equations works:

1. $\log_2(1) = 0$ *[1]*
2. $\log_2(\log_3(81)) = 2$ *[2]*
3. $\log_2(41) = \log_{1/2}\left(\frac{1}{41}\right)$ *[2]*

Solutions:

1. $a^0 = 1$ if a is any valid base, *i.e.* $a > 0$ and $a \neq 1$. In particular, $2^0 = 1$, which is what $\log_2(1) = 0$ really means. $\square$
2. First, note that $\log_3(81) = 4$ since $81 = 9^2 = 3^4$. Second, we have that $\log_2(4) = 2$ since $2^2 = 4$. Hence $\log_2(\log_3(81)) = \log_2(4) = 2$. $\square$
3. First, note that $\log_2(41) = r$ really means that r is the number such that $2^r = 41$. Second, it follows from this that $\frac{1}{41} = \frac{1}{2^r} = \left(\frac{1}{2}\right)^r$, which is what $\log_{1/2}\left(\frac{1}{41}\right) = r$ really means. Hence $\log_2(41) = r = \log_{1/2}\left(\frac{1}{41}\right)$. $\square$

Quiz #4. Tuesday, 13 May, 2008 *[10 minutes]*

Do *one* of problems 1 and 2.

1. Use the limit definition of the derivative to compute $f'(2)$ for $f(x) = x^2 + 2x - 3$. *[10]*
2. Use the limit rules to compute $\displaystyle\lim_{x\to-\infty} \frac{x^2 2^x}{(3-x)^2}$. *[10]*

Solutions:

1. By definition $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. In this case $a = 2$, so:

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{\left((2+h)^2 + 2(2+h) - 3\right) - \left(2^2 + 2\cdot 2 - 3\right)}{h} \\ &= \lim_{h\to 0} \frac{\left(2^2 + 2h + h^2 + 4 + 2h - 3\right) - (4+4-3)}{h} \\ &= \lim_{h\to 0} \frac{\left(5 + 4h + h^2\right) - 5}{h} \\ &= \lim_{h\to 0} \frac{4h + h^2}{h} = \lim_{h\to 0} (4+h) = 4 + 0 = 4 \qquad \square \end{aligned}$$

2. Here goes serious overkill!

$$\begin{aligned} \lim_{x\to-\infty} \frac{x^2 2^x}{(3-x)^2} &= \lim_{x\to-\infty} \left(\frac{x^2}{(3-x)^2}\right)(2^x) = \left(\lim_{x\to-\infty} \frac{x^2}{(3-x)^2}\right)\left(\lim_{x\to-\infty} 2^x\right) \\ &= \left(\lim_{x\to-\infty} \frac{x^2}{9-6x+x^2}\right)\cdot 0 = \left(\lim_{x\to-\infty} \frac{x^2}{9-6x+x^2}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)\cdot 0 \\ &= \left(\lim_{x\to-\infty} \frac{\frac{x^2}{x^2}}{\frac{9}{x^2} - \frac{6x}{x^2} + \frac{x^2}{x^2}}\right)\cdot 0 = \left(\lim_{x\to-\infty} \frac{1}{\frac{9}{x^2} - \frac{6}{x} + 1}\right)\cdot 0 \\ &= \left(\frac{\lim_{x\to-\infty} 1}{\lim_{x\to-\infty}\left(\frac{9}{x^2} - \frac{6}{x} + 1\right)}\right)\cdot 0 \\ &= \left(\frac{1}{\left(\lim_{x\to-\infty}\frac{9}{x^2}\right) - \left(\lim_{x\to-\infty}\frac{6}{x}\right) + \left(\lim_{x\to-\infty} 1\right)}\right)\cdot 0 \\ &= \left(\frac{1}{0-0+1}\right)\cdot 0 = 1\cdot 0 = 0 \end{aligned}$$

Note that $\lim_{x\to-\infty} 2^x = 0$ since $2 > 1$. Why do we have to evaluate $\lim_{x\to-\infty} \frac{x^2}{9-6x+x^2}$ even though it will be multiplied by 0? $\square$

Quiz #5. Thursday, 15 May, 2008 *[10 minutes]*

Do any *two* of problems 1–3.

1. Use the limit definition of the derivative to compute $f'(x)$ for $f(x) = \frac{1}{x}$. *[5]*
2. Compute $f'(1)$ if $f(x) = e^x \ln\left(x^7\right)$. *[5]*
3. Find the point(s) on the graph of $y = x^3 - x$ where the graph has slope 0. *[5]*

Solutions:

1. Note that we must have $x \neq 0$ for $f(x)$ to be defined.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h \to 0} \frac{x - x - h}{hx(x+h)} \\ &= \lim_{h \to 0} \frac{-h}{hx(x+h)} = \lim_{h \to 0} \frac{-1}{x(x+h)} = \frac{-1}{x(x+0)} = \frac{-1}{x^2} \qquad \square \end{aligned}$$

2. We'll use the basic derivatives and derivative rules done in class. Note that $f(x) = e^x \ln\left(x^7\right) = 7e^x \ln(x)$.

$$\begin{aligned} f'(x) &= \frac{d}{dx} f(x) = \frac{d}{dx}\left(7e^x \ln(x)\right) = 7\frac{d}{dx}\left(e^x \ln(x)\right) \\ &= 7\left(\left[\frac{d}{dx}e^x\right]\ln(x) + e^x\left[\frac{d}{dx}\ln(x)\right]\right) \\ &= 7\left(e^x \ln(x) + e^x \frac{1}{x}\right) = 7e^x\left(\ln(x) + \frac{1}{x}\right) \end{aligned}$$

It follows that $f'(1) = 7e^1\left(\ln(1) + \frac{1}{1}\right) = 7e(0+1) = 7e$. $\square$

3. The slope of the graph of $y = x^3 - 3x$ at x is given by

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 - 3x\right) = \left(\frac{d}{dx}x^3\right) - 3\left(\frac{d}{dx}x\right) = 3x^2 - 3 \cdot 1 = 3\left(x^2 - 1\right).$$

The slope of the graph is therefore 0 when

$$3\left(x^2 - 1\right) = 0 \iff x^2 - 1 = 0 \iff x^2 = 1 \iff x = \pm 1.$$

To find the points we also need their y-coordinates, which we can get by plugging $x = \pm 1$ into $y = x^3 - 3x$: when $x = 1$, $y = 1^3 - 3 \cdot 1 = -2$, and when $x = -1$, $y = (-1)^3 - 3(-1) = -1 + 3 = 2$. Thus the points on the graph of $y = x^3 - 3x$ where the slope of the graph is 0 are $(1, -2)$ and $(-1, 2)$. $\square$

Quiz #6. Thursday, 22 May, 2008 *[10 minutes]*

1. Find the coordinates of the point(s), if any, on the graph of $y = x^3 - 9x$ where the graph has slope 3. *[5]*

Solution. We are looking for the points where $\frac{dy}{dx} = 3$. First:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 - 9x\right) = \frac{d}{dx}\left(x^3\right) - \frac{d}{dx}(9x) = 3x^2 - 9$$

Second:

$$\frac{dy}{dx} = 3 \quad\iff\quad 3x^2 - 9 = 3 \quad\iff\quad x^2 - 3 = 1 \quad\iff\quad x^2 = 4 \quad\iff\quad x = \pm 2$$

Finally, we still need the y values corresponding to $x = \pm 3$. When $x = 2$, $y = 2^3 - 9 \cdot 2 = 8 - 18 = -10$, and when $x = -2$, $y = (-2)^3 - 9 \cdot (-2) = -8 - (-18) = -8 + 18 = 10$. Thus the coordinates of the points on the graph of $y = x^3 - 9x$ where the graph has slope 3 are $(-2, 10)$ and $(2, -10)$. $\square$

6

Quiz #7. Tuesday, 27 May, 2008 *[10 minutes]*

1. Find the domain and all the intercepts, vertical and horizontal asymptotes, and local minimum and maximum points of $y = \dfrac{x}{1+x^2}$, and sketch the graph of this function. *[5]*

Solution. Here goes!

Domain: $y = \dfrac{x}{1+x^2}$ is defined everywhere, except where the denominator is equal to zero. However, since $1 + x^2 \geq 1.0$ for all x, the denominator is never zero, and so the function is defined for all real values of x.

Intercepts: If $x = 0$, then $y = \frac{0}{1+0^2} = \frac{0}{1} = 0$, so the y-intercept is $(0,0)$. On the other hand, if $y = \frac{x}{1+x^2} = 0$, we must have $x = 0$, so $(0,0)$ is also the (only) x-intercept.

Asymptotes: Since $y = \dfrac{x}{1+x^2}$ is a rational function, it is continuous everywhere it is defined, which is everywhere. It follows that there are no vertical asymptotes.

We check for horizontal asymptotes:

$$\lim_{x\to\infty} \frac{x}{1+x^2} = \lim_{x\to\infty} \frac{x}{1+x^2} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{1}{x} + \frac{x^2}{x}} = \lim_{x\to\infty} \frac{1}{\frac{1}{x} + x} = \frac{1}{0+\infty} = 0^+$$

$$\lim_{x\to-\infty} \frac{x}{1+x^2} = \lim_{x\to-\infty} \frac{x}{1+x^2} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to-\infty} \frac{\frac{x}{x}}{\frac{1}{x} + \frac{x^2}{x}} = \lim_{x\to-\infty} \frac{1}{\frac{1}{x} + x} = \frac{1}{0-\infty} = 0^-$$

It follows that $y = 0$ is a horizontal asymtote in both directions, approached from above as $x \to \infty$ and from below as $x \to -\infty$.

Local extrema: Using the Quotient Rule:

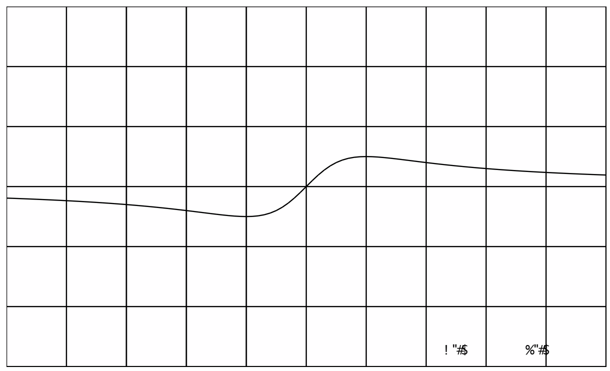
$$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{1+x^2}\right) &= \frac{\left(\frac{d}{dx}x\right)\cdot\left(1+x^2\right) - x \cdot \frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2} \\ &= \frac{1\cdot\left(1+x^2\right) - x\cdot 2x}{\left(1+x^2\right)^2} = \frac{1-x^2}{\left(1+x^2\right)^2} \end{aligned}$$

Note that $\left(1+x^2\right)^2 \geq 1^2 = 1 > 0$ for all x, so $\frac{dy}{dx}$ is < 0, $= 0$, or > 0 according to whether the numerator, $1 - x^2$, is < 0, $= 0$, or > 0:

$$\begin{array}{lllllll} 1-x^2 < 0 & \Longrightarrow & 1 < x^2 & \Longrightarrow & |x| > 1 & \Longrightarrow & x < -1 \text{ or } x > 1 \\ 1-x^2 = 0 & \Longrightarrow & x^2 = 1 & \Longrightarrow & x = \pm 1 & & \\ 1-x^2 > 0 & \Longrightarrow & 1 > x^2 & \Longrightarrow & |x| < 1 & \Longrightarrow & -1 < x < 1 \end{array}$$

When $x = -1$, $y = \frac{-1}{1+(-1)^2} = -\frac{1}{2}$, and when $x = 1$, $y = \frac{1}{1+1^2} = \frac{1}{2}$, so the critical points are $\left(-1, 1\frac{1}{2}\right)$ and $\left(1, \frac{1}{2}\right)$. Since $\frac{dy}{dx} < 0$ when $x < -1$ and > 0 when $-1 < x < 1$, $\left(-1, 1\frac{1}{2}\right)$ is a local minimum point. Similarly, since $\frac{dy}{dx} > 0$ when $-1 < x < 1$ and < 0 when $x > 1$, $\left(-1, 1\frac{1}{2}\right)$ is a local maximum point.

The graph: A couple of small observations that can be used as sanity checks first. Note that becaose $1 + x^2 \geq 1 > 0$ for all x, $y = \frac{x}{1+x^2}$ is > 0 when $x > 0$ and is < 0 when $x < 0$. In addition, it follows from the work above that y is increasing when $-1 < x < 1$ and decreasing when $-1 < x$ or $x > 1$.



The end. □

Quiz #8. Tuesday, 29 May, 2008 *[10 minutes]*

1. A spherical balloon is being filled at a rate of 1 m^3/s. How is the radius of the balloon changing at the instant that the balloon's radius is 2 m? *[5]*

   The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Solution. We are asked to find $\frac{dr}{dt}$ at the instant in question. Since $V = \frac{4}{3}\pi r^3$,

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi \frac{d}{dt}r^3 = \frac{4}{3}\pi \cdot \left(\frac{d}{dr}r^3\right) \cdot \left(\frac{dr}{dt}\right) = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} == 4\pi r^2 \cdot \frac{dr}{dt}\,.$$

Plugging in the information that $\frac{dV}{dt} = 1\ m^3/s$ and that $r = 2\ m$ at the instant in question, we get

$$1 = \frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} = 4\pi \cdot 2^2 \cdot \frac{dr}{dt} = 16\pi \cdot \frac{dr}{dt}\,.$$

It follows that at the instant in question,

$$\frac{dr}{dt} = \frac{1}{16\pi}\ m/\mathrm{s}\,. \qquad \Box$$

9

Quiz #9. Tuesday, 3 June, 2008 *[10 minutes]*

Compute *two* of the following three indefinite integrals. *[10 = 2 × 5 each]*

1. $\displaystyle\int (w-1)^2\,dw$ **2.** $\displaystyle\int \frac{\ln(x)}{x}\,dx$ **3.** $\displaystyle\int \frac{1-\sqrt{x}}{\sqrt{x}}\,dx$

Solutions. We'll use the various integration rules.

1.
$$\begin{aligned}\int (w-1)^2\,dw &= \int \left(w^2-2w+1\right)\,dw \\ &= \int w^2\,dw - 2\int w\,dw + \int 1\,dw \\ &= \frac{1}{3}w^3 - 2\cdot\frac{1}{2}w^2 + w + C \\ &= \frac{1}{3}w^3 - w^2 + w + C \qquad \Box\end{aligned}$$

2.
$$\begin{aligned}\int \frac{\ln(x)}{x}\,dx &= \int \frac{1}{u}\,du \qquad \text{(Where } u=\ln(x) \text{ and so } du = \frac{1}{x}\,dx.\text{)} \\ &= \ln(u) + C \\ &= \ln\left(\ln(x)\right) + C \qquad \Box\end{aligned}$$

3.
$$\begin{aligned}\int \frac{1-\sqrt{x}}{\sqrt{x}}\,dx &= \int \left(\frac{1}{\sqrt{x}} - \frac{\sqrt{x}}{\sqrt{x}}\right)\,dx \\ &= \int \left(x^{-1/2}-1\right)\,dx \\ &= \int x^{-1/2}\,dx - \int 1\,dx \\ &= \frac{x^{1/2}}{1/2} - x + C \\ &= 2x^{1/2} - x + C \\ &= 2\sqrt{x} - x + C \qquad \Box\end{aligned}$$

10

Quiz #10. Thursday, 5 June, 2008 *[10 minutes]*

1. Find the area of the region between the x-axis and the curve $y = x^3 - x = x(x+1)(x-1)$ for $-1 \le x \le 1$. *[5]*

Solution. The curve $y = x^3 - x = x(x+1)(x-1)$ meets the x-axis when $y = 0$, *i.e.* when $x = -1$, $x = 0$, and $x = 1$. When $-1 \le x \le 0$, the curve is above the x-axis since $\left(-\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right) = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8} > 0$, and when $0 \le x \le 1$, the curve is below the x-axis since $\left(\frac{1}{2}\right)^3 - \frac{1}{2} = \frac{1}{8} - \frac{1}{2} = -\frac{3}{8} < 0$. Thus the area of the region between the x-axis and the curve $y = x^3 - x = x(x+1)(x-1)$ for $-1 \le x \le 1$ is given by:

$$\begin{aligned}
\int_{-1}^{1} \left|x^3 - x\right| \, dx &= \int_{-1}^{0} \left(x^3 - x\right) \, dx + \int_{0}^{1} -\left(x^3 - x\right) \, dx \\
&= \int_{-1}^{0} \left(x^3 - x\right) \, dx - \int_{0}^{1} \left(x^3 - x\right) \, dx \\
&= \left(\frac{1}{4}x^4 - \frac{1}{2}x^2\right)\Big|_{-1}^{0} - \left(\frac{1}{4}x^4 - \frac{1}{2}x^2\right)\Big|_{0}^{1} \\
&= \left[\left(\frac{1}{4}0^4 - \frac{1}{2}0^2\right) - \left(\frac{1}{4}(-1)^4 - \frac{1}{2}(-1)^2\right)\right] \\
&\qquad - \left[\left(\frac{1}{4}1^4 - \frac{1}{2}1^2\right) - \left(\frac{1}{4}0^4 - \frac{1}{2}0^2\right)\right] \\
&= \left[0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right] - \left[\left(\frac{1}{4} - \frac{1}{2}\right) - 0\right] \\
&= -\left(-\frac{1}{4}\right) - \left(-\frac{1}{4}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \qquad \square
\end{aligned}$$

Mathematics 105H – Applied calculus
TRENT UNIVERSITY, Summer 2008

Solutions to the Test

1. Do *all* of **a**–**d**.

a. Find the equation of the line through $(0,3)$ with slope 2. *[2]*

Solution. A line with slope 2 must have an equation of the form $y = 2x + b$. Plugging in $x = 0$ and $y = 3$ gives $3 = 2 \cdot 0 + b$, *i.e.* $b = 3 - 0 = 3$. Thus the line in question has the equation $y = 2x + 3$. ■

b. Find the tip (*i.e.* vertex) of the parabola $y = -x^2 + 2x + 2$. *[2]*

Solution. We will find the tip of the parabola by completing the square in the equation $y = -x^2 + 2x + 2$:

$$\begin{aligned} y &= -x^2 + 2x + 2 = -\left(x^2 - 2x\right) + 2 = -\left(x^2 - 2x + 1 - 1\right) + 2 \\ &= -\left(x^2 - 2x + 1\right) - (-1) + 2 = -(x-1)^2 + 3 \end{aligned}$$

It follows that the tip of the parabola is at $(1,3)$ and that it opens downward. ■

c. Find the coordinates of the point(s) where the line and the parabola meet. *[3]*

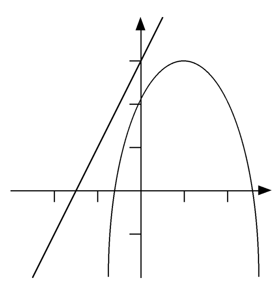
Solution. At any such point the two eqtuations would have to give the same y value for the same x:

$$-x^2 + 2x + 2 = y = 2x + 3 \quad \Longrightarrow \quad -x^2 = 1 \quad \Longrightarrow \quad x^2 = -1$$

Since there is no real number whose square is -1, the line and the parabola have no points in common, *i.e.* they do not intersect. ■

d. Sketch a graph of the line and the parabola. *[3]*

Solution.



■

1

2. Let $f(x) = 1 - e^x$. Do *all* of **a**–**d**.

a. Find the x- and y-intercepts, if any, of $y = f(x)$. *[2]*

Solution. $f(0) = 1 - e^0 = 1 - 1 = 0$, so the y-intercept is $(0, 0)$. $f(x) = 1 - e^x = 0$ when $e^x = 1$, which occurs only when $x = 0$, so $(0, 0)$ is also the only x-intercept. ■

b. Find the domain and range of $f(x)$. *[2]*

Solution. $f(x) = 1 - e^x$ is defined for all x since e^x is, so the domain of $f(x)$ is $\mathbb{R} = (-\infty, \infty)$.

Since e^x takes on all the values between 0 and ∞ as x runs through the real numbers, $f(x) = 1 - e^x$ takes on all the values in between $1 - 0 = 1$ and $1 - \infty = -\infty$ as x runs through the real numbers, so the range of $f(x)$ is $(-\infty, 1)$. ■

c. Determine whether $y = f(x)$ has any horizontal asymptotes. *[4]*

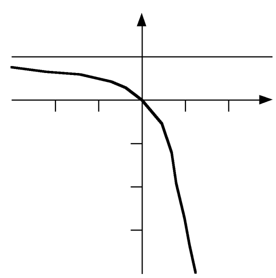
Solution. We check to see what $f(x)$ does as $x \to \pm\infty$:

$$\lim_{x\to\infty}(1 - e^x) = \left(\lim_{x\to\infty} 1\right) - \left(\lim_{x\to\infty} e^x\right) = 1 - \infty = -\infty$$
$$\lim_{x\to-\infty}(1 - e^x) = \left(\lim_{x\to-\infty} 1\right) - \left(\lim_{x\to-\infty} e^x\right) = 1 - 0^+ = 1^-$$

Thus $y = f(x)$ has a horizontal asymptote at $y = 1$ as $x \to \infty$, which it approaches from below, but no horizontal asymptote as $x \to \infty$. ■

d. Sketch the graph of $y = f(x)$. *[2]*

Solution.



■

2

3. Do any *two (2)* of **a**–**c**.

a. Compute $\displaystyle\lim_{x \to -\infty} \frac{1-x}{1+2x} 3^x$. *[5]*

Solution.

$$\begin{aligned}
\lim_{x \to -\infty} \frac{1-x}{1+2x} 3^x &= \left(\lim_{x \to -\infty} \frac{1-x}{1+2x} \right) \cdot \left(\lim_{x \to -\infty} 3^x \right) = \left(\lim_{x \to -\infty} \frac{1-x}{1+2x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \right) \cdot 0 \\
&= \left(\lim_{x \to -\infty} \frac{\frac{1}{x} - \frac{x}{x}}{\frac{1}{x} + \frac{2x}{x}} \right) \cdot 0 = \left(\lim_{x \to -\infty} \frac{\frac{1}{x} - 1}{\frac{1}{x} + 2} \right) \cdot 0 = \left(\frac{\lim_{x \to -\infty} \left(\frac{1}{x} - 1\right)}{\lim_{x \to -\infty} \left(\frac{1}{x} + 2\right)} \right) \cdot 0 \\
&= \frac{0-1}{0+2} \cdot 0 = -\frac{1}{2} \cdot 0 = 0 \qquad \blacksquare
\end{aligned}$$

b. Compute $\displaystyle\lim_{t \to 3} \frac{(t-3)(t+2)}{t^2-2t-3}$. *[5]*

Solution. Note that $t^2 - 2t - 3 = (t-3)(t+1)$.

$$\begin{aligned}
\lim_{t \to 3} \frac{(t-3)(t+2)}{t^2-2t-3} &= \lim_{t \to 3} \frac{(t-3)(t+2)}{(t-3)(t+1)} = \lim_{t \to 3} \frac{t+2}{t+1} \\
&= \frac{\lim_{t \to 3}(t+2)}{\lim_{t \to 3}(t+1)} = \frac{3+2}{3+1} = \frac{5}{4} \qquad \blacksquare
\end{aligned}$$

c. Use the limit definition of the derivative to compute $f'(1)$ for $f(x) = 2x^2 - 4x + 8$. *[5]*

Solution.

$$\begin{aligned}
f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\left(2(1+h)^2 - 4(1+h) + 8\right) - \left(2 \cdot 1^2 - 4 \cdot 1 + 8\right)}{h} \\
&= \lim_{h \to 0} \frac{2\left(1 + 2h + h^2\right) - 4 - 4h + 8 - 6}{h} = \lim_{h \to 0} \frac{2 + 4h + 2h^2 - 4h - 2}{h} \\
&= \lim_{h \to 0} \frac{2h^2}{h} = \lim_{h \to 0} 2h = 0 \qquad \blacksquare
\end{aligned}$$

4. Do any *two (2)* of **a**–**c**.

a. Find $h'(x)$ if $h(x) = e^{\left(x^2+3\right)} - \ln(5x)$. *[5]*

Solution. First, note that

$$h(x) = e^{\left(x^2+3\right)} - \ln(5x) = e^{\left(x^2+3\right)} - [\ln(5) + \ln(x)] = e^{\left(x^2+3\right)} - \ln(5) - \ln(x)\,.$$

Second, using the various rules for differentiation:

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(e^{\left(x^2+3\right)} - \ln(5) - \ln(x)\right) = \left(\frac{d}{dx}e^{\left(x^2+3\right)}\right) - \left(\frac{d}{dx}\ln(5)\right) - \left(\frac{d}{dx}\ln(x)\right) \\ &= e^{\left(x^2+3\right)} \cdot \left[\frac{d}{dx}\left(x^2+3\right)\right] - 0 - \frac{1}{x} = e^{\left(x^2+3\right)} \cdot [2x+0] - \frac{1}{x} = 2xe^{\left(x^2+3\right)} - \frac{1}{x} \qquad \blacksquare \end{aligned}$$

b. Find $g'(w)$ if $g(w) = \dfrac{w\ln(w)}{(w+1)^2}$ *[5]*

Solution. A job for the Quotient Rule if ever there was one!

$$\begin{aligned} g'(w) &= \frac{d}{dw}\left(\frac{w\ln(w)}{(w+1)^2}\right) = \frac{\left(\frac{d}{dw}w\ln(w)\right)\cdot(w+1)^2 - w\ln(w)\cdot\left(\frac{d}{dw}(w+1)^2\right)}{\left((w+1)^2\right)^2} \\ &= \frac{\left(\left[\frac{d}{dw}w\right]\cdot\ln(w) + w\cdot\left[\frac{d}{dw}\ln(w)\right]\right)\cdot(w+1)^2 - w\ln(w)\cdot\left(2(w+1)\cdot\left[\frac{d}{dw}(w+1)\right]\right)}{(w+1)^4} \\ &= \frac{\left(1\cdot\ln(w) + w\cdot\frac{1}{w}\right)\cdot(w+1)^2 - w\ln(w)\cdot(2(w+1)\cdot 1)}{(w+1)^4} \\ &= \frac{(w+1)^3\ln(w) - 2w(w+1)\ln(w)}{(w+1)^4} = \frac{(w+1)^2\ln(w) - 2w\ln(w)}{(w+1)^3} \\ &= \frac{\left((w+1)^2 - 2w\right)\ln(w)}{(w+1)^3} = \frac{\left(w^2+2w+1-2w\right)\ln(w)}{(w+1)^3} = \frac{\left(w^2+1\right)\ln(w)}{(w+1)^3} \qquad \blacksquare \end{aligned}$$

c. Find the equation of the tangent line to $y = \frac{1}{2}x^2 + x - 1$ at $x = 3$. *[5]*

Solution. At $x = 3$, $y = \frac{1}{2}3^2 + 3 - 1 = \frac{9}{2} + 2 = \frac{13}{2}$, so the tangent line passes through the point $\left(3, \frac{13}{2}\right)$. Since

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 + x - 1\right) = 2x + 1\,,$$

the slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{x=3} = 2\cdot 3 + 1 = 7\,.$$

Plugging the point $\left(3, \frac{13}{2}\right)$ into the equation $y = 7x + b$ yields

$$\frac{13}{2} = 7\cdot 3 + b \quad\Longrightarrow\quad b = \frac{13}{2} - 21 = \frac{13}{2} - \frac{42}{2} = -\frac{29}{2}\,.$$

Thus the equation of the tangent line in question is $y = 7x - \frac{29}{2}$. $\blacksquare$

4

Mathematics 1100Y
Calculus I: Calculus of one variable
Summer 2010
Solutions

Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2010

Quiz Solutions

Quiz #1 *Wednesday, 12 May, 2010. [10 minutes]*

1. Suppose the graph of $y = x^2$ is stretched vertically by a factor of 3, and then shifted by 2 units to the right and 1 unit down. Find the formula of the parabola with this curve as its graph. *[5]*
2. Use the Limit Laws to evaluate $\lim_{x \to 0} \frac{x^2 - 1}{x^2 + 1}$. *[5]*

SOLUTION TO 1. To stretch the graph of $y = x^2$ vertically by a factor of 3, we simply multiply the output by 3 to get $y = 3x^2$. Shifting the graph by 2 units to the right corresponds to replacing x by $x - 2$ to get $y = 3(x-2)^2$. To shift the graph down by 1, we just subtract 1 to get $y = 3(x-2)^2 - 1$. It follows that the formula of the desired parabola is $y = 3(x-2)^2 - 1 = 3x^2 - 12x + 11$. ■

SOLUTION TO 2. Here goes – you should be able to identify the Limit Law(s) used at each step for yourself pretty readily:

$$\lim_{x \to 0} \frac{x^2 - 1}{x^2 + 1} = \frac{\lim_{x \to 0} \left(x^2 - 1\right)}{\lim_{x \to 0} \left(x^2 + 1\right)} = \frac{\left(\lim_{x \to 0} x^2\right) - \left(\lim_{x \to 0} 1\right)}{\left(\lim_{x \to 0} x^2\right) + \left(\lim_{x \to 0} 1\right)} = \frac{0^2 - 1}{0^2 + 1} = \frac{-1}{1} = -1 \qquad ■$$

Quiz #2 *Monday, 17 May, 2010. [12 minutes]*

Do *one* (1) of the following two questions.

1. Find all the vertical and horizontal asymptotes of $f(x) = \frac{x}{x-1}$ and give a rough sketch of its graph. *[10]*
2. Use the ε–δ definition of limits to verify that $\lim_{x \to 1} (3x - 1) = 2$. *[10]*

SOLUTION TO 1. To find the horizontal asymptotes, we need only compute the limits of $f(x)$ as x tends to $+\infty$ and $-\infty$, respectively, and see what happens:

$$\lim_{x \to +\infty} \frac{x}{x-1} = \lim_{x \to +\infty} \frac{x}{x-1} \cdot \frac{1/x}{1/x} = \lim_{x \to +\infty} \frac{1}{1 - 1/x} = \frac{1}{1-0} = 1^+$$
$$\lim_{x \to -\infty} \frac{x}{x-1} = \lim_{x \to -\infty} \frac{x}{x-1} \cdot \frac{1/x}{1/x} = \lim_{x \to -\infty} \frac{1}{1 - 1/x} = \frac{1}{1+0} = 1^-$$

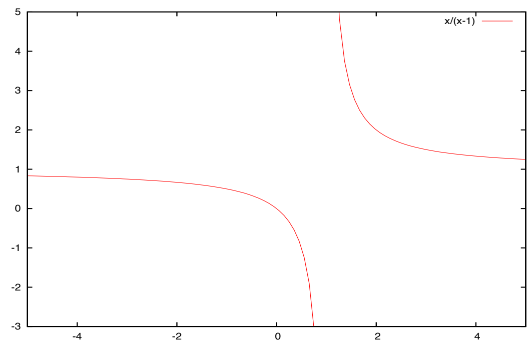
Thus $f(x)$ has $y = 1$ as an asymptote in both directions. Note that it approaches this asymptote from above when $x \to +\infty$ and from below when $x \to -\infty$.

To find any vertical asymptotes, we first need to find the points at which $f(x)$ is undefined. Since the expression $\frac{x}{x-1}$ makes sense for any x unless the denominator is 0, *i.e.* when $x = 1$. To actually check for vertical asymptotes at, we now compute the limits $f(x)$ as x tends to 1 from the left and the right, respectively, and see what happens:

$$\lim_{x \to 1^-} \frac{x}{x-1} = \frac{1}{1^- - 1} = \frac{1}{0^-} = -\infty$$
$$\lim_{x \to 1^+} \frac{x}{x-1} = \frac{1}{1^+ - 1} = \frac{1}{0^+} = +\infty$$

Thus $f(x)$ has a vertical asymptote at $x = 1$; $f(x)$ shoots down to $-\infty$ as x approaches 1 from the left and $f(x)$ shoots up to $+\infty$ as x approaches 1 from the right.

To graph $f(x)$ it's also convenient to note that $f(0) = \frac{0}{0-1} = 0$, so $(0, 0)$ is both and x- and y-intercept. Here's a graph of $f(x) = \dfrac{x}{x-1}$:



This graph was created using `Yacas` ("Yet Another Computer Algebra System"), a free program that can do much of what `Maple` and `Mathematica` can. ■

Solution to 2. We need to show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x - 1| < \delta$, then $|(3x - 1) - 2| < \varepsilon$.

Suppose, then, that an $\varepsilon > 0$ is given. We will find a corresponding $\delta > 0$ by reverse-engineering $|(3x - 1) - 2| < \varepsilon$ to look a much as possible like $|x - 1| < \delta$:

$$\begin{aligned} |(3x-1)-2| < \varepsilon &\iff |3x-3| < \varepsilon \\ &\iff |3(x-1)| < \varepsilon \\ &\iff 3\,|x-1| < \varepsilon \\ &\iff |x-1| < \frac{\varepsilon}{3} \end{aligned}$$

Since the steps are all reversible, it follows that $\delta = \frac{\varepsilon}{3}$ does the job: if $|x - 1| < \delta$, then $|(3x - 1) - 2| < \varepsilon$.

It follows by the ε–δ definition of limits that $\lim_{x \to 1}(3x - 1) = 2$. ■

2

Quiz #3 *Wednesday, 19 May, 2010. [10 minutes]*

1. Compute the derivative of $f(x) = \dfrac{x^2 - 2x}{x - 1}$. *[5]*
2. Compute the derivative of $g(x) = \arctan(e^x)$. *[5]*

SOLUTION TO 1. Our main tool here is the Quotient Rule:

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\frac{x^2 - 2x}{x - 1}\right) \\
&= \frac{\left[\frac{d}{dx}\left(x^2 - 2x\right)\right] \cdot (x-1) - \left(x^2 - 2x\right) \cdot \left[\frac{d}{dx}(x-1)\right]}{(x-1)^2} \\
&= \frac{(2x-2) \cdot (x-1) - \left(x^2 - 2x\right) \cdot 1}{(x-1)^2} \\
&= \frac{2x^2 - 4x + 2 - x^2 + 2x}{(x-1)^2} \\
&= \frac{x^2 - 2x + 2}{(x-1)^2}
\end{aligned}$$

For those determined to simplify further, one could rewrite $x^2 - 2x + 2$ as $(x-1)^2 + 1$ and take it from there, but one doesn't really gain much by this. ■

SOLUTION TO 2. The main tool for this one is the Chain Rule:

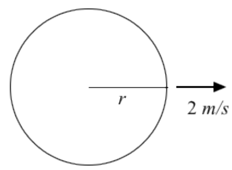
$$\begin{aligned}
g'(x) &= \frac{d}{dx} \arctan(e^x) \\
&= \arctan'(e^x) \cdot \frac{d}{dx} e^x \\
&= \frac{1}{1 + (e^x)^2} \cdot e^x \\
&= \frac{e^x}{1 + e^{2x}}
\end{aligned}$$

Those who really want to could also rewrite this as $\frac{1}{e^{-x} + e^x}$. ■

3

Quiz #4 *Wednesday, 26 May, 2010. [12 minutes]*

1. Use logarithmic differentiation to compute the derivative of $g(x) = x^x$. *[5]*
2. A pebble is dropped into a still pond, creating a circular ripple that moves outward from its centre at 2 m/s. How is the area enclosed by the ripple changing at the instant that the radius of the ripple is 3 m? *[5]*



(*Just in case:* The area of a circle with radius r is πr^2.)

Solution to 1. We will take the derivative of $\ln(g(x))$ and then solve for $g'(x)$. On the one hand,

$$\begin{aligned}\frac{d}{dx}\ln(g(x)) = \frac{d}{dx}\ln(x^x) &= \frac{d}{dx}(x\ln(x)) \qquad \text{[Using properties of logarithms.]}\\ &= \left(\frac{d}{dx}x\right)\cdot\ln(x) + x\cdot\frac{d}{dx}\ln(x) \qquad \text{[Using the Product Rule.]}\\ &= 1\cdot\ln(x) + x\cdot\frac{1}{x} = \ln(x)+1\,,\end{aligned}$$

and on the other hand, it follows from the Chain Rule that

$$\frac{d}{dx}\ln(g(x)) = \frac{1}{g(x)}\cdot g'(x)\,.$$

Hence

$$g'(x) = g(x)\cdot\frac{d}{dx}\ln(g(x)) = x^x\cdot(\ln(x)+1)\ .$$

(Those with a taste for perversity may rearrange this as

$$g'(x) = x^x\cdot(\ln(x)+1) = x^x\ln(x) + x^x = \ln\left(x^{(x^x)}\right) + x^x\,,$$

but that's just a little sickening . . . :-) ■

Solution to 2. The area of a circle of radius r is $A = \pi r^2$; we wish to know $\frac{dA}{dt}$ at the instant in question. Using the Chain Rule,

$$\frac{dA}{dt} = \frac{d}{dt}\pi r^2 = \pi\left(\frac{d}{dr}r^2\right)\cdot\frac{dr}{dt} = \pi\cdot 2r\frac{dr}{dt} = 2\pi r\frac{dr}{dt}\,.$$

Plugging in the given values, that $r = 3$ at the instant we're interested in and that $\frac{dr}{dt} = 2$, we thus get:

$$\frac{dA}{dt} = 2\pi\cdot 3\cdot 2 = 12\pi$$

Thus the area enclosed by the ripple is changing at a rate of 12π m^2/s at the instant in question. ■

4

Quiz #5 *Monday, 31 May, 2010. [15 minutes]*

1. Let $f(x) = \dfrac{x}{x^2+1}$. Find the domain and all the intercepts, vertical and horizontal asymptotes, and local maxima and minima of $f(x)$, and sketch its graph using this information. *[10]*

SOLUTION. Here goes!

i. (*Domain.*) The expression $\frac{x}{x^2+1}$ makes sense for all x. (Note that the denominator is always ≥ 1.) Thus the domain of $f(x)$ is all of $\mathbb{R}$. $\square$

ii. (*Intercepts.*) $f(0) = 0$, so $(0,0)$ is the only x-intercept. Since $\frac{x}{x^2+1}$ can only equal 0 if the numerator, x, is 0, $(0,0)$ is also the only y-intercept. $\square$

iii. (*Vertical asymptotes.*) Since there are no points at which $f(x)$ is not defined and continuous, it has no vertical asymptotes. $\square$

iv. (*Horizontal asymptotes.*) Since

$$\lim_{x\to+\infty} \frac{x}{x^2+1} = \lim_{x\to+\infty} \frac{x}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{1+\frac{1}{x^2}} = \frac{0}{1+0} = 0 \quad \text{and}$$
$$\lim_{x\to-\infty} \frac{x}{x^2+1} = \lim_{x\to-\infty} \frac{x}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{\frac{1}{x}}{1+\frac{1}{x^2}} = \frac{0}{1+0} = 1\,,$$

$f(x)$ has the horizontal asymptote $y = 0$ in both directions. Note that as $x \to +\infty$, $\frac{x}{x^2+1} > 0$, since the numerator and denominator are both positive when $x > 0$. Similarly, as $x \to -\infty$, $\frac{x}{x^2+1} < 0$, since the numerator is negative and the denominator is positive when $x < 0$. It follows that $f(x)$ approaches the horizontal asymptote $y = 0$ from above when heading out to $+\infty$, and from below when heading out to $-\infty$. $\square$

v. (*Local maxima and minima.*) We'll need the derivative of $f(x)$, which we compute using the Quotient Rule:

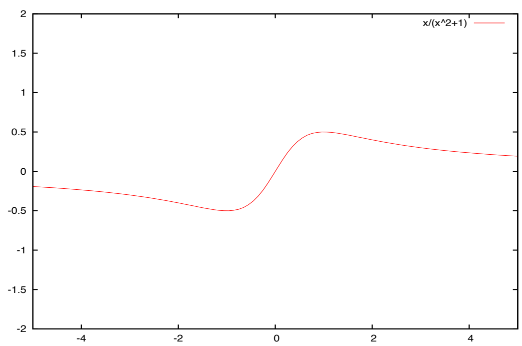
$$f'(x) = \frac{\frac{dx}{dx} \cdot \left(x^2+1\right) - x \cdot \frac{d}{dx}(x^2+1)}{\left(x^2+1\right)^2} = \frac{1 \cdot \left(x^2+1\right) - x \cdot 2x}{\left(x^2+1\right)^2} = \frac{1-x^2}{\left(x^2+1\right)^2}$$

To fond the critical points, observe that $f'(x) = 0$ exactly when $1 - x^2 = 0$. Since $1 - x^2 = (1+x)(1-x)$, this means that $f'(x) = 0$ for $x = -1$ and $x+1$. We construct the usual table to determine if these are local maxima, minima, or neither:

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$f'(x)$	<0	0	>0	0	<0
$f(x)$	$\downarrow$	$-\frac{1}{2}$	$\uparrow$	$\frac{1}{2}$	$\downarrow$

It follows that $f(-1) - -\frac{1}{2}$ is a local minimum and $f(1) = \frac{1}{2}$ is a local maximum of $f(x)$. Note that $f'(x)$, like $f(x)$, is defined and continuous everywhere, so the critical points are all we need to check when looking for local maxima and minima. $\square$

vi. (*Graph.*) Here's a graph of $f(x) = \dfrac{x}{x^2+1}$:



This graph was created using `Yacas` ("Yet Another Computer Algebra System"), a free program that can do much of what `Maple` and `Mathematica` can. □

That's all, folks! ■

Quiz #6 *Wednesday, 2 June, 2010. [10 minutes]*

1. Use the Left-Hand Rule to compute $\displaystyle\int_0^1 (x+1)\,dx$, the area between the line $y = x+1$ and the x-axis for $0 \le x \le 1$. *[10]*

   Hint: You may need the formula $\displaystyle\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$.

