# 1 Preliminaries

Welcome to the study of intermediate algebra. In this first chapter, we will quickly review the skills that are prerequisite for a successful completion of a course in intermediate algebra.

We begin by defining the various number systems that are an integral part of the study of intermediate algebra, then we move to review the skills and tools that are used to solve linear equations and formulae. Finally, we'll spend some serious effort on the logic of the words " and" and "or," and their application to linear inequalities.

As all of the material in this "preliminary" chapter is prerequisite material, the pace with which we travel these opening pages will be much quicker than that spent on the chapters that follow. Indeed, if you have an opportunity to work on this material before the first day of classes, it will be time well spent.

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# 1.1 Number Systems

In this section we introduce the number systems that we will work with in the remainder of this text.

## The Natural Numbers

We begin with a definition of the *natural numbers*, or the *counting numbers*.

**Definition 1.** *The set of* **natural numbers** *is the set*

$$\mathbb{N} = \{1, 2, 3, \ldots\}. \tag{2}$$

The notation in **equation (2)**[2] is read "$\mathbb{N}$ is the set whose members are 1, 2, 3, and so on." The ellipsis (the three dots) at the end in **equation (2)** is a mathematician's way of saying "et-cetera." We list just enough numbers to establish a recognizable pattern, then write "and so on," assuming that a pattern has been sufficiently established so that the reader can intuit the rest of the numbers in the set. Thus, the next few numbers in the set $\mathbb{N}$ are 4, 5, 6, 7, "and so on."

Note that there are an infinite number of natural numbers. Other examples of natural numbers are 578,736 and 55,617,778. The set $\mathbb{N}$ of natural numbers is unbounded; i.e., there is no largest natural number. For any natural number you choose, adding one to your choice produces a larger natural number.

For any natural number $n$, we call $m$ a *divisor* or *factor* of $n$ if there is another natural number $k$ so that $n = mk$. For example, 4 is a divisor of 12 (because $12 = 4 \times 3$), but 5 is not. In like manner, 6 is a divisor of 12 (because $12 = 6 \times 2$), but 8 is not.

We next define a very special subset of the natural numbers.

**Definition 3.** *If the only divisors of a natural number $p$ are 1 and itself, then $p$ is said to be* **prime**.

For example, because its only divisors are 1 and itself, 11 is a prime number. On the other hand, 14 is not prime (it has divisors other than 1 and itself, i.e., 2 and 7). In like manner, each of the natural numbers 2, 3, 5, 7, 11, 13, 17, and 19 is prime. Note that 2 is the only even natural number that is prime.[3]

If a natural number other than 1 is not prime, then we say that it is *composite*. Note that any natural number (except 1) falls into one of two classes; it is either prime, or it is composite.

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[2] In this textbook, definitions, equations, and other labeled parts of the text are numbered consecutively, regardless of the type of information. Figures are numbered separately, as are Tables.

[3] Although the natural number 1 has only 1 and itself as divisors, mathematicians, particularly number theorists, don't consider 1 to be prime. There are good reasons for this, but that might take us too far afield. For now, just note that 1 is not a prime number. Any number that is prime has exactly two factors, namely itself and 1.

We can factor the composite number 36 as a product of prime factors, namely

$$36 = 2 \times 2 \times 3 \times 3.$$

Other than rearranging the factors, this is the only way that we can express 36 as a product of prime factors.

**Theorem 4.** The *Fundamental Theorem of Arithmetic* says that every natural number has a *unique prime factorization.*

No matter how you begin the factorization process, all roads lead to the same prime factorization. For example, consider two different approaches for obtaining the prime factorization of 72.

$$\begin{aligned} 72 &= 8 \times 9 \\ &= (4 \times 2) \times (3 \times 3) \\ &= 2 \times 2 \times 2 \times 3 \times 3 \end{aligned} \qquad \begin{aligned} 72 &= 4 \times 18 \\ &= (2 \times 2) \times (2 \times 9) \\ &= 2 \times 2 \times 2 \times 3 \times 3 \end{aligned}$$

In each case, the result is the same, $72 = 2 \times 2 \times 2 \times 3 \times 3$.

## *Zero*

The use of zero as a placeholder and as a number has a rich and storied history. The ancient Babylonians recorded their work on clay tablets, pressing into the soft clay with a stylus. Consequently, tablets from as early as 1700 BC exist today in museums around the world. A photo of the famous Plimpton_322 is shown in **Figure 1**, where the markings are considered by some to be Pythagorean triples, or the measures of the sides of right triangles.

**Figure 1.** Plimpton_322

The people of this ancient culture had a sexagesimal (base 60) numbering system that survived without the use of zero as a placeholder for over 1000 years. In the early Babylonian system, the numbers 216 and 2106 had identical recordings on the clay tablets of the authors. One could only tell the difference between the two numbers based upon the context in which they were used. Somewhere around the year 400 BC, the Babylonians started using two wedge symbols to denote a zero as a placeholder (some tablets show a single or a double-hook for this placeholder).

The ancient Greeks were well aware of the Babylonian positional system, but most of the emphasis of Greek mathematics was geometrical, so the use of zero as a placeholder was not as important. However, there is some evidence that the Greeks used a symbol resembling a large omicron in some of their astronomical tables.

It was not until about 650 AD that the use of zero as a number began to creep into the mathematics of India. Brahmagupta (598-670?), in his work *Brahmasphutasiddhanta*, was one of the first recorded mathematicians who attempted arithmetic operations with the number zero. Still, he didn't quite know what to do with division by zero when he wrote

> *Positive or negative numbers when divided by zero is a fraction with zero as denominator.*

Note that he states that the result of division by zero is a fraction with zero in the denominator. Not very informative. Nearly 200 years later, Mahavira (800-870) didn't do much better when he wrote

> *A number remains unchanged when divided by zero.*

It seems that the Indian mathematicians could not admit that division by zero was impossible.

The Mayan culture (250-900 AD) had a base 20 positional system and a symbol they used as a zero placeholder. The work of the Indian mathematicians spread into the Arabic and Islamic world and was improved upon. This work eventually made its way to the far east and also into Europe. Still, as late as the 1500s European mathematicians were still not using zero as a number on a regular basis. It was not until the 1600s that the use of zero as a number became widespread.

Of course, today we know that adding zero to a number leaves that number unchanged and that division by zero is meaningless,[4] but as we struggle with these concepts, we should keep in mind how long it took humanity to come to grips with this powerful abstraction (zero as a number).

If we add the number zero to the set of natural numbers, we have a new set of numbers which are called the *whole numbers*.

**Definition 5.** *The set of* **whole numbers** *is the set*

$$\mathbb{W} = \{0, 1, 2, 3, \ldots\}.$$

## The Integers

Today, much as we take for granted the fact that there exists a number zero, denoted by 0, such that

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[4] It makes no sense to ask how many groups of zero are in five. Thus, 5/0 is undefined.

$$a + 0 = a \tag{6}$$

for any whole number $a$, we similarly take for granted that for any whole number $a$ there exists a unique number $-a$, called the "negative" or "opposite" of $a$, so that

$$a + (-a) = 0. \tag{7}$$

In a natural way, or so it seems to modern-day mathematicians, this easily introduces the concept of a *negative* number. However, history teaches us that the concept of negative numbers was not embraced wholeheartedly by mathematicians until somewhere around the 17th century.