SOLUTION. We plug $f(x) = x+1$, $a = 0$, and $b = 1$ into the Left-Hand Rule formula and compute the resulting limit:

$$\begin{aligned}
\int_0^1 (x+1)\,dx &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{b-a}{n} f\left(a + i\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1-0}{n} f\left(0 + i\frac{1-0}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{n} f\left(\frac{i}{n}\right) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{n}\left(\frac{i}{n}+1\right) = \lim_{n\to\infty} \sum_{i=0}^{n-1}\left(\frac{i}{n^2}+\frac{1}{n}\right) \\
&= \left(\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{i}{n^2}\right) + \left(\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{n}\right) = \left(\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=0}^{n-1} i\right) + \left(\lim_{n\to\infty} n\frac{1}{n}\right) \\
&= \left(\lim_{n\to\infty} \frac{1}{n^2} \frac{(n-1)n}{2}\right) + \left(\lim_{n\to\infty} 1\right) = \left(\lim_{n\to\infty} \frac{n-1}{2n}\right) + 1 \\
&= \left(\lim_{n\to\infty} \frac{n-1}{2n} \cdot \frac{1/n}{1/n}\right) + 1 = \left(\lim_{n\to\infty} \frac{1-\frac{1}{n}}{2}\right) + 1 = \frac{1-0}{2} + 1 = \frac{3}{2} \qquad \blacksquare
\end{aligned}$$

6

Quiz #7 *Monday, 7 June, 2010. [10 minutes]*

1. Compute $\displaystyle\int_0^2 \left(x^2 - 2x + 1\right)\, dx$. *[10]*

SOLUTION. Here goes, in entirely excessive detail!

$$\begin{aligned}
\int_0^2 \left(x^2 - 2x + 1\right)\, dx &= \int_0^2 x^2\, dx - \int_0^2 2x\, dx + \int_0^2 1\, dx \quad \text{[Linearity.]} \\
&= \left.\frac{x^3}{3}\right|_0^2 - 2\left.\frac{x^2}{2}\right|_0^2 + \left.x\right|_0^2 \quad \text{[Power Rule.]} \\
&= \left(\frac{2^3}{3} - \frac{0^3}{3}\right) - \left(\frac{2^2}{2} - \frac{0^2}{2}\right) + (2-0) \quad \text{[Putting in the numbers.]} \\
&= \frac{8}{3} - 2 + 2 = \frac{8}{3} \quad \text{[Arithmetic.]} \qquad \blacksquare
\end{aligned}$$

Quiz #8 *Wednesday, 9 June, 2010. [10 minutes]*

1. Find the area between $y = x\cos\left(x^2\right)$ and the x-axis for $-\sqrt{\frac{\pi}{2}} \le x \le \sqrt{\frac{\pi}{2}}$. *[10]*

SOLUTION. Note that the x-axis is the line $y = 0$. Observe that when $-\sqrt{\frac{\pi}{2}} \le x \le 0$, $0 \le x^2 \le \frac{\pi}{2}$, so $\cos\left(x^2\right) \ge 0$ and hence $x\cos\left(x^2\right) \le 0$. Similarly, when $0 \le x \le \sqrt{\frac{\pi}{2}}$, $0 \le x^2 \le \frac{\pi}{2}$, so $\cos\left(x^2\right) \ge 0$ and hence $x\cos\left(x^2\right) \ge 0$. The area we want, therefore, is the sum of two definite integrals, which we evaluate with the help of the Substitution Rule:

$$\text{Area} = \int_{-\sqrt{\pi/2}}^{0} \left[0 - x\cos\left(x^2\right)\right]\, dx + \int_0^{\sqrt{\pi/2}} \left[x\cos\left(x^2\right) - 0\right]\, dx$$

Using $u = x^2$, so $du = 2x\, dx$, and thus $x\, dx = \frac{1}{2}\, du$, and

changing limits, $\begin{array}{cccc} x & -\sqrt{\pi/2} & 0 & \sqrt{\pi/2} \\ u & \pi/2 & 0 & \pi/2 \end{array}$ we get:

$$= \int_{\pi/2}^{0} -\frac{1}{2}\cos(u)\, du + \int_0^{\pi/2} \frac{1}{2}\cos(u)\, du$$

Since $\displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$, we now get:

$$= \int_0^{\pi/2} \frac{1}{2}\cos(u)\, du + \int_0^{\pi/2} \frac{1}{2}\cos(u)\, du$$

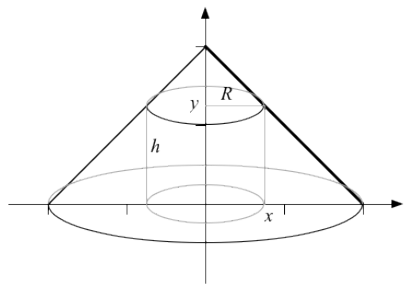
$$= \int_0^{\pi/2} \cos(u)\, du = \left.\sin(u)\right|_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1 \qquad \blacksquare$$

7

Quiz #9 *Monday, 14 June, 2010. [10 minutes]*

The region between $y = 2 - x$ and the x-axis, for $0 \leq x \leq 2$, is rotated about the y-axis. Find the volume of the resulting solid of revolution using both

1. the disk method *[5]* and
2. the method of cylindrical shells. *[5]*



Solution to 1. Note that if x is between 0 and 2, $y = 2 - x$ is also between 0 and 2. The disk at height $y = 2 - x$ would have radius $R = x = 2 - y$. Thus the volume of the solid of revolution in this case is:

$$\begin{aligned}
\int_0^2 \pi R^2\,dy &= \int_0^2 \pi x^2\,dy = \int_0^2 (2-y)^2\,dy \\
&= \int_0^2 \left(4 - 4y + y^2\right)\,dy = \pi\left(4y - 4\frac{y^2}{2} + \frac{y^3}{3}\right)\Bigg|_0^2 \\
&= \pi\left(4\cdot 2 - 4\frac{4}{2} + \frac{8}{3}\right) - \pi\left(4\cdot 0 - 4\frac{0}{2} + \frac{0}{3}\right) \\
&= \pi\cdot\frac{8}{3} - \pi\cdot 0 = \frac{8}{3}\pi \qquad \blacksquare
\end{aligned}$$

Solution to 2. The cylindrical shell at x would have radius $R = x$ and height $h = y = 2 - x$. Thus the volume of the solid of revolution in this case is:

$$\begin{aligned}
\int_0^2 2\pi Rh\,dx &= \int_0^2 2\pi xy\,dx = \int_0^2 2\pi x(2-x)\,dx \\
&= \int_0^2 2\pi\left(2x - x^2\right)\,dx = 2\pi\left(2\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^2 \\
&= 2\pi\left(2\frac{4}{2} - \frac{8}{3}\right) - 2\pi\left(2\frac{0}{2} - \frac{0}{3}\right) \\
&= 2\pi\cdot\frac{4}{3} - 2\pi\cdot 0 = \frac{8}{3}\pi \qquad \blacksquare
\end{aligned}$$

8

Quiz #10 *Wednesday, 16 June, 2010. [10 minutes]*

1. Compute $\int_1^e (\ln(x))^2\,dx$. *[10]*

SOLUTION. We will use integration by parts with $u = (\ln(x))^2$ and $v' = 1$, so $u' = 2\ln(x)\cdot\frac{1}{x}$ and $v = x$. Then:

$$\begin{aligned}
\int_1^e (\ln(x))^2\,dx &= \int_1^e u\cdot v'\,dx = u\cdot v|_1^e - \int_1^e v\cdot u'\,dx \\
&= (\ln(x))^2\cdot x\Big|_1^e - \int_1^e x\cdot 2\ln(x)\cdot\frac{1}{x}\,dx \\
&= \left[(\ln(e))^2\cdot e - (\ln(1))^2\cdot 1\right] - 2\int_1^e \ln(x)\,dx
\end{aligned}$$

To solve the remaining integral we use parts again, with $s = \ln(x)$ and $t' = 1$, so $s' = \frac{1}{x}$ and $t = x$.

$$\begin{aligned}
&= \left[1^2\cdot e - 0^2\cdot 1\right] - 2\left[\ln(x)\cdot x|_1^e - \int_1^e x\cdot\frac{1}{x}\,dx\right] \\
&= e - 2\left[(\ln(e)\cdot e - \ln(1)\cdot 1) - \int_1^e 1\,dx\right] \\
&= e - 2\left[(1\cdot e - 0\cdot 1) - x|_1^e\right] \\
&= e - 2\left[e - (e-1)\right] \\
&= e - 2 \qquad \blacksquare
\end{aligned}$$

Quiz #11 *Monday, 21 June, 2010. [12 minutes]*

Compute each of the following integrals:

1. $\int_0^{\pi/2} \cos^3(x)\sin^2(x)\,dx$ *[5]* 2. $\int \sec^3(x)\,dx$ *[5]*

SOLUTION TO 1.

$$\begin{aligned}
\int_0^{\pi/2} \cos^3(x)\sin^2(x)\,dx &= \int_0^{\pi/2} \cos^2(x)\sin^2(x)\cos(x)\,dx \\
&= \int_0^{\pi/2} \left(1-\sin^2(x)\right)\sin^2(x)\cos(x)\,dx \\
&= \int_0^{\pi/2} \left(\sin^2(x)-\sin^4(x)\right)\cos(x)\,dx = \int_0^1 \left(u^2-u^4\right)\,du
\end{aligned}$$

Using the substitution $u = \sin(x)$, so $du = \cos(x)\,dx$, and $\begin{array}{ccc} x & 0 & \pi/2 \\ u & 0 & 1 \end{array}$.

$$\begin{aligned}
&= \left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right)\Big|_0^1 = \left(\frac{1}{3}1^3 - \frac{1}{5}1^5\right) - \left(\frac{1}{3}0^3 - \frac{1}{5}0^5\right) \\
&= \frac{1}{3} - \frac{1}{5} = \frac{2}{15} \qquad \blacksquare
\end{aligned}$$

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Solution i to 2. We'll use the reduction formula for $\int \sec^n(x)\,dx$,

$$\begin{aligned}\int \sec^3(x)\,dx &= \frac{1}{3-1}\sec^{3-2}(x)\tan(x) + \frac{3-2}{3-1}\int \sec^{3-2}(x)\,dx \\ &= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx \\ &= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln\left(\sec(x)+\tan x\right) + C\,,\end{aligned}$$

as well as having memorized a certain notoriously nasty anti-derivative. ■

Solution ii to 2. We'll use integration by parts, with

$$\begin{array}{ll} u = \sec(x) & v' = \sec^2(x) \\ u' = \sec(x)\tan(x) & v = \tan(x) \end{array}$$

rather than apply the reduction formula, and also do a certain notoriously nasty anti-derivative from scratch.

$$\begin{aligned}\int \sec^3(x)\,dx &= \sec(x)\tan(x) - \int \tan(x)\sec(x)\tan(x)\,dx \\ &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx \\ &= \sec(x)\tan(x) - \int \sec(x)\left(\sec^2(x)-1\right)\,dx \\ &= \sec(x)\tan(x) - \int \left(\sec^3(x)-\sec(x)\right)\,dx \\ &= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx\end{aligned}$$

Moving all copies of $\int \sec^3(x)\,dx$ to the left in this equation gives

$$2\int \sec^3(x)\,dx = \sec(x)\tan(x) + \int \sec(x)\,dx\,,$$

so

$$\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx\,.$$

We still need to compute $\int \sec(x)\,dx$:

$$\int \sec(x)\,dx = \int \sec(x)\cdot\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx$$

We now use the substitution $u = \sec(x)+\tan(x)$, so $du = \left(\sec(x)\tan(x)+\sec^2(x)\right)\,dx$.

$$= \int \frac{1}{u}\,du = \ln(u) + C = \ln\left(\sec(x)+\tan(x)\right) + C$$

Hence

$$\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln\left(\sec(x)+\tan(x)\right) + C\,.$$

Whew! ■

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Quiz #12 *Wednesday, 23 June, 2010. [15 minutes]*

Compute each of the following integrals:

1. $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$ *[5]* 2. $\displaystyle\int_1^2 x\sqrt{x^2-1}\,dx$ *[5]*

SOLUTION TO 1. Since we see see an expression of the form $\frac{1}{\sqrt{4-x^2}}$, we will use the trig substitution $x = 2\sin(\theta)$, so $dx = 2\cos(\theta)\,d\theta$ and $\theta = \arcsin\left(\frac{x}{2}\right)$.

$$\begin{aligned}\int \frac{1}{\sqrt{4-x^2}}\,dx &= \int \frac{1}{\sqrt{4-4\sin^2(\theta)}} 2\cos(\theta)\,d\theta = \int \frac{2\cos(\theta)}{\sqrt{4\cos^2(\theta)}}\,d\theta \\ &= \int \frac{2\cos(\theta)}{2\cos(\theta)}\,d\theta = \int 1\,d\theta = \theta + C = \arcsin\left(\frac{x}{2}\right) + C \qquad \blacksquare\end{aligned}$$

SOLUTION I TO 2. *[The hard way.]* Since we see an expression of the form $\sqrt{x^2-1}$, we will use the trig substitution $x = \sec(\theta)$, so $dx = \sec(\theta)\tan(\theta)\,d\theta$ and $\begin{matrix} x & 1 & 2 \\ \theta & 0 & \frac{\pi}{3}\end{matrix}$. (Recall that $\cos(0) = 1$, so $\sec(0) = 1$, and $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, so $\sec\left(\frac{\pi}{3}\right) = 2$.)

$$\begin{aligned}\int_1^2 x\sqrt{x^2-1}\,dx &= \int_0^{\pi/3} \sec(\theta)\sqrt{\sec^2(\theta)-1}\,\sec(\theta)\tan(\theta)\,d\theta \\ &= \int_0^{\pi/3} \sec(\theta)\sqrt{\tan^2(\theta)}\,\sec(\theta)\tan(\theta)\,d\theta \\ &= \int_0^{\pi/3} \sec(\theta)\tan(\theta)\sec(\theta)\tan(\theta)\,d\theta \\ &= \int_0^{\pi/3} \tan^2(\theta)\sec^2(\theta)\,d\theta = \int_0^{\sqrt{3}} u^2\,du\end{aligned}$$

Using the substitution $u = \tan(\theta)$,

so $du = \sec^2(\theta)\,d\theta$ and $\begin{matrix} \theta & 0 & \pi/3 \\ u & 0 & \sqrt{3}\end{matrix}$.

$$= \frac{1}{3}u^3\Big|_0^{\sqrt{3}} = \frac{1}{3}\left(\sqrt{3}\right)^3 - \frac{1}{3}0^3 = \frac{1}{3}3\sqrt{3} - 0 = \sqrt{3} \qquad \blacksquare$$

SOLUTION II TO 2. *[The easier way.]* Note that the derivative of $w = x^2 - 1$ is $\frac{dw}{dx} = 2x$ and that we have an x outside the square root. We will therefore use the substitution $w = x^2 - 1$, so $dw = 2x\,dx$ and $\begin{matrix} x & 1 & 2 \\ w & 0 & 3\end{matrix}$. Note that $\frac{1}{2}dw = x\,dx$.

$$\begin{aligned}\int_1^2 x\sqrt{x^2-1}\,dx &= \int_0^3 \sqrt{w}\,\frac{1}{2}dw = \frac{1}{2}\int_0^3 w^{1/2}\,dw = \frac{1}{2}\cdot\frac{w^{3/2}}{3/2}\Big|_0^3 \\ &= \frac{1}{3}w^{3/2}\Big|_0^3 = \frac{1}{3}\left(\sqrt{3}\right)^3 - \frac{1}{3}0^3 = \frac{1}{3}3\sqrt{3} - 0 = \sqrt{3} \qquad \blacksquare\end{aligned}$$

Quiz #13 *Monday, 28 June, 2010. [12 minutes]*

1. Compute $\displaystyle\int \frac{2x^2+3}{(x^2+4)(x-1)}\,dx$

Solution. This is a job for partial fractions. Note that the polynomial in the numerator has lower degree than the polynomial in the denominator, and that the latter is already factored into an irreducible quadratic (as $x^2+4>0$ for all x) and a linear term.

First, we rewrite the rational function using partial fractions:

$$\begin{aligned}\frac{2x^2+3}{(x^2+4)(x-1)} &= \frac{Ax+B}{x^2+4}+\frac{C}{x-1}\\ &= \frac{(Ax+B)(x-1)+C\left(x^2+4\right)}{(x^2+4)(x-1)}\\ &= \frac{Ax^2-Ax+Bx-B+Cx^2+4C}{(x^2+4)(x-1)}\\ &= \frac{(A+C)x^2+(B-A)x+(4C-B)}{(x^2+4)(x-1)}\end{aligned}$$

Comparing the coefficients in the numerators at the beginning and the end gives us a system of linear equations,

$$\begin{array}{rrrrrrrrr} & A & & & + & C & = & 2 \\ - & A & + & B & & & = & 0 \\ & & - & B & + & 4C & = & 3\,, \end{array}$$

which we solve. From the second equation, we know that $A=B$. Substituting this in gives us a system of two equations:

$$\begin{array}{rrrrrr} & A & + & C & = & 2 \\ - & A & + & 4C & = & 3 \end{array}$$

If we add these two, A disappears and we are left with $5C=5$, so $C=1$. Substituting this into $A+C=2$ gives us $A=1$, and since $A=B$, it follows also that $B=1$. Thus

$$\frac{2x^2+3}{(x^2+4)(x-1)} = \frac{x+1}{x^2+4}+\frac{1}{x-1}\,.$$

Second, we split the integral up accordingly,

$$\begin{aligned}\frac{2x^2+3}{(x^2+4)(x-1)}\,dx &= \int \frac{x+1}{x^2+4}\,dx + \int \frac{1}{x-1}\,dx\\ &= \int \frac{x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx + \int \frac{1}{x-1}\,dx\,,\end{aligned}$$

and work on the pieces.

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For $\int \frac{x}{x^2+4}\,dx$ we use the substitution $u = x^2+4$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. Then

$$\int \frac{x}{x^2+4}\,dx = \int \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\ln(u) + K = \frac{1}{2}\ln\left(x^2+4\right) + K\,,$$

where K is the generic constant.

For $\int \frac{1}{x^2+4}\,dx$ we use the trig substitution $x = 2\tan(\theta)$, so $dx = 2\sec^2(\theta)\,d\theta$ and $\theta = \arctan\left(\frac{x}{2}\right)$. Then, using the identity $1+\tan^2(\theta) = \sec^2(\theta)$ at the key step,

$$\begin{aligned}\int \frac{1}{x^2+4}\,dx &= \int \frac{1}{4\tan^2(\theta)+4}\cdot 2\sec^2(\theta)\,d\theta = \frac{2}{4}\int \frac{\sec^2(\theta)}{\sec^2(\theta)}\,d\theta \\ &= \frac{1}{2}\int 1\,d\theta = \frac{1}{2}\theta + L = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + L\,,\end{aligned}$$

where L is the generic constant.

For $\int \frac{1}{x-1}\,dx$ we use the subsitution $w = x-1$, so $dw = dx$. Then

$$\int \frac{1}{x-1}\,dx = \int \frac{1}{w}\,dw = \ln(w) + M = \ln(x-1) + M\,,$$

where M is the generic constant.

It follows that

$$\begin{aligned}\int \frac{2x^2+3}{(x^2+4)(x-1)}\,dx &= \int \frac{x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx + \int \frac{1}{x-1}\,dx \\ &= \frac{1}{2}\ln\left(x^2+4\right) + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + \ln(x-1) + N\,,\end{aligned}$$

where $N = K+L+M$ is the combined generic constant. ■

Quiz #14 *Wednesday, 30 June, 2010. [10 minutes]*

1. Compute $\displaystyle\int_0^\infty \frac{1}{x^2+1}\,dx$. *[10]*

Solution. This is obviously an improper integral, so we need to take a limit:

$$\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{x^2+1}\,dx$$

Using the trig substitution $x = \tan(\theta)$, so that $dx = \sec^2(\theta)\,d\theta$, and keeping the limits for x gives:

$$= \lim_{t\to\infty}\int_{x=0}^{x=t} \frac{1}{\tan^2(\theta)+1}\sec^2(\theta)\,d\theta$$

Using the identity $\tan^2(\theta)+1 = \sec^2(\theta)$ gives:

$$= \lim_{t\to\infty}\int_{x=0}^{x=t} \frac{1}{\sec^2(\theta)}\sec^2(\theta)\,d\theta$$

$$= \lim_{t\to\infty}\int_{x=0}^{x=t} 1\,d\theta = \lim_{t\to\infty} \theta\Big|_{x=0}^{x=t}$$

Substituting back:

$$= \lim_{t\to\infty} \arctan(x)\Big|_{x=0}^{x=t}$$

$$= \lim_{t\to\infty} \left(\arctan(t) - \arctan(0)\right)$$

Since $\tan(0) = 0$ we also have $\arctan(0) = 0$.

$$= \lim_{t\to\infty} \arctan(t) = \frac{\pi}{2}$$

Since $\tan(\theta)$ has a vertical asymptote at $\frac{\pi}{2}$. ■

Quiz #15 *Monday, 5 July, 2010. [10 minutes]*

1. Compute the arc-length of the curve $y = \frac{2}{3}x^{3/2}$, where $0 \le x \le 1$.

Solution. First, $\frac{dy}{dx} = \frac{2}{3}\cdot\frac{3}{2}x^{1/2} = x^{1/2}$. Plugging this into the arc-length formula gives:

$$\text{Arc-length} = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+\left(x^{1/2}\right)^2}\,dx = \int_0^1 \sqrt{1+x}\,dx$$

Substitute $u = x+1$, so $du = dx$ and $\begin{matrix} x & 0 & 1 \\ u & 1 & 2 \end{matrix}$.

$$= \int_1^2 \sqrt{u}\,du = \int_1^2 u^{1/2}\,du = \left.\frac{u^{3/2}}{3/2}\right|_1^2 = \frac{2}{3}2^{3/2} - \frac{2}{3}1^{3/2}$$

$$= \frac{2}{3}2\sqrt{2} - \frac{2}{3}1 = \frac{2}{3}\left(2\sqrt{2}-1\right) \qquad ■$$

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Quiz #16 *Wednesday, 7 July, 2010. [15 minutes]*

1. Find the arc-length of the parametric curve $x = t\cos(t)$ and $y = t\sin(t)$, where $0 \le t \le 1$. *[10]*

SOLUTION. First, using the Chain Rule, we compute

$$\frac{dx}{dt} = 1 \cdot \cos(t) + t \cdot (-\sin(t)) = \cos(t) - t\sin(t) \quad \text{and}$$
$$\frac{dy}{dt} = 1 \cdot \sin(t) + t \cdot \sin(t) = \sin(t) + t\cos(t)\,.$$

Second, we plug this into the arc-length formula for parametric curves:

$$\begin{aligned}
\text{Arc-length} &= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_0^1 \sqrt{(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2}\,dt \\
&= \int_0^1 \sqrt{\begin{array}{l}\cos^2(t) - 2t\cos(t)\sin(t) + t^2\sin^2(t) \\ \quad + \sin^2(t) + 2t\cos(t)\sin(t) + t^2\cos^2(t)\end{array}}\,dt \\
&= \int_0^1 \sqrt{\left(\cos^2(t) + \sin^2(t)\right)(1+t^2)}\,dt = \int_0^1 \sqrt{1+t^2}\,dt \\
&\qquad \text{Substitute } t = \tan(\theta)\text{, so } dt = \sec^2(\theta)\,d\theta \\
&\qquad \text{and } \begin{array}{ccc} t & 0 & 1 \\ \theta & 0 & \pi/4 \end{array}. \\
&= \int_0^{\pi/4} \sqrt{1+\tan^2(\theta)}\,\sec^2(\theta)\,d\theta \\
&= \int_0^{\pi/4} \sqrt{\sec^2(\theta)}\,\sec^2(\theta)\,d\theta = \int_0^{\pi/4} \sec^3(\theta)\,d\theta \\
&\qquad \text{This can be done by parts, or looking it up,} \\
&\qquad \text{or even doing Quiz \#11 over again. :-)} \\
&= \left[\frac{1}{2}\tan(\theta)\sec(\theta) - \frac{1}{2}\ln\left(\tan(\theta) + \sec(\theta)\right)\right]\Bigg|_0^{\pi/4} \\
&\qquad \text{Recall that } \tan(\pi/4) = 1 \text{ and } \sec(\pi/4) = \sqrt{2}, \\
&\qquad \text{while } \tan(0) = 0 \text{ and } \sec(0) = 1\text{, and } \ln(1) = 0. \\
&= \left[\frac{1}{2}1\sqrt{2} - \frac{1}{2}\ln\left(1+\sqrt{2}\right)\right] - \left[\frac{1}{2}0 \cdot 1 - \frac{1}{2}\ln(0+1)\right] \\
&= \frac{1}{2}\sqrt{2} - \frac{1}{2}\ln\left(1+\sqrt{2}\right) \qquad \blacksquare
\end{aligned}$$

Quiz #17 *Monday, 12 July, 2010. [15 minutes]*

1. Sketch the curve given by $r = \sin(\theta)$, $0 \le \theta \le \pi$, in polar coordinates. *[2]*
2. Sketch the curve given by $r = \sin(\theta)$, $\pi \le \theta \le 2\pi$, in polar coordinates. *[2]*
3. Find the area of the region enclosed by the curve given by $r = \sin(\theta)$, $0 \le \theta \le \pi$, in polar coordinates. *[6]*

Bonus: Find an equation in Cartesian coordinates for the curve given by $r = \sin(\theta)$, $0 \le \theta \le \pi$, in polar coordinates. *[2]*

Solution to 1. Note that as θ changes from 0 to π, r increases from 0 to 1 (at $\theta = \pi/2$), and then decreases to 0 again. See Figure 1 below. ■

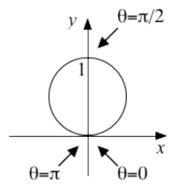

Figure 1

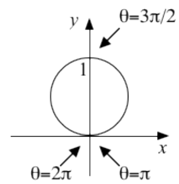

Figure 2

Solution to 2. Note that as θ changes from π to 2π, r decreases from 0 to -1 (at $\theta = \pi/2$), and then increases to 0 again. Recall that a negative r is interpreted as being $|r|$ units from the origin in the opposite direction. See Figure 2 above. ■

Solution to 3. We plug $r = \sin(\theta)$ into the area formula for polar coordinates and chug away:

$$\begin{aligned}\int_0^\pi \frac{1}{2} r^2 \, d\theta &= \frac{1}{2}\int_0^\pi \sin^2(\theta)\, d\theta = \frac{1}{2}\int_0^\pi \frac{1-\cos(2\theta)}{2}\, d\theta = \frac{1}{4}\int_0^\pi (1-\cos(2\theta))\, d\theta \\ &= \frac{1}{4}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Bigg|_0^\pi = \frac{1}{4}\left(\pi - \frac{1}{2}\sin(2\pi)\right) - \frac{1}{4}\left(0 - \frac{1}{2}\sin(2\cdot 0)\right) \\ &= \frac{1}{4}(\pi - 0) - \frac{1}{4}(0-0) = \frac{\pi}{4} \qquad \blacksquare\end{aligned}$$

Solution to the Bonus. Observe that for this curve $x = r\cos(\theta) = \sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$ and $y = r\sin(\theta) = \sin(\theta)\sin(\theta) = \sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$, so

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\sin(2\theta)\right)^2 + \left(\frac{1}{2} - \frac{1}{2}\cos(2\theta) - \frac{1}{2}\right)^2 = \frac{1}{4}\sin^2(2\theta) + \frac{1}{4}\cos^2(2\theta) = \frac{1}{4}\,.$$

That is, the curve is a circle of radius $\frac{1}{2}$ and centre $\left(0, \frac{1}{2}\right)$, which is compatible with the sketch in Figure 1 above. ■

Quiz #18 *Wednesday, 14 July, 2010. [12 minutes]*

1. Use the definition of convergence of a series to compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. *[10]*

Hint: Note that $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$.

Solution. By definition, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to a number L if the partial sums $S_n = \displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)}$ have a limit of L as $n \to \infty$. We check to see what happens when we take the limit of the partial sums:

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i(i+1)} \\
&= \lim_{n\to\infty} \left(\frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \frac{1}{3(3+1)} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)} \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)} \right) \\
&\qquad \text{(Now the hint comes in at last!)} \\
&= \lim_{n\to\infty} \left(\left[\frac{1}{1} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \cdots + \left[\frac{1}{n-1} - \frac{1}{n}\right] + \left[\frac{1}{n} - \frac{1}{n+1}\right] \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{1} + \left[-\frac{1}{2} + \frac{1}{2}\right] + \left[-\frac{1}{3} + \cdots + \frac{1}{n-1}\right] + \left[-\frac{1}{n} + \frac{1}{n}\right] - \frac{1}{n+1} \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{1} - \frac{1}{n+1} \right) = 1 - 0 = 1 \qquad \text{since } \frac{1}{n+1} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

It follows that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and $= 1$. ■

Quiz #19 *Monday, 19 July, 2010. [10 minutes]*

1. Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges or diverges. *[10]*

Solution. Observe that $\dfrac{1}{n^2+1}$ is a ratio of polynomials in n with the degree of the denominator being $2 = 2 - 0$ more than the degree of the numerator. Since $2 > 1$, it follows by the Generalized p-Test that the series converges. ■

Note: This series can also be shown to converge by using the Integral Test or the Comparison Test (or one of its several variants and extensions).

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Quiz #20 *Wednesday, 14 July, 2010. [12 minutes]*

1. Determine whether the series $\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{1+n}$ converges conditionally, converges absolutely, or diverges. *[10]*

SOLUTION. This series converges conditionally. To see that it does converge, we apply the Alternating Series Test.

Zeroth, observe that $\cos(0) = 1$, $\cos(\pi) = -1$, $\cos(2\pi) = 1$, $\cos(3\pi) = -1$, and so on. In general, $\cos(n\pi) = (-1)^n$.

First, note that the series survives the Divergence Test:

$$\lim_{n\to\infty} \left| \frac{\cos(n\pi)}{1+n} \right| = \lim_{n\to\infty} \frac{|(-1)^n|}{1+n} = \lim_{n\to\infty} \frac{1}{1+n} = 0\,,$$

since $1 + n \to \infty$ as $n \to \infty$.

Second, since $n + 2 > n + 1$ for $n \geq 0$,

$$\left| \frac{\cos(n\pi)}{1+n} \right| = \frac{1}{n+1} > \frac{1}{n+2} = \left| \frac{\cos((n+1)\pi))}{1+n} \right|,$$

so the terms of the series are decreasing in absolute value.

Third, since $\cos(n\pi) = (-1)^n$ and since $\frac{1}{1+n} > 0$ when $n \geq 0$, it follows that $\frac{\cos(n\pi)}{1+n}$ alternates sign as n increases, *i.e.* $\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{1+n}$ is an alternating series.

It follows that $\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{1+n}$ converges by the Alternating Series Test. To see that it does not converge absolutely, consider the corresponding series of absolute values, $\sum_{n=0}^{\infty} \left| \frac{\cos(n\pi)}{1+n} \right| = \sum_{n=0}^{\infty} \frac{1}{1+n}$. Since $\frac{1}{1+n}$ is a ratio of polynomials in n with the degree of the denominator being $1 = 1 - 0$ more than the degree of the numerator. Since $1 \leq 1$, it follows by the Generalized p-Test that the series of absolute values diverges.

Thus $\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{1+n}$ converges conditionally. ■

Quiz #21 *Monday, 26 July, 2010. [15 minutes]*

1. Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{n3^n}{2^{n+1}} x^n$. *[10]*

SOLUTION. We will use the Ratio Test to find the radius of convergence:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{(n+1)3^{n+1}}{2^{n+2}} x^{n+1}}{\frac{n3^n}{2^{n+1}} x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)3^{n+1}2^{n+1}x^{n+1}}{n3^n2^{n+2}x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{3(n+1)x}{2n} \right| = \frac{3|x|}{2} \lim_{n\to\infty} \frac{n+1}{n} \\
&= \frac{3|x|}{2} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = \frac{3|x|}{2}(1+0) = \frac{3|x|}{2}
\end{aligned}$$

It follows by the Ratio Test that $\displaystyle\sum_{n=0}^{\infty} \frac{n3^n}{2^{n+1}} x^n$ converges absolutely when $\dfrac{3|x|}{2} < 1$, *i.e.* when $|x| < \dfrac{2}{3}$, and diverges when $\dfrac{3|x|}{2} > 1$, *i.e.* when $|x| > \dfrac{2}{3}$. Thus the radius of convergence of the series is $R = \dfrac{2}{3}$.

As $R < \infty$, we need to determine whether the series converges at $x = \pm R = \pm\frac{2}{3}$ to find the interval of convergence. That is, we need to determine whether the series

$$\sum_{n=0}^{\infty} \frac{n3^n}{2^{n+1}} \left(-\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{n}{2} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{n3^n}{2^{n+1}} \left(\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} \frac{n}{2}$$

converge or diverge. Since

$$\left.\begin{matrix} \lim_{n\to\infty} \left|(-1)^n \frac{n}{2}\right| \\ \lim_{n\to\infty} \frac{n}{2} \end{matrix}\right\} = \frac{1}{2} \lim_{n\to\infty} n = \infty \neq 0\,,$$

both of these series diverge by the Divergence Test. Thus the interval of convergence of the given series is $\left(-\frac{2}{3}, \frac{2}{3}\right)$. ■

Quiz #22 *Wednesday, 28 July, 2010. [15 minutes]*

1. Find the Taylor series of $f(x) = \ln(x)$ at $a = 1$. *[10]*

SOLUTION I. *(Using Taylor's formula.)* We take the successive derivatives of $f(x) = \ln(x)$ and evaluate them at $a = 1$:

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln(x)$	0
1	x^{-1}	1
2	$-x^{-2}$	-1
3	$2x^{-3}$	2
4	$-6x^{-4}$	-6
5	$24x^{-5}$	24
$\vdots$	$\vdots$	$\vdots$

In general, when $n \geq 1$, we have $f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$ and so $f^{(n)}(1) = (-1)^{n-1}(n-1)!$. Note that $f^{(0)}(1) = \ln(1) = 0$.

It follows that the Taylor series of $f(x) = \ln(x)$ at $a = 1$ is

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots . \quad \blacksquare\end{aligned}$$

SOLUTION II. *(Using underhanded cunning.)* Observe that $\frac{d}{dx}\ln(x) = \frac{1}{x} = \frac{1}{1-(1-x)}$. Using the formula for the sum of a geometric series, we get:

$$\begin{aligned}\frac{1}{x} = \frac{1}{1-(1-x)} &= 1 + (1-x) + (1-x)^2 + (1-x)^3 + \cdots \\ &= 1 - (x-1) + (x-1)^2 - (x-3)^3 + \cdots = \sum_{k=0}^{\infty}(-1)^k(x-1)^k\end{aligned}$$

By the uniqueness of Taylor series, this must be the Taylor series at $a = 1$ of $\frac{1}{x} = \frac{d}{dx}\ln(x)$. Integrating this series term-by-term gives the Taylor series at $a = 0$ of $\ln(x)$, at least up to a constant:

$$\begin{aligned}\sum_{k=0}^{\infty} \int (-1)^k(x-1)^k \, dx &= C + \sum_{k=0}^{\infty} \frac{(-1)^k(x-1)^{k+1}}{k+1} \\ &= C + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} - \cdots\end{aligned}$$

Since this series should equal $\ln(1) = 0$ when $x = 1$, we must have $C = 0$.

Thus the Taylor series of $f(x) = \ln(x)$ at $a = 1$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$. (Here we've changed indices, with $n = k+1$, to make it look like the previous solution.) $\blacksquare$

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Mathematics 1100Y – Calculus I: Calculus of one variable
Trent University, Summer 2010
Solutions to Test 1

1. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

a. Find the slope of the tangent line to $y = \tan(x)$ at $x = 0$.

Solution. The slope of the tangent line at a given point is given by evaluating the derivative at the given point. In this case, $\frac{dy}{dx} = \frac{d}{dx}\tan(x) = \sec^2(x)$. At $x = 0$ this gives $\sec^2(0) = \frac{1}{\cos^2(0)} = \frac{1}{1} = 1$, so the tangent line to $y = \tan(x)$ at $x = 0$ has slope 1. ■

b. Use the limit definition of the derivative to compute $f'(1)$ for $f(x) = x^2$.

Solution. Here goes:

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{(1+h)^2 - 1^1}{h} \\ &= \lim_{h\to 0} \frac{1^1 + 2\cdot 1\cdot h + h^2 - 1^2}{h} \\ &= \lim_{h\to 0} \frac{2h + h^2}{h} \\ &= \lim_{h\to 0}(2+h) = 2 + 0 = 2 \qquad \blacksquare \end{aligned}$$

c. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x\to 1}(2x - 1) = 1$.

Solution. We need to show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x - 1| < \delta$, then $|(2x - 1) - 1| < \varepsilon$. Given a $\varepsilon > 0$, we reverse-engineer the $\delta > 0$ we need:

$$\begin{aligned} |(2x-1) - 1| < \varepsilon &\iff |2x - 2| < \varepsilon \\ &\iff |2(x-1)| < \varepsilon \\ &\iff 2\,|x - 1| < \varepsilon \\ &\iff |x - 1| < \frac{\varepsilon}{2} \end{aligned}$$

Since each step above is reversible, it follows that that if $\delta = \frac{\varepsilon}{2}$, then $|(2x - 1) - 1| < \varepsilon$ whenever $|x - 1| < \delta = \frac{\varepsilon}{2}$. Thus $\lim_{x\to 1}(2x - 1) = 1$ by the $\varepsilon - \delta$ definition of limits. ■

2. Find $\frac{dy}{dx}$ in any *three* (3) of **a**–**d**. *[9 = 3 × 3 each]*

a. $y = \dfrac{x}{x+1}$

SOLUTION. Apply the Quotient Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{\left[\frac{d}{dx}x\right](x+1) - x\left[\frac{d}{dx}(x+1)\right]}{(x+)^2} = \frac{1(x+1) - x1}{(x+)^2} = \frac{1}{(x+)^2} \quad \blacksquare$$

b. $x^2 + y^2 = 4$

SOLUTION I. Use implicit differentiation and the Chain Rule:

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}4 \Longrightarrow \frac{d}{dx}x^2 + \frac{d}{dx}t^2 = 0 \Longrightarrow 2x + \left(\frac{d}{dy}y^2\right)\frac{dy}{dx} = 0$$
$$\Longrightarrow 2x + 2y\frac{dy}{dx} = 0 \Longrightarrow 2y\frac{dy}{dx} = -2x \Longrightarrow \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y} \quad \blacksquare$$

SOLUTION II. Solve for y and then differentiate using the Chain Rule. First,

$$x^2 + y^2 = 4 \Longrightarrow y^2 = 4 - x^2 \Longrightarrow y = \pm\sqrt{(4-x^2)}\,.$$

Second,

$$\frac{dy}{dx} = \frac{d}{dx} \pm \sqrt{(4-x^2)} = \frac{1}{\pm 2\sqrt{(4-x^2)}} \cdot \frac{d}{dx}\left(4-x^2\right) = \frac{1}{\pm 2\sqrt{(4-x^2)}} \cdot (0-2x)$$
$$= \frac{-2x}{\pm 2\sqrt{(4-x^2)}} = \frac{-x}{\pm\sqrt{(4-x^2)}} = -\frac{x}{y}\,. \quad \blacksquare$$

c. $y = \displaystyle\int_0^x t\cos(3t)\,dt$

SOLUTION. By the Fundamental Theorem of Calculus:

$$\frac{dy}{dx} = \frac{d}{dx}\int_0^x t\cos(3t)\,dt = x\cos(3t) \quad \blacksquare$$

d. $y = \ln\left(x^3\right)$

SOLUTION I. Simplify, then differentiate. First, $y = \ln\left(x^3\right) = 3\ln(x)$. Second,

$$\frac{dy}{dx} = \frac{d}{dx}3\ln(x) = 3 \cdot \frac{1}{x} = \frac{3}{x}\,. \quad \blacksquare$$

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Solution ii. Differentiate using the Chain Rule, then simplify:

$$\frac{dy}{dx} = \frac{d}{dx}\ln\left(x^3\right) = \frac{1}{x^3} \cdot \frac{d}{dx}x^3 = \frac{1}{x^3} \cdot 3x^2 = \frac{3}{x} \quad \blacksquare$$

3. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

a. Explain why $\lim_{x\to 0} \frac{x}{|x|}$ doesn't exist.

Solution. Note that when $x > 0$, $|x| = x$, so $\frac{x}{|x|} = 1$, and when $x < 0$, $x = -|x|$, so $\frac{x}{|x|} = -1$. It follows that $\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} -1 = -1$ and $\lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^+} 1 = 1$, so $\lim_{x\to 0} \frac{x}{|x|}$ can't exist since $-1 \neq 1$. $\blacksquare$

b. A spherical balloon is being inflated at a rate of 1 m^3/s. How is its radius changing at the instant that it is equal to 2 m? [The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.]

Solution. On the one hand, we are given that $\frac{dV}{dt} = 1$; on the other hand, using the Chain Rule,

$$\frac{dV}{dt} = \frac{d}{dt}\frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{d}{dr}r^3\right)\frac{dr}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}\,.$$

It follows that $1 = \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{1}{4\pi r^2}$. Thus, at the instant that $r = 2$ m, we have $\frac{dr}{dt} = \frac{1}{4\pi 2^2} = \frac{1}{16\pi}$ m/s. $\blacksquare$

c. Use the Left-Hand Rule to find $\int_1^3 x\,dx$. $\left[\ \sum_{i=0}^{n-1} i = 0 + 1 + \cdots + (n-1) = \frac{n(n-1)}{2}\ \right]$

Solution. Not letting the right hand know what the left hand is doing:

$$\begin{aligned}
\int_1^3 x\,dx &= \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{3-1}{n}\cdot\left(1+i\frac{3-1}{n}\right) \qquad \text{[Since our function is just } f(x) = x.\text{]}\\
&= \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{2}{n}\left(1+i\frac{2}{n}\right) = \lim_{n\to\infty}\frac{2}{n}\sum_{i=0}^{n-1}\left(1+i\frac{2}{n}\right)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\left[\sum_{i=0}^{n-1}1\right]+\left[\sum_{i=0}^{n-1}i\frac{2}{n}\right]\right) = \lim_{n\to\infty}\frac{2}{n}\left(n+\left[\frac{2}{n}\sum_{i=0}^{n-1}i\right]\right)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(n+\frac{2}{n}\cdot\frac{n(n-1)}{2}i\right) = \lim_{n\to\infty}\frac{2}{n}\left(n+(n-1)\right)\\
&= \lim_{n\to\infty}\frac{2}{n}(2n-1) = \lim_{n\to\infty}\left(4-\frac{2}{n}\right) = 4-0 = 4 \qquad \blacksquare
\end{aligned}$$

4. Let $f(x) = \dfrac{x^2}{x^2+1}$. Find the domain and all the intercepts, vertical and horizontal asymptotes, and maxima and minima of $f(x)$, and sketch its graph using this information. *[11]*

SOLUTION. We run through the checklist:

i. Domain. $f(x) = \dfrac{x^2}{x^2+1}$ always makes sense because the denominator $x^2+1 \geq 1 > 0$ for all x. Thus the domain of $f(x)$ is all of $\mathbb{R}$; note that $f(x)$ must also be continuous everywhere. $\square$

ii. Intercepts. $f(0) = 0$, so $(0,0)$ is the y-intercept. Since $f(x) = \dfrac{x^2}{x^2+1} = 0$ is only possible when the numerator is 0, any x-intercepts occur when $x^2 = 0$, *i.e.* when $x = 0$. Thus $(0,0)$ is the only x-intercept, as well as the y-intercept. $\square$

iii. Vertical asymptotes. Since $f(x)$ is defined and continuous on all of $\mathbb{R}$ it has no vertical asymptotes. (As noted in i above, this is because the denominator is never 0.) $\square$

iv. Horizontal asymptotes. We check how $f(x)$ behaves as $x \to \pm\infty$:

$$\lim_{x\to\infty} \frac{x^2}{x^2+1} = \lim_{x\to\infty} \frac{x^2}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+0^+} = 1^-$$
$$\lim_{x\to-\infty} \frac{x^2}{x^2+1} = \lim_{x\to-\infty} \frac{x^2}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+0+)} = 1^-$$

Thus $f(x)$ has $x = 1$ as a horizontal asymptote in both directions. Note that because $\frac{x^2}{x^2+1} = \frac{1}{1+1/x^2} < 1$ for all x, $f(x)$ approaches this asymptote from below in both directions. $\square$

v. Maxima and minima. Since $f(x)$ is defined and continuous on all of $\mathbb{R}$, we only have to check any critical points to find any local maxima and minima. We first compute the derivative:

$$\begin{aligned} f'(x) = \frac{d}{dx}\left(\frac{x^2}{x^2+1}\right) &= \frac{\left[\frac{d}{dx}x^2\right]\left(x^2+1\right) - x^2\left[\frac{d}{dx}\left(x^2+1\right)\right]}{\left(x^2+1\right)^2} \\ &= \frac{2x\left(x^2+1\right) - x^2\left(2x+0\right)}{\left(x^2+1\right)^2} \\ &= \frac{2x^3+2x-2x^3}{\left(x^2+1\right)^2} = \frac{2x}{\left(x^2+1\right)^2} \end{aligned}$$

Since the denominator is never 0, $f'(x)$ is defined for all x and $f'(x) = 0$ only when the numerator, $2x$, is 0, *i.e.* when $x = 0$. Thus $x = 0$ is the only critical point. From the behaviour around the critical point,

$$\begin{array}{cccc} x & (-\infty,0) & 0 & (0,\infty) \\ f'(x) & <0 & 0 & >0 \\ f(x) & \downarrow & 0 & \uparrow \end{array},$$

$f(0) = 0$ is a local (and absolute!) minimum. Note that $f(x)$ has no local maxima. $\square$

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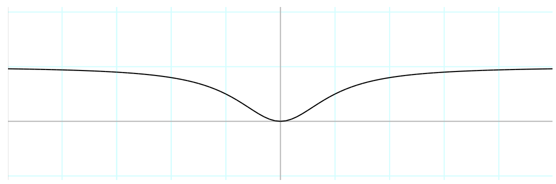
vi. Graph.



This graph was drawn using a program called `EdenGraph`. □

Whew! ■

Bonus. Find any inflection points of $f(x) = \dfrac{x^2}{x^2+1}$ as well. *[3]*

Solution. We add one more item to the checklist above:

vii. Inflection points. Note that $f'(x)$ is defined and differentiable for all x. We first compute the second derivative:

$$\begin{aligned} f''(x) &= \frac{d}{dx}f'(x) = \frac{d}{dx}\left(\frac{2x}{\left(x^2+1\right)^2}\right) = \frac{\left[\frac{d}{dx}2x\right]\left(x^2+1\right)^2 - 2x\left[\frac{d}{dx}\left(x^2+1\right)^2\right]}{\left(\left(x^2+1\right)^2\right)^2} \\ &= \frac{2\left(x^2+1\right)^2 - 2x\left[2\left(x^2+1\right)\cdot\frac{d}{dx}\left(x^2+1\right)\right]}{\left(x^2+1\right)^4} \\ &= \frac{2\left(x^2+1\right)^2 - 2x\left[2\left(x^2+1\right)\cdot(2x+0)\right]}{\left(x^2+1\right)^4} = \frac{2\left(x^2+1\right)^2 - 2x\left[4x\left(x^2+1\right)\right]}{\left(x^2+1\right)^4} \\ &= \frac{2\left(x^2+1\right)^2 - 8x^2\left(x^2+1\right)}{\left(x^2+1\right)^4} = \frac{2\left(x^2+1\right) - 8x^2}{\left(x^2+1\right)^3} = \frac{2-6x^2}{\left(x^2+1\right)^3} \end{aligned}$$

Since the denominator is never 0, $f''(x)$ is defined for all x and $f''(x) = 0$ only when the numerator, $2-6x^2$, is 0, *i.e.* when $x = \pm\frac{1}{\sqrt{3}}$. Thus the potential inflection points of $f(x)$ are $x = -\frac{1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$. From the behaviour around these points,

x	$\left(-\infty, -\frac{1}{\sqrt{3}}\right)$	$-\frac{1}{\sqrt{3}}$	$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{1}{\sqrt{3}}$	$\left(\frac{1}{\sqrt{3}}, \infty\right)$
$f''(x)$	<0	0	>0	0	<0
$f'(x)$	$\downarrow$		$\uparrow$		$\downarrow$
$f(x)$	concave down	$\frac{1}{4}$	concave up	$\frac{1}{4}$	concave down

,

it follows that $f(x)$ has two inflection points, at $x = -\frac{1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$. □

Bonus whew! ■

5

Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2010
Solutions to Test 2

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Compute any *four* (4) of the integrals in parts **a-f**. *[16 = 4 × 4 each]*

a. $\displaystyle\int \frac{1}{4-x^2}\,dx$ **b.** $\displaystyle\int \tan(x)\,dx$ **c.** $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

d. $\displaystyle\int \frac{x^3+x+1}{x^2+1}\,dx$ **e.** $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2(x)\,dx$ **f.** $\displaystyle\int x\ln(x)\,dx$

SOLUTION I TO **a**. *(Partial fractions.)* Since $\dfrac{1}{4-x^2} = \dfrac{1}{(2-x)(2+x)} = \dfrac{A}{2-x} + \dfrac{B}{2+x}$, we have $1 = A(2+x) + B(2-x) = (A-B)x + (2A+2B)$. This boils down to solving the linear equations $A - B = 0$, *i.e.* $A = B$, and $2A + 2B = 1$, *i.e.* $A + B = \frac{1}{2}$. Plugging the first into the second gives $2A = \frac{1}{2}$; it follows that $B = A = \frac{1}{4}$. Now

$$\int \frac{1}{4-x^2}\,dx = \frac{1}{4}\int \frac{1}{2-x}\,dx + \frac{1}{4}\int \frac{1}{2+x}\,dx$$

Substitute $u = 2-x$, so $du = -dx$ and $dx = (-1)\,du$, in the first integral, and $w = 2+x$, so $dw = dx$, in the second.

$$= \frac{1}{4}\int \frac{-1}{u}\,du + \frac{1}{4}\int \frac{1}{w}\,dw = -\frac{1}{4}\ln(u) + \frac{1}{4}\ln(w) + C$$

$$= -\frac{1}{4}\ln(2-x) + \frac{1}{4}\ln(2+x) + C\,. \qquad \blacksquare$$

SOLUTION II TO **a**. *(Trig substitution.)* We'll use the trigonometric substitution $x = 2\sin(\theta)$, so $dx = 2\cos(\theta)\,d\theta$. Note that it follows that $\sin(\theta) = \frac{x}{2}$ and $\cos(\theta) = \sqrt{1-\frac{x^2}{4}}$. Now

$$\int \frac{1}{4-x^2}\,dx = \int \frac{2\cos(\theta)}{4-4\sin^2(\theta)}\,d\theta = \int \frac{2\cos(\theta)}{4\cos^2(\theta)}\,d\theta = \int \frac{1}{2\cos(\theta)}\,d\theta = \frac{1}{2}\int \sec(\theta)\,d\theta$$

$$= \frac{1}{2}\ln\left(\sec(\theta)+\tan(\theta)\right) + C = \frac{1}{2}\ln\left(\frac{1}{\cos(\theta)} + \frac{\sin(\theta)}{\cos(\theta)}\right) + C$$

$$= \frac{1}{2}\ln\left(\frac{1}{\sqrt{1-\frac{x^2}{4}}} + \frac{\frac{x}{2}}{\sqrt{1-\frac{x^2}{4}}}\right) + C = \frac{1}{2}\ln\left(\frac{1+\frac{x}{2}}{\sqrt{1-\frac{x^2}{4}}}\right) + C\,. \qquad \blacksquare$$

Exercise: Show that solutions I and II to **a** actually give the same answer.

Solution to **b**. We'll use the definition of $\tan(x)$:

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx \qquad \text{Substitute } u=\cos(x)\text{, so } du=-\sin(x)\,dx \text{ and } (-1)\,du=\sin(x)\,dx.$$
$$= \int \frac{-1}{u}\,du = -\ln(u)+C = -\ln(\cos(x))+C = \ln\left(\frac{1}{\cos(x)}\right)+C$$
$$= \ln(\sec(x))+C \qquad \blacksquare$$

Solution to **c**. We'll use the Power Rule:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 x^{-1/2}\,dx = \left.\frac{x^{1/2}}{1/2}\right|_0^1 = 2\sqrt{x}\big|_0^1 = 2\sqrt{1}-2\sqrt{0} = 2 \qquad \blacksquare$$

Exercise: Explain why the method used in **c** is not quite correct, even though it gives the right answer.

Solution to **d**. Note that $\dfrac{x^3+x+1}{x^2+1}$ is a rational function in which the degree of the denominator is less that the degree of the numerator. Since $x^3+x+1 = x\left(x^2+1\right)+1$ (you can do long division to get this, or just use the "eyeball theorem"), it follows that

$$\frac{x^3+x+1}{x^2+1} = \frac{x\left(x^2+1\right)+1}{x^2+1} = x+\frac{1}{x^2+1}\,.$$

Hence

$$\int \frac{x^3+x+1}{x^2+1}\,dx = \int\left(x+\frac{1}{x^2+1}\right)\,dx = \int x\,dx + \int \frac{1}{x^2+1}\,dx$$
$$= \frac{1}{2}x^2+\arctan(x)+C\,.$$

Those who haven't yet memorized that the antiderivative of $\dfrac{1}{x^2+1}$ is $\arctan(x)$ can get it with the trig substitution $x=\tan(\theta)$. $\blacksquare$

Solution to **e**.

$$\int_{-\pi/4}^{\pi/4} \sec^2(x)\,dx = \tan(x)|_{-\pi/4}^{\pi/4} = \tan(\pi/4)-\tan(-\pi/4) = 1-(-1) = 2$$

since $\sin(\pi/4)=\cos(\pi/4)=\cos(-\pi/4)=\frac{1}{\sqrt{2}}$ and $\sin(-\pi/4)=-\frac{1}{\sqrt{2}}$. $\blacksquare$

Solution to **f**. We'll do this one using integration by parts; let $u = \ln(x)$ and $v' = x$, so $u' = \frac{1}{x}$ and $v = \frac{1}{2}x^2$.

$$\begin{aligned}\int x\ln(x)\,dx &= \ln(x)\cdot\frac{1}{2}x^2 - \int \frac{1}{x}\cdot\frac{1}{2}x^2\,dx = \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int x\,dx \\ &= \frac{1}{2}x^2\ln(x) - \frac{1}{2}\cdot\frac{1}{2}x^2 + C = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C \qquad \blacksquare\end{aligned}$$

2. Do any *two* (2) of parts **a-e**. *[12 = 2 × 6 each]*

a. Compute $\int_0^2 (x+1)\,dx$ using the Right-hand Rule.

Solution. We plug into the formula and chug away:

$$\begin{aligned}\int_0^2 (x+1)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2-0}{n}\cdot\left[\left(0+i\frac{2-0}{n}\right)+1\right] = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\cdot\left(i\frac{2}{n}+1\right) \\ &= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left(i\frac{2}{n}+1\right) = \lim_{n\to\infty}\frac{2}{n}\left(\left[\sum_{i=1}^{n} i\frac{2}{n}\right]+\left[\sum_{i=1}^{n}1\right]\right) \\ &= \lim_{n\to\infty}\frac{2}{n}\left(\left[\frac{2}{n}\sum_{i=1}^{n} i\right]+n\right) = \lim_{n\to\infty}\frac{2}{n}\left(\frac{2}{n}\cdot\frac{n(n+1)}{2}+n\right) \\ &= \lim_{n\to\infty}\frac{2}{n}\left((n+1)+n\right) = \lim_{n\to\infty}\frac{2}{n}(2n+1) \\ &= \lim_{n\to\infty}\left(4+\frac{2}{n}\right) = 4+0 = 4\,,\end{aligned}$$

since $\frac{2}{n} \to 0$ as $n \to \infty$. $\blacksquare$

b. Find the area of the region bounded by $y = 2 + x$ and $y = x^2$ for $-1 \le x \le 1$.

Solution. A little experimentation with values at different points or a very quick sketch suffices to show that the two curves touch at $x = -1$ and that $2 + x > x^2$ until somewhere to the right of $x = 1$. Thus the area between the two curves for $-1 \le x \le 1$ is:

$$\begin{aligned}\int_{-1}^{1}\left(2+x-x^2\right)\,dx &= \left(2x+\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_{-1}^{1} \\ &= \left(2\cdot 1+\frac{1^2}{2}-\frac{1^3}{3}\right)-\left(2\cdot(-1)+\frac{(-1)^2}{2}-\frac{(-1)^3}{3}\right) \\ &= \left(2+\frac{1}{2}-\frac{1}{3}\right)-\left(-2+\frac{1}{2}+\frac{1}{3}\right) \\ &= \frac{13}{6}-\frac{-5}{6}=\frac{18}{6}=3 \qquad \blacksquare\end{aligned}$$

c. Without actually computing $\displaystyle\int_0^{10/\pi} \arctan(x)\,dx$, find as small an upper bound as you can on the value of this integral.

SOLUTION. Note that $\arctan(x) < \dfrac{\pi}{2}$ for all x. (Just look at the graph!) It follows that

$$\int_0^{10/\pi} \arctan(x)\,dx < \int_0^{10/\pi} \frac{\pi}{2}\,dx = \left.\frac{\pi}{2}x\right|_0^{10/\pi} = \frac{\pi}{2}\cdot\frac{10}{\pi} - \frac{\pi}{2}\cdot 0 = 5\,.$$

Good enough for me! ■

d. Compute the arc-length of the curve $y = \ln(\cos(x))$, $0 \le x \le \pi/6$.

SOLUTION. We plug into the arc-length formula and chug away. Note that

$$\frac{dy}{dx} = \frac{1}{\cos x}\cdot\frac{d}{dx}\cos(x) = \frac{1}{\cos x}\cdot(-\sin(x)) = -\frac{\sin(x)}{\cos x} = -\tan(x)\,,$$

and that $\cos(\pi/6) = \frac{1}{2}$ and $\sin(\pi/6) = \frac{\sqrt{3}}{2}$, so $\sec(\pi/6) = 2$ and $\tan(\pi/6) = \sqrt{3}$. Then

$$\begin{aligned}
\text{arc length} &= \int_0^{\pi/6} \sqrt{1 + (-\tan(x))^2}\,dx = \int_0^{\pi/6} \sqrt{1+\tan^2(x)}\,dx \\
&= \int_0^{\pi/6} \sqrt{\sec^2(x)}\,dx = \int_0^{\pi/6} \sec(x)\,dx = \left.\ln(\sec(x)+\tan(x))\right|_0^{\pi/6} \\
&= \ln(\sec(\pi/6)+\tan(\pi/6)) - \ln(\sec(0)+\tan(0)) \\
&= \ln\left(2+\sqrt{3}\right) - \ln(1+0) = \ln\left(2+\sqrt{3}\right)
\end{aligned}$$

since $\ln(1) = 0$. ■

NOTE: If you want to know where $y = \ln(\cos(x))$ came from, look at the solution to **1b**.

e. Give a example of a function $f(x)$ such that $f(x) = 1 + \displaystyle\int_0^x f(t)\,dt$ for all x.

SOLUTION. Suppose $f(x)$ satisfied the given equation. Then, by the Fundamental Theorem of Calculus, we would have that

$$f'(x) = \frac{d}{dx}\left(1 + \int_0^x f(t)\,dt\right) = 0 + f(x) = f(x)\,.$$

One well-known function satisfying this condition is $f(x) = 0$, but it fails to satisfy the original equation since

$$0 \neq 1 = 1 + 0 = 1 + \int_0^0 0\,dt\,.$$

The other well-known function satisfying $f'(x) = f(x)$ is $f(x) = e^x$. Since

$$e^0 = 1 = 1 + 0 = 1 + \int_0^0 e^t\,dt\,,$$

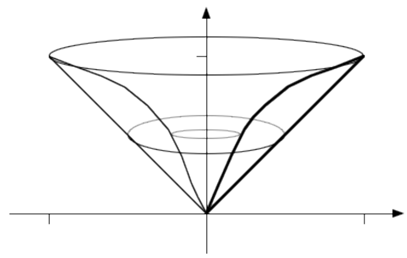
it has chance. We verify that it does satisfy the original equation:

$$1 + \int_0^x e^t\,dt = 1 + \left.e^t\right|_0^x = 1 + \left(e^x - e^0\right) = 1 + e^x - 1 = e^x \qquad ■$$

3. Do *one* (1) of parts **a** or **b**. *[12]*

a. Sketch the solid obtained by rotating the region bounded by $y = \sqrt{x}$ and $y = x$, where $0 \le x \le 1$, about the y-axis, and find its volume.

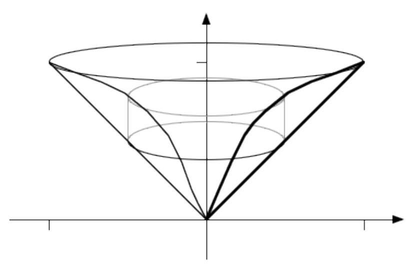
SOLUTION I TO **a**. *(Disks/Washers)* Note that we have $x \le \sqrt{x}$ for $0 \le x \le 1$. Here's a sketch of the solid, with a typical "washer" cross-section in the picture.



Considering the sketch, it is easy to see that the washer at height y has an out radius of $R = x = y$ and an inner radius of $r = x = y^2$ (since $y = \sqrt{x}$), and hence area $\pi\left(R^2 - r^2\right) = \pi\left(y^2 - \left(y^2\right)^2\right) = \pi\left(y^2 - y^4\right)$. Since we rotated the region about the y-axis, the washers are stacked vertically, so we must integrate over y to get the volume of the solid. Note that $0 \le y \le 1$ over the given region.

$$\begin{aligned}\text{Volume} &= \int_0^1 \pi\left(R^2 - r^2\right)\,dy = \pi\int_0^1 \left(y^2 - y^4\right)\,dy = \pi\left(\frac{y^3}{3} - \frac{y^5}{5}\right)\Bigg|_0^1 \\ &= \pi\left(\frac{1^3}{3} - \frac{1^5}{5}\right) - \pi\left(\frac{0^3}{3} - \frac{0^5}{5}\right) = \pi\left(\frac{1}{3} - \frac{1}{5}\right) - \pi(0-0) = \frac{2}{15}\pi \qquad \blacksquare\end{aligned}$$

SOLUTION II TO **a**. *(Cylindrical shells)* Note that we have $x \le \sqrt{x}$ for $0 \le x \le 1$. Here's a sketch of the solid, with a typical cylindrical shell in the picture.



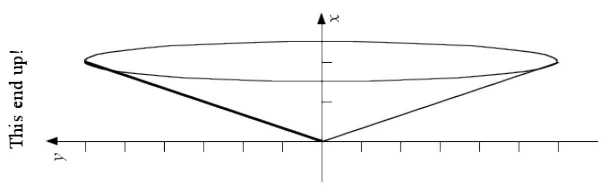
5

Considering the sketch, it is easy to see that the cylinder centred on the y-axis with radius $r = x$ has height $h = \sqrt{x} - x$, and hence area $2\pi rh = 2\pi x\left(\sqrt{x} - x\right)$. Since we rotated the region about the y-axis, the cylinders are vertical and so nested horizontally, so we must integrate over x to get the volume of the solid.

$$\begin{aligned}
\text{Volume} &= \int_0^1 2\pi rh\,dx = 2\pi \int_0^1 x\left(\sqrt{x} - x\right)\,dx = 2\pi \int_0^1 \left(x^{3/2} - x^2\right)\,dx \\
&= 2\pi \left(\frac{x^{5/2}}{5/2} - \frac{x^3}{3}\right)\Bigg|_0^1 = 2\pi\left(\frac{1^{5/2}}{5/2} - \frac{1^3}{3}\right) - 2\pi\left(\frac{0^{5/2}}{5/2} - \frac{0^3}{3}\right) \\
&= 2\pi\left(\frac{2}{5} - \frac{1}{3}\right) - 2\pi(0-0) = 2\pi\frac{1}{15} = \frac{2}{15}\pi \qquad \blacksquare
\end{aligned}$$

b. Sketch the cone obtained by rotating the line $y = 3x$, where $0 \le x \le 2$, about the x-axis, and find its surface area.

Solution. Here's a sketch of the cone:



The cross-section of the cone at x has radius $r = y = 3x$ and $\dfrac{dy}{dx} = 3$. Hence

$$\begin{aligned}
\text{Surface Area} &= \int_0^2 2\pi r\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^2 3x\sqrt{1+3^2}\,dx \\
&= 6\sqrt{10}\,\pi\int_0^2 x\,dx = 6\sqrt{10}\,\pi\,\frac{x^2}{2}\Bigg|_0^2 = 6\sqrt{10}\,\pi\left(\frac{2^2}{2} - \frac{0^2}{2}\right) \\
&= 6\sqrt{10}\,\pi(2-0) = 12\sqrt{10}\,\pi\,. \qquad \blacksquare
\end{aligned}$$

[Total = 40]

6

Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2010
Solutions to the Final Examination

Part I. Do all three (3) of **1–3**.

1. Compute $\frac{dy}{dx}$ as best you can in any *three* (3) of **a–f**. *[15 = 3 × 5 each]*

a. $x^2 + 3xy + y^2 = 23$ **b.** $y = \ln(\tan(x))$ **c.** $y = \int_x^3 \ln(\tan(t))\, dt$

d. $y = \frac{e^x}{e^x - e^{-x}}$ **e.** $\begin{array}{l} x = \cos(2t) \\ y = \sin(3t) \end{array}$ **f.** $y = (x+2)e^x$

SOLUTIONS TO **1**. Using various tricks!

a. Implicit differentiation and some algebra:

$$\begin{aligned} x^2 + 3xy + y^2 = 23 \quad &\implies \quad \frac{d}{dx}\left(x^2 + 3xy + y^2\right) = \frac{d}{dx}23 \\ &\implies \quad 2x + 3y + 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \\ &\implies \quad (2x+3y) + (3x+2y)\frac{dy}{dx} = 0 \\ &\implies \quad \frac{dy}{dx} = -\frac{2x+3y}{3x+2y} \quad \blacksquare \end{aligned}$$

b. Chain Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\ln(\tan(x)) = \frac{1}{\tan(x)} \cdot \frac{d}{dx}\tan(x) = \cot(x)\sec^2(x) \quad \blacksquare$$

c. The Fundamental Theorem of Calculus:

$$\frac{dy}{dx} = \frac{d}{dx}\int_x^3 \ln(\tan(t))\, dt = \frac{d}{dx}(-1)\int_3^x \ln(\tan(t))\, dt = -\ln(\tan(x)) \quad \blacksquare$$

d. Quotient Rule and some algebra with e^x:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{e^x}{e^x - e^{-x}}\right) = \frac{\left(\frac{d}{dx}e^x\right)\left(e^x - e^{-x}\right) - e^x\frac{d}{dx}\left(e^x - e^{-x}\right)}{\left(e^x - e^{-x}\right)^2} \\ &= \frac{e^x\left(e^x - e^{-x}\right) - e^x\left(e^x + e^{-x}\right)}{\left(e^x - e^{-x}\right)^2} = \frac{e^{2x} - e^0 - e^{2x} - e^0}{\left(e^x - e^{-x}\right)^2} \\ &= \frac{-2}{\left(e^x - e^{-x}\right)^2} \qquad \text{Note that } \frac{d}{dx}e^{-x} = -e^{-x}. \ \blacksquare \end{aligned}$$

e. As usual with parametric functions:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\cos(2t)}{\frac{d}{dt}\sin(3t)} = \frac{\sin(2t)\cdot(-2)}{\cos(3t)\cdot 3} = -\frac{2\sin(2t)}{3\cos(3t)}$$

... and there's not much one can try to do to simplify this that doesn't make it worse. ■

f. Product Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\left((x+2)e^x\right) = \left(\frac{d}{dx}(x+2)\right)\cdot e^x + (x+2)\cdot\frac{d}{dx}e^x = 1e^x + (x+2)e^x = (x+3)e^x \quad \blacksquare$$

2. Evaluate any *three* (3) of the integrals **a**–**f**. *[15 = 3 × 5 each]*

$$\textbf{a.} \int_{-\pi/4}^{\pi/4} \tan(x)\,dx \quad \textbf{b.} \int \frac{1}{t^2-1}\,dt \quad \textbf{c.} \int_0^{\pi} x\cos(x)\,dx$$

$$\textbf{d.} \int \sqrt{w^2+9}\,dw \quad \textbf{e.} \int_1^e \ln(x)\,dx \quad \textbf{f.} \int \frac{e^x}{e^{2x}+2e^x+1}\,dx$$

SOLUTIONS TO **2**. Using various tricks!

a. We'll write $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and take it from there.

$$\int_{-\pi/4}^{\pi/4} \tan(x)\,dx = \int_{-\pi/4}^{\pi/4} \frac{\sin(x)}{\cos(x)}\,dx$$

Substitute $u = \cos(x)$, so $du = -\sin(x)\,dx$ and $(-1)\,du = \sin(x)\,dx$. Also, $\begin{array}{ccc} x & -\pi/4 & \pi/4 \\ u & 1/\sqrt{2} & 1/\sqrt{2} \end{array}$.

$$= \int_{1/\sqrt{2}}^{1/\sqrt{2}} \frac{-1}{u}\,du = 0 \quad \blacksquare$$

b. This one can be done with the trig substitution $t = \tan(\theta)$, but that approach requires integrating $\csc(\theta)$ along the way. We will use partial fractions instead. Note first that $t^2 - 1 = (t-1)(t+1)$. Then

$$\frac{1}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1},$$

which requires that $1 = A(t+1) + B(t-1) = (A+B)t + (A-B)$, *i.e.* $A + B = 0$ and $A - B = 1$. Adding the last two equations gives $2A = 1$, so $A = \frac{1}{2}$, and substituting back into either equation and solving for B gives $B = -\frac{1}{2}$. Hence

$$\begin{aligned}\int \frac{1}{t^2-1}\,dt &= \int \left(\frac{1/2}{t-1} - \frac{1/2}{t+1}\right)dt \\ &= \frac{1}{2}\int \frac{1}{t-1}\,dt - \frac{1}{2}\int \frac{1}{t+1}\,dt \\ &= \frac{1}{2}\ln(t-1) - \frac{1}{2}\ln(t+1) + C = \frac{1}{2}\ln\left(\frac{t-1}{t+1}\right) + C. \quad \blacksquare\end{aligned}$$

2

c. This is a job for integration by parts. We'll use $u = x$ and $v' = \cos(x)$, so $u' = 1$ and $v = \sin(x)$. Thus

$$\begin{aligned}
\int_0^{\pi} x\cos(x)\,dx &= \int_0^{\pi} uv'\,dx = uv|_0^{\pi} - \int_0^{\pi} u'v\,dx \\
&= x\sin(x)|_0^{\pi} - \int_0^{\pi} \sin(x)\,dx \\
&= \pi\sin(\pi) - 0\sin(0) - (-\cos(x))|_0^{\pi} \\
&= \pi \cdot 0 - 0 \cdot 0 + \cos(\pi) - \cos(0) = 0 - 0 - 1 - 1 = -2\,. \quad \blacksquare
\end{aligned}$$

d. This is a job for a trig substitution, namely $w = 3\tan(\theta)$, so $dw = 3\sec^2(\theta)\,d\theta$.

$$\begin{aligned}
\int \sqrt{w^2+9}\,dw &= \int \sqrt{9\tan^2(\theta)+9} \cdot 3\sec^2(\theta)\,d\theta \\
&= \int 3\sqrt{\tan^2(\theta)+1} \cdot 3\sec^2(\theta)\,d\theta \\
&= 9\int \sqrt{\sec^2(\theta)} \cdot \sec^2(\theta)\,d\theta = 9\int \sec^3(\theta)\,d\theta \\
&\qquad \text{This last we look up rather than do it from scratch} \ldots \\
&= \frac{9}{2}\sec(\theta)\tan(\theta) + \frac{9}{2}\ln\left(\sec(\theta)+\tan(\theta)\right) + C \\
&= \qquad \text{Substituting back, } \tan(\theta) = \frac{w}{3} \text{ and } \sec(\theta) = \sqrt{1+\frac{w^2}{9}}. \\
&= \frac{9}{2}\cdot\frac{w}{3}\sqrt{1+\frac{w^2}{9}} + \frac{9}{2}\ln\left(\frac{w}{3}+\sqrt{1+\frac{w^2}{9}}\right) + C \\
&= \frac{3w}{2}\sqrt{1+\frac{w^2}{9}} + \frac{9}{2}\ln\left(\frac{w}{3}+\sqrt{1+\frac{w^2}{9}}\right) + C \quad \blacksquare
\end{aligned}$$

e. Integration by parts again, with $u = \ln(x)$ and $v' = 1$, so $u' = \dfrac{1}{x}$ and $v = x$.

$$\begin{aligned}
\int_1^{e} \ln(x)\,dx &= \int_1^{e} uv'\,dx = uv|_1^{e} - \int_1^{e} u'v\,dx \\
&= x\ln(x)|_1^{e} - \int_1^{e} \frac{1}{x}\cdot x\,dx \\
&= e\ln(e) - 1\ln(1) - \int_1^{e} 1\,dx \\
&= e\cdot 1 - 1\cdot 0 - x|_1^{e} \\
&= e - (e-1) = e - e + 1 = 1 \quad \blacksquare
\end{aligned}$$

3

f. Substitute $u = e^x$, so $du = e^x\,dx$ and $e^{2x} = (e^x)^2 = u^2$, and see what happens:

$$\int \frac{e^x}{e^{2x}+2e^x+1}\,dx = \int \frac{1}{u^2+2u+1}\,du = \int \frac{1}{(u+1)^2}\,du$$

Substitute again, with $w = u+1$ and $dw = du$.

$$= \int \frac{1}{w^2}\,dw = \frac{-2}{w^3} + C$$

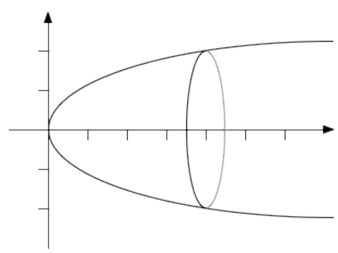
Now we undo the substitutions.

$$= -\frac{2}{(u+1)^3} + C = -\frac{2}{(e^x+1)^3} + C \quad \blacksquare$$

3. Do any *five* (5) of **a**–**i**. *[25 = 5 × 5 ea.]*

a. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $0 \le x \le 4$, the x-axis, and $x = 4$, about the x-axis.

SOLUTION. Here's a crude sketch of the solid:



We'll use the disk/washer method. The disk at x has radius $R = \sqrt{x} - 0 = \sqrt{x}$; since it is a disk rather than a washer, we need not worry about an inner radius. The the volume of the solid is

$$\int_0^4 \pi R^2\,dx = \int_0^4 \pi\left(\sqrt{x}\right)^2\,dx = \pi\int_0^4 x\,dx$$
$$= \pi\left.\frac{x^2}{2}\right|_0^4 = \pi\frac{4^2}{2} - \pi\frac{0^2}{2} = 8\pi - 0\pi = 8\pi\,,$$

cube units of whatever sort. $\blacksquare$

b. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x\to 1} 3x = 3$.

SOLUTION. We need to show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that, for all x, if $|x-1| < \delta$, then $|3x-3| < \varepsilon$.

Given an $\varepsilon > 0$, we obtain the required $\delta > 0$ with some reverse-engineering:

$$|3x-3| < \varepsilon \quad \iff \quad |3(x-1)| < \varepsilon \quad \iff \quad |x-1| < \frac{\varepsilon}{3}$$

Since each step is reversible, it follows that if we let $\delta = \frac{\varepsilon}{3}$, then $|3x-3| < \varepsilon$.

c. Find the Taylor series of $f(x) = \dfrac{x^2}{1-x^2}$ at $a = 0$ without taking any derivatives.

Solution. Recall that the formula for the sum of the geometric series $\sum_{n=0}^{\infty} ar^n$ with first term s and common ratio $r < 1$ is $\dfrac{s}{1-r}$. If we set $s = x^2$ and $r = x^2$, it now follows that

$$\frac{x^2}{1-x^2} = \sum_{n=0}^{\infty} x^2 \left(x^2\right)^n = \sum_{n=0}^{\infty} x^{2n+2} \,,$$

at least when $\left|x^2\right| < 1$, *i.e.* when $|x| < 1$. By the uniqueness of power series representations, it follows that $\sum_{n=0}^{\infty} x^{2n+2}$ is the Taylor series at 0 of $f(x) = \dfrac{x^2}{1-x^2}$. ■

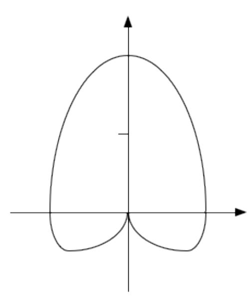
d. Sketch the polar curve $r = 1 + \sin(\theta)$ for $0 \le \theta \le 2\pi$.

Solution. The simplest way to do this is to compute some points on the curve and connect up the dots.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin(\theta)$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
r	1	$3/2$	$1+1/\sqrt{2}$	$1+\sqrt{3}/2$	2	$1+\sqrt{3}/2$	$1+1/\sqrt{2}$	$3/2$	1

$7\pi/6$	$5\pi/4$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$7\pi/4$	$11\pi/6$	2π
$-1/2$	$-1/\sqrt{2}$	$-\sqrt{3}/2$	-1	$-\sqrt{3}/2$	$-1/\sqrt{2}$	$-1/2$	0
$1/2$	$1-1/\sqrt{2}$	$1-\sqrt{3}/2$	0	$1-\sqrt{3}/2$	$1-1/\sqrt{2}$	$1/2$	1

Here's a rough sketch of the curve:



■

e. Use the limit definition of the derivative to compute $f'(1)$ for $f(x) = x^2$.

Solution. Here goes:

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} \\ &= \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} = \lim_{h \to 0} \frac{2h + h^2}{h} \\ &= \lim_{h \to 0} (2 + h) = 2 \quad \blacksquare \end{aligned}$$

f. Use the Right-hand Rule to compute the definite integral $\displaystyle\int_1^2 \frac{x}{2}\, dx$.

Solution. We plug into the Right-hand Rule formula and chug away:

$$\begin{aligned} \int_1^2 \frac{x}{2}\, dx &= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2-1}{n} \cdot \frac{1 + i\frac{2-1}{n}}{2} = \lim_{n \to \infty} \frac{1}{2n} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right) \\ &= \lim_{n \to \infty} \frac{1}{2n} \left[\left(\sum_{i=1}^{n} 1 \right) + \left(\frac{1}{n} \sum_{i=1}^{n} i \right) \right] = \lim_{n \to \infty} \frac{1}{2n} \left[n + \frac{1}{n} \cdot \frac{n(n+1)}{2} \right] \\ &= \lim_{n \to \infty} \left[\frac{1}{2n} \cdot n + \frac{1}{2n} \cdot \frac{n+1}{2} \right] = \lim_{n \to \infty} \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{4n} \right] \\ &= \frac{3}{4} + 0 = \frac{3}{4} \quad \blacksquare \end{aligned}$$

g. Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges absolutely, converges conditionally, or diverges.

Solution. The series converges by the Alternating Series Test: First,

$$\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} \left| \frac{(-1)^n}{\ln(n)} \right| = \lim_{n \to \infty} \frac{1}{\ln(n)} = 0\,,$$

since $\ln(n) \to \infty$ as $n \to \infty$. Second, since $\ln(n)$ is an increasing function of n, we have that

$$|a_{n+1}| = \left| \frac{(-1)^{n+1}}{\ln(n+1)} \right| = \frac{1}{\ln(n+1)} < \frac{1}{\ln(n)} = \left| \frac{(-1)^n}{\ln(n)} \right| = |a_n|\,.$$

Third, since $\ln(n) > 0$ when $n \geq 2$ and $(-1)^n$ alternates sign, this is an alternating series.

On the other hand, the Comparison Test shows the series does not converge absolutely. Note that $n > \ln(n)$ for $n \geq 2$, so

$$\frac{1}{n} < \frac{1}{\ln(n)} = \left| \frac{(-1)^n}{\ln(n)} \right|\,.$$

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Since the harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, it follows that the series $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln(n)} \right|$ diverges as well. Thus the given series does not converge absolutely.

Since it converges, but not absolutely, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges conditionally. ■

h. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n^2}{\pi^n} x^n$.

Solution. We will use the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{(n+1)^2}{\pi^{n+1}} x^{n+1}}{\frac{n^2}{\pi^n} x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{x}{\pi} \right| \\
&= \frac{|x|}{\pi} \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} = \frac{|x|}{\pi} \lim_{n\to\infty} \left(1 + \frac{2}{n} + \frac{1}{n^2} \right) \\
&= \frac{|x|}{\pi} (1+0+0) = \frac{|x|}{\pi}
\end{aligned}$$

It follows by the Ratio Test that $\sum_{n=0}^{\infty} \frac{n^2}{\pi^n} x^n$ converges if $\frac{|x|}{\pi} < 1$, *i.e.* if $|x| < \pi$, and diverges if $\frac{|x|}{\pi} > 1$, *i.e.* if $|x| > \pi$, so the radius of convergence of the series is $R = \pi$. ■

i. Compute the arc-length of the polar curve $r = \theta$, $0 \le \theta \le 1$.

Solution. We plug the given curve into the polar version of the arc-length formula and chug away. Note that $\frac{dr}{d\theta} = 1$ if $r = \theta$.

$$\text{Length} = \int_0^1 \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^1 \sqrt{\theta^2 + 1^2}\, d\theta = \int_0^1 \sqrt{\theta^2+1}\, d\theta$$

We use the trig substitution $\theta = \tan(t)$,

so $d\theta = \sec^2(t)\, dt$ and $\begin{array}{ccc} \theta & 0 & 1 \\ t & 0 & \pi/4 \end{array}$

$$= \int_0^{\pi/4} \sqrt{\tan^2(t)+1}\, \sec^2(t)\, dt = \int_0^{\pi/4} \sqrt{\sec^2(t)}\, \sec^2(t)\, dt = \int_0^{\pi/4} \sec^3(t)\, dt$$

As in the solution to **2d**, we look this up.

$$\begin{aligned}
&= \frac{1}{2}\sec(t)\tan(t) + \frac{1}{2}\ln\left(\sec(t)+\tan(t)\right)\Big|_0^{\pi/4} \\
&= \frac{1}{2}\sec(\pi/4)\tan(\pi/4) + \frac{1}{2}\ln\left(\sec(\pi/4)+\tan(\pi/4)\right) \\
&\quad - \frac{1}{2}\sec(0)\tan(0) - \frac{1}{2}\ln\left(\sec(0)+\tan(0)\right) \\
&= \frac{1}{2}\cdot\sqrt{2}\cdot 1 + \frac{1}{2}\ln\left(\sqrt{2}+1\right) - \frac{1}{2}\cdot 1\cdot 0 - \frac{1}{2}\ln(1+0) = \frac{1}{\sqrt{2}} + \frac{1}{2}\ln\left(\sqrt{2}+1\right) \quad ■
\end{aligned}$$

Part II. Do any *two* (2) of **4**–**6**.