In his work *Arithmetica* (c. 250 AD?), the Greek mathematician Diophantus (c. 200-284 AD?), who some call the "Father of Algebra," described the equation $4 = 4x + 20$ as "absurd," for how could one talk about an answer less than nothing? Girolamo Cardano (1501-1576), in his seminal work *Ars Magna* (c. 1545 AD) referred to negative numbers as "numeri ficti," while the German mathematician Michael Stifel (1487-1567) referred to them as "numeri absurdi." John Napier (1550-1617) (the creator of logarithms) called negative numbers "defectivi," and Rene Descartes (1596-1650) (the creator of analytic geometry) labeled negative solutions of algebraic equations as "false roots."

On the other hand, there were mathematicians whose treatment of negative numbers resembled somewhat our modern notions of the properties held by negative numbers. The Indian mathematician Brahmagupta, whose work with zero we've already mentioned, described arithmetical rules in terms of fortunes (positive number) and debts (negative numbers). Indeed, in his work *Brahmasphutasiddhanta*, he writes "a fortune subtracted from zero is a debt," which in modern notation would resemble $0 - 4 = -4$. Further, "a debt subtracted from zero is a fortune," which resonates as $0 - (-4) = 4$. Further, Brahmagupta describes rules for multiplication and division of positive and negative numbers:

- *The product or quotient of two fortunes is one fortune.*
- *The product or quotient of two debts is one fortune.*
- *The product or quotient of a debt and a fortune is a debt.*
- *The product or quotient of a fortune and a debt is a debt.*

In modern-day use we might say that "like signs give a positive answer," while "unlike signs give a negative answer." Modern examples of Brahmagupta's first two rules are $(5)(4) = 20$ and $(-5)(-4) = 20$, while examples of the latter two are $(-5)(4) = -20$ and $(5)(-4) = -20$. The rules are similar for division.

In any event, if we begin with the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, add zero, then add the negative of each natural number, we obtain the set of *integers*.

**Definition 8.** *The set of* **integers** *is the set*

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}. \qquad (9)$$

The letter $\mathbb{Z}$ comes from the word *Zahl*, which is a German word for "number."

It is important to note that an *integer* is a "whole" number, either positive, negative, or zero. Thus, $-11\,456$, $-57$, 0, 235, and $41\,234\,576$ are integers, but the numbers $-2/5$, 0.125, $\sqrt{2}$ and $\pi$ are not. We'll have more to say about the classification of the latter numbers in the sections that follow.

## Rational Numbers

You might have noticed that every natural number is also a whole number. That is, every number in the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ is also a number in the set $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$. Mathematicians say that "$\mathbb{N}$ is a subset of $\mathbb{W}$," meaning that each member of the set $\mathbb{N}$ is also a member of the set $\mathbb{W}$. In a similar vein, each whole number is also an integer, so the set $\mathbb{W}$ is a subset of the set $\mathbb{Z} = \{\ldots, -2, -2, -1, 0, 1, 2, 3, \ldots\}$.

We will now add fractions to our growing set of numbers. Fractions have been used since ancient times. They were well known and used by the ancient Babylonians and Egyptians.

In modern times, we use the phrase *rational number* to describe any number that is the *ratio* of two integers. We will denote the set of rational numbers with the letter $\mathbb{Q}$.

**Definition 10.** *The set of* **rational numbers** *is the set*

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \text{ are integers}, \ n \neq 0 \right\}. \qquad (11)$$

This notation is read "the set of all ratios $m/n$, such that $m$ and $n$ are integers, and $n$ is not 0." The restriction on $n$ is required because division by zero is undefined.

Clearly, numbers such as $-221/31$, $-8/9$, and $447/119$, being the ratios of two integers, are rational numbers (fractions). However, if we think of the integer 6 as the ratio $6/1$ (or alternately, as $24/4$, $-48/-8$, etc.), then we note that 6 is also a rational number. In this way, any integer can be thought of as a rational number (e.g., $12 = 12/1$, $-13 = -13/1$, etc.). Therefore, the set $\mathbb{Z}$ of integers is a subset of the set $\mathbb{Q}$ of rational numbers.

But wait, there is more. Any decimal that terminates is also a rational number. For example,

$$0.25 = \frac{25}{100}, \qquad 0.125 = \frac{125}{1000}, \qquad \text{and} \quad -7.6642 = -\frac{76642}{10000}.$$

The process for converting a terminating decimal to a fraction is clear; count the number of decimal places, then write 1 followed by that number of zeros for the denominator.

For example, in 7.638 there are three decimal places, so place the number over 1000, as in

$$\frac{7638}{1000}.$$

But wait, there is still more, for any decimal that repeats can also be expressed as the ratio of two integers. Consider, for example, the repeating decimal

$$0.0\overline{21} = 0.0212121\ldots.$$

Note that the sequence of integers under the "repeating bar" are repeated over and over indefinitely. Further, in the case of $0.0\overline{21}$, there are precisely two digits[5] under the repeating bar. Thus, if we let $x = 0.0\overline{21}$, then

$$x = 0.0212121\ldots,$$

and multiplying by 100 moves the decimal two places to the right.

$$100x = 2.12121\ldots$$

If we align these two results

$$\begin{aligned} 100x &= 2.12121\ldots \\ -x &= 0.02121\ldots \end{aligned}$$

and subtract, then the result is

$$\begin{aligned} 99x &= 2.1 \\ x &= \frac{2.1}{99}. \end{aligned}$$

However, this last result is not a ratio of two integers. This is easily rectified by multiplying both numerator and denominator by 10.

$$x = \frac{21}{990}$$

We can reduce this last result by dividing both numerator and denominator by 3. Thus, $0.0\overline{21} = 7/330$, being the ratio of two integers, is a rational number.

Let's look at another example.

▶ **Example 12.** *Show that $0.\overline{621}$ is a rational number.*

In this case, there are three digits under the repeating bar. If we let $x = 0.\overline{621}$, then multiply by 1000 (three zeros), this will move the decimal three places to the right.

$$\begin{aligned} 1000x &= 621.621621\ldots \\ x &= \phantom{62}0.621621\ldots \end{aligned}$$

Subtracting,

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[5] The singletons 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 are called *digits*.

$$\begin{aligned} 999x &= 621 \\ x &= \frac{621}{999}. \end{aligned}$$

Dividing numerator and denominator by 27 (or first by 9 then by 3), we find that $0.\overline{621} = 23/37$. Thus, $0.\overline{621}$, being the ratio of two integers, is a rational number.

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At this point, it is natural to wonder, "Are all numbers rational?" Or, "Are there other types of numbers we haven't discussed as yet?" Let's investigate further.

## *The Irrational Numbers*

If a number is not rational, mathematicians say that it is *irrational.*

**Definition 13.** *Any number that cannot be expressed as a ratio of two integers is called an* **irrational number***.*

Mathematicians have struggled with the concept of irrational numbers throughout history. Pictured in **Figure 2** is an ancient Babylonian artifact called *The Square Root of Two Tablet.*

**Figure 2.** The *Square Root of Two Tablet.*

There is an ancient fable that tells of a disciple of Pythagoras who provided a geometrical proof of the irrationality of $\sqrt{2}$. However, the Pythagoreans believed in the absoluteness of numbers, and could not abide the thought of numbers that were not rational. As a punishment, Pythagoras sentenced his disciple to death by drowning, or so the story goes.

But what about $\sqrt{2}$? Is it rational or not? A classic proof, known in the time of Euclid (the "Father of Geometry," c. 300 BC), uses *proof by contradiction.* Let us assume that $\sqrt{2}$ is indeed rational, which means that $\sqrt{2}$ can be expressed as the ratio of two integers $p$ and $q$ as follows.

$$\sqrt{2} = \frac{p}{q}$$

Square both sides,

$$2 = \frac{p^2}{q^2},$$

then clear the equation of fractions by multiplying both sides by $q^2$.

$$p^2 = 2q^2 \tag{14}$$

Now $p$ and $q$ each have their own unique prime factorizations. Both $p^2$ and $q^2$ have an *even* number of factors in their prime factorizations.[6] But this contradicts **equation 14**, because the left side would have an even number of factors in its prime factorization, while the right side would have an odd number of factors in its prime factorization (there's one extra 2 on the right side).

Therefore, our assumption that $\sqrt{2}$ was rational is false. Thus, $\sqrt{2}$ is irrational.

There are many other examples of irrational numbers. For example, $\pi$ is an irrational number, as is the number $e$, which we will encounter when we study exponential functions. Decimals that neither repeat nor terminate, such as

$$0.14114111411114\ldots,$$

are also irrational. Proofs of the irrationality of such numbers are beyond the scope of this course, but if you decide on a career in mathematics, you will someday look closely at these proofs. Suffice it to say, there are a lot of irrational numbers out there. Indeed, there are many more irrational numbers than there are rational numbers.

## *The Real Numbers*

If we take all of the numbers that we have discussed in this section, the natural numbers, the whole numbers, the integers, the rational numbers, and the irrational numbers, and lump them all into one giant set of numbers, then we have what is known as the set of *real numbers*. We will use the letter $\mathbb{R}$ to denote the set of all real numbers.

**Definition 15.**

$$\mathbb{R} = \{x : \ \ x \textit{ is a real number}\}.$$

This notation is read "the set of all $x$ such that $x$ is a real number." The set of real numbers $\mathbb{R}$ encompasses all of the numbers that we will encounter in this course.

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[6] For example, if $p = 2 \times 3 \times 3 \times 5$, then $p^2 = 2 \times 2 \times 3 \times 3 \times 3 \times 3 \times 5 \times 5$, which has an even number of factors.

## 1.1 Exercises

In **Exercises 1-8**, find the prime factorization of the given natural number.

**1.** 80

**2.** 108

**3.** 180

**4.** 160

**5.** 128

**6.** 192

**7.** 32

**8.** 72

In **Exercises 9-16**, convert the given decimal to a fraction.

**9.** 0.648

**10.** 0.62

**11.** 0.240

**12.** 0.90

**13.** 0.14

**14.** 0.760

**15.** 0.888

**16.** 0.104

In **Exercises 17-24**, convert the given repeating decimal to a fraction.

**17.** $0.\overline{27}$

**18.** $0.\overline{171}$

**19.** $0.\overline{24}$

**20.** $0.\overline{882}$

**21.** $0.\overline{84}$

**22.** $0.\overline{384}$

**23.** $0.\overline{63}$

**24.** $0.\overline{60}$

**25.** Prove that $\sqrt{3}$ is irrational.

**26.** Prove that $\sqrt{5}$ is irrational.

In **Exercises 27-30**, copy the given table onto your homework paper. In each row, place a check mark in each column that is appropriate. That is, if the number at the start of the row is rational, place a check mark in the rational column. *Note: Most (but not all) rows will have more than one check mark.*

**27.**

|  | $\mathbb{N}$ | $\mathbb{W}$ | $\mathbb{Z}$ | $\mathbb{Q}$ | $\mathbb{R}$ |
|---|---|---|---|---|---|
| $0$ |  |  |  |  |  |
| $-2$ |  |  |  |  |  |
| $-2/3$ |  |  |  |  |  |
| $0.15$ |  |  |  |  |  |
| $0.\overline{2}$ |  |  |  |  |  |
| $\sqrt{5}$ |  |  |  |  |  |

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**28.**

| | $\mathbb{N}$ | $\mathbb{W}$ | $\mathbb{Z}$ | $\mathbb{Q}$ | $\mathbb{R}$ |
|---|---|---|---|---|---|
| $10/2$ | | | | | |
| $\pi$ | | | | | |
| $-6$ | | | | | |
| $0.\overline{9}$ | | | | | |
| $\sqrt{2}$ | | | | | |
| $0.37$ | | | | | |

**29.**

| | $\mathbb{N}$ | $\mathbb{W}$ | $\mathbb{Z}$ | $\mathbb{Q}$ | $\mathbb{R}$ |
|---|---|---|---|---|---|
| $-4/3$ | | | | | |
| $12$ | | | | | |
| $0$ | | | | | |
| $\sqrt{11}$ | | | | | |
| $1.\overline{3}$ | | | | | |
| $6/2$ | | | | | |

**30.**

| | $\mathbb{N}$ | $\mathbb{W}$ | $\mathbb{Z}$ | $\mathbb{Q}$ | $\mathbb{R}$ |
|---|---|---|---|---|---|
| $-3/5$ | | | | | |
| $\sqrt{10}$ | | | | | |
| $1.625$ | | | | | |
| $10/2$ | | | | | |
| $0/5$ | | | | | |
| $11$ | | | | | |

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In **Exercises 31-42**, consider the given statement and determine whether it is true or false. Write a sentence explaining your answer. In particular, if the statement is false, try to give an example that contradicts the statement.

**31.** All natural numbers are whole numbers.

**32.** All whole numbers are rational numbers.

**33.** All rational numbers are integers.

**34.** All rational numbers are whole numbers.

**35.** Some natural numbers are irrational.

**36.** Some whole numbers are irrational.

**37.** Some real numbers are irrational.

**38.** All integers are real numbers.

**39.** All integers are rational numbers.

**40.** No rational numbers are natural numbers.

**41.** No real numbers are integers.

**42.** All whole numbers are natural numbers.

# 1.1 *Answers*

**1.** $2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$

**3.** $2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$

**5.** $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

**7.** $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

**9.** $\dfrac{81}{125}$

**11.** $\dfrac{6}{25}$

**13.** $\dfrac{7}{50}$

**15.** $\dfrac{111}{125}$

**17.** $\dfrac{3}{11}$

**19.** $\dfrac{8}{33}$

**21.** $\dfrac{28}{33}$

**23.** $\dfrac{7}{11}$

**25.** Suppose that $\sqrt{3}$ is rational. Then it can be expressed as the ratio of two integers $p$ and $q$ as follows:

$$\sqrt{3} = \frac{p}{q}$$

Square both sides,

$$3 = \frac{p^2}{q^2},$$

then clear the equation of fractions by multiplying both sides by $q^2$:

$$p^2 = 3q^2 \tag{16}$$

Now $p$ and $q$ each have their own unique prime factorizations. Both $p^2$ and $q^2$ have an *even* number of factors in their prime factorizations. But this contradicts equation (**14**), because the left side would have an even number of factors in its prime factorization, while the right side would have an odd number of factors in its prime factorization (there's one extra 3 on the right side).