4. Find the domain, all maximum, minimum, and inflection points, and all vertical and horizontal asymptotes of $f(x) = e^{-x^2}$, and sketch its graph. *[15]*

SOLUTION. We'll run through the usual checklist and then graph $f(x) = e^{-x^2}$.

i. *Domain.* Note that both $g(x) = e^x$ and $h(x) = -x^2$ are defined and continuous for all x. It follows that $f(x) = g(h(x)) = e^{-x^2}$ is also defined and continuous for all x. It follows that the domain of $f(x)$ is all of $\mathbb{R}$ and that it has no vertical asymptotes. □

ii. *Intercepts.* Since $g(x) = e^x$ is never 0, $f(x) = e^{-x^2}$ can never equal 0 either, so it has no x-intercepts. For the y-intercept, simply note that $f(0) = e^{-0^2} = e^0 = 1$. □

iii. *Asymptotes.* As noted above, $f(x) = e^{-x^2}$ has no vertical asymptotes, so we only need to check for horizontal asymptotes.

$$\lim_{x\to\infty} e^{-x^2} = \lim_{x\to\infty} \frac{1}{e^{x^2}} = 0 \quad \text{and} \quad \lim_{x\to-\infty} e^{-x^2} = \lim_{x\to-\infty} \frac{1}{e^{x^2}} = 0\,,$$

since $e^{x^2} \to \infty$ as $x^2 \to \infty$, which happens as $x \to \pm\infty$. Thus $f(x) = e^{-x^2}$ has the horizontal asymptote $y = 0$ in both directions. □

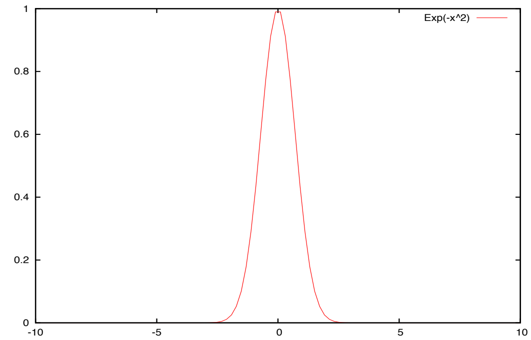
iv. *Maxima and minima.* $f'(x) = e^{-x^2} \dfrac{d}{dx}\left(-x^2\right) = -2xe^{-x^2}$, which equals 0 exactly when $x = 0$ because $-2e^{-x^2} \neq 0$ for all x. Note that this is the only critical point. Since $e^{-x^2} > 0$ for all x, $f'(x) = -2xe^{-x^2} > 0$ when $x < 0$ and < 0 when $x > 0$, so $f(x) = e^{-x^2}$ is increasing for $x < 0$ and decreasing for $x > 0$. Thus $x = 0$ is an (absolute!) maximum point of $f(x)$, which has no minimum points. □

v. *Inflection points.*

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(-2xe^{-x^2}\right) = -2e^{-x^2} - 2x\frac{d}{dx}\left(-x^2\right) \\ &= -2e^{-x^2} - 2x \cdot \left(-2xe^{-x^2}\right) = \left(4x^2 - 2\right)e^{-x^2}\,, \end{aligned}$$

which equals which equals 0 exactly when $4x^2 - 2 = 0$, *i.e.* when $x = \pm\frac{1}{\sqrt{2}}$, because $-2e^{-x^2} \neq 0$ for all x. Since $e^{-x^2} > 0$ for all x, $f''(x) = \left(4x^2 - 2\right)e^{-x^2} > 0$ exactly when $4x^2 - 2 > 0$, *i.e.* when $|x| > \frac{1}{\sqrt{2}}$, and is < 0 exactly when $4x^2 - 2 < 0$, *i.e.* when $|x| < \frac{1}{\sqrt{2}}$. Thus $f(x) = e^{-x^2}$ is concave up on $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$ and concave down on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Thus $f(x) = e^{-x^2}$ has two inflection points, at $x = \pm\frac{1}{\sqrt{2}}$. □

v. *Graph.* $f(x) = e^{-x^2}$ is essentially the classic "bell curve" without some small adjustments that are made to have the total area under the "bell curve" be equal to 1.

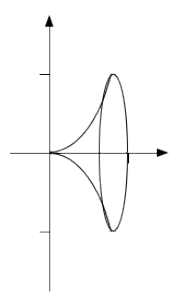


This graph was generated with the command tt Plot2D(Exp(-xˆ2),-10:10) in `Yacas` ("Yet Another Computer Algebra System"). □

That's all for this one, folks! ■

5. Find the area of the surface obtained by rotating the curve $y = \tan(x)$, $0 \le x \le \dfrac{\pi}{4}$, about the x-axis. *[15]*

SOLUTION. This is, quite unintentionally, by far the hardest problem on the exam. [I hallucinated my way to a fairly simple "solution" when making up the exam, and the error survived all my checks ...] Here's a crude sketch of the surface:



Note that $\dfrac{dy}{dx} = \dfrac{d}{dx}\tan(x) = \sec^2(x)$. Plugging this into the appropriate surface area

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formula gives:

$$\int_0^{\pi/4} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{\pi/4} 2\pi\tan(x)\sqrt{1+\sec^4(x)}\,dx$$

Let $u = \sec^2(x)$, so $du = 2\sec^(x)\tan(x)\,dx$ and $2\tan(x)\,dx = \frac{1}{sec^2(x)}\,du = \frac{1}{u}\,du$; also $\begin{array}{ccc} x & 0 & \pi/4 \\ u & 1 & 2 \end{array}$.

$$= \pi\int_1^2 \sqrt{1+u^2}\cdot\frac{1}{u}\,du = \pi\int_1^2 \frac{1}{u}\sqrt{1+u^2}\,du$$

Now let $u = \tan(\theta)$, so $du = \sec^2(\theta)\,d\theta$ and $\begin{array}{ccc} u & 1 & 2 \\ \theta & \pi/4 & \arctan(2) \end{array}$.

$$= \int_{\pi/4}^{\arctan(2)} \frac{1}{\tan(\theta)}\sqrt{1+\tan^2(\theta)}\,\sec^2(\theta)\,d\theta$$

$$= \int_{\pi/4}^{\arctan(2)} \frac{\sec^3(\theta)}{\tan(\theta)}\,d\theta = \int_{\pi/4}^{\arctan(2)} \frac{1}{\sin(\theta)\cos^2(\theta)}\,d\theta$$

At this point – if they even got this far – most people would get stuck. We have one last desperate option, though, namely the Weierstrauss substitution: $t = \tan\left(\frac{\theta}{2}\right)$, so $\cos(\theta) = \frac{1-t^2}{1+t^2}$, $\sin(\theta) = \frac{2t}{1+t^2}$, and $d\theta = \frac{2}{1+t^2}\,dt$. The limits get pretty ugly here, though: $\begin{array}{ccc} \theta & \pi/4 & \arctan(2) \\ t & \tan(\pi/8) & \tan\left(\arctan(2)/2\right) \end{array}$. [There may be some way to simplify the limits, but by now I can't be bothered …] Resuming integration:

$$\int_0^{\pi/4} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{\pi/4}^{\arctan(2)} \frac{1}{\sin(\theta)\cos^2(\theta)}\,d\theta$$

$$= \int_{\tan(\pi/8)}^{\tan(\arctan(2)/2)} \frac{1+t^2}{2t}\cdot\left(\frac{1+t^2}{1-t^2}\right)^2\cdot\frac{2}{1+t^2}\,dt$$

After some algebra, which I'll let you do, we get

$$= \int_{\tan(\pi/8)}^{\tan(\arctan(2)/2)} \frac{t^4+2t^2+1}{t(t-1)^2(t+1)^2}\,dt$$

… which we can do using partial fractions.

To continue we need to find the constants A–E such that

$$\frac{t^4+2t^2+1}{t(t-1)^2(t+1)^2} = \frac{A}{t} + \frac{B}{(t-1)^2} + \frac{C}{t-1} + \frac{D}{(t+1)^2} + \frac{E}{t+1}$$

$$= \frac{\begin{array}{r} A(t-1)^2(t+1)^2 + Bt(t+1)^2 \\ + Ct(t-1)(t+1)^2 + Dt(t-1)^2 \\ + Et(t-1)^2(t+1) \end{array}}{t(t-1)^2(t+1)^2}$$

$$= \frac{\begin{array}{r} (A+C+E)t^4 + (B+C+D-E)t^3 \\ + (-2A+2B-C-2D-E)t^2 \\ + (B-C+D+E)t + A \end{array}}{t(t-1)^2(t+1)^2},$$

that is, satisfying the system of linear equations:

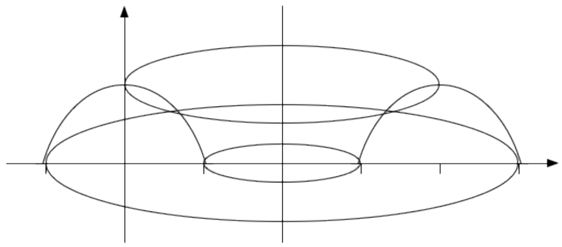
$$\begin{array}{rrrrrrrrrrrr} & A & & & + & C & & & + & E & = & 1 \\ & & & B & + & C & + & D & - & E & = & 0 \\ - & 2A & + & 2B & - & C & - & 2D & - & E & = & 2 \\ & & & B & - & C & + & D & + & E & = & 0 \\ & A & & & & & & & & & = & 1 \end{array}$$

Solving this [more work for you!] gives us: $A = 1$, $B = 1$, $C = 0$, $D = -1$, and $E = 0$. Resuming integration again [and leaving some more routine work for you]:

$$\begin{aligned} \int_0^{\pi/4} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx &= \int_{\tan(\pi/8)}^{\tan(\arctan(2)/2)} \frac{t^4+2t^2+1}{t(t-1)^2(t+1)^2}\, dt \\ &= \int_{\tan(\pi/8)}^{\tan(\arctan(2)/2)} \left(\frac{1}{t} + \frac{1}{(t-1)^2} - \frac{1}{(t+1)^2}\right) dt \\ &= \left(\ln(t) - \frac{1}{t-1} + \frac{1}{t+1}\right)\Big|_{\tan(\pi/8)}^{\tan(\arctan(2)/2)} \\ &= \Bigg(\ln\left(\tan\left(\arctan(2)/2\right)\right) - \frac{1}{\tan\left(\arctan(2)/2\right) - 1} \\ &\qquad + \frac{1}{\tan\left(\arctan(2)/2\right) + 1}\Bigg) \\ &\qquad - \left(\ln\left(\tan(\pi/8)\right) - \frac{1}{\tan(\pi/8) - 1} + \frac{1}{\tan(\pi/8) + 1}\right) \end{aligned}$$

Simplify if you can — and dare! ■

6. Find the volume of the solid obtained by rotating the region below $y = 1 - x^2$, $-1 \leq x \leq 1$, and above the x-axis about the line $x = 2$. *[15]*



Solution. Here's a crude sketch of the solid:

We will use the method of cylindrical shells to find the volume of this solid. Note that the shell at x, where $-1 \leq x \leq 1$, has radius $r = 2 - x$ and height $h = y - 0 = 1 - x^2$. Plugging this into the formula for the volume gives:

$$\begin{aligned}
\int_{-1}^{1} 2\pi r h \, dx &= \int_{-1}^{1} 2\pi(2-x)\left(1-x^2\right)\, dx \\
&= 2\pi \int_{-1}^{1} \left(2 - x - 2x^2 + x^3\right)\, dx \\
&= 2\pi \left(2x - \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4\right)\Big|_{-1}^{1} \\
&= 2\pi \left(2 - \frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) - 2\pi\left(-2 - \frac{1}{2} + \frac{2}{3} + \frac{1}{4}\right) \\
&= 2\pi \cdot \frac{13}{12} - 2\pi \cdot \frac{-19}{12} = 2\pi \cdot \frac{32}{12} = \frac{16}{3}\pi \quad \blacksquare
\end{aligned}$$

Part III. Do *one* (1) of **7** or **8**.

7. Do all three (3) of **a**–**c**.

a. Use Taylor's formula to find the Taylor series of e^x centred at $a = -1$. *[7]*

Solution. If $f(x) = e^x$, then $f'(x) = e^x$, $f''(x) - e^x$, and so on; it is pretty easy to see that $f^{(n)}(x) = e^x$ for all $n \geq 0$. It follows that $f^{(n)}(-1) = e^{-1} = \dfrac{1}{e}$ for all $n \geq 0$. Hence the Taylor series of e^x centred at $a = -1$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}\left(x - (-1)\right)^n = \sum_{n=0}^{\infty} \frac{e^{-1}}{n!}(x+1)^n = \sum_{n=0}^{\infty} \frac{1}{n!e}(x+1)^n \quad \blacksquare$$

b. Determine the radius and interval of convergence of this Taylor series. *[4]*

Solution. We'll use the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!e}(x+1)^{n+1}}{\frac{1}{n!e}(x+1)^n}\right| = \lim_{n\to\infty}\left|\frac{1}{n+1}(x+1)\right|$$
$$= |x+1|\lim_{n\to\infty}\frac{1}{n+1} = |x+1|\cdot 0 = 0$$

It follows that the series converges for any x whatsoever, *i.e.* it has radius of convergence $R=\infty$ and hence has interval of convergence $(-\infty,\infty)$. ■

c. Find the Taylor series of e^x centred at $a=-1$ using the fact that the Taylor series of e^x centred at 0 is $\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots$. *[4]*

Solution. We plug $x-(-1)=x+1$ in for x in e^x and in its Taylor series:

$$e^{x+1} = \sum_{n=0}^{\infty}\frac{(x+1)^n}{n!}$$

Since $e^{x+1}=e^x e$, it follows that

$$e^x = \frac{1}{e}\sum_{n=0}^{\infty}\frac{(x+1)^n}{n!} = \sum_{n=0}^{\infty}\frac{(x+1)^n}{n!e}.$$

Since Taylor series are unique this must be the Taylor series of e^x centred at $a=-1$. ■

8. Do all three (3) of **a**–**c**. You may assume that the Taylor series of $f(x)=\ln(1+x)$ centred at $a=0$ is $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\cdots$.

a. Find the radius and interval of convergence of this Taylor series. *[6]*

Solution. We'll use the Ratio Test to find the radius of convergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}}{n+1}x^{n+1}}{\frac{(-1)^{n+1}}{n}x^n}\right| = \lim_{n\to\infty}\left|-\frac{n}{n+1}x\right|$$
$$= |x|\lim_{n\to\infty}\frac{n}{n+1} = |x|\lim_{n\to\infty}\frac{n/n}{(n+1)/n}$$
$$= |x|\lim_{n\to\infty}\frac{1}{1+1/n} = |x|\cdot\frac{1}{1+0} = |x|\cdot 1 = |x|$$

It follows by the Ratio Test that the given Taylor series converges absolutely when $|x|<1$ and diverges when $|x|>1$, so the radius of convergence is $R=1$.

To determine the interval of convergence, we need to check what happens at $x = \pm R = \pm 1$. Plugging in $x = -1$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(-1)^n = \sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$$

(*i.e.* the negative of the harmonic series), which diverges by the p-Test. Plugging in $x = 1$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} 1^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

(*i.e.* the alternating harmonic series), which converges by the Alternating Series Test, as we've seen in class. Therefore the interval of convergence of the given Taylor series is $(-1, 1]$. ■

b. Use this series to show that $\ln\left(\frac{3}{2}\right) = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}$. *[3]*

Solution. Since a function is equal to its Taylor series within the latter's radius of convergence and $\left|\frac{3}{2} - 1\right| = \frac{1}{2} < 1$, we must have

$$\ln\left(\frac{3}{2}\right) = \ln\left(1 + \frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n},$$

as desired. ■

c. Find an n such that $T_n\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \cdots + \frac{(-1)^{n+1}}{n2^n}$ is guaranteed to be within $0.01 = \frac{1}{100}$ of $\ln\left(\frac{3}{2}\right)$. *[6]*

Solution. We need to find an n such that

$$\left|\ln\left(\frac{3}{2}\right) - T_n\left(\frac{1}{2}\right)\right| = \left|\sum_{i=n+1}^{\infty} \frac{(-1)^{i+1}}{i2^i}\right| < 0.01 .$$

One could, with some effort, accomplish this by considering the nth remainder term, $R_n\left(\frac{1}{2}\right)$, of the given Taylor series, but in this case there is a simpler approach available. Note that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}$ is an alternating series. It follows from the proof of the Alternating Series Test that

$$\left|\sum_{i=n+1}^{\infty} \frac{(-1)^{i+1}}{i2^i}\right| < \left|\frac{(-1)^{n+2}}{(n+1)2^{n+1}}\right| ,$$

so all we need to do is ensure that $\left|\frac{(-1)^{n+2}}{(n+1)2^{n+1}}\right| = \frac{1}{(n+1)2^{n+1}} < 0.01 = \frac{1}{100}$. A little brute force goes a long way here:

n	1	2	3	4	5	$\cdots$
$\frac{1}{(n+1)2^{n+1}}$	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{64}$	$\frac{1}{160}$	$\frac{1}{768}$	$\cdots$

Thus $n = 4$ does the job. (Note that any larger n would serve too.) ■

[Total = 100]

Part IV - Something different. Bonus!

$\mathbf{e^{i\pi}}$**.** Write a haiku touching on caclulus or mathmatics in general. *[2]*

haiku?

seventeen in three:
five and seven and five of
syllables in lines

I hope that you enjoyed the course. Enjoy the rest of the summer!

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Mathematics 1101Y
Calculus I: Functions and calculus of one variable
2010-2011
Solutions

Mathematics 1101Y – Calculus I: functions and calculus of one variable
TRENT UNIVERSITY, 2010–2011

Solutions to the Quizzes

Quiz #1. ~~Friday, 24~~ Monday, 27 September, 2010. *(10 minutes)*

1. Find the location of the tip of the parabola $y = 2x^2 + 2x - 12$, as well as its x- and y-intercepts. *[5]*

SOLUTION. Note that since x^2 has a positive coefficient, this parabola opens upwards.

To find the location of the tip of the parabola, we complete the square in the quadratic expression defining the parabola:

$$\begin{aligned} y &= 2x^2 + 2x - 12 \\ &= 2\left(x^2 + x\right) - 12 \\ &= 2\left[x^2 + 2\frac{1}{2}x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right] - 12 \\ &= 2\left[x^2 + 2\frac{1}{2}x + \left(\frac{1}{2}\right)^2\right] - 2\left(\frac{1}{2}\right)^2 - 12 \\ &= 2\left(x + \frac{1}{2}\right)^2 - \frac{1}{2} - 12 \\ &= 2\left(x + \frac{1}{2}\right)^2 - \frac{25}{2} \end{aligned}$$

It follows that the tip of the parabola occurs when $x + \frac{1}{2} = 0$, *i.e.* when $x = -\frac{1}{2}$, at which point $y = -\frac{25}{2}$. Thus thus the tip of the parabola is at the point $\left(-\frac{1}{2}, -\frac{25}{2}\right)$.

To find the y-intercept of the parabola, we simply plug $x = 0$ into the quadratic expression defining the parabola:

$$y = 2 \cdot 0^2 + 2 \cdot 0 - 12 = 0 + 0 - 12 = 12$$

Thus the y-intercept of the parabola is the point $(0, -12)$.

To find the x-intercept(s) of the parabola, we apply the quadratic formula to find the roots of the quadratic expression defining the parabola: $2x^2 + 2x - 12 = 0$ exactly when

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot (-12)}}{2 \cdot 2} = \frac{-2 \pm \sqrt{4 - (-96)}}{4} \\ &= \frac{-2 \pm \sqrt{100}}{4} = \frac{-2 \pm 10}{4} = \frac{-1 \pm 5}{2}, \end{aligned}$$

i.e. exactly when $x = \frac{4}{2} = 2$ or $x = -\frac{6}{2} = -3$. It follows that the parabola has its x-intercepts at $x = 2$ and $x = -3$, *i.e.* at the points $(-3, 0)$ and $(2, 0)$. ■

Quiz #2. Friday, 1 October, 2010. *(6 minutes)*

1. Solve the equation $e^{2x} - 2e^x + 1 = 0$ for x.

Hint: Solve for e^x first ...

SOLUTION. Recall that $e^{2x} = (e^x)^2$, so we can rewrite the given equation as $(e^x)^2 - 2e^x + 1 = 0$. Following the hint, we solve for e^x using the quadratic equation:

$$e^x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{2 \pm \sqrt{4-4}}{2} = \frac{2 \pm 0}{2} = 1$$

Thus $x = \ln(e^x) = \ln(1) = 0$. ■

Quiz #3. Friday, 8 October, 2010. *(10 minutes)*

1. Evaluate the limit $\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1}$, if it exists. *[5]*

SOLUTION. We factor the numerator and simplify:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+2)}{x-1} = \lim_{x \to 1} (x+2) = 1 + 2 = 3$$

In case of problems factoring this by sight, one could always apply the quadratic formula. The roots of $x^2 + x - 2$ are:

$$\frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-2)}}{2 \cdot 1} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2} = +1 \text{ or } -2$$

It follows that $x^2 + x - 2 = (x - 1)(x - (-2)) = (x - 1)(x + 2)$. ■

Quiz #4. Friday, 15 October, 2010. *(10 minutes)*

1. Use the limit definition of the derivative to compute $f'(2)$ if $f(x) = x^2 + 3x + 1$. *[5]*

SOLUTION. Here goes!

$$\begin{aligned}
f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h \to 0} \frac{\left[(2+h)^2 + 3(2+h) + 1\right] - \left[2^2 + 3 \cdot 2 + 1\right]}{h} \\
&= \lim_{h \to 0} \frac{\left[4 + 4h + h^2 + 6 + 3h + 1\right] - \left[4 + 6 + 1\right]}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 7h}{h} \\
&= \lim_{h \to 0} (h + 7) = 0 + 7 = 7 \qquad ■
\end{aligned}$$

Quiz #5. ~~Friday, 22 October~~ Monday, 1 November, 2010. *(10 minutes)*

1. Find $f'(x)$ if $f(x) = \frac{x^2+2x}{x^2+2x+1}$. Simplify your answer as much as you reasonably can. *[5]*

SOLUTION. The Quotient Rule followed by algebra:

$$\begin{aligned}
f'(x) &= \frac{\frac{d}{dx}\left(x^2+2x\right)\cdot\left(x^2+2x+1\right) - \left(x^2+2x\right)\cdot\frac{d}{dx}\left(x^2+2x+1\right)}{\left(x^2+2x+1\right)^2} \\
&= \frac{(2x+2)\left(x^2+2x+1\right) - \left(x^2+2x\right)(2x+2)}{\left(x^2+2x+1\right)^2} \\
&= \frac{2(x+1)(x+1)^2 - \left(x^2+2x\right)2(x+1)}{\left((x+1)^2\right)^2} \\
&= \frac{2(x+1)^3 - \left(x^2+2x\right)2(x+1)}{(x+1)^4} \\
&= \frac{2(x+1)^2 - 2\left(x^2+2x\right)}{(x+1)^3} \\
&= \frac{2\left(x^2+2x+1\right) - 2\left(x^2+2x\right)}{(x+1)^3} \\
&= \frac{2}{(x+1)^3} \qquad \blacksquare
\end{aligned}$$

Quiz #6. Friday, 5 November, 2010. *(10 minutes)*

1. Find $\dfrac{dy}{dx}$ if $y = \sqrt{x + \arctan(x)}$. *[5]*

SOLUTION. This is a job for the Chain Rule. Note first that, using the Power Rule, $\frac{d}{dt}\sqrt{t} = \frac{d}{dt}t^{1/2} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}}$. Letting $t = x + \arctan(x)$ and applying the Chain Rule gives:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dt}\cdot\frac{dt}{dx} = \left(\frac{d}{dt}\sqrt{t}\right)\cdot\frac{dt}{dx} = \frac{1}{2\sqrt{t}}\cdot\frac{dt}{dx} \\
&= \frac{1}{2\sqrt{x+\arctan(x)}}\cdot\frac{d}{dx}\left(x+\arctan(x)\right) \\
&= \frac{1}{2\sqrt{x+\arctan(x)}}\cdot\left(\frac{dx}{dx}+\frac{d}{dx}\arctan(x)\right) \\
&= \frac{1}{2\sqrt{x+\arctan(x)}}\cdot\left(1+\frac{1}{1+x^2}\right)
\end{aligned}$$

There's not much one can to do to meaningfully simplify this. A little algebra could give you something like $\frac{dy}{dx} = \frac{2+x^2}{2(1+x^2)\sqrt{x+\arctan(x)}}$, but it's not clear that's an improvement. ■

Quiz #7. Friday, 12 November, 2010. *(10 minutes)*

1. Find the maximum and minimum of $f(x) = \dfrac{x}{1+x^2}$ on the interval $[-2, 2]$. *[5]*

SOLUTION. We compute $f'(x)$ using the Quotient Rule:

$$f'(x) = \frac{\left(\frac{d}{dx}x\right)\left(1+x^2\right) - x\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2} = \frac{1\left(1+x^2\right) - x \cdot 2x}{\left(1+x^2\right)^2} = \frac{1-x^2}{\left(1+x^2\right)^2}$$

Note that the denominator of $f'(x)$ is never 0 because $1 + x^2 \geq 1$ for all x, so $f'(x)$ is defined for all x in the interval $[-2, 2]$. $f'(x) = \dfrac{1-x^2}{\left(1+x^2\right)^2} = 0$ exactly when $1 - x^2 = 0$. It follows that the critical points of $f(x)$ are $x = \pm 1$, both of which in the interval $[-2, 2]$.

We compare the values of $f(x)$ at the critical points and the endpoints of the interval:

x	$f(x) = \dfrac{x}{1+x^2}$
-2	$\dfrac{-2}{1+(-2)^2} = -\dfrac{2}{5}$
-1	$\dfrac{-1}{1+(-1)^2} = -\dfrac{1}{2}$
1	$\dfrac{1}{1+1^2} = \dfrac{1}{2}$
2	$\dfrac{2}{1+2^2} = \dfrac{2}{5}$

Since $-\dfrac{1}{2} < -\dfrac{2}{5} < \dfrac{2}{5} < \dfrac{1}{2}$, it follows that the maximum of $f(x) = \dfrac{x}{1+x^2}$ on the interval $[-2, 2]$ is $f(1) = \dfrac{1}{2}$ and the minimum is $f(-1) = -\dfrac{1}{2}$. ■

Quiz #8. Friday, 26 November, 2010. *(10 minutes)*

1. Find an antiderivative of $f(x) = 4x^3 - 3\cos(x) + \dfrac{1}{x}$. *[5]*

SOLUTION. This is mainly an exercise in memorizing basic rules about antiderivatives and the antiderivatives of standard functions. Using the indefinite integral notation for antiderivatives we get:

$$\begin{aligned}\int \left(4x^3 - 3\cos(x) + \frac{1}{x}\right) dx &= 4\int x^3\,dx - 3\int \cos(x)\,dx + \int \frac{1}{x}\,dx \\ &= 4 \cdot \frac{x^{3+1}}{3+1} - 3\sin(x) + \ln(x) + C \\ &= x^4 - 3\sin(x) + \ln(x) + C\end{aligned}$$

Since we just asked for *an* antiderivative, any value of C – including 0 – is fine here. ■

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Quiz #9. Friday, 3 December, 2010. *(10 minutes)*

1. Compute the definite integral $\int_0^1 (2x+1)\,dx$ using the Right-hand Rule. *[5]*

Hint: You may assume that $\displaystyle\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$.

SOLUTION. Recall that the Right-hand Rule formula is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{b-a}{n} f\left(a + k\frac{b-a}{n}\right)$$

We plug $a=0$, $b=1$, and $f(x)=2x+1$ into this formula and grind away:

$$\begin{aligned}
\int_0^1 (2x+1)\,dx &= \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1-0}{n} f\left(0 + k\frac{1-0}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{n} f\left(\frac{k}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} f\left(\frac{k}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} \left(2\frac{k}{n} + 1\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\left(\sum_{k=1}^{\infty} \frac{2k}{n}\right) + \left(\sum_{k=1}^{\infty} 1\right)\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\left(\frac{2}{n} \sum_{k=1}^{\infty} k\right) + \left(\sum_{k=1}^{\infty} 1\right)\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\frac{2}{n} \cdot \frac{n(n+1)}{2} + n\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[(n+1) + n\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[2n+1\right] = \lim_{n\to\infty} \left[\frac{2n}{n} + \frac{1}{n}\right] \\
&= \lim_{n\to\infty} \left[2 + \frac{1}{n}\right] = 2 + 0 = 2 \qquad \blacksquare
\end{aligned}$$

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Quiz #10. Friday, 10 December, 2010. *(10 minutes)*

1. Find the area between the graphs of $f(x) = \sin(x)$ and $g(x) = \dfrac{2x}{\pi}$ for $0 \leq x \leq \dfrac{\pi}{2}$. *[5]*

SOLUTION. Note that $f(0) = 0 = g(0)$ and $f\left(\frac{\pi}{2}\right) = g\left(\frac{\pi}{2}\right) = 1$. Between these points $f(x) \geq g(x)$ – sketch the graphs to convince yourself, if necessary – so the area between them is:

$$\begin{aligned}
\int_0^{\pi/2} (f(x) - g(x))\, dx &= \int_0^{\pi/2} \left(\sin(x) - \frac{2x}{\pi}\right) dx \\
&= \left(-\cos(x) - \frac{2}{\pi}\cdot\frac{x^2}{2}\right)\Bigg|_0^{\pi/2} \\
&= \left(-\cos(\pi/2) - \frac{(\pi/2)^2}{\pi}\right) - \left(-\cos(0) - \frac{0^2}{\pi}\right) \\
&= \left(-0 - \frac{\pi}{4}\right) - (-1 - 0) \\
&= -\frac{\pi}{4} + 1 = 1 - \frac{\pi}{4}
\end{aligned}$$

Quick sanity check: $\pi < 4$, so $\frac{\pi}{4} < 1$, so $1 - \frac{\pi}{4}$ is positive, as an area should be. ■

Quiz #11. Friday, 14 January, 2011. *(10 minutes)*

1. Compute $\displaystyle\int_0^{\pi/2} \cos^3(x)\, dx$. *[5]*

SOLUTION. This can be done pretty quickly using the appropriate reduction formula or integration by parts, but it's also easy to do by a combination of the trig identity $\cos^2(x) = 1 - \sin^2(x)$ and substitution:

$$\begin{aligned}
\int_0^{\pi/2} \cos^3(x)\, dx &= \int_0^{\pi/2} \cos^2(x)\cos(x)\, dx \\
&= \int_0^{\pi/2} \left(1 - \sin^2(x)\right)\cos(x)\, dx \\
&\qquad \text{Substitute } u = \sin(x)\text{, so } du = \cos(x)\, dx\text{, and} \\
&\qquad \text{change the limits: } \begin{array}{ccc} x & 0 & \pi/2 \\ u & 0 & 1 \end{array}. \\
&= \int_0^1 \left(1 - u^2\right) du \\
&= \left(u - \frac{u^3}{3}\right)\Bigg|_0^1 \\
&= \left(1 - \frac{1^3}{3}\right) - \left(0 - \frac{0^3}{3}\right) \\
&= \frac{2}{3} - 0 = \frac{2}{3} \qquad \blacksquare
\end{aligned}$$

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Quiz #12. Friday, 21 January, 2011. *(10 minutes)*

1. Compute $\int \tan^3(x)\sec(x)\,dx$. *[5]*

SOLUTION. There are other ways to pull this off, but the following use of the identity $\tan^2(x) = \sec^2(x) - 1$ and substitution is pretty quick:

$$\begin{aligned}
\int \tan^3(x)\sec(x)\,dx &= \int \tan^2(x)\tan(x)\sec(x)\,dx \\
&= \int \left(\sec^2(x) - 1\right)\sec(x)\tan(x)\,dx \\
&\qquad \text{Substitute } u = \sec(x), \text{ so } du = \sec(x)\tan(x)\,dx. \\
&= \int \left(u^2 - 1\right)\,dx \\
&= \frac{1}{3}u^3 - u + C \\
&= \frac{1}{3}\sec^3(x) - \sec(x) + C \qquad \blacksquare
\end{aligned}$$

Quiz #13. Friday, 28 January, 2011. *(10 minutes)*

1. Compute $\int \frac{1}{\sqrt{4+x^2}}\,dx$. *[5]*

SOLUTION. We'll use the trigonometric substitution $x = 2\tan(\theta)$, so $dx = 2\sec^2(\theta)\,d\theta$, and also $\tan(\theta) = \frac{x}{2}$ and $\sec(\theta) = \sqrt{\sec^2(\theta)} = \sqrt{1+\tan^2(\theta)} = \sqrt{1+\frac{x^2}{4}}$. (We'll need these last when substituting back.)

$$\begin{aligned}
\int \frac{1}{\sqrt{4+x^2}}\,dx &= \int \frac{1}{\sqrt{4+(2\tan(\theta))^2}}\,2\sec^2(\theta)\,d\theta \\
&= \int \frac{2\sec^2(\theta)}{\sqrt{4+4\tan^2(\theta)}}\,d\theta = \int \frac{2\sec^2(\theta)}{\sqrt{4\left(1+\tan^2(\theta)\right)}}\,d\theta \\
&= \frac{2}{\sqrt{4}}\int \frac{\sec^2(\theta)}{\sqrt{1+\tan^2(\theta)}}\,d\theta = \frac{2}{2}\int \frac{\sec^2(\theta)}{\sqrt{\sec^2(\theta)}}\,d\theta \\
&= \int \frac{\sec^2(\theta)}{\sec(\theta)}\,d\theta = \int \sec(\theta)\,d\theta \\
&= \ln\left(\sec(\theta) + \tan(\theta)\right) + C \\
&= \ln\left(\sqrt{1+\frac{x^2}{4}} + \frac{x}{2}\right) + C \qquad \blacksquare
\end{aligned}$$

Quiz #14. Friday, 4 February, 2011. *(15 minutes)*

1. Compute $\displaystyle\int \frac{4x^2+3x}{(x+2)\left(x^2+1\right)}\,dx$. *[5]*

Solution. This is a job for partial fractions. Note that the denominator of the integrand, $(x+2)\left(x^2+1\right)$, come pre-factored into linear factor and irreducible quadratic factors. (x^2+1 doesn't factor any further because it has no roots, since $x^2+1 \geq 1 > 0$ for all x.) It follows that

$$\frac{4x^2+3x}{(x+2)\left(x^2+1\right)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

for some constants A, B, and C. To determine these constants we put the right-hand side of the above equation over the common denominator

$$\begin{aligned}
\frac{A}{x+2} + \frac{Bx+C}{x^2+1} &= \frac{A\left(x^2+1\right)+(Bx+C)(x+2)}{(x+2)\left(x^2+1\right)} \\
&= \frac{Ax^2+A+Bx^2+2Bx+Cx+2C}{(x+2)\left(x^2+1\right)} \\
&= \frac{(A+B)x^2+(2B+C)x+(A+2C)}{(x+2)\left(x^2+1\right)},
\end{aligned}$$

and then equate numerators

$$4x^2+3x = (A+B)x^2+(2B+C)x+(A+2C)$$

to obtain a set of three linear equations in A, B, and C:

$$\begin{array}{ccccccc}
A & + & B & & & = & 4 \\
 & & 2B & + & C & = & 3 \\
A & & & + & 2C & = & 0
\end{array}$$

These equations can be solved in a variety of ways; we will do so using substitution. Solving the third equation for A gives $A = -2C$. Substituting this into the first equation gives $B - 2C = 4$; solving this for B now gives $B = 4 + 2C$. Substituting the last into the second equation now gives $2(4+2C)+C = 3$, *i.e.* $8+5C = 3$, so $5C = 3-8 = -5$, so $C = -5/5 = -1$. It follows that $B = 4+2C = 4+2(-1) = 2$ and $A = -2C = -2(-1) = 2$.

Thus

$$\begin{aligned}
\int \frac{4x^2+3x}{(x+2)\left(x^2+1\right)}\,dx &= \int \left(\frac{2}{x+2} + \frac{2x-1}{x^2+1}\right)dx \\
&= \int \frac{2}{x+2}\,dx + \int \frac{2x-1}{x^2+1}\,dx \\
&= \int \frac{2}{x+2}\,dx + \int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx
\end{aligned}$$

We handle the three parts separately. In the first, we use the substitution $u = x + 2$, so $du = dx$. In the second, we use the substitution $w = x^2 + 1$, so $dw = 2x\,dx$. In the third, we recollect that $\frac{1}{x^2+1}$ is the derivative of $\arctan(x)$. Now

$$\begin{aligned}
\int \frac{4x^2+3x}{(x+2)\left(x^2+1\right)}\,dx &= \int \frac{2}{x+2}\,dx + \int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx \\
&= 2\int \frac{1}{u}\,du + \int \frac{1}{w}\,dw + \arctan(x) \\
&= 2\ln(u) + \ln(w) + \arctan(x) + C \\
&\qquad \text{Since the last integral sign has disappeared, the generic} \\
&\qquad \text{constant must now show up. Substituting back gives:} \\
&= 2\ln(x+2) + \ln\left(x^2+1\right) + \arctan(x) + C
\end{aligned}$$

Whew! ■

~~**Quiz #15.** Friday, 18 February, 2011. *(15 minutes)*~~

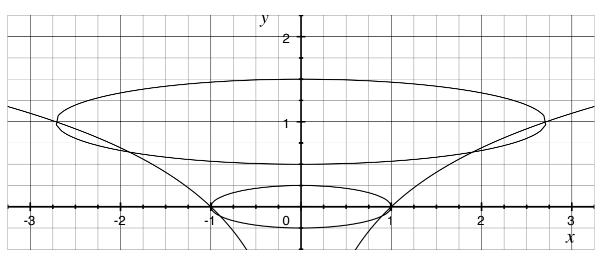
1. Sketch the surface obtained by revolving the curve $y = \ln(x)$, $1 \le x \le e$, about the y-axis, and find its area. *[5]*

Hint: You may find it convenient to just use the fact that

$$\int \sec^3(\theta)\,d\theta = \frac{1}{2}\tan(\theta)\sec(\theta) + \frac{1}{2}\ln\left(\tan(\theta) + \sec(\theta)\right) + C\,,$$

instead of having to work it out from scratch.

Solution. Here's a sketch of the surface, albeit I cheated a little by starting with a graph of $y = \ln(x)$ drawn by a computer.



Note that we only want the part that come from revolving the part of $y = \ln(x)$ for $1 \le x \le e$.

To find the area of this surface, we use the usual formula for the area of a surface of revolution obtained by revolving a curve about the y-axis:

$$\begin{aligned}
\int_1^e 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx &= 2\pi\int_1^e x\sqrt{1+\left(\frac{d}{dx}\ln(x)\right)^2}\,dx = 2\pi\int_1^e x\sqrt{1+\left(\frac{1}{x}\right)^2}\,dx \\
&= 2\pi\int_1^e x\sqrt{1+\frac{1}{x^2}}\,dx = 2\pi\int_1^e \sqrt{x^2\left(1+\frac{1}{x^2}\right)}\,dx \\
&= 2\pi\int_1^e \sqrt{x^2+1}\,dx \\
&\qquad \text{Substitute } x=\tan(\theta)\text{, so } dx=\sec^2(\theta)\,d\theta, \\
&\qquad \text{keep the old limits and substitute back.} \\
&= 2\pi\int_{x=1}^{x=e} \sqrt{\tan^2(\theta)+1}\,\sec^2(\theta)\,d\theta \\
&= 2\pi\int_{x=1}^{x=e} \sqrt{\sec^2(\theta)}\,\sec^2(\theta)\,d\theta = 2\pi\int_{x=1}^{x=e}\sec^3(\theta)\,d\theta \\
&\qquad \text{Use the hint! Note that } \sec(\theta)=\sqrt{\tan^2(\theta)+1}=\sqrt{x^2+1}. \\
&= 2\pi\left[\frac{1}{2}\tan(\theta)\sec(\theta)+\frac{1}{2}\ln\left(\tan(\theta)+\sec(\theta)\right)\right]\Bigg|_{x=1}^{x=e} \\
&= \pi\left[x\sqrt{x^2+1}+\ln\left(x+\sqrt{x^2+1}\right)\right]\Bigg|_{x=1}^{x=e} \\
&= \pi\left[e\sqrt{e^2+1}+\ln\left(e+\sqrt{e^2+1}\right)-\sqrt{2}-\ln\left(1+\sqrt{2}\right)\right]
\end{aligned}$$

This doesn't seem to simplify nicely … ■

Quiz #15. Some time or other, 2011. *(15 minutes)*

1. Find the area of the surface obtained by revolving the curve $y=\sqrt{1-x^2}$, where $0\le x\le 1$, about the y-axis. *[5]*

SOLUTION. In this case

$$\frac{dy}{dx}=\frac{d}{dx}\sqrt{1-x^2}=\frac{1}{2\sqrt{1-x^2}}\cdot\frac{d}{dx}\left(1-x^2\right)=\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)=\frac{-x}{\sqrt{1-x^2}}\,.$$

We plug this into the surface area formula $\int_a^b 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$, giving:

$$\begin{aligned}
\text{Area} &= \int_0^1 2\pi x\sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx = \int_0^1 2\pi x\sqrt{1+\frac{x^2}{1-x^2}}\,dx \\
&= \int_0^1 2\pi x\sqrt{\frac{1-x^2}{1-x^2}+\frac{x^2}{1-x^2}}\,dx = \int_0^1 2\pi x\sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx \\
&= \int_0^1 2\pi x\sqrt{\frac{1}{1-x^2}}\,dx = \int_0^1 2\pi x\frac{1}{\sqrt{1-x^2}}\,dx
\end{aligned}$$

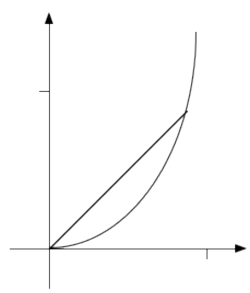
We will compute the last integral using the substitution $u = 1 - x^2$, so $du = -2x\,dx$ and $(-1)\,du = 2x\,dx$, changing the limits as we go along: $\begin{array}{ccc} x & 0 & 1 \\ u & 1 & 0 \end{array}$. Then:

$$\begin{aligned}\text{Area} &= \int_0^1 2\pi x \frac{1}{\sqrt{1-x^2}}\,dx = \int_1^0 \pi \frac{1}{\sqrt{u}}(-1)\,du = \pi \int_0^1 u^{1/2}\,du \\ &= \pi \frac{u^{3/2}}{3/2}\bigg|_0^1 = \pi \frac{2}{3} u^{3/2}\bigg|_0^1 = \pi\frac{2}{3}1^{3/2} - \pi\frac{2}{3}0^{3/2} = \frac{2}{3}\pi \qquad \blacksquare\end{aligned}$$

Quiz #16. Some time or other, 2011. *(12 minutes)*

1. Sketch the region bounded by $r = \tan(\theta)$, $\theta = 0$, and $\theta = \frac{\pi}{4}$ in polar coordinates and find its area. *[5]*

Solution. Note that when $\theta = 0$, $r = \tan(0) = 0$, and when $\theta = \frac{\pi}{4}$, $r = \tan(\pi/4) = \frac{\sin(\pi/4)}{\cos(\pi/4)} = \frac{1/\sqrt{2}}{1/\sqrt{2}} = 1$. In between, $\sin(\theta)$ is increasing and $\cos(\theta)$ is decreasing as θ is increasing, so $r = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ is increasing. The region therefore looks something like this:



To find its area, we need to plug $r = \tan(\theta)$ into the polar area formula for $0 \le \theta \le \frac{\pi}{4}$ and integrate away:

$$\begin{aligned}\text{Area} &= \int_\alpha^\beta \frac{1}{2} r^2\,d\theta = \int_0^{\pi/4} \frac{1}{2}\tan^2(\theta)\,d\theta = \frac{1}{2}\int_0^{\pi/4} \left(\sec^2(\theta) - 1\right)\,d\theta = \frac{1}{2}\left(\tan(\theta) - \theta\right)\bigg|_0^{\pi/4} \\ &= \frac{1}{2}\left(\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right) - \frac{1}{2}\left(\tan(0) - 0\right) = \frac{1}{2}\left(1 - \frac{\pi}{4}\right) - \frac{1}{2}0 = \frac{1}{2} - \frac{\pi}{8} \qquad \blacksquare\end{aligned}$$

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Quiz #17. Friday, 11 March, 2011. *(12 minutes)*

1. Find the arc-length of the parametric curve $x = \sec(t)$, $y = \ln(\sec(t) + \tan(t))$, where $0 \leq t \leq \frac{\pi}{4}$.