Therefore, our assumption that $\sqrt{3}$ was rational is false. Thus, $\sqrt{3}$ is irrational.

**27.**

| | $\mathbb{N}$ | $\mathbb{W}$ | $\mathbb{Z}$ | $\mathbb{Q}$ | $\mathbb{R}$ |
|---|---|---|---|---|---|
| $0$ | | x | x | x | x |
| $-2$ | | | x | x | x |
| $-2/3$ | | | | x | x |
| $0.15$ | | | | x | x |
| $0.\overline{2}$ | | | | x | x |
| $\sqrt{5}$ | | | | | x |

**29.**

| | $\mathbb{N}$ | $\mathbb{W}$ | $\mathbb{Z}$ | $\mathbb{Q}$ | $\mathbb{R}$ |
|---|---|---|---|---|---|
| $-4/3$ | | | | x | x |
| $12$ | x | x | x | x | x |
| $0$ | | x | x | x | x |
| $\sqrt{11}$ | | | | | x |
| $1.\overline{3}$ | | | | x | x |
| $6/2$ | x | x | x | x | x |

**31.** True. The only difference between the two sets is that the set of whole numbers contains the number 0.

**33.** False. For example, $\frac{1}{2}$ is not an integer.

**35.** False. All natural numbers are rational, and therefore not irrational.

**37.** True. For example, $\pi$ and $\sqrt{2}$ are real numbers which are irrational.

**39.** True. Every integer $b$ can be written as a fraction $b/1$.

**41.** False. For example, 2 is a real number that is also an integer.

# 1.2 Solving Equations

In this section, we review the equation-solving skills that are prerequisite for successful completion of the material in this text. Before we list the tools used in the equation-solving process, let's make sure that we understand what is meant by the phrase "solve for $x$."

**Solve for $x$.** Using the properties that we provide, you must "isolate $x$," so that your final solution takes the form

$$x = \text{"Stuff,"}$$

where "Stuff" can be an expression containing numbers, constants, other variables, and mathematical operators such as addition, subtraction, multiplication, division, square root, and the like.

"Stuff" can even contain other mathematical functions, such as exponentials, logarithms, or trigonometric functions. However, it is essential that you understand that there is one thing "Stuff" must not contain, and that is the variable you are solving for, in this case, $x$. So, in a sense, you want to isolate $x$ on one side of the equation, and put all the other "Stuff" on the other side of the equation.

Now, let's provide the tools to help you with this task.

**Property 1.** Let $a$ and $b$ be any numbers such that $a = b$. Then, if $c$ is any number,

$$a + c = b + c,$$

and,

$$a - c = b - c.$$

In words, the first of these tools allows us to add the same quantity to both sides of an equation without affecting equality. The second statement tells us that we can subtract the same quantity from both sides of an equation and still have equality.

Let's look at an example.

▶ **Example 2.** *Solve the equation $x + 5 = 7$ for $x$.*

The goal is to "isolate $x$ on one side of the equation. To that end, let's subtract 5 from both sides of the equation, then simplify.

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$$\begin{aligned} x + 5 &= 7 \\ x + 5 - 5 &= 7 - 5 \\ x &= 2. \end{aligned}$$

It is important to check your solution by showing that $x = 2$ “satisfies” the original equation. To that end, substitute $x = 2$ in the original equation and simplify both sides of the result.

$$\begin{aligned} x + 5 &= 7 \\ 2 + 5 &= 7 \\ 7 &= 7 \end{aligned}$$

This last statement (i.e., $7 = 7$) is a true statement, so $x = 2$ is a solution of the equation $x + 5 = 7$.

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An important concept is the idea of *equivalent equations*.

**Equivalent Equations**. Two equations are said to be **equivalent** if and only if they have the same solution set. That is, two equations are equivalent if each of the solutions of the first equation is also a solution of the second equation, and vice-versa.

Thus, in **Example 2**, the equations $x+5 = 7$ and $x = 2$ are equivalent, because they both have the same solution set $\{2\}$. It is no coincidence that the tools in **Property 1** produce equivalent equations. Whenever you add the same amount to both sides of an equation, the resulting equation is equivalent to the original equation (they have the same solution set). This is also true for subtraction. When you subtract the same amount from both sides of an equation, the resulting equation has the same solutions as the original equation.

Let’s look at another example.

► **Example 3.** *Solve the equation $x - 7 = 12$ for $x$.*

We want to “isolate $x$” on one side of the equation, so we add 7 to both sides of the equation and simplify.

$$\begin{aligned} x - 7 &= 12 \\ x - 7 + 7 &= 12 + 7 \\ x &= 19 \end{aligned}$$

We will leave it to our readers to check that $x = 19$ is a solution of $x - 7 = 12$.

◆

Let’s pause for a moment and define what is meant by a *monomial*.

**Definition 4.** *A monomial is an algebraic expression that is the product of a number and zero or more variables, each raised to some arbitrary exponent.*

Examples of monomials are:

$$3x^2, \qquad \text{or} \qquad -4ab^2, \qquad \text{or} \qquad 25x^3y^5, \qquad \text{or} \qquad 17, \qquad \text{or} \qquad -11x.$$

Monomials are commonly referred to as "terms." We often use algebraic expressions that are the sum of two or more terms. For example, the expression

$$3x^3 + 2x^2 - 7x + 8 \qquad \text{or equivalently} \qquad 3x^3 + 2x^2 + (-7x) + 8,$$

is the sum of four terms, namely, $3x^3$, $2x^2$, $-7x$, and 8. Note that the terms are thos parts of the expression that are separated by addition symbols.

Some mathematicians prefer use the word "term" in a more relaxed manner, simply stating that the terms of an algebraic expression are those components of the expression that are separated by addition symbols. For example, the terms of the expression

$$3x^2 - \frac{1}{x} + \frac{2x^2}{x+3} \qquad \text{or equivalenty} \qquad 3x^2 + \left(-\frac{1}{x}\right) + \frac{2x^2}{x+3},$$

are $3x^2$, $-1/x$, and $2x^2/(x+3)$. This is the meaning we will use in this text.

Having made the definition of what is meant by a "term," let's return to our discussion of solving equations.

▶ **Example 5.** *Solve the equation* $3x - 3 = 2x + 4$ *for* $x$.