SOLUTION. We're going to need to know $\frac{dx}{dt}$ and $\frac{dy}{dt}$, so we'll compute them first:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{d}{dt}\sec(t) = \sec(t)\tan(t) \\
\frac{dy}{dt} &= \frac{d}{dt}\ln\left(\sec(t) + \tan(t)\right) \\
&= \frac{1}{\sec(t) + \tan(t)} \cdot \frac{d}{dt}\left(\sec(t) + \tan(t)\right) \\
&= \frac{1}{\sec(t) + \tan(t)} \cdot \left(\sec(t)\tan(t) + \sec^2(t)\right) \\
&= \frac{1}{\sec(t) + \tan(t)} \cdot \sec(t)\left(\tan(t) + \sec(t)\right) = \sec(t)
\end{aligned}$$

We can now compute the arc-length of the given curve:

$$\begin{aligned}
\text{arc-length} &= \int_C ds = \int_0^{\pi/4} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^{\pi/4} \sqrt{\left(\sec(t)\tan(t)\right)^2 + \left(\sec(t)\right)^2}\, dt \\
&= \int_0^{\pi/4} \sqrt{\sec^2(t)\left[\tan^2(t) + 1\right]}\, dt = \int_0^{\pi/4} \sqrt{\sec^2(t)\sec^2(t)}\, dt \\
&= \int_0^{\pi/4} \sqrt{\left(\sec^2(t)\right)^2}\, dt = \int_0^{\pi/4} \sec^2(t)\, dt = \left.\tan(t)\right|_0^{\pi/4} \\
&= \tan\left(\frac{\pi}{4}\right) - \tan(0) = 1 - 0 = 1 \qquad \blacksquare
\end{aligned}$$

Quiz #18. Friday, 18 March, 2011. *(10 minutes)*

1. Compute $\lim\limits_{n\to\infty} \dfrac{n^2}{e^n}$. *[5]*

Solution. Observe that $f(x) = \dfrac{x^2}{e^x}$ is defined and differentiable (and hence continuous) on $[0, \infty)$, and such that $f(n) = \dfrac{n^2}{e^n}$. It follows that:

$$
\begin{aligned}
\lim_{n\to\infty} \frac{n^2}{e^n} &= \lim_{x\to\infty} \frac{x^2}{e^x} && \text{Since } x^2 \to \infty \text{ and } e^x \to \infty \text{ as } x \to \infty, \\
& && \text{we apply L'Hôpital's Rule.} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}x^2}{\frac{d}{dx}e^x} \\
&= \lim_{x\to\infty} \frac{2x}{e^x} && \text{Since } 2x \to \infty \text{ and } e^x \to \infty \text{ as } x \to \infty, \\
& && \text{we apply L'Hôpital's Rule again.} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}2x}{\frac{d}{dx}e^x} \\
&= \lim_{x\to\infty} \frac{2}{e^x} \\
&= 0 && \ldots \text{ because 2 is constant and } e^x \to \infty \text{ as } x \to \infty. \blacksquare
\end{aligned}
$$

Quiz #19. Friday, 25 March, 2011. *(10 minutes)*

1. Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2 + 2^n}$ converges or diverges. *[5]*

Solution. Note that $0 < \dfrac{1}{n^2+2^n} \leq \dfrac{1}{2^n} = \left(\dfrac{1}{2}\right)^n$ for all $n \geq 0$, and that the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges, because it is a geometric series with common ratio $\frac{1}{2} < 1$. It follows that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+2^n}$ converges by the Comparison Test. $\blacksquare$

Note: One could also compare the given series to the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2}$, which converges by the p-Test.

Quiz #20. Friday, 1 April, 2011. *(15 minutes)*

1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ converges absolutely, converges conditionally, or diverges.

SOLUTION. This series converges conditionally.

First, we check if the given series converges absolutely. The corresponding series of positive terms is $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n \ln(n)}{n}\right| = \sum_{n=2}^{\infty} \frac{\ln(n)}{n}$. Since $f(x) = \dfrac{\ln(x)}{x}$ is defined, continuous, and positive for $x \geq 2$, we can use the Integral Test. Since the improper integral

$$\begin{aligned}\int_2^{\infty} \frac{\ln(x)}{x}\,dx &= \lim_{t\to\infty} \int_2^t \frac{\ln(x)}{x}\,dx \qquad \text{(Substitute } u = \ln(x)\text{, so } du = \tfrac{1}{x}\,dx\text{, and change limits accordingly.)}\\ &= \lim_{t\to\infty} \int_{\ln(2)}^{\ln(t)} u\,du = \lim_{t\to\infty} u^2\Big|_{\ln(2)}^{\ln(t)} = \lim_{t\to\infty} \left[(\ln(t))^2 - (\ln(2))^2\right]\\ &= \infty \qquad \text{(Since } \ln(t) \to \infty \text{ as } t \to \infty.)\end{aligned}$$

does not converge, it follows by the Integral Test that neither does $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$. (One could also do this part very quickly by comparison with the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$.) Thus the given series does not converge absolutely.

Second, to check that the given series converges, we will apply the Alternating Series Test.

- The series is indeed alternating: once $n > 1$, $\dfrac{\ln(n)}{n}$ is always positive, so the $(-1)^n$ forces successive terms to switch sign.
- $\lim_{n\to\infty} \left|\frac{(-1)^n \ln(n)}{n}\right| = \lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$, as required. (Note the use of l'Hôpital's Rule at the key step.)
- Successive terms decrease in absolute value once $n > 1$ since the function $f(x) = \frac{\ln(x)}{x}$ is decreasing for $x > e$ because (Quotient Rule!) $f'(x) = \frac{\frac{1}{x}x - \ln(x)1}{x^2} = \frac{1-\ln(x)}{x^2} < 0$ as soon as $\ln(x) > 1$.

It follows that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ converges by the Alternating Series Test. Since it does converge, but not absolutely, the series converges conditionally. ■

0

TRENT UNIVERSITY

MATH 1101Y Test 1

19 November, 2010

Time: 50 minutes

Name: Steffi Graph

STUDENT NUMBER: 01234567

Question	Mark
1	____
2	____
3	____
4	____
Total	____

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Find $\frac{dy}{dx}$ in any *three* (3) of **a-e**. *[12 = 3 × 4 each]*

a. $y = x^x$ **b.** $y = \frac{1}{1+x^2}$ **c.** $y = \cos\left(\sqrt{x}\right)$ **d.** $y^2 + x = 1$ **e.** $y = x^2 e^{-x}$

SOLUTIONS. **a.** $y = x^x = \left(e^{\ln(x)}\right)^x = e^{x\ln(x)}$ so, using the Chain and Product Rules:

$$\frac{dy}{dx} = \frac{d}{dx}x^x = \frac{d}{dx}e^{x\ln(x)} = e^{x\ln(x)} \cdot \frac{d}{dx}\left(x\ln(x)\right) = e^{x\ln(x)} \cdot \left[\left(\frac{d}{dx}x\right) \cdot \ln(x) + x \cdot \frac{d}{dx}\ln(x)\right]$$
$$= e^{x\ln(x)} \cdot \left[1 \cdot \ln(x) + x \cdot \frac{1}{x}\right] = x^x \cdot (\ln(x) + 1)$$

This can also be done using logarithmic differentiation. □

b. Using the Quotient Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{\left(\frac{d}{dx}1\right) \cdot \left(1+x^2\right) - 1 \cdot \frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$
$$= \frac{0 \cdot \left(1+x^2\right) - 1 \cdot (2x)}{\left(1+x^2\right)^2} = \frac{-2x}{\left(1+x^2\right)^2} \qquad \square$$

c. Using the Chain Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\cos\left(\sqrt{x}\right) = -\sin\left(\sqrt{x}\right) \cdot \frac{d}{dx}\sqrt{x} = -\sin\left(\sqrt{x}\right) \cdot \frac{1}{2\sqrt{x}} = \frac{-\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Recall that $\sqrt{x} = x^{1/2}$, so, using the Power Rule, $\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} - \frac{1}{2\sqrt{x}}$. □

d. Using implicit differentiation and the Chain Rule:

$$y^2 + x = 1 \quad \Rightarrow \quad \frac{d}{dx}\left(y^2 + x\right) = \frac{d}{dx}1 \quad \Rightarrow \quad 2y\frac{dy}{dx} + 1 = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{2y}$$

One could also solve for y in terms of x in the original equation and then differentiate. □

e. Using the Product and Chain Rules:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 e^{-x}\right) = \left(\frac{d}{dx}x^2\right) \cdot e^{-x} + x^2 \cdot \left(\frac{d}{dx}e^{-x}\right) = 2xe^{-x} + x^2 e^{-x} \cdot \left(\frac{d}{dx}(-x)\right)$$
$$= 2xe^{-x} + x^2 e^{-x} \cdot (-1) = \left(2x - x^2\right)e^{-x} = x\,(2-x)\,e^{-x} \qquad \square$$

2. Do any *two* (2) of **a**-**d**. *[10 = 2 × 5 each]*

a. Use the limit definition of the derivative to compute $f'(0)$ for $f(x) = x^2 - 3x + \pi$.

b. Suppose $f(x) = \dfrac{x}{\sin(x)}$ for $x \neq 0$. What would $f(0)$ have to be to make $f(x)$ continuous at $a = 0$?

c. Find the equation of the tangent line to $y = x^2$ at the point $(2, 4)$.

d. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x\to 1}(2x+3) = 5$.

SOLUTIONS. **a.** Plug into the definition and chug away:

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{\left(h^2 - 3h + \pi\right) - \left(0^2 - 3\cdot 0 + \pi\right)}{h} \\ &= \lim_{h\to 0} \frac{h^2 - 3h + \pi - \pi}{h} = \lim_{h\to 0} \frac{h^2 - 3h + 0}{h} = \lim_{h\to 0} (h-3) = 0 - 3 = -3 \qquad \square \end{aligned}$$

b. To make $f(x)$ continuous at $a = 0$, we need to make $f(0) = \lim_{x\to 0} f(x)$, so we have to compute the limit:

$$\begin{aligned} \lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{x}{\sin(x)} && \text{This we compute using l'Hôpital's Rule, since both } x \to 0 \text{ and } \sin(x) \to 0 \text{ as } x \to 0. \\ &= \lim_{x\to 0} \frac{\frac{d}{dx}x}{\frac{d}{dx}\sin(x)} = \lim_{x\to 0} \frac{1}{\cos(x)} = \frac{1}{1} = 1 && \text{Since } \cos(x) \to 1 \text{ as } x \to 0. \end{aligned}$$

Thus we need to make $f(0) = 1$ to have $f(x)$ be continuous at $a = 1$. $\square$

c. The slope of the tangent line to $y = x^2$ at $(2, 4)$ is given by $\dfrac{dy}{dx} = \dfrac{d}{dx}x^2 = 2x$ evaluated at $x = 2$: $m = \left.\dfrac{dy}{dx}\right|_{x=2} = 2x|_{x=2} = 2 \cdot 2 = 4$. The equation of the line is therefore $y = 4x + b$ for some b. To find b, plug the coordinates of the point $(2, 4)$ in for x and y in the equation of the line and solve for b: $4 = 4 \cdot 2 + b$, so $b = 4 - 8 = -4$.

Thus the equation of the tangent line to $y = x^2$ at $(2, 4)$ is $y = 4x - 4$. $\square$

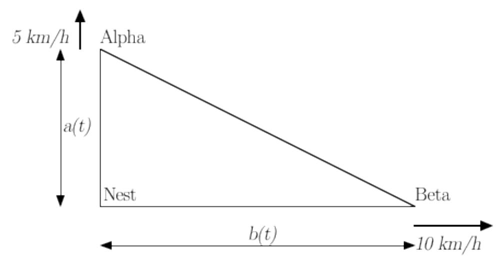
d. To verify that $\lim_{x\to 1}(2x+3) = 5$, we need to show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - 1| < \delta$, then $|(2x+3) - 5| < \varepsilon$. As usual, we reverse-engineer the required δ from what we need to achieve:

$$|(2x+3) - 5| < \varepsilon \quad \Leftrightarrow \quad |2x - 2| < \varepsilon \quad \Leftrightarrow \quad 2|x-1| < \varepsilon \quad \Leftrightarrow \quad |x - 1| < \frac{\varepsilon}{2}$$

It follows that $\delta = \dfrac{\varepsilon}{2}$ does the job: if $|x - 1| < \delta = \frac{\varepsilon}{2}$, we can traverse the chain of equivalences above backwards to obtain $|(2x+3) - 5| < \varepsilon$, as required.

Thus $\lim_{x\to 1}(2x+3) = 5$. $\square$

3. Birds Alpha and Beta leave their nest at the same time, with Alpha flying due north at 5 *km/h* and Beta flying due east at 10 *km/h*. How is the area of the triangle formed by their respective positions and the nest changing 1 *h* after their departure? *[8]*



Solution. Note that after 1 *h*, Alpha and Beta will have flown 5 *km* and 10 *km*, respectively.

Let $a(t)$ and $b(t)$ be the distances that birds Alpha and Beta, respectively, are from the nest at time t. Then most of the given information can be summarized as follows: $a(0) = b(0) = 0$, $a(1) = 5$, $b(1) = 10$, $a'(t) = 5$, and $b'(t) = 10$. Since the birds fly north and east, respectively, their positions and the position of the nest form a right triangle with base $b(t)$ and height $a(t)$ at each instant; the area of this triangle is therefore $A(t) = \frac{1}{2}a(t)b(t)$. We want to know what $A'(t)$ is at $t = 1$.

Using the Product Rule:

$$A'(t) = \frac{d}{dt}\left[\frac{1}{2}a(t)b(t)\right] = \frac{1}{2}\left[a'(t)b(t) + a(t)b'(t)\right]$$

Hence

$$\begin{aligned} A'(1) &= \frac{1}{2}\left[a'(1)b(1) + a(1)b'(1)\right] \\ &= \frac{1}{2}\left[5\ km/h \cdot 10\ km + 5\ km \cdot 10\ km/h\right] \\ &= \frac{1}{2}100\ km^2/h = 50\ km^2/h\,, \end{aligned}$$

i.e. the area of the triangle formed by the birds' respective positions and the nest is increasing at a rate of 50 km^2/h one hour after their departure from the nest. □

4. Find the domain and all intercepts, maxima and minima, and vertical and horizontal asymptotes of $f(x) = \dfrac{x^2+2}{x^2+1}$ and sketch its graph based on this information. *[10]*

SOLUTION. We run through the checklist:

Domain. $f(x) = \frac{x^2+2}{x^2+1}$ makes sense for all possible x – note that since $x^2 \geq 0$, the denominator is always $\geq 1 > 0$ – so the domain of $f(x)$ is $(-\infty, \infty)$.

Intercepts. $f(0) = \frac{0^2+2}{0^2+1} = \frac{2}{1} = 2$, so the y-intercept is $(0, 2)$. Since $x^2 + 2 \geq 2$ for all x – since, again, $x^2 \geq 0$ – $f(x)$ is never 0, so $f(x)$ has no x-intercepts.

Maxima and minima. There are no endpoints to worry about, so all we need to do is check what happens around critical points. Using the Quotient Rule,

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}\left(x^2+2\right) \cdot \left(x^2+1\right) - \left(x^2+2\right) \cdot \frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2} \\ &= \frac{2x \cdot \left(x^2+1\right) - \left(x^2+2\right) \cdot 2x}{\left(x^2+1\right)^2} = \frac{-2x}{\left(x^2+1\right)^2}, \end{aligned}$$

which $= 0$ when $x = 0$, > 0 when $x < 0$, and < 0 when $x > 0$. Note that there are no points where $f'(x)$ is undefined, since $\left(x^2+1\right)^2 \geq 1 > 0$ for all x. We build the usual table:

x	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$+$	0	$-$
$f(x)$	$\uparrow$	max	$\downarrow$

Since $f(x)$ is increasing to the left of 0 and decreasing to the right of 0, the critical point 0 (also the y-intercept!) is a maximum. Note that there are no minimum points.

Vertical asymptotes. Since $f(x)$ is defined for all x and continuous (being a rational function) wherever it is defined, $f(x)$ has no vertical asymptotes.

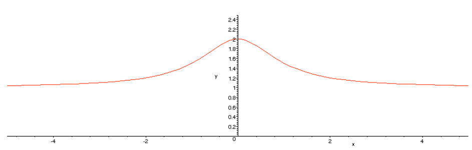
Horizontal asymptotes. We need to check what $f(x)$ does as $x \to \pm\infty$:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2+2}{x^2+1} = \lim_{x\to+\infty} \frac{x^2+2}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to+\infty} \frac{1+2/x^2}{1+1/x^2} = \frac{1+0}{1+0} = 1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2+2}{x^2+1} = \lim_{x\to-\infty} \frac{x^2+2}{x^2+1} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{1+2/x^2}{1+1/x^2} = \frac{1+0}{1+0} = 1$$

It follows that $f(x)$ has $y = 1$ as its horizontal asymptote in both directions.

The graph. `plot((x^2+2)/(x^2+1),x=-5..5,y=0..2.5);` in `Maple` gives:



That's that! $\square$

[Total = 40]

Trent University

MATH 1101Y Test 2

11 February, 2011

Time: 50 minutes

Name: Steffi Graph

Student Number: 01234567

?

1	____
2	____
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4	____
Total	____

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1

1. Compute any *four* (4) of the integrals in parts **a**-**f**. *[16 = 4 × 4 each]*

a. $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx$ **b.** $\displaystyle\int_0^{\pi/4} \sec(x)\tan(x)\,dx$ **c.** $\displaystyle\int_0^{\infty} e^{-x}\,dx$

d. $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$ **e.** $\displaystyle\int \frac{\cos(x)}{\sin(x)}\,dx$ **f.** $\displaystyle\int_1^{e} \ln(x)\,dx$

SOLUTIONS. **a.** We'll use the trig substitution $x = \tan(\theta)$, so $dx = \sec^2(\theta)\,d\theta$ and $\sqrt{x^2+1} = \sqrt{\tan^2(\theta)+1} = \sqrt{\sec^2(\theta)} = \sec(\theta)$.

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+1}}\,dx &= \int \frac{1}{\sec(\theta)}\sec^2(\theta)\,d\theta = \int \sec(\theta)\,d\theta = \ln\left(\tan(\theta)+\sec(\theta)\right)+C \\ &= \ln\left(x+\sqrt{x^2+1}\right)+C \qquad \square\end{aligned}$$

b. We'll use the substitution $u = \sec(x)$, so $du = \sec(x)\tan(x)\,dx$ and $\begin{array}{ccc} x & 0 & \pi/4 \\ u & 1 & \sqrt{2} \end{array}$. (Note that $\sec(\pi/4) = 1/\cos(\pi/4) = 1/\left(1/\sqrt{2}\right) = \sqrt{2}$.)

$$\int_0^{\pi/4} \sec(x)\tan(x)\,dx = \int_1^{\sqrt{2}} 1\,du = u\Big|_1^{\sqrt{2}} = \sqrt{2}-1 \qquad \square$$

c. We'll use the substitution $w = -x$, so $dw = (-1)dx$ and $dx = (-1)dw$, and $\begin{array}{ccc} x & 0 & t \\ w & 0 & -t \end{array}$. Note that this is an improper integral, so we'll have to take a limit first.

$$\begin{aligned}\int_0^{\infty} e^{-x}\,dx &= \lim_{t\to\infty}\int_0^{t} e^{-x}\,dx = \lim_{t\to\infty}\int_0^{-t} e^{w}\,(-1)dw = \lim_{t\to\infty}\,(-1)e^{w}\Big|_0^{-t} \\ &= \lim_{t\to\infty}\left[(-1)e^{-t}-(-1)e^{0}\right] = \lim_{t\to\infty}\left[-e^{-t}+1\right] = \lim_{t\to\infty}\left[1-\frac{1}{e^t}\right] = 1-0 = 1\end{aligned}$$

Note that $\dfrac{1}{e^t} \to 0$ as $t \to \infty$ since $e^t \to \infty$ as $t \to \infty$. $\square$

d. This is a job for partial fractions. Note first that $x^2+3x+2 = (x+1)(x+2)$. (This can be done by eyeballing, experimenting a bit, or using the quadratic formula to find the roots of x^2+3x+2. Calculators that can do some symbolic computation should be able to factor the quadratic too.) We must therefore have a partial fraction decomposition of the form

$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{A}{x+1}+\frac{B}{x+2}$$

for some constants A and B. It follows that

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1}+\frac{B}{x+2} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)} = \frac{(A+B)x+(2A+B)}{(x+1)(x+2)},$$

so $A + B = 0$ and $2A + B = 1$. Then $A = (2A + B) - (A + B) = 1 - 0 = 1$ and $B = 0 - A = -1$.

We can now integrate at last; we'll use the substitutions $u = x + 1$ and $w = x + 2$, so $du = dx$ and $dw = dx$.

$$\begin{aligned}\int \frac{1}{x^2+3x+2}\,dx &= \int \left(\frac{1}{x+1} + \frac{-1}{x+2}\right)dx = \int \frac{1}{x+1}\,dx - \int \frac{1}{x+2}\,dx \\ &= \int \frac{1}{u}\,du - \int \frac{1}{w}\,dw = \ln(u) - \ln(w) + C \\ &= \ln(x+1) - \ln(x+2) + C \qquad \square\end{aligned}$$

e. We'll use the substitution $u = \sin(x)$, so $du = \cos(x)\,dx$.

$$\int \frac{\cos(x)}{\sin(x)}\,dx = \int \frac{1}{u}\,du = \ln(u) + C = \ln(\sin(x)) + C \qquad \square$$

f. We'll use integration by parts, with $u = \ln(x)$ and $v' = 1$, so $u' = \dfrac{1}{x}$ and $v = x$.

$$\begin{aligned}\int_1^e \ln(x)\,dx &= \int_1^e uv'\,dx = uv|_1^e - \int_1^e u'v\,dx = x\ln(x)|_1^e - \int_1^e \frac{1}{x}x\,dx \\ &= (e\ln(e) - 1\ln(1)) - \int_1^e 1\,dx = (e\cdot 1 - 1\cdot 0) - x|_1^e = e - (e-1) = 1 \qquad \square\end{aligned}$$

2

2. Do any *two* (2) of parts **a-e**. *[12 = 2 × 6 each]*

a. Compute $\displaystyle\int_1^2 \frac{x^3 - x^2 - x + 1}{x+1}\,dx$

b. Find the area between $y = \cos(x)$ and $y = \sin(x)$ for $0 \le x \le \dfrac{\pi}{2}$.

c. Which of $\displaystyle\int_\pi^{41} \arctan\left(\sqrt{x}\right)\,dx$ and $\displaystyle\int_\pi^{41} \arctan\left(x^2\right)\,dx$ is larger? Explain why.

d. Use the Right-hand Rule to compute $\displaystyle\int_1^2 x\,dx$.

e. Find the area of the region bounded by $y = 0$ and $y = \ln(x)$ for $0 < x \le 1$.

Solutions. **a.** This is a rational function whose numerator has degree greater than its denominator. Observe that

$$\begin{aligned}\frac{x^3 - x^2 - x + 1}{x+1} &= \frac{\left(x^3 - x\right) + \left(-x^2 + 1\right)}{x+1} = \frac{x\left(x^2-1\right) - 1\left(x^2-1\right)}{x+1} \\ &= \frac{(x-1)\left(x^2-1\right)}{x+1} = \frac{(x-1)(x-1)(x+1)}{x+1} = (x-1)^2,\end{aligned}$$

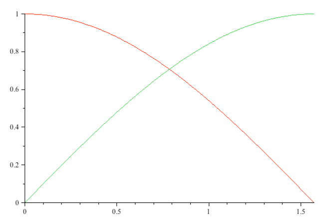
3

which we could also get by dividing $x+1$ into x^3-x^2-x+1 if we didn't spot the cheap bit of algebra above.

We can now integrate; we'll use the substitution $w = x-1$, so $dw = dx$, and we'll change limits accordingly: $\begin{array}{ccc} x & 1 & 2 \\ w & 0 & 1 \end{array}$. Thus:

$$\int_1^2 \frac{x^3-x^2-x+1}{x+1}\,dx = \int_1^2 (x-1)^2\,dx = \int_0^1 w^2\,dw = \left.\frac{w^3}{3}\right|_0^1 = \frac{1}{3} - \frac{0}{3} = \frac{1}{3} \qquad \square$$

b. Recall what the graphs of $\cos(x)$ and $\sin(x)$ look like:




```
plot( [cos(x), sin(x)], x=0..(1/2)*Pi );
```

$\cos(0) = 1$ and $\sin(0) = 0$, but $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$; the graphs of the two functions cross each other at $x = \frac{\pi}{4}$, where both are equal to $1/\sqrt{2}$. The area between the curves is therefore:

$$\begin{aligned} \text{Area} &= \int_0^{\pi/4} (\cos(x) - \sin(x))\,dx + \int_{\pi/4}^{\pi/2} (\sin(x) - \cos(x))\,dx \\ &= (\sin(x) - (-\cos(x)))|_0^{\pi/4} + (-\cos(x) - \sin(x))|_{\pi/4}^{\pi/2} \\ &= \left(\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\right) - (\sin(0) + \cos(0)) \\ &\quad + \left(-\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right) - \left(-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right) \\ &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1) + (-0-1) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) \\ &= \frac{4}{\sqrt{2}} - 2 = 2\sqrt{2} - 2 = 2\left(\sqrt{2}-1\right) \qquad \square \end{aligned}$$

c. Note that the two definite integrals are the same except for the function of x being composed with arctan. As $\arctan(t)$ is an increasing function – its derivative, $\frac{1}{1+t^2}$, is positive for all t – and $\sqrt{x} < x^2$ for all $x > 1$, we must have $\arctan\left(\sqrt{x}\right) < \arctan\left(x^2\right)$ for all x in $[\pi, 41]$. It follows that $\int_\pi^{41} \arctan\left(\sqrt{x}\right)\,dx < \int_\pi^{41} \arctan\left(x^2\right)\,dx$. $\square$

d. We throw the Right-hand Rule formula, $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{b-a}{n}\cdot f\left(a+i\frac{b-a}{n}\right)$, at the given definite integral and compute away. Note that $f(x)=x$ in this case.

$$\begin{aligned}
\int_1^2 x\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2-1}{n}\cdot\left(1+i\frac{2-1}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\cdot\left(1+\frac{i}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\left(\left[\sum_{i=1}^{n}1\right]+\left[\sum_{i=1}^{n}\frac{i}{n}\right]\right) = \lim_{n\to\infty}\frac{1}{n}\left(n+\left[\frac{1}{n}\sum_{i=1}^{n}i\right]\right)\\
&= \lim_{n\to\infty}\frac{1}{n}\left(n+\frac{1}{n}\cdot\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\frac{1}{n}\left(n+\frac{n+1}{2}\right) = \lim_{n\to\infty}\frac{1}{n}\left(\frac{3}{2}n+\frac{1}{2}\right)\\
&= \lim_{n\to\infty}\left(\frac{3}{2}+\frac{1}{2n}\right) = \frac{3}{2}+0 \quad \text{since } \frac{1}{2n}\to 0 \text{ as } n\to\infty. \ \square
\end{aligned}$$

e. Since $\ln(x)<0$ for $0<x<1$, the area of the given region is just $\int_0^1 (0-\ln(x))\,dx = -\int_0^1 \ln(x)\,dx$. However, since $\ln(x)$ has an asymptote at $x=0$, this is an improper integral, forcing us to do some additional work. To find the antiderivative of $\ln(x)$ itself, we will use integration by parts, with $u=\ln(x)$ and $v'=1$, so $u'=\frac{1}{x}$ and $v=x$.

$$\begin{aligned}
\text{Area} &= -\int_0^1 \ln(x)\,dx = \lim_{t\to 0^+}\left(-\int_t^1 \ln(x)\,dx\right) = -\lim_{t\to 0^+}\int_t^1 \ln(x)\,dx\\
&= -\lim_{t\to 0^+}\left[x\ln(x)|_t^1 - \int_t^1 \frac{1}{x}x\,dx\right] = -\lim_{t\to 0^+}\left[1\ln(1)-t\ln(t)-\int_t^1 1\,dx\right]\\
&= -\lim_{t\to 0^+}\left[1\cdot 0 - t\ln(t) - x|_t^1\right] = -\lim_{t\to 0^+}\left[-t\ln(t)-(1-t)\right]\\
&= \lim_{t\to 0^+}\left[t\ln(t)+(1-t)\right] = \lim_{t\to 0^+}\frac{\ln(t)}{1/t} + \lim_{t\to 0^+}(1-t)\\
&\qquad \text{Use l'Hôpital's Rule since } \ln(t)\to-\infty \text{ and } \frac{1}{t}\to\infty \text{ as } t\to 0^+:\\
&= \left(\lim_{t\to 0^+}\frac{1/t}{-1/t^2}\right)+(1-0) = \left(\lim_{t\to 0^+} -t\right)+1 = -0+1 = 1 \qquad \square
\end{aligned}$$

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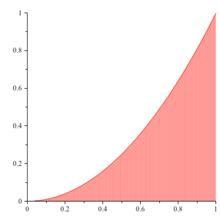
3. Do *one* (1) of parts **a** or **b**. *[12]*

a. Sketch the solid obtained by rotating the region bounded above by $y = x^2$ and below by $y = 0$, where $0 \leq x \leq 2$, about the y-axis, and find its volume.

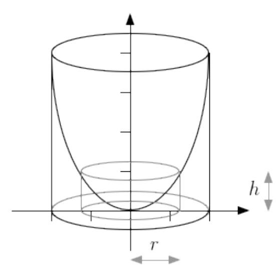
b. Sketch the solid obtained by rotating the region bounded above by $y = x^2$ and below by $y = 0$, where $0 \leq x \leq 2$, about the x-axis, and find its volume.

SOLUTIONS. Note that the region being rotated is the same in both **a** and **b**; they differ in the axis about which the region is rotated.



```
plot(x^2,x=0..1,color="Red",filled=[color="Red",transparency=.5])
```

SOLUTION TO **a**. Here is a crude sketch of the solid with a generic cylindrical shell.



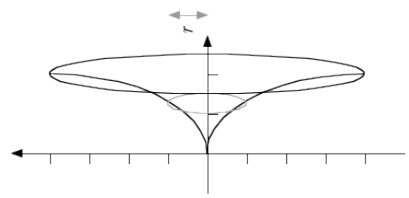
The solid with a cylindrical shell.

We will find the volume of the solid using cylindrical shells. Note that since we rotated the region about the y-axis, we will have to integrate with respect to x if we're using shells. Looking at the diagram, it is easy to see that the radius of the cylindrical shell that comes from rotating the vertical cross-section at x of the original region is just going to be $r = x - 0 = x$. It is also easy to see that its height, which is the length of the vertical cross-section at x of the original region, is going to be $h = x^2 - 0 = x^2$. The limits of integration will come from the possible x values in the original region, *i.e.* $0 \leq x \leq 2$.

Thus the volume of the solid is:

$$\begin{aligned}\text{Volume} &= \int_0^2 2\pi r h\,dx = \int_0^2 2\pi x x^2\,dx = 2\pi \int_0^2 x^3\,dx \\ &= 2\pi \frac{x^4}{4}\bigg|_0^2 = 2\pi\left(\frac{2^4}{4} - \frac{0^4}{4}\right) = 2\pi\left(\frac{16}{4} - 0\right) = 8\pi \qquad \square\end{aligned}$$

Solution to **b**. Here is a crude sketch of the solid with a generic disk.



The solid with a disk.
Rotate picture 90° *clockwise!*

We will find the volume of the solid using disks. Note that since we rotated the region about the x-axis, we will have to integrate with respect to x if we're using disks. Looking at the diagram, it is easy to see that the radius of the disk that comes from rotating the vertical cross-section at x of the original region is just going to be the length of that vertical cross-section, namely $r = x^2 - 0 = x^2$. Note that the disk has no hole because the x-axis forms part of the boundary of the give region, so we needn't worry about the inner radius: it is always 0. The limits of integration will come from the possible x values in the original region, *i.e.* $0 \le x \le 2$.

Thus the volume of the solid is:

$$\begin{aligned}\text{Volume} &= \int_0^2 \pi r^2\,dx = \pi \int_0^2 \left(x^2\right)^2\,dx = \pi \int_0^2 x^4\,dx \\ &= \pi\,\frac{x^5}{5}\bigg|_0^2 = \pi\left(\frac{2^5}{5} - \frac{0^5}{5}\right) = \frac{32}{5}\pi \qquad \square\end{aligned}$$

[Total = 40]

7

Mathematics 1101Y – Calculus I: functions and calculus of one variable
TRENT UNIVERSITY, 2010–2011
Solutions to the Final Examination

Part X. Do all three (3) of **1**–**3**.

1. Compute $\frac{dy}{dx}$ as best you can in any *three* (3) of **a**–**f**. *[15 = 3 × 5 each]*

$$\textbf{a. } y = \cos(e^x) \quad \textbf{b. } y = \int_1^x e^t \ln(t)\, dt \quad \textbf{c. } y = x\ln(x)$$

$$\textbf{d. } y = \frac{\ln(x)}{x} \quad \textbf{e. } \arctan(x+y) = 0 \quad \textbf{f. } \begin{array}{c} x = e^t \\ y = e^{2t} \end{array}$$

SOLUTIONS TO **1**.

a. Using the Chain Rule: $\frac{dy}{dx} = -\sin(e^x) \cdot \frac{d}{dx}e^x = -e^x \sin(e^x)$ □

b. Using the Fundamental Theorem of Calculus: $\frac{dy}{dx} = e^x \ln(x)$ □

c. Using the Product Rule: $\frac{dy}{dx} = \frac{dx}{dx} \cdot \ln(x) + x \cdot \frac{d}{dx}\ln(x) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$ □

d. Using the Quotient Rule:

$$\frac{dy}{dx} = \frac{\left(\frac{d}{dx}\ln(x)\right) \cdot x - \ln(x) \cdot \frac{dx}{dx}}{x^2} = \frac{\frac{1}{x} \cdot x - \ln(x) \cdot 1}{x^2} = \frac{1 - \ln(x)}{x^2} \quad \square$$

e. $\arctan(x+y) = 0 \Leftrightarrow x + y = 0 \Leftrightarrow y = -x \Rightarrow \frac{dy}{dx} = -1$ □

Note: This can also be done using implicit differentiation.

f. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}e^{2t}}{\frac{d}{dt}e^t} = \frac{e^{2t} \cdot \frac{d}{dt}(2t)}{e^t} = \frac{2e^{2t}}{e^t} = \frac{2\left(e^t\right)^2}{e^t} = 2e^t = 2x$ □

Note: This can also be done by observing that $y = e^{2t} = \left(e^t\right)^2 = x^2$ to begin with.

2. Evaluate any *three* (3) of the integrals **a**–**f**. *[15 = 3 × 5 each]*

$$\textbf{a. } \int \frac{\ln(x)}{x}\, dx \quad \textbf{b. } \int \frac{1}{\sqrt{z^2 - 1}}\, dz \quad \textbf{c. } \int_1^4 \sqrt{x}\, dx$$

$$\textbf{d. } \int_0^1 \frac{x^2 + 2}{x^2 + 1}\, dx \quad \textbf{e. } \int \tan^2(w)\, dw \quad \textbf{f. } \int_0^{\ln(2)} e^{2t}\, dt$$

SOLUTIONS TO **2**.

a. Use the substitution $u = \ln(x)$, so $du = \frac{1}{x}\, dx$:

$$\int \frac{\ln(x)}{x}\, dx = \int u\, du = u^2 + C = (\ln(x))^2 + C \quad \square$$

b. Use the trigonometric substition $z = \sec(\theta)$, so $dz = \sec(\theta)\tan(\theta)\,d\theta$:

$$\begin{aligned}\int \frac{1}{\sqrt{z^2-1}}\,dz &= \int \frac{1}{\sqrt{\sec^2(\theta)-1}}\,\sec(\theta)\tan(\theta)\,d\theta = \int \frac{1}{\sqrt{\tan^2(\theta)}}\,\sec(\theta)\tan(\theta)\,d\theta \\ &= \int \frac{1}{\tan(\theta)}\,\sec(\theta)\tan(\theta)\,d\theta = \int \sec(\theta)\,d\theta \\ &= \ln\left(\sec(\theta)+\tan(\theta)\right)+C = \ln\left(z+\sqrt{z^2-1}\right)+C \quad \square\end{aligned}$$

c. Use the Power Rule for integration:

$$\begin{aligned}\int_1^4 \sqrt{x}\,dx &= \int_1^4 x^{1/2}\,dx = \left.\frac{x^{3/2}}{3/2}\right|_1^4 = \left.\frac{2}{3}\left(\sqrt{x}\right)^3\right|_1^4 \\ &= \frac{2}{3}\left(\sqrt{4}\right)^3 - \frac{2}{3}\left(\sqrt{1}\right)^3 = \frac{2}{3}2^3 - \frac{2}{3} = \frac{14}{3} \quad \square\end{aligned}$$

d. Observe that the integrand is a rational function in which the degree of the numerator is not less than the degree of the denominator. This means that we need to divide the denominator into the numerator first, which is very easy in this case:

$$\frac{x^2+2}{x^2+1} = \frac{\left(x^2+1\right)+1}{x^2+1} = \frac{x^2+1}{x^2+1} + \frac{1}{x^2+1} = 1 + \frac{1}{x^2+1}$$

Hence

$$\begin{aligned}\int_0^1 \frac{x^2+2}{x^2+1}\,dx &= \int_0^1 \left(1+\frac{1}{x^2+1}\right) = \int_0^1 1\,dx + \int_0^1 \frac{1}{x^2+1}\,dx \\ &= \left.x\right|_0^1 + \left.\arctan(x)\right|_0^1 = [1-0] + [\arctan(1)-\arctan(0)] \\ &= 1 + \left[\frac{\pi}{4}-0\right] = 1 + \frac{\pi}{4}\,. \quad \square\end{aligned}$$

e. Use the trigonometric identity $\tan^2(w) = \sec^2(w) - 1$:

$$\int \tan^2(w)\,dw = \int \left(\sec^2(w)-1\right)\,dw = \int \sec^2(w)\,dw - \int 1\,dw = \tan(w) - w + C \quad \square$$

f. Use the substitution $u = 2t$, so $du = 2\,dt$ and $\frac{1}{2}\,du = dt$. Note that $\begin{array}{ccc} x & 0 & \ln(2) \\ u & 0 & 2\ln(2) \end{array}$.

$$\begin{aligned}\int_0^{\ln(2)} e^{2t}\,dt &= \int_0^{2\ln(2)} e^u \frac{1}{2}\,du = \frac{1}{2}\left.e^u\right|_0^{2\ln(2)} = \frac{1}{2}\left(e^{2\ln(2)} - e^0\right) \\ &= \frac{\left(e^{\ln(2)}\right)^2 - 1}{2} = \frac{2^2-1}{2} = \frac{4-1}{2} = \frac{3}{2} \quad \square\end{aligned}$$

3. Do any *five* (5) of **a**–**i**. *[25 = 5 × 5 each]*

a. Use the limit definition of the derivative to compute $g'(0)$ for $g(x) = 2x + 1$.

b. Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ converges or diverges.

c. Find the Taylor series of $f(x) = e^{x+1}$ at $a = 0$.

d. Sketch the polar curve $r = 1$, where $0 \leq \theta \leq 2\pi$, and find the area of the region it encloses.

e. Sketch the surface obtained by rotating $y = \dfrac{x^2}{2}$, $0 \leq x \leq 2$, about the y-axis, and find its area.

f. Use the Right-hand Rule to compute the definite integral $\displaystyle\int_0^1 4x\,dx$.

g. Use the $\varepsilon - \delta$ definition of limits to verify that $\displaystyle\lim_{x\to 0}(2x - 1) = -1$.

h. Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$.

i. Sketch the solid obtained by rotating the region bounded by $y = x$, $y = 0$, and $x = 2$, about the x-axis, and find its volume.

Solutions to **3**.

a. $$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{(2h-1) - (2\cdot 0 - 1)}{h}$$
$$= \lim_{h\to 0} \frac{2h - 1 - (-1)}{h} = \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = 2 \quad \square$$

b. Note that $f(x) = \dfrac{1}{x\ln(x)}$ is positive and continuous for $x \geq 2$, and also decreasing, because $x\ln(x)$ is clearly increasing to ∞ as $x \to \infty$. Since $a_n = f(n)$, it follows from the Integral Test that the given series will converge or diverge depending on whether the improper integral $\displaystyle\int_2^{\infty} \frac{1}{x\ln(x)}\,dx$ converges or diverges. Since the improper integral actually diverges,

$$\int_2^{\infty} \frac{1}{x\ln(x)}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{x\ln(x)}\,dx = \lim_{t\to\infty} \int_{\ln(2)}^{\ln(t)} \frac{1}{u}\,du = \lim_{t\to\infty} \ln(u)\Big|_{\ln(2)}^{\ln(t)}$$
$$\text{(Using the substitution } u = \ln(x)\text{, so } du = \frac{1}{x}\,dx.)$$
$$= \lim_{t\to\infty} \left[\ln(\ln(t)) - \ln(\ln(2))\right] = \infty \qquad (\text{As } \ln(z) \to \infty \text{ when } z \to \infty.)$$

the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges as well. $\square$

3

c. *(Direct.)* Note that $f(x) = e^{x+1} = e \cdot e^x$, so $f'(x) = e \cdot \frac{d}{dx} e^x = e \cdot e^x = f(x)$. It follows that $f^{(n)}(x) = f(x) = e \cdot e^x$ for every $n \geq 0$, and so $f^{(n)}(0) = f(0) = e \cdot e^0 = e \cdot 1 = e$ for every $n \geq 0$. Plugging this into Taylor's Formula gives us

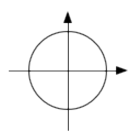
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{e}{n!} x^n \quad \square$$

c. *(Indirect.)* We know that the Taylor series of e^x at 0 is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Thus

$$f(x) = e^{x+1} = e \cdot e^x = e \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{e}{n!} x^n ,$$

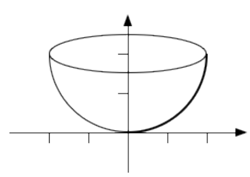
and this last must be the Taylor series for $f(x) = e^{x+1}$ at 0, because only one power series can fill this role. $\square$

d. Since $0 \leq \theta \leq 2\pi$, the curve goes all the way around the origin, and since $r = 1$ it follows that the curve consists of all the points which are a distance of 1 from the origin. This means that the curve is the circle of radius 1 centred at the origin – here's a sketch:



– and so has area $\pi 1^2 = \pi$. $\square$

e. Here's a sketch of the surface:



To find its area, we will use the formula for the area of a surface of revolution. Since

4

$y = \dfrac{x^2}{2}$, $\dfrac{dy}{dx} = \dfrac{1}{2}2x = x$. Hence

$$\text{Area} = \int_0^2 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \pi \int_0^2 2x\sqrt{1+x^2}\, dx = \pi \int_1^5 \sqrt{u}\, du$$

We substituted $u = 1 + x^2$, so $du = 2x\,dx$ and $\begin{matrix} x & 0 & 2 \\ u & 1 & 5 \end{matrix}$.

$$= \pi \int_1^5 u^{1/2}\, du = \frac{2}{3}\pi u^{3/2}\Big|_1^5 = \frac{2}{3}\pi 5\sqrt{5} - \frac{2}{3}\pi 1\sqrt{1} = \frac{2}{3}\pi\left(5\sqrt{5} - 1\right). \quad \square$$

f. Plug and chug! Recall that the Right-hand Rule formula is

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a + i\frac{b-a}{n}\right),$$

so

$$\begin{aligned}
\int_0^1 4x\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1-0}{n} \cdot 4\left(0 + i\frac{1-0}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cdot 4\frac{i}{n} \\
&= \lim_{n\to\infty} \frac{4}{n^2} \sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{4}{n^2} \cdot \frac{n(n+1)}{2} = \lim_{n\to\infty} \frac{2(n+1)}{n} \\
&= \lim_{n\to\infty} \left(\frac{2n}{n} + \frac{2}{n}\right) = \lim_{n\to\infty} \left(2 + \frac{2}{n}\right) = 2 + 0 = 2. \quad \square
\end{aligned}$$

g. We need to verify that for every $\varepsilon > 0$, there is some $\delta > 0$, such that if $|x - 0| < \delta$, then $|(2x-1) - (-1)| < \varepsilon$. As usual, we will try to reverse-engineer the necessary δ from ε. Suppose an $\varepsilon > 0$ is given. Then

$$|(2x-1) - (-1)| < \varepsilon \Leftrightarrow |2x - 1 + 1| < \varepsilon \Leftrightarrow |2x| < \varepsilon \Leftrightarrow |x| < \frac{\varepsilon}{2} \Leftrightarrow |x - 0| < \frac{\varepsilon}{2},$$

so $\delta = \dfrac{\varepsilon}{2}$ will do the job. Note that every step of our reverse-engineering process above is reversible, so if $|x - 0| < \delta = \dfrac{\varepsilon}{2}$, then $|(2x-1) - (-1)| < \varepsilon$. $\square$

h. We will use the Ratio Test to find the radius of convergence.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{2(n+1)}}{(n+1)!}}{\frac{x^{2n}}{n!}}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+2}}{x^{2n}} \cdot \frac{n!}{(n+1)!}\right| = \lim_{n\to\infty} \left|\frac{x^2}{n+1}\right| = 0,$$

since $x^2 \to x^2$ and $n+1 \to \infty$ as $n \to \infty$. It follows that $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$ converges (absolutely) for all x, *i.e.* the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$. $\square$

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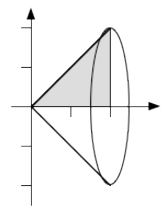
i. Here's a sketch of the solid:



We will use the disk/washer method to find its volume. Note that the cross-section of the solid for a fixed value of x is a washer outer radius $R = y - 0 = y = x$ and inner radius $r = 0$ (so it's really a disk). Thus

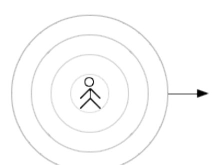
$$\begin{aligned} \text{Volume} &= \int_0^2 \pi \left(R^2 - r^2\right)\, dx = \int_0^2 \pi \left(x^2 - 0^2\right)\, dx \\ &= \int_0^2 \pi x^2 \, dx = \pi \frac{x^3}{3}\bigg|_0^2 = \pi\frac{2^3}{3} - \pi\frac{0^3}{3} = \frac{8}{3}\pi\,. \quad \square \end{aligned}$$

Part Y. Do any *three* (3) of **4**–**7**. *[45 = 3 × 15 each]*

4. A zombie is dropped into a still pool, creating a circular ripple that moves outward from the point of impact at a constant speed. After 2 s the length of the ripple is increasing at a rate of $2\pi\ m/s$. How is the area enclosed by the ripple changing at this instant?

Hint: You have (just!) enough information to work out how the radius of the ripple changes with time.

SOLUTION. Recall that the length, *i.e.* circumference, of a circle of radius r is $C = 2\pi r$, while the area it encloses is $A = \pi r^2$. In this case $r = r(t)$ changes with time: what we are given amounts to (1) $\frac{dr}{dt}$ is constant and (2) $\left.\frac{dC}{dt}\right|_{t=2} = 2\pi$, and what we are being asked to compute is $\left.\frac{dA}{dt}\right|_{t=2}$.



6

Ignoring the hint for the moment, observe that $\frac{dC}{dt} = \frac{d}{dt} 2\pi r = 2\pi \frac{dr}{dt}$. Then

$$\frac{dA}{dt} = \frac{d}{dt}\pi r^2 = \pi \frac{dr^2}{dr} \cdot \frac{dr}{dt} = \pi 2r \frac{dr}{dt} = r\left(2\pi \frac{dr}{dt}\right) = r\frac{dC}{dt},$$

so $\left.\frac{dA}{dt}\right|_{t=2} = r(2) \left.\frac{dC}{dt}\right|_{t=2} = 2\pi r(2)$. The problem is that we do not know what $r(2)$ is.

Following the hint, we will try to figure out the radius of the ripple, $r(t)$, as a function of time, and then compute $r(2)$. There is one more piece of information in the set-up that we have not made explicit yet: (0) $r(0) = 0$, since the ripple was only created at the instant that the zombie fell into the pool.

Now, since $\frac{dr}{dt}$ is constant, $r(t) = mt + b$ for some constants m and b. (Obviously, $m = \frac{dr}{dt}$.) Note that $b = m0 + b = r(0) = 0$, using the observation above, so $r(t) = mt$. Since

$$2\pi = \left.\frac{dC}{dt}\right|_{t=2} = 2\pi \left.\frac{dr}{dt}\right|_{t=2} = 2\pi m,$$

we have that $m = \frac{dr}{dt} = 1$, so $r(t) = t$. It follows that $r(2) = 2$, and thus $\left.\frac{dA}{dt}\right|_{t=2} = r(2) \left.\frac{dC}{dt}\right|_{t=2} = 2\pi r(2) = 2\pi \cdot 2 = 4\pi$. $\square$

5. Find all the intercepts, maximum, minimum, and inflection points, and all the vertical and horizontal asymptotes of $h(x) = \frac{x}{1-x^2}$, and sketch its graph.

Solution. *i. (y-intercept)* Plug in $x = 0$: $h(0) = \frac{0}{1-0^2} = \frac{0}{1} = 0$.

ii. (x-intercept) $y = h(x) = \frac{x}{1-x^2} = 0$ if and only if $x = 0$.

Note: Thus the sole x- and y-intercept is the point $(0, 0)$.

iii. (Vertical asymptotes) $h(x) = \frac{x}{1-x^2}$ is a rational function and hence defined and continuous whenever the denominator $1 - x^2 \neq 0$, so any vertical asymptotes could occur only when $1 - x^2 = 0$, *i.e.* when $x = \pm 1$. We'll take limits from both directions at $x = \pm 1$ and see what happens. Note that $1 - x^2 \to 0$ as $|x| \to 1$, so each of these limits must approach some infinity.

$$\begin{aligned}
\lim_{x\to -1^-} \frac{x}{1-x^2} &= +\infty \quad \text{since } 1 - x^2 < 0 \text{ when } x < -1; \\
\lim_{x\to -1^+} \frac{x}{1-x^2} &= -\infty \quad \text{since } 1 - x^2 > 0 \text{ when } -1 < x < 0; \\
\lim_{x\to 1^-} \frac{x}{1-x^2} &= +\infty \quad \text{since } 1 - x^2 > 0 \text{ when } 0 < x < 1; \\
\lim_{x\to 1^+} \frac{x}{1-x^2} &= -\infty \quad \text{since } 1 - x^2 < 0 \text{ when } x > 1;
\end{aligned}$$

Thus $h(x)$ has vertical asymptotes at both $x = -1$ and $x = +1$, heading up to $+\infty$ from the left and down to $-\infty$ from the right of each point.

iv. (Horizontal asymptotes) One could compute the necessary limits with the help of l'Hôpital's Rule, but it's pretty easy to do with a little algebra too:

$$\lim_{x\to-\infty} \frac{x}{1-x^2} = \lim_{x\to-\infty} \frac{x}{1-x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{\frac{x}{x^2}}{\frac{1}{x^2} - \frac{x^2}{x^2}} = \lim_{x\to-\infty} \frac{\frac{1}{x}}{\frac{1}{x^2} - 1} = \frac{0}{0-1} = 0$$

$$\lim_{x\to+\infty} \frac{x}{1-x^2} = \lim_{x\to+\infty} \frac{x}{1-x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{\frac{x}{x^2}}{\frac{1}{x^2} - \frac{x^2}{x^2}} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{\frac{1}{x^2} - 1} = \frac{0}{0-1} = 0$$

Thus $h(x)$ has the horizontal asymptote $y = 0$ in both directions.

v. (Maxima and minima) Note that $h(x)$ is defined and differentiable wherever it is defined because it is a rational function. First, we compute $h'(x)$ using the Quotient Rule:

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(\frac{x}{1-x^2}\right) = \frac{\frac{dx}{dx} \cdot \left(1-x^2\right) - x \cdot \frac{d}{dx}\left(1-x^2\right)}{\left(1-x^2\right)^2} \\ &= \frac{1 \cdot \left(1-x^2\right) - x \cdot (0-2x)}{\left(1-x^2\right)^2} = \frac{1-x^2+2x^2}{\left(1-x^2\right)^2} = \frac{1+x^2}{\left(1-x^2\right)^2} \end{aligned}$$

Second, we find all the critical points, *i.e.* the points where $h'(x)$ is 0 or undefined. $h'(x) = \dfrac{1+x^2}{\left(1-x^2\right)^2} = 0$ exactly when $1 + x^2 = 0$, which is to say never, since $1 + x^2 \geq 1 > 0$ for all x. $h'(x) = \dfrac{1+x^2}{\left(1-x^2\right)^2}$ is defined and continuous unless $1 - x^2 = 0$, *i.e.* when $x = \pm 1$. Since $h(x)$ has no critical points except where it has vertical asymptotes, $h(x)$ has no local maxima or minima.

Just for fun, here is the usual table for intervals of increase and decrease:

x	< -1	$\in (-1, 1)$	> 1
$h'(x)$	$-$	$+$	$-$
$h(x)$	$\downarrow$	$\uparrow$	$\downarrow$

vi. (Inflection points) First, we compute $h''(x)$ using the Quotient Rule:

$$\begin{aligned} h''(x) &= \frac{d}{dx}\left(\frac{1+x^2}{\left(1-x^2\right)^2}\right) = \frac{\frac{d}{dx}\left(1+x^2\right) \cdot \left(1-x^2\right)^2 - \left(1+x^2\right) \cdot \frac{d}{dx}\left(1-x^2\right)^2}{\left(1-x^2\right)^4} \\ &= \frac{2x \cdot \left(1-x^2\right)^2 - \left(1+x^2\right) \cdot 2\left(1-x^2\right)(0-2x)}{\left(1-x^2\right)^4} \\ &= \frac{2x \cdot \left(1-x^2\right) - \left(1+x^2\right) \cdot 2(-2x)}{\left(1-x^2\right)^3} = \frac{2x - 2x^3 + 4x + 4x^3}{\left(1-x^2\right)^3} \\ &= \frac{6x + 2x^3}{\left(1-x^2\right)^3} = \frac{2x\left(3+x^2\right)}{\left(1-x^2\right)^3} \end{aligned}$$

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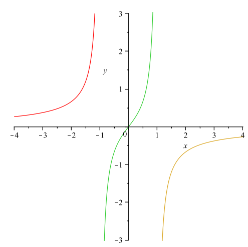
Second, we find all the potential inflection points, *i.e.* the points where $h''(x)$ is 0 or undefined. $h''(x) = \dfrac{2x\left(3+x^2\right)}{\left(1-x^2\right)^3} = 0$ exactly when either $x = 0$ or when $3 + x^2 = 0$ (which can't happen because $3 + x^2 \geq 3 > 0$ for all x). $h''(x) = \dfrac{2x\left(3+x^2\right)}{\left(1-x^2\right)^3}$ is defined and continuous unless $1 - x^2 = 0$, *i.e.* when $x = \pm 1$, which are the points where $h(x)$ has vertical asymptotes. Thus $x = 0$ is the only possible point of inflection. Third, we test it to see if it really is, building the usual table:

x	< -1	$\in (-1, 0)$	0	$\in (0, 1)$	> 1
$h''(x)$	$+$	$-$	0	$+$	1
concavity	$\uparrow$	$\downarrow$		$\uparrow$	$\downarrow$

Thus $x = 0$ is indeed an inflection point of $h(x)$.

vii. (Graph) Here is the graph of $h(x) = \frac{x}{1-x^2}$:



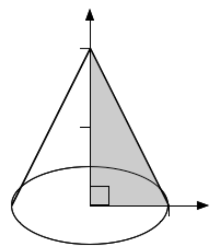
Cheating just a little, it was generated using the following `Maple` command:

```
> plot([[s,s/(1-s^2),s=-4..-1],[s,s/(1-s^2),s=-1..1],
  [s,s/(1-s^2),s=1..4]],x=-4..4,y=-3..3);
```

□

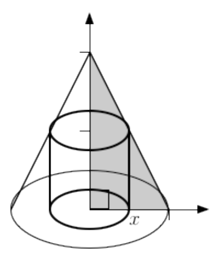
6. Show that a cone with base radius 1 and height 2 has volume $\frac{2}{3}\pi$.
Hint: It's a solid of revolution ...

Solution. Following the hint, we need to realize the cone as a solid of revolution. One can obtain a ("right-circular") cone by rotating a right-angled triangle whose base is a radius of the base of the cone and whose altitude is the axis of symmetry of the cone. The hypotenuse of the triangle then runs along the surface of the cone from the tip to the base. We will set up our coordinate system so that the origin is at the right angle of the triangle, the x-axis runs along the base (the side that is the radius of the cone), and the y-axis runs along the altitude (the side that is the axis of symmetry of the cone). Since the cone has



base radius 1 and height 2, the hypotenuse of the triangle is the line segment joining $(0,2)$ to $(1,0)$.as in the diagram. It is not hard to work out that the equation of the line forming the hypotenuse of the triangle is $y = -2x + 2$.

Thus we are trying to find the volume of the solid of revolution obtained by revolving the region bounded by $x = 0$, $y = 0$, and $y = -2x + 2$ about the y-axis, *i.e.* about the line $x = 0$. The volume can be computed readily using either the disk/washer method or the method of cylindrical shells. Since we used disks in the solution to **3i** (note that the solid of revolution there was a cone as well), we will use cylindrical shells here.



Since we revolved the region about the y-axis, a typical shell is obtained by revolving a vertical cross-section of the region, that is, one for a fixed value of x, about the y-axis. The shell for a given value of x has radius $r = x$ and height $h = y - 0 = -2x + 2$, and so has surface area $2\pi rh = 2\pi x(-2x+2) = 4\pi\left(x - x^2\right)$. Note that since we revolved the region about the y-axis and are using shells, we need to integrate with respect to x, and the possible values of x in the triangle are $0 \leq x \leq 1$. Thus the volume of the cone is:

$$\begin{aligned}\text{Volume} &= \int_0^1 2\pi rh\,dx = \int_0^1 4\pi\left(x - x^2\right)\,dx = 4\pi\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 \\ &= 4\pi\left[\left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right)\right] = 4\pi\left(\left[\frac{1}{2} - \frac{1}{3}\right] - 0\right) \\ &= 4\pi\frac{1}{6} = \frac{2}{3}\pi \quad \Box\end{aligned}$$

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7. Do all *four* (4) of **a**–**d**.

a. Use Taylor's formula to find the Taylor series of $f(x) = \sin(x)$ at $a = 0$. *[7]*

b. Determine the radius and interval of convergence of this Taylor series. *[4]*

c. Find the Taylor series of $g(x) = x\sin(x)$ at $a = 0$ by multiplying the Taylor series for $f(x) = \sin(x)$ by x. *[1]*

d. Use Taylor's formula and your series from **c** to compute $g^{(16)}(0)$. *[3]*

SOLUTION TO **a**. Recall that the Taylor series of $f(x)$ at $a = 0$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$, so we'll need to know what the values of the nth derivatives of $f(x) = \sin(x)$ are at 0:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin(x)$	0
1	$\cos(x)$	1
2	$-\sin(x)$	0
3	$-\cos(x)$	-1
4	$\sin(x)$	0
5	$\cos(x)$	1
6	$-\sin(x)$	0
$\vdots$	$\vdots$	$\vdots$
$2k$	$(-1)^k \sin(x)$	0
$2k+1$	$(-1)^k \cos(x)$	$(-1)^k$
$\vdots$	$\vdots$	$\vdots$

Note that $f^{(n)}(0) = 0$ when n is even, and alternates between 1 and -1 when n is odd. It follows that the Taylor series of $f(x) = \sin(x)$ at $a = 0$ is $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$. $\square$

SOLUTION TO **b**. We will use the Ratio Test to find the radius of convergence.

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{\frac{(-1)^{k+1}}{(2(k+1)+1)!} x^{2(k+1)+1}}{\frac{(-1)^k}{(2k+1)!} x^{2k+1}} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{(-1)^k x^{2k+1}} \right|$$
$$= \lim_{k\to\infty} \left| \frac{(-1)x^2}{(2k+3)(2k+2)} \right| = \lim_{k\to\infty} \frac{x^2}{(2k+3)(2k+2)} = 0\,,$$

since $x^2 \to x^2$ and $(2k+3)(2k+2) \to \infty$ as $k \to \infty$. It follows that the series converges (absolutely) for all x, *i.e.* the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$. $\square$

SOLUTION TO **c**. If $g(x) = x\sin(x)$ then the Taylor series at $a = 0$ of $g(x)$ should be the Taylor series for $f(x) = \sin(x)$ multiplied by by x:

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n = x \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+2} \quad \square$$

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SOLUTION TO **d**. By Taylor's formula the coefficient of x^{16} in the Taylor series of $g(x)$ is $\dfrac{g^{(16)}(0)}{16!}$. x^{16} occurs in $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+2}$ when $2k+2 = 16$, *i.e.* when $k = 7$. The coefficient of x^{16} is therefore $\dfrac{(-1)^7}{(2 \cdot 7 + 1)!} = \dfrac{-1}{15!}$. We compare this to the coefficient given by Taylor's formula, $\dfrac{g^{(16)}(0)}{16!} = \dfrac{-1}{15!}$, and solve for $g^{(16)}(0) = 16! \dfrac{-1}{15!} = -16$. $\square$

[Total = 100]

Part Z. Bonus problems! Do them (or not), if you feel like it.

$\ln\left(\frac{1}{e}\right)$**.** Does $\displaystyle\lim_{n\to\infty} \left[\left(\sum_{k=1}^{n} \frac{1}{k}\right) - \ln(n)\right]$ exist? Explain why or why not. *[2]*

SOLUTION. It does exist. The real number computed by this limit is often called the Euler-Mascheroni constant, about which surprisingly little is known. (For example, we don't even know if it is rational or irrational.) The Wikipedia article about it at

`http://en.wikipedia.org/wiki/Euler-mascheroni_constant`

is good place to start learning about it. A little searching (not in the article just referred to, although the graph there interpreting the limit as an area is a clue) should turn up a proof the limit exists. $\square$

$\ln\left(\frac{1}{1}\right)$**.** Write a haiku touching on calculus or mathematics in general. *[2]*

haiku?

seventeen in three:
five and seven and five of
syllables in lines

SOLUTION. You're on your own on this one! $\square$

Mathematics 1100Y
Calculus I: Calculus of one variable
Summer 2011
Solutions

Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2011

Solutions to the Quizzes

Quiz #1. Wednesday, 11 May, 2011. *[10 minutes]*

1. Compute $\displaystyle\lim_{x\to-3}\frac{x+3}{x^2-9}$ using the appropriate limit laws and algebra. *[5]*

SOLUTION. Note that $\lim_{x\to-3}(x+3) = -3+3 = 0$ and $\lim_{x\to-3}(x^2-9) = (-3)^2-9 = 9-9 = 0$, so we can't use the limit law for quotients lest we divide by 0. We will resort to algebra instead, using the fact that $x^2-9 = (x+3)(x-3)$. Then

$$\begin{aligned}\lim_{x\to-3}\frac{x+3}{x^2-9} &= \lim_{x\to-3}\frac{x+3}{(x+3)(x-3)} = \lim_{x\to-3}\frac{1}{x-3}\\ &= \frac{1}{\lim_{x\to-3}(x-3)} = \frac{1}{-3-3} = \frac{1}{-6} = -\frac{1}{6}\,. \quad \square\end{aligned}$$

Quiz #2. Monday, 16 May, 2011. *[10 minutes]*

Do *one* of questions 1 or 2.

1. Use the ε-δ definition of limits to verify that $\lim_{x\to1}(3x-2) = 1$. *[5]*

SOLUTION. We need to check that for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x-1| < \delta$, then $|(3x-2)-1| < \varepsilon$. As usual, we will try to reverse-engineer the required δ. Suppose $\varepsilon > 0$ is given. Then

$$\begin{aligned}|(3x-2)-1| < \varepsilon \quad &\iff \quad |3x-3| < \varepsilon\\ &\iff \quad |3(x-1)| < \varepsilon\\ &\iff \quad |x-1| < \frac{\varepsilon}{3}\,.\end{aligned}$$

Since very step above is reversible, if $|x-1| < \frac{\varepsilon}{3}$, then $|(3x-2)-1| < \varepsilon$. Hence $\delta = \frac{\varepsilon}{3}$ wil do the job, and so, by the ε-δ definition of limits, $\lim_{x\to1}(3x-2) = 1$. $\square$

2. Find the x- and y-intercepts and all the horizontal asymptotes of $f(x) = \dfrac{x^2}{x^2+1}$, and sketch its graph. *[5]*

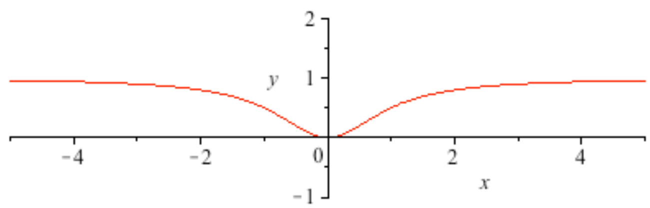
SOLUTION. *i. (Intercepts)* Since $f(0) = \dfrac{0^2}{0^2+1} = \dfrac{0}{1} = 0$, the y-intercept is the point $(0,0)$. $f(x) = \dfrac{x^2}{x^2+1} = 0$ is only possible when $x = 0$, so the point $(0,0)$ is also the only x-intercept.

ii. (Horizontal asymptotes) We check the limits of $f(x)$ as $x \to \pm\infty$:

$$\lim_{x\to+\infty} \frac{x^2}{x^2+1} = \lim_{x\to+\infty} \frac{x^2}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+0} = 1$$
$$\lim_{x\to-\infty} \frac{x^2}{x^2+1} = \lim_{x\to-\infty} \frac{x^2}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+0} = 1$$

(Note that $\frac{1}{x^2} \to 0$ as $x \to \pm\infty$.) It follows that $f(x)$ has $y = 1$ as a horizontal asymptote in both directions.

iii. (Graph) Note that since $x^2 < x^2 + 1$, $0 \le \dfrac{x^2}{x^2+1} < 1$. Together with the information obtained above, this means that the graph of $f(x)$ looks like:



I cheated a bit to draw this, of course; this graph was made by `Maple` with the command:

```
> plot(x^2/(x^2+1), x=-5..5, y=-1..2);
```

The ">" is `Maple`'s prompt. □

Quiz #3. Wednesday, 18 May, 2011. *[10 minutes]*

1. Use the limit definition of the derivative to compute $f'(a)$ for $f(x) = \dfrac{1}{x}$. (You may assume that $a \neq 0$.) *[5]*

SOLUTION. By the limit definition of the derivative,

$$\begin{aligned} f'(a) &= \lim_{h\to0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to0} \frac{\frac{1}{a+h}-\frac{1}{a}}{h} = \lim_{h\to0} \frac{1}{h} \cdot \frac{a-(a+h)}{(a+h)a} \\ &= \lim_{h\to0} \frac{1}{h} \cdot \frac{-h}{(a+h)a} = \lim_{h\to0} \frac{-1}{(a+h)a} = \frac{-1}{(a+0)a} = \frac{-1}{a^2}. \qquad \square \end{aligned}$$

Quiz #4. Wednesday, 25 May, 2011. *[10 minutes]*

1. Compute $f'(x)$ for $f(x) = \ln\left(\frac{x}{1+x^2}\right)$. *[5]*

Solution. Using the Chain Rule and Quotient Rule at the key steps, and simplifying as much as can easily be done at the end:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\ln\left(\frac{x}{1+x^2}\right)\right) \\ &= \frac{1}{\frac{x}{1+x^2}} \cdot \frac{d}{dx}\left(\frac{x}{1+x^2}\right) \\ &= \frac{1+x^2}{x} \cdot \frac{\frac{dx}{dx}\cdot\left(1+x^2\right) - x \cdot \frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2} \\ &= \frac{1}{x} \cdot \frac{1\cdot\left(1+x^2\right) - x\cdot(0+2x)}{1+x^2} \\ &= \frac{1}{x} \cdot \frac{1+x^2-2x^2}{1+x^2} \\ &= \frac{1}{x} \cdot \frac{1-x^2}{1+x^2} \\ &= \frac{1-x^2}{x\left(1+x^2\right)} \qquad \square \end{aligned}$$

Quiz #5. Monday, 30 May, 2011. *[10 minutes]*

Do *one* of questions 1 or 2.

1. Find $\frac{dy}{dx}$ at the point $(2,2)$ on the curve defined by $x = \sqrt{x+y}$. *[5]*

Solution i. We differentiate both sides of the equation defining the curve,

$$1 = \frac{dx}{dx} = \frac{d}{dx}\sqrt{x+y} = \frac{1}{2\sqrt{x+y}} \cdot \frac{d}{dx}(x+y) = \frac{1}{2\sqrt{x+y}} \cdot \left(\frac{dx}{dx} + \frac{dy}{dx}\right) = \frac{1+\frac{dy}{dx}}{2\sqrt{x+y}}\,,$$

and then solve for $\frac{dy}{dx}$,

$$\frac{1+\frac{dy}{dx}}{2\sqrt{x+y}} = 1 \quad\Longrightarrow\quad 1 + \frac{dy}{dx} = 2\sqrt{x+y} \quad\Longrightarrow\quad \frac{dy}{dx} = 2\sqrt{x+y} - 1\,.$$

It follows that $\left.\frac{dy}{dx}\right|_{(x,y)=(2,2)} = 2\sqrt{2+2} - 1 = 2\sqrt{4} - 1 = 2\cdot 2 - 1 = 4 - 1 = 3$. $\square$

Solution ii. We solve for y first,

$$x = \sqrt{x+y} \quad\Longrightarrow\quad x^2 = x + y \quad\Longrightarrow\quad y = x^2 - x\,,$$

and then differentiate, $\frac{dy}{dx} = \frac{d}{dx}\left(x^2 - x\right) = 2x - 1$. It follows that $\left.\frac{dy}{dx}\right|_{(x,y)=(2,2)} = 2 \cdot 2 - 1 = 4 - 1 = 3$. $\square$

2. Find $\frac{dy}{dx}$ at $x = e$ for $y = \ln(x\ln(x))$. *[5]*

Solution i. We differentiate directly, using the Chain and Product Rules:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\ln(x\ln(x)) = \frac{1}{x\ln(x)} \cdot \frac{d}{dx}(x\ln(x)) = \frac{1}{x\ln(x)} \cdot \left(\left[\frac{dx}{dx}\right] \cdot \ln(x) + x \cdot \left[\frac{d}{dx}\ln(x)\right]\right) \\ &= \frac{1}{x\ln(x)} \cdot \left(1 \cdot \ln(x) + x \cdot \frac{1}{x}\right) = \frac{\ln(x)+1}{x\ln(x)}\end{aligned}$$

It follows that $\left.\frac{dy}{dx}\right|_{x=e} = \frac{\ln(e)+1}{e\ln(e)} = \frac{1+1}{e \cdot 1} = \frac{2}{e}$. $\square$

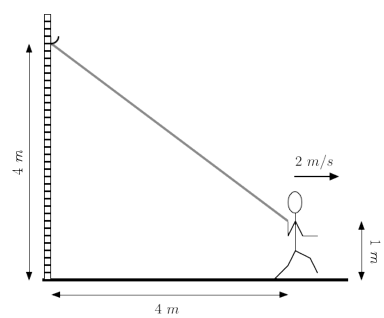
Solution ii. We rewrite y using the properties of logarithms, $y = \ln(x\ln(x)) = \ln(x) + \ln(\ln(x))$, and then differentiate, using the Sum and Chain Rules:

$$\frac{dy}{dx} = \frac{d}{dx}(\ln(x) + \ln(\ln(x))) = \frac{1}{x} + \frac{1}{\ln(x)} \cdot \frac{d}{dx}\ln(x) = \frac{1}{x} + \frac{1}{\ln(x)} \cdot \frac{1}{x} = \frac{1}{x}\left(1 + \frac{1}{\ln(x)}\right)$$

It follows that $\left.\frac{dy}{dx}\right|_{x=e} = \frac{1}{e}\left(1 + \frac{1}{\ln(e)}\right) = \frac{1}{e}\left(1 + \frac{1}{1}\right) = \frac{2}{e}$. $\square$

Quiz #6. Wednesday, 1 June, 2011. *[15 minutes]*

1. A $3\,m$ long, very stretchy, bungee cord is suspended from a hook $4\,m$ up on a wall. The other end of the cord is grabbed by a child who runs directly away from the wall at $2\,m/s$, holding the end of the cord $1\,m$ off the ground, stretching the cord in the process. How is the length of the cord changing at the instant that the child's end of the cord is $4\,m$ away from the wall? *[5]*



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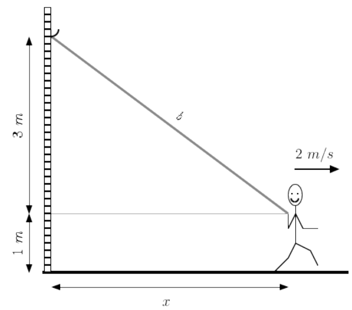
Solution. Call the distance the child is from the wall x and observe that the bungee cord forms the hypotenuse of a right triangle with base x and height $4 - 1 = 3$. (Recall that the child holds the end of the cord $1\,m$ above the ground, while the top end of the cord is attached to the wall $4\,m$ above the ground.) The fact that the child is running at $2\,m/s$ amounts to saying that $\frac{dx}{dt} = 2$.



If we denote the length of the bungee cord by b, then we have $x^2 + 3^2 = b^2$ by the Pythagorean theorem, and are trying to discover what $\frac{db}{dt}$ is when $x = 4$.

We first differentiate both sides of the equation,

$$x^2 + 3^2 = b^2 \quad \Longrightarrow \quad \frac{d}{dt}\left(x^2 + 3^2\right) = \frac{d}{dt}b^2 \quad \Longrightarrow \quad 2x\frac{dx}{dt} + 0 = 2b\frac{db}{dt}\,,$$

and solve for $\frac{db}{dt}$,

$$\frac{db}{dt} = \frac{2x}{2b} \cdot \frac{dx}{dt} = \frac{x}{b} \cdot \frac{dx}{dt}\,.$$

We know that $\frac{dx}{dt} = 2$ and that $x = 4$ at the instant in question. It follows from $x^2 + 3^2 = b^2$ that $b = \sqrt{x^2 + 3^2}$, so $b = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$ when $x = 4$. Thus, when $x = 4$, $\frac{db}{dt} = \frac{x}{b} \cdot \frac{dx}{dt} = \frac{4}{5} \cdot 2 = \frac{8}{5} = 1.6\,m/s$. $\square$

Quiz #7. Monday, 6 June, 2011. *[15 minutes]*

1. Find any and all intercepts, intervals of increase and decrease, local maxima and minima, and vertical and horizontal asymptotes, of $y = xe^{-x}$, and sketch this curve based on the information you obtained. *[5]*

Bonus: Find any and all the points of inflection of this curve too. *[1]*

Hint: You may assume that $\lim_{x\to+\infty} xe^{-x} = 0$. For $\lim_{x\to-\infty} xe^{-x}$ you're on your own.

SOLUTION. Here goes!

i. *(Domain)* The expression xe^{-x} is defined for all x. Since it is a product of continuous functions, it is continuous too. It follows, in particular, that there are no vertical asymptotes.

ii. *(Intercepts)* For the y-intercept we plug in $x = 0$, then $y = 0e^{-0} = 0 \cdot 1 = 0$. For the x-intercept, note that since $e^{-x} > 0$ for all x, $xe^{-x} = 0$ exactly when $x = 0$. Thus the origin, $(0, 0)$ is both the y- and the only x-intercept.

iii. *(Vertical asymptotes)* None, for the reasons noted in i above.

iv. *(Horizontal asymptotes)* The hint tells us that we can assume that $\lim_{x\to+\infty} xe^{-x} = 0$, so curve has the horizontal asymptote $y = 0$ in the +ve direction. On the other hand, since $e^{-x} \to +\infty$ as $x \to -\infty$, $\lim_{x\to-\infty} xe^{-x} = -\infty$, there is no horizontal asymptote in the −ve direction.

v. *(Increase, decrease, maxima, & minima)* First, we differentiate, using the Product and Chain Rules:

$$\begin{aligned}\frac{dy}{dx} = \frac{d}{dx}\left(xe^{-x}\right) &= \left(\frac{d}{dx}x\right)\cdot e^{-x} + x \cdot \left(\frac{d}{dx}e^{-x}\right) = 1 \cdot e^{-x} + x \cdot e^{-x} \cdot \frac{d}{dx}(-x) \\ &= e^{-x} + xe^{-x}(-1) = (1-x)e^{-x}\end{aligned}$$

Since $e^{-x} > 0$ for all x, it follows that:

$$\frac{dy}{dx} = (1-x)e^{-x} \begin{array}{c}>\\=\\<\end{array} 0 \iff (1-x) \begin{array}{c}>\\=\\<\end{array} 0 \iff x \begin{array}{c}<\\=\\>\end{array} 1$$

Building the usual table:

x	$(-\infty, 1)$	1	$(1, \infty)$
y	$\uparrow$	max	$\downarrow$
$\frac{dy}{dx}$	$+$	0	$-$

In words: $y = xe^{-x}$ is increasing on $(\infty, 1)$, has a maximum at 1, and is decreasing on $(1, +\infty)$. Note that since $y = xe^{-x}$ is defined and continuous everywhere, the maximum is an absolute maximum. From our work looking horizontal asymptotes in iv above, it is clear that there is no absolute minimum.

vi. *(Bonus – Points of inflection)* We differentiate $\frac{dy}{dx}$, using the Product and Chain Rules, to get $\frac{d^2y}{dx^2}$:

$$\begin{aligned}\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left((1-x)e^{-x}\right) &= \left(\frac{d}{dx}(1-x)\right)\cdot e^{-x} + (1-x)\cdot\left(\frac{d}{dx}e^{-x}\right) \\ &= (-1)\cdot e^{-x} + (1-x)\cdot e^{-x}\cdot\frac{d}{dx}(-x) \\ &= -e^{-x} + (1-x)e^{-x}(-1) = (x-2)e^{-x}\end{aligned}$$

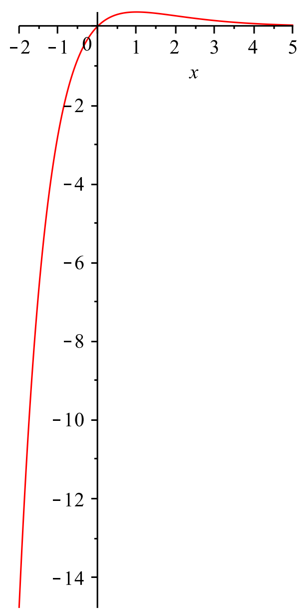
Since $e^{-x} > 0$ for all x, it follows that:

$$\frac{d^2y}{dx^2} = (x-2)e^{-x} \begin{array}{c}>\\=\\<\end{array} 0 \iff (x-2) \begin{array}{c}>\\=\\<\end{array} 0 \iff x \begin{array}{c}>\\=\\<\end{array} 2$$

Thus $y = xe^{-x}$ is concave down on $(-\infty, 2)$ and concave up on $(2, +\infty)$, so $x = 2$ gives the only point of inflection.

vii. *(Graph)* The following was drawn using `Maple` with the command

```
> plot(x*exp(-x),x=-2..5);
```



That's all, folks! □

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Quiz #8. Monday, 13 June, 2011. *[10 minutes]*

1. Compute $\lim_{x \to \infty} \frac{x^2}{e^x}$. *[5]*

SOLUTION. Since $x^2 \to \infty$ and $e^x \to \infty$ as $x \to \infty$, we can apply l'Hôpital's Rule.

$$\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{\frac{d}{dx} x^2}{\frac{d}{dx} e^x} = \lim_{x \to \infty} \frac{2x}{e^x}$$

At this point, note that $2x \to \infty$ and $e^x \to \infty$ as $x \to \infty$, so we can apply l'Hôpital's Rule again. Thus

$$\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{\frac{d}{dx}(2x)}{\frac{d}{dx} e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0,$$

since $2 \to 2$ and $e^x \to \infty$ as $x \to \infty$. $\square$

Quiz #9. Monday, 20 June, 2011. *[10 minutes]*

Do *one* of questions 1, 2, or 3.

1. Compute $\int_1^2 (x+1)\,dx$ using the Right-Hand Rule. *[5]*

   Hint: You may assume that $1 + 2 + 3 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

SOLUTION. We'll throw the Right-Hand Rule formula,

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + i\frac{b-a}{n}\right),$$

at this problem and hope we survive the algebra! Note that $f(x) = x + 1$ in this case.

$$\begin{aligned}
\int_1^2 (x+1)\,dx &= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2-1}{n} f\left(1 + i\frac{2-1}{n}\right) = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} f\left(1 + \frac{i}{n}\right) \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n}\left(\left(1 + \frac{i}{n}\right) + 1\right) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left(2 + \frac{i}{n}\right) \\
&= \lim_{n \to \infty} \frac{1}{n}\left[\left(\sum_{i=1}^{n} 2\right) + \left(\sum_{i=1}^{n} \frac{i}{n}\right)\right] = \lim_{n \to \infty} \frac{1}{n}\left[2n + \frac{1}{n}\left(\sum_{i=1}^{n} i\right)\right] \\
&= \lim_{n \to \infty} \frac{1}{n}\left[2n + \frac{1}{n} \cdot \frac{n(n+1)}{2}\right] = \lim_{n \to \infty} \frac{1}{n}\left[2n + \frac{n+1}{2}\right] \\
&= \lim_{n \to \infty} \frac{1}{n}\left[\frac{5}{2}n + \frac{1}{2}\right] = \lim_{n \to \infty}\left[\frac{5}{2} + \frac{1}{2n}\right] = \frac{5}{2} + 0 = \frac{5}{2},
\end{aligned}$$

since $\frac{1}{2n} \to 0$ as $n \to \infty$. $\square$

2. Compute $\int_{-1}^{3}(x+1)^2\,dx$. *[5]*

SOLUTION. We'll expand $(x+1)^2$ and then use the Power, Sum, and Constant Rules (all at once!) to find the antiderivative and (implicitly!) apply the Fundamental Theorem of Calculus to evaluate the definite integral.

$$\begin{aligned}\int_{-1}^{3}(x+1)^2\,dx &= \int_{-1}^{3}\left(x^2+2x+1\right)\,dx = \left(\frac{x^3}{3}+2\frac{x^2}{2}+x\right)\Bigg|_{-1}^{3}\\ &= \left(\frac{3^3}{3}+3^2+3\right)-\left(\frac{(-1)^3}{3}+(-1)^2+(-1)\right)\\ &= (9+9+3)-\left(-\frac{1}{3}+1-1\right) = 21-\left(-\frac{1}{3}\right)\\ &= \frac{63}{3}+\frac{1}{3}=\frac{64}{3}\end{aligned}$$

Note that because we were finding the antiderivative to evaluate an indefinite integral, we did not worry about adding a generic constant of C to the antiderivative. (If we had, it would have cancelled out when we evaluated the definite integral anyway.) □

3. Compute $\int \sin(x)\cos(x)\,dx$. *[5]*

SOLUTION. This is a job for the Substitution and Power Rules. We will use the substitution $u=\sin(x)$, so $\dfrac{du}{dx}=\sin(x)$ and $du=\sin(x)\,dx$:

$$\int \sin(x)\cos(x)\,dx = \int u\,du = \frac{u^2}{2}+C=\frac{\sin^2(x)}{2}+C$$

Note that since we were computing an indefinite integral, we need to find the most general possible antiderivative and so must add the generic constant C. □

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Quiz #10. Wednesday, 22 June, 2011. *[10 minutes]*

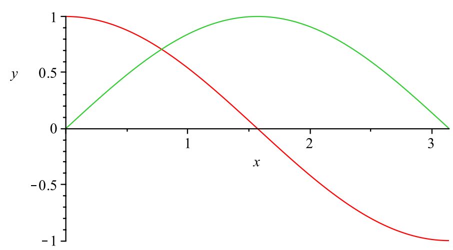
1. Find the area of the region between the curves $y = \cos(x)$ and $y = \sin(x)$, where $0 \leq x \leq \pi$. *[5]*

   Hint: Recall that $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

SOLUTION. From $x = 0$ to $x = \pi$, $\cos(x)$ starts at 1, decreases to 0 at $x = \frac{\pi}{2}$, and decreases some more to -1, while $\sin(x)$ starts at 0, increases to 1 at $x = \frac{\pi}{2}$, and then decreases to 0. The two graphs cross once in the interval $[0, \pi]$: $\cos(x) = \sin(x)$ at $x = \frac{\pi}{4}$. *(Hint!)* It follows that $\cos(x) \geq \sin(x)$ for $0 \leq x \leq \frac{\pi}{4}$, and $\sin(x) \geq \cos(x)$ for $\frac{pi}{4} \leq x \leq \pi$.