We will isolate all terms containing an $x$ on the left side of this equation (we could just as well have chosen to isolate terms containing $x$ on the right side of the equation). To this end, we don't want the $-3$ on the left side of the equation (we want it on the right), so we add 3 to both sides of the equation and simplify.

$$\begin{aligned} 3x - 3 &= 2x + 4 \\ 3x - 3 + 3 &= 2x + 4 + 3 \\ 3x &= 2x + 7 \end{aligned}$$

Remember that we have chosen to isolate all terms containing $x$ on the left side of the equation. So, for our next step, we choose to subtract $2x$ from both sides of the equation (this will "move" it from the right over to the left), then simplify.

$$\begin{aligned} 3x &= 2x + 7 \\ 3x - 2x &= 2x + 7 - 2x \\ x &= 7 \end{aligned}$$

To check the solution, substitute $x = 7$ in the original equation to obtain

$$\begin{aligned} 3x - 3 &= 2x + 4 \\ 3(7) - 3 &= 2(7) + 4 \\ 21 - 3 &= 14 + 4 \\ 18 &= 18 \end{aligned}$$

The last line is a true statement, so $x = 7$ checks and is a solution of $3x - 3 = 2x + 4$.

◆

If you use the technique of **Example 5** repeatedly, there comes a point when you tire of showing the addition or subtraction of the same amount on both sides of your equation. Here is a tool, which, if carefully used, will greatly simplify your work.[9]

**Useful Shortcut**. When you move a term from one side of an equation to the other, that is, when you move a term from one side of the equal sign to the other side, simply change its sign.

Let's see how we would apply this shortcut to the equation of **Example 5**. Start with the original equation,

$$3x - 3 = 2x + 4,$$

then move all terms containing an $x$ to the left side of the equation, and move all other terms to the right side of the equation. Remember to change the sign of a term if it moves from one side of the equals sign to the other. If a term does not move from one side of the equation to the other, leave its sign alone. The result would be

$$3x - 2x = 4 + 3.$$

Thus, $x = 7$ and you are finished.

It is important to note that when we move the $-3$ from the left-hand side of the above equation to the right-hand side of the equation and change its sign, what we are actually doing is adding 3 to both sides of the equation. A similar statement explains that moving $2x$ from the right-hand side to the left-hand side and changing its sign is simply a shortcut for subtracting $2x$ from both sides of the equation.

Here are two more useful tools for solving equations.

---

[9] You should be aware that mathematics educators seemingly divide into two distinct camps regarding this tool: some refuse to let their students use it, others are comfortable with their students using it. There are good reasons for this dichotomy which we won't go into here, but you should check to see how your teacher feels about your use of this tool in your work.

**Property 6.** Let $a$ and $b$ be any numbers such that $a = b$. Then, if $c$ is any number other than zero,

$$ac = bc.$$

If $c$ is any number other than zero, then

$$\frac{a}{c} = \frac{b}{c}.$$

In words, the first of these tools allows us to multiply both sides of an equation by the same number. A similar statement holds for division, provided we do not divide by zero (division by zero is meaningless). Both of these tools produce equivalent equations.

Let's look at an example.

▶ **Example 7.** *Solve the equation* $5x = 15$ *for* $x$.

In this case, only one term contains the variable $x$ and this term is already isolated on one side of the equation. We will divide both sides of this equation by 5, then simplify, obtaining

$$\begin{aligned} 5x &= 15 \\ \frac{5x}{5} &= \frac{15}{5} \\ x &= 3. \end{aligned}$$

We'll leave it to our readers to check this solution.

◆

▶ **Example 8.** *Solve the equation* $x/2 = 7$ *for* $x$.

Again, there is only one term containing $x$ and it is already isolated on one side of the equation. We will multiply both sides of the equation by 2, then simplify, obtaining

$$\begin{aligned} \frac{x}{2} &= 7 \\ 2\left(\frac{x}{2}\right) &= 2(7) \\ x &= 14. \end{aligned}$$

Again, we will leave it to our readers to check this solution.

◆

Let's apply everything we've learned in the next example.

▶ **Example 9.** *Solve the equation* $7x - 4 = 5 - 3x$ *for* $x$.

Note that we have terms containing $x$ on both sides of the equation. Thus, the first step is to isolate the terms containing $x$ on one side of the equation (left or right, your choice).[10] We will move the terms containing $x$ to the left side of the equation, everything else will be moved to the right side of the equation. Remember the rule, if a term moves from one side of the equal sign to the other, change the sign of the term you are moving. Thus,

$$\begin{aligned} 7x - 4 &= 5 - 3x \\ 7x + 3x &= 5 + 4. \end{aligned}$$

Simplify.

$$10x = 9$$

Divide both sides of this last result by 10.

$$\begin{aligned} 10x &= 9 \\ \frac{10x}{10} &= \frac{9}{10} \\ x &= \frac{9}{10} \end{aligned}$$

To check this solution, substitute $x = 9/10$ into both sides of the original equation and simplify.

$$\begin{aligned} 7x - 4 &= 5 - 3x \\ 7\left(\frac{9}{10}\right) - 4 &= 5 - 3\left(\frac{9}{10}\right) \\ \frac{63}{10} - 4 &= 5 - \frac{27}{10} \end{aligned}$$

We'll need a common denominator to verify that our solution is correct. That is,

$$\begin{aligned} \frac{63}{10} - \frac{40}{10} &= \frac{50}{10} - \frac{27}{10} \\ \frac{23}{10} &= \frac{23}{10}. \end{aligned}$$

Thus, $x = 9/10$ checks and is a solution of $7x - 4 = 5 - 3x$.

Note that the check can sometimes be more difficult than solving the equation. This is one of the reasons that we tend to get lazy and not check our solutions. However, we shouldn't need to tell you what will probably happen if you do not check your work.

There is a workaround that involves the use of the graphing calculator. We first store the solution for $x$ in our calculator, then calculate each side of the original equation and compare results.

1. Enter `9/10` in your calculator window, then
2. push the `STO▶` key, then
3. push the `X` key followed by the `ENTER` key.

---

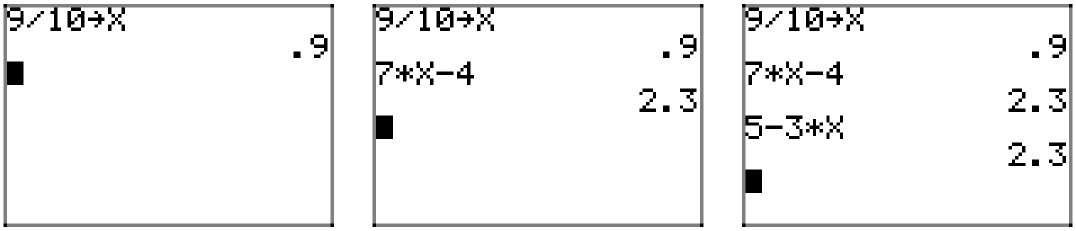
[10] Although moving all the terms containing $x$ to the right side is alright, it is often preferable to have the $x$ terms on the left side of the equation in order to end up with $x =$ "Stuff."

The result is shown in **Figure 1**(a).