It is a little easier to see all this if you can visualize or draw the graphs. Here is a graph produced by `Maple`:

```
> plot([[t,cos(t),t=0..Pi],[t,sin(t),t=0..Pi]],x=0..Pi,y=-1..1);
```



We will break up the integral according to our analysis above:

$$\begin{aligned}
\text{Area} &= \int_0^{\pi/4} [\cos(x) - \sin(x)]\; dx + \int_{\pi/4}^{\pi} [\sin(x) - \cos(x)]\; dx \\
&= [\sin(x) - (-\cos(x))]\big|_0^{\pi/4} + [(-\cos(x)) - \sin(x)]\big|_{\pi/4}^{\pi} \\
&= [\sin(x) + \cos(x)]\big|_0^{\pi/4} + [-\cos(x) - \sin(x)]\big|_{\pi/4}^{\pi} \\
&= \left[\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\right] - [\sin(0) + \cos(0)] \\
&\qquad + [-\cos(\pi) - \sin(\pi)] - \left[-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right] \\
&= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] - [0 + 1] + [-(-1) - 0] - \left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right] \\
&= \frac{2}{\sqrt{2}} - 1 + 1 - \left[-\frac{2}{\sqrt{2}}\right] = \frac{4}{\sqrt{2}} = 2\sqrt{2} \qquad \square
\end{aligned}$$

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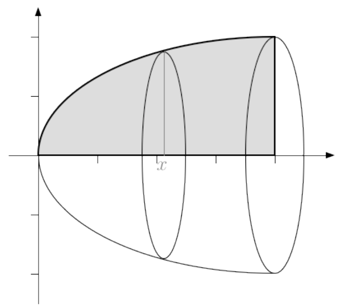
Quiz #11. Monday, 27 June, 2011. *[10 minutes]*

1. Sketch the solid obtained by rotating the region bounded by $y = \sqrt{x}$ and $y = 0$, where $0 \leq x \leq 4$, about the x-axis and find its volume. *[5]*

SOLUTION. Here's a sketch of the solid, with the original region shaded in:



We will use the washer method to find the volume of the solid. Since we are using washers and rotated about a horizontal line, we will be integrating with respect to x, the variable belonging to the horizontal axis. The cross section of the solid at x has outside radius $R = y = \sqrt{x}$ and inside radius $r = 0$ (so it is actually a disk). Plugging these into the volume formula for the washer method gives:

$$\begin{aligned}\text{Volume} &= \int_0^4 \pi \left(R^2 - r^2\right)\, dx = \pi \int_0^4 \left(\left(\sqrt{x}\right)^2 - 0^2\right)\, dx = \pi \int_0^4 (x - 0)\, dx \\ &= \pi \int_0^4 x\, dx = \pi \left.\frac{x^2}{2}\right|_0^4 = \pi \left(\frac{4^2}{2} - \frac{0^2}{2}\right) = \pi \left(\frac{16}{2} - 0\right) = 8\pi\end{aligned}$$

Note that the limits for the integral, as always, come from the original region. □

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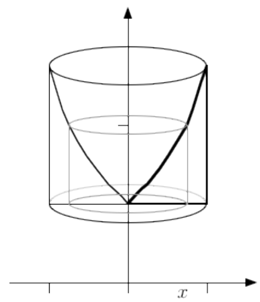
Quiz #12. Wednesday, 29 June, 2011. *[10 minutes]*

1. Sketch the solid obtained by rotating the region between $y = e^x$ and $y = 1$, where $0 \leq x \leq 1$, about the y-axis and find its volume. *[5]*

Solution. Here's a sketch of the solid, with the original region outlined in a bolder line and with a typical cylindrical shell drawn in as well:



We will use the method of cylindrical shells to find the volume of the solid. Since we are using shells and rotated about a vertical line (the y-axis, otherwise known as $x = 0$), we will be integrating with respect to x, the variable belonging to the horizontal axis. The cylindrical shell at x has radius $r = x - 0 = x$ and height $h = e^x - 1$. Plugging these into the volume formula for the shell method gives:

$$\begin{aligned}
\text{Volume} &= \int_0^1 2\pi r h \, dx = \int_0^1 2\pi x \left(e^x - 1\right) \, dx = 2\pi \int_0^1 \left(xe^x - x\right) \, dx \\
&= 2\pi \int_0^1 xe^x \, dx - 2\pi \int_0^1 x \, dx \qquad \text{Use integration by parts on the} \\
&\qquad \text{first piece, with } u = x \text{ and } v' = e^x \text{, so } u' = 1 \text{ and } v = e^x. \\
&\qquad \text{(Recall the integration by parts formula: } \int uv' \, dx = uv - \int u'v \, dx.\text{)} \\
&= 2\pi \left[xe^x \Big|_0^1 - \int_0^1 1e^x \, dx \right] - 2\pi \left. \frac{x^2}{2} \right|_0^1 = 2\pi \left[1e^e - 0e^0 - e^x \Big|_0^1 \right] - 2\pi \left[\frac{1^2}{2} - \frac{0^2}{2} \right] \\
&= 2\pi \left[e - 0 - \left(e^1 - e^0\right) \right] - 2\pi \left[\frac{1}{2} - 0 \right] = 2\pi[e - e + 1] - 2\pi \frac{1}{2} = 2\pi - \pi = \pi
\end{aligned}$$

Note that the limits for the integral, as always, come from the original region. $\square$

Quiz #13. Monday, 4 July, 2011. *[10 minutes]*

1. Compute $\displaystyle\int \sec^3(x)\tan^3(x)\,dx$. *[5]*

Solution. We'll use a combination of the trig identity $\tan^2(x) = \sec^2(x) - 1$ and the fact that $\dfrac{d}{dx}\sec(x) = \sec(x)\tan(x)$ to set up a suitable substitution:

$$\begin{aligned}
\int \sec^3(x)\tan^3(x)\,dx &= \int \sec^2(x)\tan^2(x)\sec(x)\tan(x)\,dx \\
&= \int \sec^2(x)\left(\sec^2(x)-1\right)\sec(x)\tan(x)\,dx \\
&\qquad \text{Substitute } u = \sec(x)\text{, so } du = \sec(x)\tan(x)\,dx. \\
&= \int u^2\left(u^2-1\right)\,du = \int \left(u^4-u^2\right)\,du \\
&= \frac{u^5}{5} - \frac{u^3}{5} + C = \frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{5} + C \qquad \square
\end{aligned}$$

Quiz #14. Monday, 11 July, 2011. *[15 minutes]*

1. Compute $\displaystyle\int \frac{1}{x^4+x^2}\,dx$. *[5]*

Solution. Since the integrand is a rational function, we will compute the integral using partial fractions. The first thing that we need to do is to factor the denominator. It is easy to see that $x^4 + x^2 = \left(x^2+1\right)x^2$.

Since x^2+1 is irreducible (as $x^2+1 \geq 1 > 0$ for all x, it has no roots) and $x^2 = (x-0)^2$ is a repeated linear factor, the partial fraction decomposition of the integrand will have the form:

$$\frac{1}{x^4+x^2} = \frac{1}{(x^2+1)\,x^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x^2} + \frac{D}{x}$$

Next, we need to work out what the constants A, B, C, and D are. Combining the right-hand side of the above equation over the common denominator of $\left(x^2+1\right)x^2$ and collecting like terms in the numerator gives:

$$\begin{aligned}
\frac{1}{(x^2+1)\,x^2} &= \frac{Ax+B}{x^2+1} + \frac{C}{x^2} + \frac{D}{x} = \frac{(Ax+B)x^2 + C\left(x^2+1\right) + D\left(x^2+1\right)x}{(x^2+1)\,x^2} \\
&= \frac{Ax^3 + Bx^2 + Cx^2 + C + Dx^3 + Dx}{(x^2+1)\,x^2} = \frac{(A+D)x^3 + (B+C)x^2 + Dx + C}{(x^2+1)\,x^2}
\end{aligned}$$

Comparing numerators, we have $1 = (A+D)x^3 + (B+C)x^2 + Dx + C$, so we must have $A+D=0$, $B+C=0$, $D=0$, and $C=1$. It follows that $A+0=0$, so $A=0$, and $B+1=0$, so $B=-1$. Thus the partial fraction decomposition of the given integrand is:

$$\frac{1}{x^4+x^2} = \frac{0x+(-1)}{x^2+1} + \frac{1}{x^2} + \frac{0}{x} = \frac{-1}{x^2+1} + \frac{1}{x^2}$$

Hence, using K for the generic constant to avoid confusing it with the C used above,

$$\begin{aligned}\int \frac{1}{x^4+x^2}\,dx &= \int \left(\frac{-1}{x^2+1}+\frac{1}{x^2}\right)dx = \int \frac{-1}{x^2+1}\,dx + \int \frac{1}{x^2}\,dx \\ &= -\int \frac{1}{x^2+1}\,dx + \int x^{-2}\,dx = -\arctan(x) + (-1)x^{-1} + K \\ &= -\arctan(x) - \frac{1}{x} + K\,. \quad \square\end{aligned}$$

Quiz #15. Wednesday, 13 July, 2011. *[10 minutes]*

1. Compute $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$. *[5]*

Solution. This is an improper integral, so we need to take a limit:

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t x^{-2}\,dx = \lim_{t\to\infty} -x^{-1}\Big|_1^t = \lim_{t\to\infty} -\frac{1}{x}\Bigg|_1^t \\ &= \lim_{t\to\infty}\left[-\frac{1}{t}-\left(-\frac{1}{1}\right)\right] = \lim_{t\to\infty}\left[-\frac{1}{t}+1\right] = -0+1 = 1\end{aligned}$$

since $\frac{1}{t}\to 0$ as $t\to\infty$. $\square$

Quiz #16. Monday, 18 July, 2011. *[12 minutes]*

Do *one* of questions 1 or 2.

1. Find the arc-length of the curve $y=\frac{2}{3}x^{3/2}$, where $0\le x\le 3$. *[5]*

Solution. First, $\frac{dy}{dx} = \frac{d}{dx}\frac{2}{3}x^{3/2} = \frac{2}{3}\cdot\frac{3}{2}x^{1/2} = x^{1/2} = \sqrt{x}$. Plugging this into the arc-length formula gives:

$$\text{arc-length} = \int_0^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^3 \sqrt{1+\left(\sqrt{x}\right)^2}\,dx = \int_0^3 \sqrt{1+x}\,dx$$

Substitute $u=x+1$, so $du=dx$ and $\begin{matrix} x & 0 & 3 \\ u & 1 & 4\end{matrix}$.

$$\begin{aligned}&= \int_1^4 \sqrt{u}\,du = \int_1^4 u^{1/2}\,du = \frac{2}{3}u^{3/2}\Bigg|_1^4 \\ &= \frac{2}{3}\left(4^{3/2}-1^{3/2}\right) = \frac{2}{3}(8-1) = \frac{14}{3} \quad \square\end{aligned}$$

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2. Find the area of the surface of revolution obtained by rotating the curve $y = 1 - \frac{1}{2}x^2$, where $0 \leq x \leq \sqrt{3}$, about the y-axis. *[5]*

Solution. First, $\frac{dy}{dx} = \frac{d}{dx}\left(1 - \frac{1}{2}x^2\right) = 0 - \frac{1}{2}2x = -x$. Second, since we are rotating the curve about the y-axis, $r = x - 0 = x$. Plugging these into the formula for the area of a surface of revolution gives:

$$\text{area} = \int_0^{\sqrt{3}} 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\sqrt{3}} 2\pi x \sqrt{1 + (-x)^2}\, dx = \pi \int_0^{\sqrt{3}} 2x\sqrt{1 + x^2}\, dx$$

Substitute $u = 1 + x^2$, so $du = 2x\,dx$ and $\begin{matrix} x & 0 & \sqrt{3} \\ u & 1 & 4 \end{matrix}$.

$$= \pi \int_1^4 \sqrt{u}\, du = \pi \int_1^4 u^{1/2}\, du = \pi \left.\frac{2}{3}u^{3/2}\right|_1^4$$

$$= \frac{2}{3}\pi\left(4^{3/2} - 1^{3/2}\right) = \frac{2}{3}\pi(8-1) = \frac{14}{3}\pi \quad \square$$

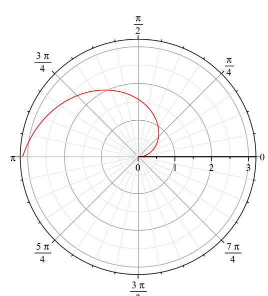
Quiz #17. Wednesday, 19 July, 2011. *[12 minutes]*

Do *one* of questions 1 or 2.

1. Sketch the curve $r = \theta$, $0 \leq \theta \leq \pi$, in polar coordinates and the area of the region between the curve and the origin. *[5]*

Solution. Cheating a bit, we use `Maple` to graph the curve:

```
> plots[polarplot](theta,theta=0..Pi);
```



To find the area of the region between curve and the origin, we plug the curve into the area formula for polar regions:

$$A = \int_C \frac{1}{2}r^2 d\theta = \int_0^\pi \frac{1}{2}\theta^2 d\theta = \frac{1}{2}\cdot\left.\frac{1}{3}\theta^3\right|_0^\pi = \frac{1}{6}\pi^3 - \frac{1}{6}0^3 = \frac{1}{6}\pi^3 \quad \square$$

2. For which values of x does the series $\sum_{n=0}^{\infty} x^{n+2} = x^2 + x^3 + x^4 + \cdots$ converge? What is the sum when it does converge? *[5]*

Solution. $\sum_{n=0}^{\infty} x^{n+2} = x^2 + x^3 + x^4 + \cdots$ is a geometric series with first term $a = x^2$ and common ratio $r = x$ between successive terms. It follows that the series converges if either $a = x^2 = 0$, *i.e.* $x = 0$, or $|r| = |x| < 0$, *i.e.* $-1 < x < 1$. Since $-1 < 0 < 1$, this means that the series converges exactly when $-1 < x < 1$.

When the series converges, it must converge to the value given by the formula for the sum of a geometric series, namely, $\frac{a}{1-r} = \frac{x^2}{1-x}$. □

Quiz #18. Monday, 25 July, 2011. *[12 minutes]*

1. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 2^n}$ converges or diverges. *[5]*

Solution. We will use the (basic form of the) Comparison Test to show that the given series converges. Since increasing a denominator makes the fraction smaller, we have both

$$\frac{1}{n^2 + 2^n} < \frac{1}{n^2} \qquad \text{and} \qquad \frac{1}{n^2 + 2^n} < \frac{1}{2^n}$$

for all n. (Note that all the terms above will always be positive for $n \geq 1$.) Either comparison will do the trick:

Comparison i. The series $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ converges by the Integral Test. As $f(x) = \frac{1}{x^2}$ is a positive, decreasing, and continuous function for $1 \leq x < \infty$ such that $\frac{1}{n^2} = f(n)$, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges if and only if the improper integral $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges. Since

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t = \lim_{t\to\infty} \left[\left(-\frac{1}{t}\right) - \left(-\frac{1}{1}\right)\right] = (-0) - (-1) = 1$$

(note that $\frac{1}{t} \to 0$ as $t \to \infty$), it follows that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. In turn, it follows by the Comparison Test that $\sum_{n=1}^{\infty} \frac{1}{n^2 + 2^n}$ converges.

Comparison ii. The series $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ converges because it is a geometric series with common ratio $r = \frac{1}{2}$ and $\left|\frac{1}{2}\right| = \frac{1}{2} < 1$. It follows by the Comparison Test that $\sum_{n=1}^{\infty} \frac{1}{n^2 + 2^n}$ converges. □

Quiz #19. Wednesday, 27 July, 2011. *[12 minutes]*

Do *one* of questions 1 or 2.

1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{(n+1)!}$ converges absolutely, converges conditionally, or diverges. *[5]*

SOLUTION. We will use the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-2)^{n+1}}{((n+1)+1)!}}{\frac{(-2)^n}{(n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}}{(n+2)!}\cdot\frac{(n+1)!}{(-2)^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{-2}{(n+2)}\right| = \lim_{n\to\infty}\frac{2}{n+2} = 0\,,$$

since $n+2\to\infty$ as $n\to\infty$. $0<1$, so it follows from the Ratio Test that the given series converges absolutely. $\square$

2. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^n}{n}$ converges absolutely, converges conditionally, or diverges. *[5]*

SOLUTION. We will try the Alternating Series Test, which requires checking that the series is indeed alternating, that the terms are decreasing in absolute value, and that the limit of the terms is 0.

First, we check if the series is indeed alternating. Since $n>0$ and $e^n>0$ for $n\geq 1$, the $(-1)^n$ forces the terms of the given series to alternate in sign. So far, so good.

Second, we check whether the terms decrease in absolute value. Is it the case that $\left|\frac{(-1)^{n+1}e^{n+1}}{n+1}\right| = \frac{e^{n+1}}{n+1} < \frac{e^n}{n} = \left|\frac{(-1)^n e^n}{n}\right|$ for all n (past some point)? Observe that

$$\frac{e^{n+1}}{n+1} < \frac{e^n}{n} \iff ne^{n+1} < (n+1)e^n \iff ne < n+1 \iff e < 1+\frac{1}{n}\,,$$

which last is *not* true for *any* $n\geq 1$ because $e>2\geq 1+\frac{1}{n}$. It follows that the Alternating Series Test cannot be used to conclude that the series converges. (Which, unfortunately, is not the same as being able to conclude that the series diverges.) Nevertheless, it pays to check the last condition required by the Alternating Series Test.

Third, we check whether the limit of the (absolute values of) the terms is 0:

$$\lim_{n\to\infty}\left|\frac{(-1)^n e^n}{n}\right| = \lim_{n\to\infty}\frac{e^n}{n} = \lim_{x\to\infty}\frac{e^x}{x}\begin{matrix}\to\infty\\ \to\infty\end{matrix} \quad \text{as } x\to\infty\text{, so we use l'Hôpital's Rule.}$$
$$= \lim_{x\to\infty}\frac{\frac{d}{dx}e^x}{\frac{d}{dx}x} = \lim_{x\to\infty}\frac{e^x}{1} = \infty$$

Not only does this mean that the Alternating Series Test cannot be used to conclude that the series converges, but that the series diverges. Failing this part of the Alternating Series Test amounts to failing the Divergence Test, so it follows that the given series diverges. $\square$

17

Trent University

MATH 1100Y Test #1

Wednesday, 8 June, 2011

Time: 50 minutes

Name: Steffie Graph

Student Number: 01234567

Question	Mark
1	____
2	____
3	____
4	____
Total	____

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

0

1. Find $\frac{dy}{dx}$ in any *three* (3) of **a**–**d**. *[9 = 3 × 3 each]*

a. $y = \left(x^2+1\right)^3$ **b.** $\ln(x+y) = 0$ **c.** $y = x^2 e^x$ **d.** $y = \dfrac{\tan(x)}{\sec(x)}$

Solution to **a**. Power and Chain Rules:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2+1\right)^3 = 3\left(x^2+1\right)^2 \cdot \frac{d}{dx}\left(x^2+1\right) = 3\left(x^2+1\right)^2(2x+0) = 6x\left(x^2+1\right)^2 \quad \square$$

Solution i to **b**. Solve for y first, then differentiate:

$$\begin{aligned} \ln(x+y) = 0 \quad &\Longrightarrow \quad x+y = e^{\ln(x+y)} = e^0 = 1 \\ &\Longrightarrow \quad y = 1-x \quad \Longrightarrow \quad \frac{dy}{dx} = 0-1 = -1 \quad \square \end{aligned}$$

Solution ii to **b**. Implicit differentiation:

$$\begin{aligned} \ln(x+y) = 0 \quad &\Longrightarrow \quad \frac{d}{dx}\ln(x+y) = \frac{d}{dx}0 \quad \Longrightarrow \quad \frac{1}{x+y} \cdot \frac{d}{dx}(x+y) = 0 \\ &\Longrightarrow \quad \frac{1}{x+y} \cdot \left(1+\frac{dy}{dx}\right) = 0 \quad \Longrightarrow \quad 1+\frac{dy}{dx} = (x+y)\cdot 0 = 0 \\ &\Longrightarrow \quad \frac{dy}{dx} = 0-1 = -1 \quad \square \end{aligned}$$

Solution to **c**. Product Rule:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 e^x\right) = \left(\frac{d}{dx}x^2\right)\cdot e^x + x^2 \cdot \left(\frac{d}{dx}e^x\right) = 2xe^x + x^2e^x = x(2+x)e^x \quad \square$$

Solution i to **d**. Simplify first, $y = \dfrac{\tan(x)}{\sec(x)} = \dfrac{\frac{\sin(x)}{\cos(x)}}{\frac{1}{\cos(x)}} = \dfrac{\sin(x)}{\cos(x)} \cdot \dfrac{\cos(x)}{1} = \sin(x)$, then differentiate, so $\dfrac{dy}{dx} = \dfrac{d}{dx}\sin(x) = \cos(x)$. $\square$

Solution ii to **d**. Quotient Rule first, then simplify:

$$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\tan(x)}{\sec(x)}\right) &= \frac{\left(\frac{d}{dx}\tan(x)\right)\cdot\sec(x) - \tan(x)\cdot\left(\frac{d}{dx}\sec(x)\right)}{\sec^2(x)} \\ &= \frac{\sec^2(x)\cdot\sec(x) - \tan(x)\cdot\sec(x)\tan(x)}{\sec^2(x)} = \frac{\sec^2(x)-\tan^2(x)}{\sec(x)} \\ &= \frac{\sec^2(x) - \left(\sec^2(x)-1\right)}{\sec(x)} = \frac{1}{\sec(x)} = \frac{1}{\frac{1}{\cos(x)}} = \cos(x) \quad \square \end{aligned}$$

1

2. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

a. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim\limits_{x\to 2}(x+1) = 3$.

b. Use the limit definition of the derivative to compute $f'(0)$ for $f(x) = x^3 + x$.

c. Compute $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3}$.

Solution to **a**. Suppose an $\varepsilon > 0$ is given. As usual, we attempt to reverse-engineer the required δ.

$$|(x+1) - 3| < \varepsilon \iff |x-2| < \varepsilon$$

Since the step taken above is reversible, it follows that if we set $\delta = \varepsilon$, then whenever $|x-2| < \delta$, we will have $|(x+1)-3| < \varepsilon$ also, as required.

Hence $\lim\limits_{x\to 2}(x+1) = 3$ by the $\varepsilon - \delta$ definition of limits. □

Solution to **b**. Here goes:

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{\left(h^3 + h\right) - \left(0^3 + 0\right)}{h} \\ &= \lim_{h\to 0} \frac{h^3+h}{h} = \lim_{h\to 0} \left(h^2 + 1\right) = 0^2 + 1 = 1 \quad \square \end{aligned}$$

Solution to **c**. Here goes:

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{x-3} = \lim_{x\to 3}(x+3) = 3+3 = 6 \quad \square$$

2

3. Do any *two* (2) of **a**–**c**. *[12 = 2 × 6 each]*

a. Each side of a square is increasing at a rate of $3\,cm/s$. At what rate is the area of the square increasing at the instant that the sides are $6\,cm$ long?

b. $f(x) = e^{-1/x^2} = e^{-(x^{-2})}$ has a removable discontinuity at $x = 0$. What should the value of $f(0)$ be to make the function continuous at $x = 0$?

c. What is the smallest possible perimeter of a rectangle with area $36\,cm^2$?

SOLUTION TO **a**. Suppose the we denote the length of a side of the square by s, so its area will be $A = s^2$. We are given that $\frac{ds}{dt} = 3$ and we wish to know $\left.\frac{dA}{dt}\right|$ at the instant that $s = 6$. We differentiate A, plug in, and then solve. $\frac{dA}{dt} = \frac{d}{dt}s^2 = 2s \cdot \frac{ds}{dt}$, so when $s = 6$, we get $\frac{dA}{dt} = 2 \cdot 6 \cdot 3 = 36\,cm^2/s$. □

SOLUTION TO **b**. $f(x)$ being continuous at $x = 0$ amounts to having $f(0) = \lim_{x\to 0} f(x)$, so we need to compute this limit.

As $x \to 0$, $\frac{1}{x^2} \to +\infty$ (note that $x^2 > 0$ for all $x \neq 0$), and so $-\frac{1}{x^2} \to -\infty$. It follows that $\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{-1/x^2} = \lim_{t\to -\infty} e^t = 0$. Thus the value of $f(0)$ should be 0 to make $f(x)$ continuous at $x = 0$. □

SOLUTION TO **c**. Suppose a rectangle has height h and base b. Then its perimeter is $P = 2h + 2b$ and its area is $A = bh$. Note that both b and h need to be > 0 for any real rectangle with positive area.

In this case $A = bh = 36$, so $h = \frac{36}{b}$ and $P = 2\frac{36}{b} + 2b = \frac{72}{b} + 2b$, where $0 \leq b < \infty$. We first find the derivative, $\frac{dP}{db} = \frac{d}{db}\left(\frac{72}{b} + 2b\right) = -\frac{72}{b^2} + 2$, and then build the usual table. $\frac{dP}{db} = -\frac{72}{b^2} + 2 = 0$ exactly when $2b^2 = 72$, *i.e.* $b^2 = 36$, so that $b = 6$. (Recall that b must be > 0.) Similarly, $\frac{dP}{db} = -\frac{72}{b^2} + 2 > 0$ exactly when $2b^2 > 72$, *i.e.* $b^2 > 36$, so that $b > 6$, and $\frac{dP}{db} = -\frac{72}{b^2} + 2 < 0$ exactly when $2b^2 < 72$, *i.e.* $b^2 < 36$, so that $b < 6$. This gives the table:

b	$(0, 6)$	6	$(6, \infty)$
P	$\downarrow$	min	$\uparrow$
$\frac{dP}{db}$	$-$	0	$+$

It follows that P has its only minimum when $b = 6$, so the smallest possible perimeter of a rectangle of area $36\,cm^2$ is $P = \frac{72}{6} + 2 \cdot 6 = 12 + 12 = 24\,cm$. Note that this rectangle is the square with sides of length $6\,cm$. □

3

4. Let $f(x) = \sqrt{x^2+1}$. Find any and all intercepts, vertical and horizontal asymptotes, and maxima and minima of $f(x)$, and sketch its graph using this information. *[9]*

SOLUTION. *i. (Domain)* x^2+1 is defined, continuous, differentiable, and ≥ 1 for all x. Since $\sqrt{t}$ is defined, continuous, and differentiable when $t > 0$, it follows that $f(x) = \sqrt{x^2+1}$ is defined, continuous, and differentiable for all x.

ii. (Intercepts) $f(0) = \sqrt{0^2+1} = \sqrt{1} = 1$, so the y-intercept is the point $(0,1)$. Since $\sqrt{x^2+1} \geq \sqrt{1} = 1 > 0$ for all x, there is no x such that $f(x) = 0$, *i.e.* $f(x)$ has no x-intercepts.

iii. (Vertical asymptotes) Since $f(x)$ is defined and continuous for all x, as noted in *i* above, it has no vertical asymptotes.

iv. (Horizontal asymptotes) To compute the relevant limits, observe that as $x \to \pm\infty$, $x^2+1 \to +\infty$, and hence $\sqrt{x^2+1} \to +\infty$. Since $\lim_{x\to+\infty} \sqrt{x^2+1} = +\infty = \lim_{x\to-\infty} \sqrt{x^2+1}$, $f(x) = \sqrt{x^2+1}$ has no horizontal asymptotes.

v. (Maxima & minima, etc.) Using the Chain and Power Rules,

$$f'(x) = \frac{d}{dx}\sqrt{x^2+1} = \frac{1}{2\sqrt{x^2+1}} \cdot \frac{d}{dx}\left(x^2+1\right) = \frac{1}{2\sqrt{x^2+1}} \cdot (2x+0) = \frac{x}{\sqrt{x^2+1}}\,.$$

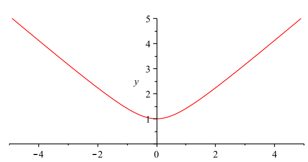
It follows that $f'(x) = 0$ if and only if $x = 0$. Moreover, since $\sqrt{x^2+1} \geq 1 > 0$ for all x, $f'(x)$ is < 0 or > 0 exactly when $x < 0$ or $x > 0$, respectively. Here is the usual table:

x	$(-\infty,0)$	0	$(0,+\infty)$
$f(x)$	$\downarrow$	min	$\uparrow$
$f'(x)$	$+$	0	$-$

Thus $f(x)$ must have a minimum at the sole critical point of $x = 0$.

vi. (Graph) Cheating a bit and using `Maple`:

```
> plot(sqrt(x^2+1),x=-5..5,y=0..5);
```



$\square$

[Total = 40]

Bonus. Simplify $\cos(\arcsin(x))$ as much as you can. *[1]*

SOLUTION. $\cos(\arcsin(x)) = \sqrt{1-\sin^2(\arcsin(x))} = \sqrt{1-x^2}$ ought to do. $\square$

Trent University

MATH 1100Y Test 2

6 July, 2011

Time: 50 minutes

Name: Steffie Graph

Student Number: 01234567

?

1	____
2	____
3	____
Total	____

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the extra page and the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Compute any *four* (4) of the integrals in parts **a**-**f**. *[16 = 4 × 4 each]*

a. $\displaystyle\int \tan^2(x)\,dx$ **b.** $\displaystyle\int_0^{3/2} 2(2x+1)^{3/2}\,dx$ **c.** $\displaystyle\int xe^x\,dx$

d. $\displaystyle\int_0^{\pi} x\cos(x)\,dx$ **e.** $\displaystyle\int \sec^3(x)\tan(x)\,dx$ **f.** $\displaystyle\int_0^1 \left(x^2+2x+3\right)\,dx$

Solution to **a**. We'll rewrite it using the trig identity $\tan^2(x) = \sec^2(x) - 1$:

$$\int \tan^2(x)\,dx = \int \left(\sec^2(x) - 1\right)\,dx = \int \sec^2(x)\,dx - \int 1\,dx = \tan(x) - x + C \quad \square$$

Solution to **b**. We'll use the substitution $u = 2x+1$, so $du = 2\,dx$ and $\begin{array}{ccc} x & 0 & 3/2 \\ u & 1 & 4 \end{array}$.

$$\begin{aligned}\int_0^{3/2} 2(2x+1)^{3/2}\,dx &= \int_1^4 u^{3/2}\,du = \left.\frac{u^{5/2}}{5/2}\right|_1^4 = \frac{1}{5/2}\left(4^{5/2} - 1^{5/2}\right) \\ &= \frac{2}{5}\left(2^5 - 1^5\right) = \frac{2}{5}(32-1) = \frac{2}{5}31 = \frac{62}{5} \quad \square\end{aligned}$$

Solution to **c**. We'll use integration by parts, with $u = x$ and $v' = e^x$, so $u' = 1$ and $v = e^x$.

$$\int xe^x\,dx = \int uv'\,dx = uv - \int u'v\,dx = xe^x - \int 1e^x\,dx = xe^x - e^x + C \quad \square$$

Solution to **d**. We will also use integration by parts here, with $u = x$ and $v' = \cos(x)$, so $u' = 1$ and $v = \sin(x)$.

$$\begin{aligned}\int_0^{\pi} x\cos(x)\,dx &= \int_0^{\pi} uv'\,dx = uv|_0^{\pi} - \int_0^{\pi} u'v\,dx = x\sin(x)|_0^{\pi} - \int_0^{\pi} 1\sin(x)\,dx \\ &= (\pi\sin(\pi) - 0\sin(0)) - (-\cos(x))|_0^{\pi} = \pi 0 - 0 + \cos(x)|_0^{\pi} \\ &= \cos(\pi) - \cos(0) = -1 - 1 = -2 \quad \square\end{aligned}$$

Solution to **e**. We'll use the substitution $u = \sec(x)$, so $du = \sec(x)\tan(x)\,dx$.

$$\int \sec^3(x)\tan(x)\,dx = \int \sec^2(x)\sec(x)\tan(x)\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{1}{3}\sec^3(x) + C \quad \square$$

Solution to **f**. The Power Rule is the main tool:

$$\begin{aligned}\int_0^1 \left(x^2+2x+3\right)\,dx &= \int_0^1 x^2\,dx + \int_0^1 2x\,dx + \int_0^1 3\,dx = \left.\frac{x^3}{3}\right|_0^1 + x^2\big|_0^1 + 3x|_0^1 \\ &= \left(\frac{1^3}{3} - \frac{0^3}{3}\right) + \left(1^2 - 0^2\right) + (3\cdot 1 - 3\cdot 0) = \frac{1}{3} + 1 + 3 = \frac{13}{3} \quad \square\end{aligned}$$

1

2. Do any *two* (2) of parts **a**-**e**. *[12 = 2 × 6 each]*

a. Compute $\int_0^3 \sqrt{9-x^2}\,dx$. What does this integral represent?

b. Sketch the solid obtained by rotating the region bounded by $y = x$, $y = 0$, and $x = 2$ about the y-axis, and find its volume.

c. Give an example of a function $f(x)$ with $f'(x) = 1 - \int_0^x f(t)\,dt$ for all x.

d. Sketch the region between $y = \sin(x)$ and $y = -\sin(x)$ for $0 \le x \le 2\pi$, and find its area.

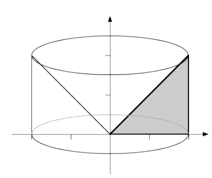
e. Compute $\int_1^2 x\,dx$ using the Right-hand Rule.

SOLUTION TO **a**. We will use the substitution $x = 3\sin(\theta)$, so $dx = 3\cos(\theta)\,d\theta$ and $\begin{array}{ccc} x & 0 & 3 \\ \theta & 0 & \pi/2 \end{array}$.

$$\begin{aligned}\int_0^3 \sqrt{9-x^2}\,dx &= \int_0^{\pi/2} \sqrt{9-9\sin^2(\theta)}\,3\cos(\theta)\,d\theta = \int_0^{\pi/2} 3\sqrt{1-\sin^2(\theta)}\,3\cos(\theta)\,d\theta \\ &= \int_0^{\pi/2} 3\sqrt{\cos^2(\theta)}\,3\cos(\theta)\,d\theta = \int_0^{\pi/2} 3\cos(\theta)\,3\cos(\theta)\,d\theta \\ &= \int_0^{\pi/2} 9\cos^2(\theta)\,d\theta = 9\int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)\,d\theta \\ &\qquad \text{Substitute } u = 2\theta, \text{ so } du = 2\,d\theta \text{ and } \frac{1}{2}\,du = d\theta, \text{ and } \begin{array}{ccc} \theta & 0 & \pi/2 \\ u & 0 & \pi \end{array}. \\ &= 9\int_0^{\pi} \left(\frac{1}{2} + \frac{1}{2}\cos(u)\right)\frac{1}{2}\,du = \frac{9}{4}\int_0^{\pi} (1+\cos(u))\,du \\ &= \frac{9}{4}(u+\sin(u))\Big|_0^{\pi} = \frac{9}{4}(\pi+\sin(\pi)) - \frac{9}{4}(0+\sin(0)) \\ &= \frac{9}{4}(\pi+0) - \frac{9}{4}(0+0) = \frac{9}{4}\pi\end{aligned}$$

Since $y = \sqrt{9-x^2}$ implies that $x^2 + y^2 = 3^2$, since we're taking the positive square root, and since $0 \le x \le 3$, the integral gives the area of the quarter of the circle of radius 3 centred at the origin that lies in the first quadrant (*i.e.* where $x \ge 0$ and $y \ge 0$). □

SOLUTION TO **b**. Here's a sketch of the solid, with the original region shaded in:



The volume is about as easy to compute with either the washer or the cylindrical shell method. We'll do it with shells; since we rotated about a vertical line and are using shells, we have to integrate with respect to x. The cylindrical shell at x has radius $r = x - 0 = x$ and height $h = x - 0 = x$, so its area is $2\pi rh = 2\pi xx = 2\pi x^2$. We plug this into the volume formula for shells:

$$V = \int_0^2 2\pi rh\,dx = \int_0^2 2\pi x^2\,dx = 2\pi \frac{x^3}{3}\bigg|_0^2 = 2\pi\frac{2^3}{3} - 2\pi\frac{0^3}{3} = 2\pi\frac{8}{3} - 0 = \frac{16}{3}\pi$$

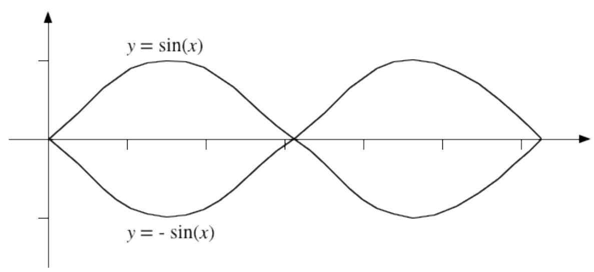
Note that, as always, the limits for the integral come from the original region. □

Solution to **c**. First, note that $f'(0) = 1 - \int_0^0 f(x)\,dx = 1 - 0 = 1$. Second, note that it follows from the Fundamental Theorem of Calculus that

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\left(1 - \int_0^x f(t)\,dt\right) = 0 - f(x) = -f(x)\,.$$

So, how many functions do you know such that $f''(x) = -f(x)$ and $f'(0) = 1$? Both $\sin(x)$ and $\cos(x)$ meet the first requirement. Since $\frac{d}{dx}\sin(x) = \cos(x)$ and $\cos(0) = 1$, $f(x) = \sin(x)$ does the job. □

Solution to **d**. Here's a crude sketch:



Note that between 0 and π, $\sin(x) \geq 0$, so $\sin(x) \geq -\sin(x)$, and between π and 2π, $\sin(x) \leq 0$, so $-\sin(x) \geq \sin(x)$. It follows that the area of the region is:

$$\begin{aligned}
A &= \int_0^\pi (\sin(x) - (-\sin(x)))\ dx + \int_\pi^{2\pi} ((-\sin(x)) - \sin(x))\ dx \\
&= \int_0^\pi 2\sin(x)\,dx - \int_\pi^{2\pi} 2\sin(x)\,dx = -2\cos(x)|_0^\pi - (-2\cos(x))|_\pi^{2\pi} \\
&= [-2\cos(\pi) - (-2\cos(0))] - [-2\cos(2\pi) - (-2\cos(\pi))] \\
&= [-2(-1) - (-2\cdot 1)] - [-2\cdot 1 - (-2(-1))] = [2+2] - [-2-2] = 8 \quad \square
\end{aligned}$$

SOLUTION TO **e**. We plug into the Right-hand Rule formula, namely $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + i\frac{b-a}{n}\right)$, and chug away. In this case $a = 1$, $b = 2$, and $f(x) = x$.

$$\begin{aligned}
\int_1^2 x\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2-1}{n} f\left(1 + i\frac{2-1}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} f\left(1 + \frac{i}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \left[\left(\sum_{i=1}^{n} 1\right) + \left(\sum_{i=1}^{n} \frac{i}{n}\right)\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[n + \frac{1}{n} \sum_{i=1}^{n} i\right] = \lim_{n\to\infty} \frac{1}{n} \left[n + \frac{1}{n} \cdot \frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[n + \frac{n+1}{2}\right] = \lim_{n\to\infty} \frac{1}{n} \left[\frac{3}{2}n + \frac{1}{2}\right] \\
&= \lim_{n\to\infty} \left[\frac{3}{2} + \frac{1}{2n}\right] = \frac{3}{2} + 0 = \frac{3}{2} \quad \square
\end{aligned}$$

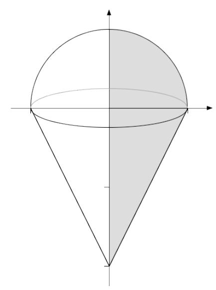
3. The region between $y = \sqrt{1-x^2}$ and $y = 2x - 2$, where $0 \le x \le 1$, is rotated about the y-axis to make a solid. Do part **a** and *one* (1) of parts **b** or **c**.

a. Sketch the solid of revolution described above. *[3]*

b. Find the volume of the solid using the disk/washer method. *[9]*

c. Find the volume of the solid using the method of cylindrical shells. *[9]*

Solution to **a**. Here's a sketch of the solid, with the original region shaded in:



Anyone for ice cream? □

Solution to **b**. Since we rotated about a vertical line and are using washers, we have to integrate with respect to y. y runs from -2 – the y-intercept of $y = 2x - 2$ – to 1 – the y-intercept of $y = \sqrt{1-x^2}$ – for the given region. The problem is that the outer radius of the washer at y is $R = x = \frac{1}{2}y + 1$ for $-2 \le y \le 0$, but is $R = x = \sqrt{1-y^2}$ for $0 \le y \le 1$, so we will have to break the integral up accordingly. Note that the inner radius of each washer is $r = 0$ either way, so every washer is actually a disk. We plug all this into the volume formula for the washer method:

$$
\begin{aligned}
V &= \int_{-2}^{0} \pi\left(R^2 - r^2\right)\,dy + \int_{0}^{1} \pi\left(R^2 - r^2\right)\,dy \\
&= \pi \int_{-2}^{0} \left(\left(\frac{1}{2}y + 1\right)^2 - 0^2\right) dy + \pi \int_{0}^{1} \left(\left(\sqrt{1-y^2}\right)^2 - 0^2\right) dy \\
&= \pi \int_{-2}^{0} \left(\frac{1}{4}y^2 + y + 1\right) dy + \pi \int_{0}^{1} \left(1 - y^2\right) dy \\
&= \pi \left(\frac{1}{4} \cdot \frac{y^3}{3} + \frac{y^2}{2} + y\right)\Bigg|_{-2}^{0} + \pi \left(y - \frac{y^3}{3}\right)\Bigg|_{0}^{1} \\
&= \pi \left(\frac{0^3}{12} + \frac{0^2}{2} + 0\right) - \pi \left(\frac{(-2)^3}{12} + \frac{(-2)^2}{2} + (-2)\right) + \pi \left(1 - \frac{1^3}{3}\right) - \pi \left(0 - \frac{0^3}{3}\right) \\
&= 0 - \pi \left(\frac{-8}{12} + \frac{4}{2} - 2\right) + \pi \frac{2}{3} - 0 = -\pi \frac{-2}{3} + \pi \frac{2}{3} = \frac{4}{3}\pi \quad \square
\end{aligned}
$$

SOLUTION TO **c**. Since we rotated about a vertical line and are using shells, we have to integrate with respect to x. The cylindrical shell at x has radius $r = x$ and height $h = \sqrt{1-x^2} - (2x-2) = \sqrt{1-x^2} - 2x + 2$. We plug these into the volume formula for the shell method:

$$V = \int_0^1 2\pi r h \, dx = 2\pi \int_0^1 x \left(\sqrt{1-x^2} - 2x + 2 \right) \, dx$$

$$= 2\pi \int_0^1 x\sqrt{1-x^2} \, dx - 2\pi \int_0^1 2x^2 \, dx + 2\pi \int_0^1 2x \, dx$$

In the first integral, substitute $u = 1 - x^2$, so $du = -2x\,dx$ and $(-1)\,du = 2x\,dx$, and change limits accordingly: $\begin{array}{ccc} x & 0 & 1 \\ u & 1 & 0 \end{array}$

$$= \pi \int_1^0 \sqrt{u}(-1) \, du - 4\pi \left. \frac{x^3}{3} \right|_0^1 + 2\pi \left. x^2 \right|_0^1$$

$$= \pi \int_0^1 u^{1/2} \, du - 4\pi \left[\frac{1^3}{3} - \frac{0^3}{3} \right] + 2\pi \left[1^2 - 0^2 \right] = \pi \left. \frac{u^{3/2}}{3/2} \right|_0^1 - \frac{4}{3}\pi + 2\pi$$

$$= \pi \left[\frac{2}{3} 1^{3/2} - \frac{2}{3} 0^{3/2} \right] + \frac{2}{3}\pi = \frac{2}{3}\pi + \frac{2}{3}\pi = \frac{4}{3}\pi \quad \square$$

Mathematics 1100Y – Calculus I: Calculus of one variable
TRENT UNIVERSITY, Summer 2011
Solutions to the Final Examination

Time: 09:00–12:00, on Wednesday, 3 August, 2011. *Brought to you by* Стефан.