Now that we've stored $x = 9/10$ in the calculator's memory, let's evaluate each side of the equation $7x - 4 = 5 - 3x$ at this value of $x$. Enter `7*X-4` in your calculator and press `ENTER`. The result is shown in **Figure 1**(b), where we see that $7x - 4$, evaluated at $x = 9/10$, equals 2.3.

Next, enter `5-3*X` and press `ENTER`. The result is shown in **Figure 1**(c), where we see that $5 - 3x$, evaluated at $x = 9/10$, also equals 2.3 (by the way, this is equivalent to the 23/10 we found in our hand check above).

Because the expressions on each side of the equation are equal when $x = 9/10$ (both equal 2.3), the solution checks.



```
9/10→X
                .9
█
```

```
9/10→X
                .9
7*X-4
               2.3
█
```

```
9/10→X
                .9
7*X-4
               2.3
5-3*X
               2.3
█
```

(a) (b) (c)

**Figure 1.** Checking the solution of $7x - 4 = 5 - 3x$ with the graphing calculator.

◆

If you need to solve an equation that contains fractions, one very useful strategy is to clear the equations of fractions by multiplying both sides of the equation by the least common denominator.

▶ **Example 10.** *Solve the equation*

$$\frac{2}{3}x - \frac{3}{4} = \frac{1}{4} - \frac{3}{2}x$$

*for $x$.*

The least common denominator is 12, so we multiply both sides of this equation by 12.

$$12\left(\frac{2}{3}x - \frac{3}{4}\right) = 12\left(\frac{1}{4} - \frac{3}{2}x\right)$$

Distribute the 12 and simplify.

$$\begin{aligned} 12\left(\frac{2}{3}x\right) - 12\left(\frac{3}{4}\right) &= 12\left(\frac{1}{4}\right) - 12\left(\frac{3}{2}x\right) \\ 8x - 9 &= 3 - 18x \end{aligned}$$

Move all terms containing $x$ to the left side of the equation, everything else to the right, then simplify.

$$\begin{aligned} 8x + 18x &= 3 + 9 \\ 26x &= 12 \end{aligned}$$

Divide both sides of this last result by 26 and simplify (always reduce to lowest terms — in this case we can divide both numerator and denominator by 2).

$$\begin{aligned} \frac{26x}{26} &= \frac{12}{26} \\ x &= \frac{6}{13} \end{aligned}$$

We leave it to our readers to check this solution. Use your graphing calculator as demonstrated in **Example 9**.

◇

You can clear decimals from an equation by multiplying by the appropriate power of 10.[11]

▶ **Example 11.** *Solve the equation* $1.23x - 5.46 = 3.72x$ *for* $x$*.*

Let's multiply both sides of this equation by 100, which moves the decimal two places to the right, which is enough to clear the decimals from this problem.

$$100(1.23x - 5.46) = 100(3.72x)$$

Distribute and simplify.

$$\begin{aligned} 100(1.23x) - 100(5.46) &= 100(3.72x) \\ 123x - 546 &= 372x \end{aligned}$$

Move each term containing an $x$ to the right side of the equation (the first time we've chosen to do this — it avoids a negative sign in the coefficient of $x$) and simplify.

$$\begin{aligned} -546 &= 372x - 123x \\ -546 &= 249x \end{aligned}$$

Divide both sides of the equation by 249 and simplify (in this case we can reduce, dividing numerator and denominator by 3).

$$\begin{aligned} \frac{-546}{249} &= \frac{249x}{249} \\ -\frac{182}{83} &= x \end{aligned}$$

Rewrite your answer, placing $x$ on the left side of the equation.

$$x = -\frac{182}{83}$$

Check your result with your calculator. It is important to be sure that you always use the original problem when you check your result. The steps are shown in **Figure 2**(a), (b), and (c).

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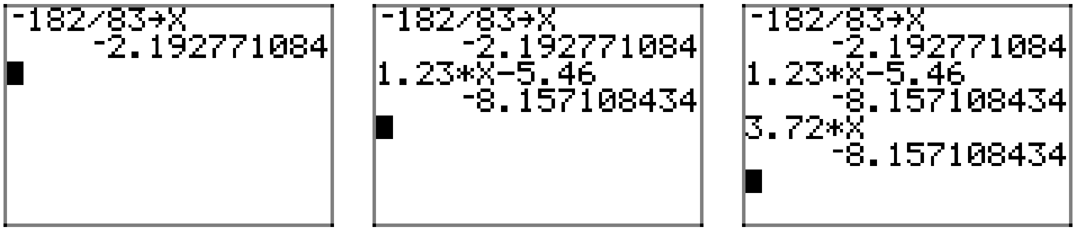
[11] Multiplying by 10 moves the decimal one place to the right, multiplying by 100 moves the decimal two places to the right, etc.



```
-182/83→X
     -2.192771084
```

```
-182/83→X
     -2.192771084
1.23*X-5.46
     -8.157108434
```

```
-182/83→X
     -2.192771084
1.23*X-5.46
     -8.157108434
3.72*X
     -8.157108434
```

(a) (e) (f)

**Figure 2.** Checking that $x = -182/83$ is a solution of $1.23x - 5.46 = 3.72x$.

## Formulae

Science is filled with formulae that involve more than one variable and a number of constants. In chemistry and physics, the instructor will expect that you can manipulate these equations, solving for one variable or constant in terms of the others in the equation.

There is nothing new to say here, as you should follow the same rules that we've given heretofore when the only variable was $x$. However, students usually find these a bit intimidating because of the presence of multiple variables and constants, so let's take our time and walk through a couple of examples.

▶ **Example 12.** *Isaac Newton is credited with the formula that determines the magnitude $F$ of the force of attraction between two planets. The formula is*

$$F = \frac{GmM}{r^2},$$

*where $m$ is the mass of the smaller planet, $M$ is the mass of the larger planet, $r$ is the distance between the two planets, and $G$ is a universal gravitational constant. Solve this equation for $G$.*

First, a word of caution.

**Warning 13.** *When using formulae of science, never change the case of a variable or constant. If it is uppercase, write it in uppercase on your homework. The same directive applies if the variable or constant is presented in lowercase. Write it in lowercase on your homework.*

This equation has fractions in it, so we will begin by multiplying both sides of the equation by the common denominator, which in this case is $r^2$.

$$r^2(F) = r^2\left(\frac{GmM}{r^2}\right)$$

This gives us

$$r^2F = GmM.$$

In this case, there is only one term with G, and that term is already isolated on one side of the equation. The next step is to divide both sides of the equation by the coefficient of $G$, then simplify.

$$\begin{aligned}\frac{r^2F}{mM} &= \frac{GmM}{mM}\\ \frac{r^2F}{mM} &= G\end{aligned}$$

Hence,

$$G = \frac{r^2F}{mM}.$$

Note that we have $G =$ "Stuff", and most importantly, the "Stuff" has no occurrence of the variable $G$. This is what it means to "solve for $G$."

◆

Let's look at a final example.

▶ **Example 14.** *Water freezes at 0° Celsius and boils at 100° Celsius. Americans are probably more familiar with Fahrenheit temperature, where water freezes at 32° Fahrenheit and boils at 212° Fahrenheit. The formula to convert Celsius temperature $C$ into Fahrenheit temperature $F$ is*

$$F = \frac{9}{5}C + 32.$$

*Solve this equation for $C$.*

Once again, the equation has fractions in it, so our first move will be to eliminate the fractions by multiplying both sides of the equation by the common denominator (5 in this case).

$$\begin{aligned}5F &= 5\left(\frac{9}{5}C + 32\right)\\ 5F &= 5\left(\frac{9}{5}C\right) + 5(32)\\ 5F &= 9C + 160\end{aligned}$$

We're solving for $C$, so move all terms containing a $C$ to one side of the equation, and all other terms to the other side of the equation.

$$5F - 160 = 9C$$

Divide both sides of this last equation by 9.

$$\begin{aligned}\frac{5F-160}{9} &= \frac{9C}{9}\\ \frac{5F-160}{9} &= C\end{aligned}$$

Thus,

$$C = \frac{5F-160}{9}.$$

Note that we have $C =$ "Stuff," and most importantly, the "Stuff" has no occurrence of the variable $C$. This is what it means to solve for $C$.

◇

Once you've solved a formula from science for a particular variable, you can use the result to make conversions or predictions.

▶ **Example 15.** *In* **Example 14**, *the relationship between Fahrenheit and Celsius temperatures is given by the result*

$$C = \frac{5F-160}{9}. \tag{16}$$

*Above the bank in Eureka, California, a sign proclaims that the Fahrenheit temperature is 40° F. What is the Celsius temperature?*

Substitute the Fahrenheit temperature into **formula (16)**. That is, substitute $F = 40$.

$$C = \frac{5F-160}{9} = \frac{5(40)-160}{9} = \frac{40}{9} \approx 4.44$$

Hence, the Celsius temperature is approximately 4.44° C. Note that you should always include units with your final answer.

◇

## 1.2 Exercises

In **Exercises 1-12**, solve each of the given equations for $x$.

**1.** $45x + 12 = 0$

**2.** $76x - 55 = 0$

**3.** $x - 7 = -6x + 4$

**4.** $-26x + 84 = 48$

**5.** $37x + 39 = 0$

**6.** $-48x + 95 = 0$

**7.** $74x - 6 = 91$

**8.** $-7x + 4 = -6$

**9.** $-88x + 13 = -21$

**10.** $-14x - 81 = 0$

**11.** $19x + 35 = 10$

**12.** $-2x + 3 = -5x - 2$

In **Exercises 13-24**, solve each of the given equations for $x$.

**13.** $6 - 3(x + 1) = -4(x + 6) + 2$

**14.** $(8x + 3) - (2x + 6) = -5x + 8$

**15.** $-7 - (5x - 3) = 4(7x + 2)$

**16.** $-3 - 4(x + 1) = 2(x + 4) + 8$

**17.** $9 - (6x - 8) = -8(6x - 8)$

**18.** $-9 - (7x - 9) = -2(-3x + 1)$

**19.** $(3x - 1) - (7x - 9) = -2x - 6$

**20.** $-8 - 8(x - 3) = 5(x + 9) + 7$

**21.** $(7x - 9) - (9x + 4) = -3x + 2$

**22.** $(-4x - 6) + (-9x + 5) = 0$

**23.** $-5 - (9x + 4) = 8(-7x - 7)$

**24.** $(8x - 3) + (-3x + 9) = -4x - 7$

In **Exercises 25-36**, solve each of the given equations for $x$. Check your solutions using your calculator.

**25.** $-3.7x - 1 = 8.2x - 5$

**26.** $8.48x - 2.6 = -7.17x - 7.1$

**27.** $-\frac{2}{3}x + 8 = \frac{4}{5}x + 4$

**28.** $-8.4x = -4.8x + 2$

**29.** $-\frac{3}{2}x + 9 = \frac{1}{4}x + 7$

**30.** $2.9x - 4 = 0.3x - 8$

**31.** $5.45x + 4.4 = 1.12x + 1.6$

**32.** $-\frac{1}{4}x + 5 = -\frac{4}{5}x - 4$

**33.** $-\frac{3}{2}x - 8 = \frac{2}{5}x - 2$

**34.** $-\frac{4}{3}x - 8 = -\frac{1}{4}x + 5$

**35.** $-4.34x - 5.3 = 5.45x - 8.1$

**36.** $\frac{2}{3}x - 3 = -\frac{1}{4}x - 1$

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In **Exercises 37-50**, solve each of the given equations for the indicated variable.

**37.** $P = IRT$ for $R$

**38.** $d = vt$ for $t$

**39.** $v = v_0 + at$ for $a$

**40.** $x = v_0 + vt$ for $v$

**41.** $Ax + By = C$ for $y$

**42.** $y = mx + b$ for $x$

**43.** $A = \pi r^2$ for $\pi$

**44.** $S = 2\pi r^2 + 2\pi rh$ for $h$

**45.** $F = \dfrac{kqq_0}{r^2}$ for $k$

**46.** $C = \dfrac{Q}{mT}$ for $T$

**47.** $\dfrac{V}{t} = k$ for $t$

**48.** $\lambda = \dfrac{h}{mv}$ for $v$

**49.** $\dfrac{P_1V_1}{n_1T_1} = \dfrac{P_2V_2}{n_2T_2}$ for $V_2$

**50.** $\pi = \dfrac{nRT}{V}i$ for $n$

---

**51.** Tie a ball to a string and whirl it around in a circle with constant speed. It is known that the acceleration of the ball is directly toward the center of the circle and given by the formula

$$a = \frac{v^2}{r}, \qquad (17)$$

where $a$ is acceleration, $v$ is the speed of the ball, and $r$ is the radius of the circle of motion.

i. Solve **formula (17)** for $r$.
ii. Given that the acceleration of the ball is $12\,\mathrm{m/s^2}$ and the speed is $8\,\mathrm{m/s}$, find the radius of the circle of motion.

**52.** A particle moves along a line with constant acceleration. It is known the velocity of the particle, as a function of the amount of time that has passed, is given by the equation

$$v = v_0 + at, \qquad (18)$$

where $v$ is the velocity at time $t$, $v_0$ is the initial velocity of the particle (at time $t = 0$), and $a$ is the acceleration of the particle.

i. Solve **formula (18)** for $t$.
ii. You know that the current velocity of the particle is $120\,\mathrm{m/s}$. You also know that the initial velocity was $40\,\mathrm{m/s}$ and the acceleration has been a constant $a = 2\,\mathrm{m/s^2}$. How long did it take the particle to reach its current velocity?