Instructions: Show all your work and justify all your answers. *If in doubt,* **ask!**

Aids: Calculator; two (2) aid sheets [all 12 sides]; one (1) brain [may be caffeinated].

Part I. Do *all* three (3) of **1**–**3**.

1. Compute $\frac{dy}{dx}$ as best you can in any *three* (3) of **a**–**f**. *[15 = 3 × 5 each]*

a. $x = e^{x+y}$ **b.** $y = \int_0^{-x} te^t\, dt$ **c.** $y = x^2 \ln(x)$

d. $y = \frac{x}{\cos(x)}$ **e.** $y = \sec^2(\arctan(x))$ **f.** $y = \sin(e^x)$

SOLUTION *i* TO **a**. We will use implicit differentiation. (The alternative is to solve for y first, which isn't too hard in this case; see solution *ii* below.) Differentiating both sides of $x = e^{x+y}$ gives

$$\begin{aligned} 1 = \frac{dx}{dx} &= \frac{d}{dx}e^{x+y} = e^{x+y} \cdot \frac{d}{dx}(x+y) = e^{x+y} \cdot \left(\frac{dx}{dx} + \frac{dy}{dx}\right) \\ &= e^{x+y} \cdot \left(1 + \frac{dy}{dx}\right) = e^{x+y} + e^{x+y}\frac{dy}{dx}\,, \end{aligned}$$

from which it follows that

$$e^{x+y}\frac{dy}{dx} = 1 - e^{x+y} \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{e^{x+y}} - \frac{e^{x+y}}{e^{x+y}} = e^{-(x+y)} - 1\,. \qquad \square$$

SOLUTION *ii* TO **a**. We will solve for y as a function of x first, and then differentiate.

$$\begin{aligned} & x = e^{x+y} = e^x e^y \quad \Longrightarrow \quad e^y = \frac{x}{e^x} = xe^{-x} \\ \Longrightarrow \quad & y = \ln(e^y) = \ln\left(xe^{-x}\right) = \ln(x) + \ln\left(e^{-x}\right) = \ln(x) - x\ln(e) = \ln(x) - x \end{aligned}$$

It follows that

$$\frac{dy}{dx} = \frac{d}{dx}(\ln(x) - x) = \frac{1}{x} - 1\,.$$

(If you plug $y = \ln(x) - x$ into $\frac{dy}{dx} = e^{-(x+y)} - 1$ it works out to $\frac{1}{x} - 1$ too.) $\square$

SOLUTION TO **b**. We will use the Chain Rule and the Fundamental Theorem of Calculus. Let $u = -x$; then $y = \int_0^u te^t\, dt$ and so

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left(\frac{d}{du}\int_0^u te^t\, dt\right)\left(\frac{d}{dx}(-x)\right) = ue^u(-1) = (-x)e^{-x}(-1) = xe^{-x}\,.$$

One could also do this by computing the definite integral first using integration by parts, and then differentiating. $\square$

Solution to **c**. We will do this one using the Product Rule as the main tool:

$$\frac{dy}{dx} = \frac{d}{dx}x^2 \ln(x) = \left(\frac{d}{dx}x^2\right) \cdot \ln(x) + x^2 \cdot \left(\frac{d}{dx}\ln(x)\right) = 2x\ln(x) + x^2\frac{1}{x} = 2x\ln(x) + x \quad \square$$

Solution to **d**. We will do this one using the Quotient Rule as the main tool:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x}{\cos(x)}\right) = \frac{\frac{dx}{dx} \cdot \cos(x) - x \cdot \frac{d}{dx}\cos(x)}{\cos^2(x)} \\ &= \frac{1\cos(x) - x\left(-\sin(x)\right)}{\cos^2(x)} = \frac{\cos(x) + x\sin(x)}{\cos^2(x)}\end{aligned}$$

Given the multitude of trig identities, which could be applied before or after differentiating, there are lots of equivalent ways of writing the answer. For one nice example, $\frac{dy}{dx} = \sec(x) + x\sec(x)\tan(x)$. $\square$

Solution to **e**. This can be done using the Chain Rule and following up with a trig identity or two, but it's easier if one simplifies y using a trig identity or two first:

$$y = \sec^2\left(\arctan(x)\right) = 1 + \tan^2\left(\arctan(x)\right) = 1 + \left(\tan\left(\arctan(x)\right)\right)^2 = 1 + x^2$$

It then follows that $\frac{dy}{dx} = \frac{d}{dx}\left(1 + x^2\right) = 0 + 2x = 2x$. $\square$

Solution to **f**. There is no avoiding the Chain Rule in this one:

$$\frac{dy}{dx} = \frac{d}{dx}\sin\left(e^x\right) = \cos\left(e^x\right) \cdot \frac{d}{dx}e^x = \cos\left(e^x\right) \cdot e^x = e^x\cos\left(e^x\right) \quad \square$$

2. Evaluate any *three* (3) of the integrals **a**–**f**. *[15 = 3 × 5 each]*

a. $\displaystyle\int \frac{2x}{\sqrt{4-x^2}}\,dx$ **b.** $\displaystyle\int_0^{\pi/2} \sin(z)\cos(z)\,dz$ **c.** $\displaystyle\int x^2\ln(x)\,dx$

d. $\displaystyle\int_{-\infty}^{\ln(3)} e^s\,ds$ **e.** $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx$ **f.** $\displaystyle\int_1^2 \frac{1}{w^2+w}\,dw$

Solution to **a**. This can be done using a trig substitution, namely $x = 2\sin(\theta)$, but it's faster to use the substitution $u = 4 - x^2$, so that $du = -2x\,dx$ and $2x\,dx = (-1)\,du$:

$$\begin{aligned}\int \frac{2x}{\sqrt{4-x^2}}\,dx &= \int \frac{1}{\sqrt{u}}(-1)\,du = -\int u^{-1/2}\,du \\ &= -\frac{u^{1/2}}{1/2} + C = -2\sqrt{u} + C = -2\sqrt{4-x^2} + C \quad \square\end{aligned}$$

Solution to **b**. This is probably easiest using the substitution $u = \sin(z)$, so $du = \cos(z)\,dz$ and $\begin{matrix} z & 0 & \pi/2 \\ u & 0 & 1 \end{matrix}$.

$$\int_0^{\pi/2} \sin(z)\cos(z)\,dz = \int_0^1 u\,du = \left.\frac{u^2}{2}\right|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2} - 0 = \frac{1}{2} \quad \square$$

Solution to **c**. We will use integration by parts with $u = \ln(x)$ and $v' = x^2$, so $u' = \frac{1}{x}$ and $v = \frac{x^3}{3}$:

$$\begin{aligned} \int x^2 \ln(x)\,dx &= \frac{x^3}{3}\ln(x) - \int \frac{1}{x} \cdot \frac{x^3}{3}\,dx \\ &= \frac{x^3}{3}\ln(x) - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C \quad \square \end{aligned}$$

Solution to **d**. This is an improper integral, so we need to set up and compute a limit:

$$\begin{aligned} \int_{-\infty}^{\ln(3)} e^s\,ds &= \lim_{t\to-\infty} \int_t^{\ln(3)} e^s\,ds = \lim_{t\to-\infty} e^s\big|_t^{\ln(3)} \\ &= \lim_{t\to-\infty} \left(e^{\ln(3)} - e^t\right) = \lim_{t\to-\infty} \left(3 - e^t\right) = 3 - 0 = 3\,, \end{aligned}$$

since $e^t \to 0$ as $t \to -\infty$. $\square$

Solution to **e**. We will use the trig substitution $x = \tan(\theta)$, so $dx = \sec^2(\theta)\,d\theta$.

$$\begin{aligned} \int \frac{1}{\sqrt{1+x^2}}\,dx &= \int \frac{1}{\sqrt{1+\tan^2(\theta)}}\,\sec^2(\theta)\,d\theta = \int \frac{1}{\sqrt{\sec^2(\theta)}}\,\sec^2(\theta)\,d\theta \\ &= \int \frac{1}{\sec(\theta)}\,\sec^2(\theta)\,d\theta = \int \sec(\theta)\,d\theta = \ln\left(\tan(\theta) + \sec(\theta)\right) + C \\ &= \ln\left(x + \sqrt{1+x^2}\right) + C \end{aligned}$$

Note that $\sec(\theta) = \sqrt{sec^2(\theta)} = \sqrt{1+\tan^2(\theta)} = \sqrt{1+x^2}$ if $x = \tan(\theta)$. $\square$

Solution to **f**. This integral requires the use of partial fractions. First, if

$$\frac{1}{w^2+w} = \frac{1}{w(w+1)} = \frac{A}{w} + \frac{B}{w+1} = \frac{A(w+1)+Bw}{w(w+1)} = \frac{(A+B)w+A}{w^2+w}\,,$$

then, comparing numerators, we must have $A + B = 0$ and $A = 1$, so $B = -1$. It follows that

$$\begin{aligned} \int_1^2 \frac{1}{w^2+w}\,dw &= \int_1^2 \left(\frac{1}{w} + \frac{-1}{w+1}\right)\,dw = \int_1^2 \frac{1}{w}\,dw - \int_1^2 \frac{1}{w+1}\,dw \\ &= \ln(w)|_1^2 - \ln(w+1)|_1^2 = [\ln(2) - \ln(1)] - [\ln(2+1) - \ln(1+1)] \\ &= [\ln(2) - 0] - [\ln(3) - \ln(2)] = \ln(2) - \ln(3) + \ln(2) \\ &= 2\ln(2) - \ln(3) \end{aligned}$$

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Those who feel compelled to simplify further may do so:

$$2\ln(2) - \ln(3) = \ln\left(2^2\right) - \ln(3) = \ln(4) - \ln(3) = \ln\left(\frac{4}{3}\right) \quad \square$$

3. Do any *five* (5) of **a**–**i**. *[25 = 5 × 5 ea.]*

a. Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n^2}{3^n}$ converges absolutely, converges conditionally, or diverges.

b. Why must the arc-length of $y = \arctan(x)$, $0 \le x \le 13$, be less than $13 + \dfrac{\pi}{2}$?

c. Find a power series equal to $f(x) = \dfrac{x}{1+x}$ (when the series converges) without using Taylor's formula.

d. Find the area of the region between the origin and the polar curve $r = \dfrac{\pi}{2} + \theta$, where $0 \le \theta \le \pi$.

e. Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$.

f. Use the limit definition of the derivative to compute $f'(0)$ for $f(x) = 2x - 1$.

g. Compute the area of the surface obtained by rotating the the curve $y = \dfrac{x^2}{2}$, where $0 \le x \le \sqrt{3}$, about the y-axis.

h. Use the Right-hand Rule to compute the definite integral $\displaystyle\int_0^3 (x+1)\,dx$.

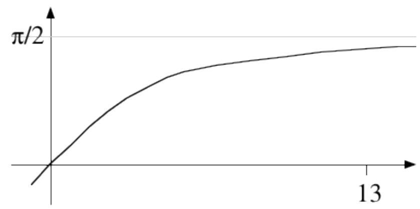
i. Use the $\varepsilon - \delta$ definition of limits to verify that $\displaystyle\lim_{x\to 2}(x+1) = 3$.

Solution to **a**. We will apply the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+1)^2}{3^{n+1}}}{\frac{(-1)^n n^2}{3^n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^2}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n^2}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)\left(n^2+2n+1\right)}{3n^2}\right| = \lim_{n\to\infty}\frac{n^2+2n+1}{3n^2} \\
&= \lim_{n\to\infty}\left(\frac{n^2}{3n^2} + \frac{2n}{3n^2} + \frac{1}{3n^2}\right) = \lim_{n\to\infty}\left(\frac{1}{3} + \frac{2}{3n} + \frac{1}{3n^2}\right) = \frac{1}{3} + 0 + 0 = \frac{1}{3}
\end{aligned}$$

Since $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3} < 1$, the given series converges absolutely by the Ratio Test. $\square$

SOLUTION TO **b**. The easiest way to see – and explain – why the arc-length of $y = \arctan(x)$, $0 \le x \le 13$, must be less than $13 + \frac{\pi}{2}$ is to draw a picture; even a crude sketch will suffice:



(Note that since $\arctan(13)$ is pretty close to $\frac{\pi}{2}$, the point $\left(13, \frac{\pi}{2}\right)$ is pretty close to the point $(13, \arctan(13))$.) It should be pretty clear from the picture that going from the origin by way of the y-axis to the point $\left(0, \frac{\pi}{2}\right)$ and then on to the point $\left(13, \frac{\pi}{2}\right)$ by way of the line $y = \frac{\pi}{2}$ is a longer trip than going from the origin to $(13, \arctan(13))$ by way of the curve $y = \arctan(x)$. It follows that $\frac{\pi}{2} + 13$ is greater than the arc-length of $y = \arctan(x)$ for $0 \le x \le 13$. □

SOLUTION TO **c**. Using the formula for the sum of a geometric series in reverse,

$$\begin{aligned} f(x) = \frac{x}{1+x} &= \frac{x}{1-(-x)} = \sum_{n=0}^{\infty} x(-x)^n = \sum_{n=0}^{\infty} x(-1)^n x^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{n+1} = x - x^2 + x^3 - x^4 + x^5 - \cdots \quad \square \end{aligned}$$

SOLUTION TO **d**. We plug the polar curve $r = \frac{\pi}{2} + \theta$, $0 \le \theta \le \pi$, into the area formula in polar coordinates:

$$\begin{aligned} \int_0^{\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi} \frac{1}{2}\left(\frac{\pi}{2} + \theta\right)^2 d\theta &= \frac{1}{2}\int_0^{\pi} \left(\left(\frac{\pi}{2}\right)^2 + 2\frac{\pi}{2}\theta + \theta^2\right) d\theta \\ &= \frac{1}{2}\left(\frac{\pi^2}{4}\theta + \frac{\pi}{2}\theta^2 + \frac{1}{3}\theta^3\right)\Big|_0^{\pi} \\ &= \frac{1}{2}\left(\frac{\pi^2}{4}\pi + \frac{\pi}{2}\pi^2 + \frac{1}{3}\pi^3\right) - \frac{1}{2}\left(\frac{\pi^2}{4}0 + \frac{\pi}{2}0^2 + \frac{1}{3}0^3\right) = \frac{13}{24}\pi^3 \quad \square \end{aligned}$$

SOLUTION TO **e**. We will first find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$

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using the Ratio Test.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{n+1}{2^{n+1}}x^{n+1}}{\frac{n}{2^n}x^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{2^{n+1}}x\cdot\frac{2^n}{n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{x}{2}\right| \\ &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{|x|}{2} = (1+0)\frac{|x|}{2} = \frac{|x|}{2}\end{aligned}$$

It follows by the Ratio Test that the series converges absolutely when $\frac{|x|}{2} < 1$, *i.e.* when $-2 < x < 2$, and diverges when $\frac{|x|}{2} > 1$, *i.e.* when $x < -2$ or $x > 2$. Thus the radius of convergence of the given power series is $R = 2$.

It remains to determine what happens at the endpoints of the interval, namely at $x = -2$ and $x = 2$. Since

$$\begin{aligned}&\lim_{n\to\infty}\left|\frac{n}{2^n}(-2)^n\right| = \lim_{n\to\infty}|(-1)^n n| = \lim_{n\to\infty} n = \infty \neq 0 \\ \text{and}\quad &\lim_{n\to\infty}\left|\frac{n}{2^n}2^n\right| = \lim_{n\to\infty} n = \infty \neq 0\,,\end{aligned}$$

the Divergence Test tells us that the series diverges for both $x = -2$ and $x = 2$. The interval of convergence is therefore $(-2, 2)$. □

Solution to **f**. We will plug $f(x) = 2x - 1$ into the limit definition of the derivative to compute $f'(0)$:

$$\begin{aligned}f'(0) &= \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{(2(0+h)-1)-(2\cdot 0-1)}{h} \\ &= \lim_{h\to 0}\frac{2h-1-(-1)}{h} = \lim_{h\to 0}\frac{2h}{h} = \lim_{h\to 0} 2 = 2 \quad \square\end{aligned}$$

Solution to **g**. We will plug the curve $y = \frac{x^2}{2}$, $0 \leq x \leq \sqrt{3}$, into the formula for the area of a surface of revolution, $\int 2\pi r\, ds$. First, note that since we are rotating the curve about the y-axis, $r = x - 0 = x$. Second, since $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{2}\right) = \frac{1}{2}2x = x$, we have that $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+x^2}\,dx$. Thus

$$\begin{aligned}\text{SA} &= \int 2\pi r\, ds = \int_0^{\sqrt{3}} 2\pi x\sqrt{1+x^2}\,dx \quad \text{Let } u = 1+x^2\text{, so } du = 2x\,dx \text{ and } \begin{array}{ccc} x & 0 & \sqrt{3} \\ u & 1 & 4\end{array}. \\ &= \pi\int_1^4 \sqrt{u}\,du = \pi\int_1^4 u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big|_1^4 = \frac{2}{3}\left(4^{3/2}-1^{3/2}\right) = \frac{2}{3}(8-1) = \frac{14}{3} \quad \square\end{aligned}$$

6

SOLUTION TO **h**. We plug the definite integral $\int_0^3 (x+1)\,dx$ into the Right-hand Rule formula $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + \frac{b-a}{n} i\right)$ and chug away:

$$\begin{aligned}
\int_0^3 (x+1)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3-0}{n}\left[\left(0+\frac{3-0}{n}i\right)+1\right] = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\frac{3}{n}i+1\right] \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{3}{n}i+1\right] = \lim_{n\to\infty} \frac{3}{n}\left[\left(\sum_{i=1}^{n} \frac{3}{n}i\right)+\left(\sum_{i=1}^{n} 1\right)\right] \\
&= \lim_{n\to\infty} \frac{3}{n}\left[\frac{3}{n}\left(\sum_{i=1}^{n} i\right)+n\right] = \lim_{n\to\infty} \frac{3}{n}\left[\frac{3}{n}\cdot\frac{n(n+1)}{2}+n\right] \\
&= \lim_{n\to\infty} \frac{3}{n}\left[\frac{3}{2}(n+1)+n\right] = \lim_{n\to\infty} \frac{3}{n}\left[\frac{5}{2}n+\frac{3}{2}\right] = \lim_{n\to\infty}\left[\frac{15}{2}+\frac{9}{2n}\right] = \frac{15}{2}
\end{aligned}$$

since $\frac{9}{2n} \to 0$ as $n \to \infty$. □

SOLUTION TO **i**. We need to show that for any $\varepsilon > 0$, there is a corresponding $\delta > 0$ such that whenever $|x-2| < \delta$, we have $|(x+1)-3| < \varepsilon$.

Suppose, then, that $\varepsilon > 0$. As usual, we will reverse-engineer the corresponding $\delta > 0$:

$$|(x+1)-3| < \varepsilon \iff |x-2| < \varepsilon\,,$$

so $\delta = \varepsilon$ will do the job. (One step of reverse-engineering is as easy as it gets!)

Thus $\lim_{x\to 2}(x+1) = 3$ by the $\varepsilon - \delta$ definition of limits. □

Part II. Do any *three* (3) of **4**–**8**.

4. Find the domain, all maximum, minimum, and inflection points, and all vertical and horizontal asymptotes of $f(x) = \frac{x^2}{x^2+1}$, and sketch its graph. *[15]*

SOLUTION. We'll run through the usual check list of items and then sketch the graph:

i. (Domain) Since $x^2+1 \geq 1 > 0$ for all x, the denominator is never 0. It follows that the rational function $f(x) = \frac{x^2}{x^2+1}$ is defined (and continuous and differentiable, too) for all x, *i.e.* its domain is $\mathbb{R} = (-\infty, \infty)$.

ii. (Intercepts) $f(0) = \frac{0^2}{0^2+1} = 0$, so the y-intercept is the origin, *i.e.* at $y = 0$. On the other hand, $f(x) = \frac{x^2}{x^2+1} = 0$ is only possible if the numerator $x^2 = 0$, *i.e.* $x = 0$. It follows that the origin is also the only x-intercept.

iii. (Vertical asymptotes) Since $f(x)$ is defined and continuous for all x, it cannot have any vertical asymptotes.

iv. (Horizontal asymptotes) We compute the relevant limits to check for horizontal asymptotes:

$$\lim_{x\to-\infty}\frac{x^2}{x^2+1}=\lim_{x\to-\infty}\frac{x^2}{x^2+1}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}=\lim_{x\to-\infty}\frac{1}{1+\frac{1}{x^2}}=\frac{1}{1+0}=1$$
$$\lim_{x\to+\infty}\frac{x^2}{x^2+1}=\lim_{x\to+\infty}\frac{x^2}{x^2+1}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}=\lim_{x\to+\infty}\frac{1}{1+\frac{1}{x^2}}=\frac{1}{1+0}=1$$

(Note that $\frac{1}{x^2}\to 0$ both as $x\to-\infty$ and as $x\to+\infty$.) It follows that $f(x)$ has the horizontal asymptote $y=1$ in both directions.

v. (Maxima and minima) First, with a little help from the Quotient Rule,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{x^2}{x^2+1}\right)=\frac{\left(\frac{d}{dx}x^2\right)\cdot\left(x^2+1\right)-x^2\cdot\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}\\ &= \frac{2x\cdot\left(x^2+1\right)-x^2\cdot 2x}{\left(x^2+1\right)^2}=\frac{2x^3+2x-2x^3}{\left(x^2+1\right)^2}=\frac{2x}{\left(x^2+1\right)^2}\,. \end{aligned}$$

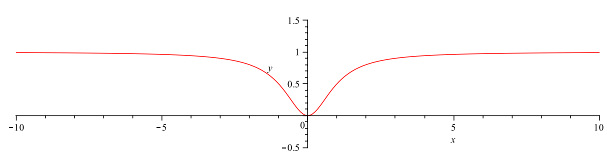
Second, note that $f'(x)$ is also a rational function that is defined and continuous for all x, using reasoning very similar to that used in i above. This means that the only kind of critical point that can occur is the sort where $f'(x)=0$. $f'(0)=\dfrac{2x}{\left(x^2+1\right)^2}=0$ can only occur when the numerator, $2x$, is 0, which happens only when $x=0$. Moreover, since the denominator, $\left(x^2+1\right)^2\geq 1>0$ for all x (and $2>0$, too) $f'(x)<0$ when $x<0$ and $f'(x)>0$ when $x>0$. We can summarize this information and its effect on $f(x)$ with the usual sort of table:

x	$(-\infty,0)$	0	$(0,+\infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	$\downarrow$	min	$\uparrow$

Note that $x=0$ gives a local (and absolute) minimum point for $f(x)$ and that $f(x)$ has no maximum point. (This makes particular sense if you do *iv* above a little more carefully and notice that $f(x)$ approaches the horizontal asymptote from below in both directions.)

vi. (Graph) We cheat ever so slightly and let `Maple` do the drawing:

```
> plot(x^2/(x^2+1),x=-10..10,y=-0.5..1.5);
```



Whew! □

5. Do both of **a** and **b**.

a. Verify that $\int \sqrt{x^2-1}\,dx = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right) + C$. *[7]*

b. Find the arc-length of $y = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right)$ for $1 \le x \le 3$. *[8]*

SOLUTION TO **a**. We can compute the indefinite integral using the trig substitution $x = \sec(\theta)$, so $dx = \sec(\theta)\tan(\theta)\,d\theta$, but it's often easier to differentiate the antiderivative and check that the result is equal to the integrand. Trying this here

$$
\begin{aligned}
&\frac{d}{dx}\left[\frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right) + C\right] \\
&= \frac{1}{2}\frac{d}{dx}\left[x\sqrt{x^2-1}\right] - \frac{1}{2}\frac{d}{dx}\ln\left(x+\sqrt{x^2-1}\right) + \frac{d}{dx}C \\
&= \frac{1}{2}\left[\frac{dx}{dx}\cdot\sqrt{x^2-1} + x\cdot\frac{d}{dx}\sqrt{x^2-1}\right] - \frac{1}{2}\frac{1}{x+\sqrt{x^2-1}}\cdot\frac{d}{dx}\left(x+\sqrt{x^2-1}\right) + 0 \\
&= \frac{1}{2}\left[1\cdot\sqrt{x^2-1} + x\cdot\frac{1}{2\sqrt{x^2-1}}\cdot\frac{d}{dx}\left(x^2-1\right)\right] \\
&\qquad - \frac{1}{2}\frac{1}{x+\sqrt{x^2-1}}\cdot\left(1+\frac{1}{2\sqrt{x^2-1}}\cdot\frac{d}{dx}\left(x^2-1\right)\right) \\
&= \frac{1}{2}\left[\sqrt{x^2-1} + \frac{x}{2\sqrt{x^2-1}}\cdot 2x\right] - \frac{1}{2}\frac{1}{x+\sqrt{x^2-1}}\cdot\left(1+\frac{1}{2\sqrt{x^2-1}}\cdot 2x\right) \\
&= \frac{1}{2}\left[\sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}}\right] - \frac{1}{2}\frac{1}{x+\sqrt{x^2-1}}\cdot\left(1+\frac{x}{\sqrt{x^2-1}}\right) \\
&= \frac{1}{2}\left[\sqrt{x^2-1}\cdot\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}} + \frac{x^2}{\sqrt{x^2-1}}\right] - \frac{1}{2}\frac{1}{x+\sqrt{x^2-1}}\cdot\left(\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}}+\frac{x}{\sqrt{x^2-1}}\right) \\
&= \frac{1}{2}\frac{\left(x^2-1\right)+x^2}{\sqrt{x^2-1}} - \frac{1}{2}\frac{1}{x+\sqrt{x^2-1}}\cdot\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}} \\
&= \frac{1}{2}\frac{2x^2-1}{\sqrt{x^2-1}} - \frac{1}{2}\frac{\sqrt{x^2-1}+x}{x+\sqrt{x^2-1}}\cdot\frac{1}{\sqrt{x^2-1}} \\
&= \frac{1}{2}\frac{2x^2-1}{\sqrt{x^2-1}} - \frac{1}{2}1\cdot\frac{1}{\sqrt{x^2-1}} = \frac{1}{2}\frac{2x^2-1-1}{\sqrt{x^2-1}} = \frac{1}{2}\frac{2\left(x^2-1\right)}{\sqrt{x^2-1}} = \frac{x^2-1}{\sqrt{x^2-1}} = \sqrt{x^2-1}
\end{aligned}
$$

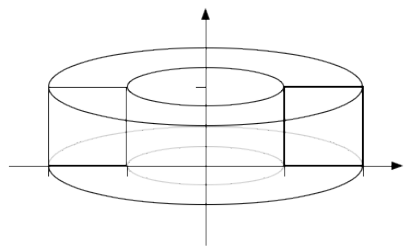
... makes for an algebra-fest of epic proportions. Maybe it would have been easier to integrate ... :-) □

Solution to **b**. It follows from **a** and the Fundamental Theorem of Calculus that $\frac{dy}{dx} = \sqrt{x^2-1}$. Plugging this into the arc-length formula gives:

$$\begin{aligned}\text{arc-length} &= \int_1^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^3 \sqrt{1+\left(\sqrt{x^2-1}\right)^2}\,dx = \int_1^3 \sqrt{1+x^2-1}\,dx \\ &= \int_1^3 \sqrt{x^2}\,dx = \int_1^3 x\,dx = \left.\frac{x^2}{2}\right|_1^3 = \frac{3^2}{2} - \frac{1^2}{2} = \frac{9-1}{2} = \frac{8}{2} = 4 \quad \square\end{aligned}$$

6. Sketch the solid obtained by rotating the square with corners at $(1,0)$, $(1,1)$, $(2,0)$, and $(2,1)$ about the y-axis and find its volume and surface area. *[15]*

Solution. Here is sketch of the solid:



Note that the original region has as its borders (pieces of) the vertical lines $x=1$ and $x=2$ and the horizontal lines $y=0$ and $y=1$, *i.e.* the region consists of all points (x,y) with $1 \le x \le 2$ and $0 \le y \le 1$.

We can compute the volume of the solid in several easy ways:

i. (Shells) Note that since we revolved the original region about the y-axis and are using shells, we will have to integrate with respect to x. The cylindrical shell at x, for some $1 \le x \le 2$, has radius $r = x - 0 = x$ and height $h = 1 - 0 = 1$. Thus the volume of the solid is:

$$V = \int_1^2 2\pi r h\,dx = \int_1^2 2\pi x \cdot 1\,dx = 2\pi \frac{1}{2} x^2 \Big|_1^2 = \pi x^2\Big|_1^2 = \pi\left(2^2 - 1^2\right) = 3\pi$$

ii. (Washers) Note that since we revolved the original region about the y-axis and are using washers, we will have to integrate with respect to y. The washer at y, for some $0 \le y \le 1$, has outer radius $R = 2 - 0 = 2$ and inner radius $r = 1 - 0 = 1$. (Note that the washers are all identical ...) Thus the volume of the solid is:

$$V = \int_0^1 \pi\left(R^2 - r^2\right)\,dy = \int_0^1 \pi\left(2^2 - 1^2\right)\,dy = \int_0^1 3\pi\,dy = 3\pi y\big|_0^1 = 3\pi(1-0) = 3\pi$$

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iii. (Look, Ma! No calculus!) The solid is a cylinder of radius $r = 2$ and height $h = 1$ (and hence volume $\pi r^2 h = 4\pi$) with a cylinder of radius $r = 1$ and height $h = 1$ (and hence volume $\pi r^2 h = \pi$) removed from it. It follows that the solid has volume $4\pi - \pi = 3\pi$.

Any correct method, correctly and completely worked out, would do, of course. :-)

It remains to find the surface area of the solid. The complication is that the surface of the solid consists of four distinct pieces: the upper and lower faces of the solid, which are both washers with outside radius $R = 2$ and inside radius $r = 1$, the outside face, which is a cylinder of radius $R = 2$ and height $h = 1$, and the inside face (*i.e.* the hole in the middle), which is a cylinder of radius $r = 1$ and height $h = 1$. While the area of each of these can be computed pretty quickly as the area of a surface of revolution, it is pointless to work so hard when we should have formulas for the areas of these objects at our fingertips from our knowledge of the washer and shell methods for computing volume:

The areas of the upper and lower faces are each $\pi\left(R^2 - r^2\right) = \pi\left(2^2 - 1^2\right) = 3\pi$, the area of the outside face is $\pi R^2 h = \pi 2^2 1 = 4\pi$, and the area of the inside face is $\pi r^2 h = \pi 1^2 1 = \pi$. The total surface area of the solid is therefore $2 \cdot 3\pi + 4\pi + \pi = 11\pi$. □

7. Do all three (3) of **a**–**c**.

a. Use Taylor's formula to find the Taylor series at 0 of $f(x) = \ln(x+1)$. *[7]*

b. Determine the radius and interval of convergence of this Taylor series. *[4]*

c. Use your answer to part **a** to find the Taylor series at 0 of $\dfrac{1}{x+1}$ without using Taylor's formula. *[4]*

SOLUTION TO **a**. We first differentiate and evaluate away to figure out what $f^{(n)}(0)$ must be for each n. $f^{(0)}(x) = f(x) = \ln(x+1)$, so $f^{(0)}(0) = \ln(0+1) = 0$; $f^{(1)}(x) = f'(x) = \frac{d}{dx}\ln(x+1) = \frac{1}{x+1} \cdot \frac{d}{dx}(x+1) = \frac{1}{x+1}1 = \frac{1}{x+1}$, so $f^{(1)}(0) = \frac{1}{0+1} = 1$; $f^{(2)}(x) = f''(x) = \frac{d}{dx}\frac{1}{x+1} = \frac{-1}{(x+1)^2} \cdot \frac{d}{dx}(x+1) = \frac{-1}{(x+1)^2}1 = \frac{-1}{(x+1)^2}$, so $f^{(2)}(0) = \frac{-1}{(0+1)^2} = -1$; $f^{(3)}(x) = f'''(x) = \frac{d}{dx}\frac{-1}{(x+1)^2} = \frac{(-1)(-2)}{(x+1)^3} \cdot \frac{d}{dx}(x+1) = \frac{(-1)^2 2}{(x+1)^3}1 = \frac{(-1)^2 2}{(x+1)^3}$, so $f^{(3)}(0) = \frac{2}{(0+1)^3} = (-1)^2 2$; and so on:

n	0	1	2	3	4	5	6	$\cdots$
$f^{(n)}(x)$	$\ln(x+1)$	$\frac{1}{x+1}$	$\frac{-1}{(x+1)^2}$	$\frac{(-1)^2 2}{(x+1)^3}$	$\frac{(-1)^3 6}{(x+1)^4}$	$\frac{(-1)^4 24}{(x+1)^5}$	$\frac{(-1)^5 120}{(x+1)^6}$	$\cdots$
$f^{(n)}(0)$	0	1	-1	$(-1)^2 2$	$(-1)^3 6$	$(-1)^4 24$	$(-1)^5 120$	$\cdots$

A little reflection on this pattern shows us that at stage $n > 0$, $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(x+1)^n}$ and so $f^{(n)}(0) = \frac{(-1)^{n-1}(n-1)!}{(0+1)^n} = (-1)^{n-1}(n-1)!$.

Applying Taylor's formula, and noting that $n = 0$ is the exception to the pattern noted above, the Taylor series at 0 of $f(x) = \ln(x+1)$ is therefore:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \frac{f^{(0)}(0)}{0!}x^0 + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \frac{0}{1}1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}x^n$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \quad \square$$

Solution to **b**. To find the radius of convergence of the power series obtained in **a**, we use the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{(n+1)-1}}{n+1}x^{n+1}}{\frac{(-1)^{n-1}}{n}x^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n}{(-1)^{n-1}}\cdot\frac{n}{n=1}\cdot\frac{x^{n+1}}{x^n}\right| \\
&= \lim_{n\to\infty}\left|(-1)\frac{n}{n+1}x\right| = \lim_{n\to\infty}\frac{n}{n+1}|x| = |x|\lim_{n\to\infty}\frac{n}{n+1} \\
&= |x|\lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{1/n}{1/n} = |x|\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = |x|\frac{1}{1+0} = |x|
\end{aligned}$$

It follows that the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$, so the radius of convergence is $R = 1$.

To find the interval of convergence, we have to determine whether the series converges or diverges at each of $x = -1$ and $x = 1$. That is, we have to determine whether each of the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(-1)^n = \sum_{n=1}^{\infty}\frac{-1}{n} = -1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\cdots$$

$$\text{and}\quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}1^n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

converges or not. The first diverges: $-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\cdots = (-1)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)$, so it is a non-zero multiple of the harmonic series, which diverges. (We showed this in class using the Integral Test, though it is even quicker to use the p-Test.) The second converges: it is just the alternating harmonic series, which converges conditionally. (We showed this in class using the Alternating Series Test.) It follows that the interval of convergence is $(-1,1)\cup\{1\} = (-1,1]$. $\square$

Solution to **c**. Since $\frac{d}{dx}\ln(x+1) = \frac{1}{x+1}$ (as noted in the solution to **a** above), we can get the Taylor series at 0 of $f'(x) = \frac{1}{x+1}$ by differentiating the Taylor series at 0 of $f(x) = \ln(x+1)$ term-by-term:

$$\begin{aligned}
&\frac{d}{dx}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots\right) \\
&= \frac{d}{dx}x-\frac{d}{dx}\left(\frac{x^2}{2}\right)+\frac{d}{dx}\left(\frac{x^3}{3}\right)-\frac{d}{dx}\left(\frac{x^4}{4}\right)+\frac{d}{dx}\left(\frac{x^5}{5}\right)-\cdots \\
&= 1-\frac{2x}{2}+\frac{3x^2}{3}-\frac{4x^3}{4}+\frac{5x^4}{5}-\cdots \\
&= 1-x+x^2-x^3+x^4-\cdots
\end{aligned}$$

This is the geometric series with first term $a = 1$ and common ratio $r = -x$. $\square$

8. A spherical balloon is being inflated at a rate of 1 m^3/s. How is its surface area changing at the instant that its volume is 36 m^3? *[15]*

[Recall that a sphere of radius r has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$.]

SOLUTION. We need to relate $1 = \frac{dV}{dt}$ to $\frac{dA}{dt}$, where A is the surface area of the balloon.

First, observe that

$$\frac{dA}{dt} = \frac{d}{dt}4\pi r^2 = \frac{d}{dr}4\pi r^2 \cdot \frac{dr}{dt} = 8\pi r \cdot \frac{dr}{dt} .$$

This means that we will need to know $\frac{dr}{dt}$ at the instant in question.

Second, we have

$$1 = \frac{dV}{dt} = \frac{d}{dt}\frac{4}{3}\pi r^3 = \frac{d}{dr}\frac{4}{3}\pi r^3 \cdot \frac{dr}{dt} = \frac{4}{3}\pi 3r^2 \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} ,$$

so $\frac{dr}{dt} = \frac{1}{4\pi r^2}$.

It follows that $\frac{dA}{dt} = 8\pi r \cdot \frac{1}{4\pi r^2} = \frac{2}{r}$.

Third, it still remains to determine what the value of r is at the instant in question. Since $V = \frac{4}{3}\pi r^3 = 36$ at the instant in question, $r = \left(\frac{36}{\frac{4}{3}\pi}\right)^{1/3} = \left(\frac{27}{\pi}\right)^{1/3} = \frac{3}{\pi^{1/3}}$.

Thus, at the instant that the volume is 36 m^3, the surface area is changing at a rate of $\frac{dA}{dt} = \frac{2}{\frac{3}{\pi^{1/3}}} = \frac{2}{3}\pi^{1/3}$ m^2/s. □

[Total = 100]

Part MMXI - Bonus problems.

13. Show that $\ln(\sec(x) - \tan(x)) = -\ln(\sec(x) + \tan(x))$. *[2]*

SOLUTION. Here goes:

$$\begin{aligned}
\ln(\sec(x) - \tan(x)) &= \ln([\sec(x) - \tan(x)] \cdot 1) = \ln\left([\sec(x) - \tan(x)] \cdot \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\right) \\
&= \ln\left(\frac{\sec^2(x) - \tan^2(x)}{\sec(x) + \tan(x)}\right) = \ln\left(\frac{\left(1 + \tan^2(x)\right) - \tan^2(x)}{\sec(x) + \tan(x)}\right) \\
&= \ln\left(\frac{1}{\sec(x) + \tan(x)}\right) = \ln\left([\sec(x) + \tan(x)]^{-1}\right) \\
&= (-1)\ln(\sec(x) + \tan(x)) = -\ln(\sec(x) + \tan(x))
\end{aligned}$$

Note that the key trick is the same one we used in class to compute $\int \sec(x)\,dx$. □

41. Write an original poem touching on calculus or mathematics in general. *[2]*

SOLUTION. Write your own! □

I HOPE THAT YOU HAD SOME FUN WITH THIS!
GET SOME REST NOW ...

13

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