**53.** Like Newton's *Universal Law of Gravitation*, the force of attraction (repulsion) between two unlike (like) charged particles is proportional to the product of the charges and inversely proportional to the distance between them.

$$F = k_C\frac{q_1q_2}{r^2} \qquad (19)$$

In this formula, $k_C \approx 8.988\times10^9\,\mathrm{Nm^2/C^2}$ and is called the *electrostatic constant*. The variables $q_1$ and $q_2$ represent the charges (in Coulombs) on the particles (which could either be positive or negative numbers) and $r$ represents the distance (in meters) between the charges. Finally, $F$ represents the force of the charge, measured in Newtons.

i. Solve **formula (19)** for $r$.
ii. Given a force $F = 2.0 \times 10^{12}\,\mathrm{N}$, two equal charges $q_1 = q_2 = 1\,\mathrm{C}$, find the approximate distance between the two charged particles.

## 1.2 Answers

---

**1.** $-\dfrac{4}{15}$

**3.** $\dfrac{11}{7}$

**5.** $-\dfrac{39}{37}$

**7.** $\dfrac{97}{74}$

**9.** $\dfrac{17}{44}$

**11.** $-\dfrac{25}{19}$

**13.** $-25$

**15.** $-\dfrac{4}{11}$

**17.** $\dfrac{47}{42}$

**19.** 7

**21.** 15

**23.** $-1$

**25.** $\dfrac{40}{119}$

**27.** $\dfrac{30}{11}$

**29.** $\dfrac{8}{7}$

**31.** $-\dfrac{280}{433}$

**33.** $-\dfrac{60}{19}$

**35.** $\dfrac{280}{979}$

**37.** $R = \dfrac{P}{IT}$

**39.** $a = \dfrac{v - v_0}{t}$

**41.** $y = \dfrac{C - Ax}{B}$

**43.** $\pi = \dfrac{A}{r^2}$

**45.** $k = \dfrac{Fr^2}{qq_0}$

**47.** $t = \dfrac{V}{k}$

**49.** $V_2 = \dfrac{n_2P_1V_1T_2}{n_1P_2T_1}$

**51.** $r = v^2/a$, $r = 16/3$ meters.

**53.** $r \approx 0.067$ meters.

# 1.3 Logic

Two of the most subtle words in the English language are the words "and" and "or." One has only three letters, the other two, but it is absolutely amazing how much confusion these two tiny words can cause. Our intent in this section is to clear the mystery surrounding these words and prepare you for the mathematics that depends upon a thorough understanding of the words "and" and "or."

## *Set Notation*

We begin with the definition of a *set*.

**Definition 1.** *A* **set** *is a collection of objects.*

The objects in the set could be anything at all: numbers, letters, first names, cities, you name it. In this section we will focus on sets of *numbers*, but it is important to understand that the objects in a set can be whatever you choose them to be.

If the number of objects in a set is finite and small enough, we can describe the set simply by listing the elements (objects) in the set. This is usually done by enclosing the list of objects in the set with curly braces. For example, let

$$A = \{1, 3, 5, 7, 9, 11\}. \qquad (2)$$

Now, when we refer to the set $A$ in the narrative, everyone should know we're talking about the set of numbers 1, 3, 5, 7, 9, and 11.

It is also possible to describe the set $A$ with words. Although there are many ways to do this, one possible description might be "Let $A$ be the set of odd natural numbers between 1 and 11, inclusive." This descriptive technique is particularly efficient when the set you are describing is either infinite or too large to enumerate in a list.

For example, we might say "let $A$ be the set of all real numbers that are greater than 4." This is much better than trying to list each of the numbers in the set $A$, which would be futile in this case. Another possibility is to combine the curly brace notation with a textual description and write something like

$$A = \{\text{real numbers that are greater than } 4\}.$$

If we're called upon to read this notation aloud, we would say "$A$ is the set of all real numbers that are greater than 4," or something similar.

There are a number of more sophisticated methods we can use to describe a set. One description that we will often employ is called *set-builder notation* and has the following appearance.

$$A = \{x : \text{some statement describing } x\}$$

---

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It is standard to read the notation $\{x : \qquad\quad \}$ aloud as follows: "The set of all $x$ such that." That is, the colon is pronounced "such that." Then you would read the description that follows the colon. For example, the set

$$A = \{x : x < 3\}$$

is read aloud "$A$ is the set of all $x$ such that $x$ is less than 3." Some people prefer to use a "bar" instead of a colon and they write

$$A = \{x | \text{some statement describing } x\}.$$

This is also pronounced "$A$ is the set of all $x$ such that," and then you would read the text description that follows the "bar." Thus, the notation

$$A = \{x | x < 3\}$$

is identical to the notation $A = \{x : x < 3\}$ used above and is read in exactly the same manner, "$A$ is the set of all $x$ such that $x$ is less than 3." We prefer the colon notation, but feel free to use the "bar" if you like it better. It means the same thing.

A moment's thought will reveal the fact that the notation $A = \{x : x < 3\}$ is not quite descriptive enough. It's probably safe to say, since the description of $x$ is "$x < 3$," that this notation is referring to *numbers* that are less than 3, but what kind of numbers? Natural numbers? Integers? Rational numbers? Irrational numbers? Real numbers? The notation $A = \{x : x < 3\}$ doesn't really tell the whole story.

We'll fix this deficiency in a moment, but first recall that in our preliminary chapter, we used specific symbols to represent certain sets of numbers. Indeed, we used the following:

$$\begin{aligned} \mathbb{N} &= \{\text{natural numbers}\} \\ \mathbb{Z} &= \{\text{integers}\} \\ \mathbb{Q} &= \{\text{rational numbers}\} \\ \mathbb{R} &= \{\text{real numbers}\} \end{aligned}$$

We can use these symbols to help denote the type of number described with our set-builder notation. For example, if we write

$$A = \{x \in \mathbb{N} : x < 3\},$$

then we say "$A$ is the set of all $x$ in the natural numbers such that $x$ is less than 3," or more simply, "the set of all natural numbers that are less than 3." The symbol $\in$ is the Greek letter "epsilon," and when used in set-builder notation, it is pronounced "is an element of," or "is in." Of course, the only natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ that are less than 3 are the natural numbers 1 and 2. Thus, $A = \{1, 2\}$, the "set whose members are 1 and 2."

On the other hand, if we write

$$A = \{x \in \mathbb{Z} : x < 3\},$$

then we say that "$A$ is the set of $x$ in the set of integers such that $x$ is less than 3," or more informally, "$A$ is the set of all integers less than 3." Of course, the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$[14] less than 3 are infinite in number. We cannot list all of them unless we appeal to the imagination with something like

$$A = \{\ldots, -3, -2, -1, 0, 1, 2\}.$$

The ellipsis ... means "etc." We've listed enough of the numbers to establish a pattern, so we're permitted to say "and so on." The reader intuits that the earlier numbers in the list are $-4$, $-5$, etc.

Let's look at another example. Suppose that we write

$$A = \{x \in \mathbb{R} : x < 3\}.$$

Then we say "$A$ is the set of all $x$ in the set of real numbers such that $x$ is less than 3," or more informally, "$A$ is the set of all real numbers less than 3." Of course, this is another infinite set and it's not hard to imagine that the notation $\{x \in \mathbb{R} : x < 3\}$ used above is already optimal for describing this set of real numbers.

In this text, we will mostly deal with sets of real numbers. Thus, from this point forward, if we write

$$A = \{x : x < 3\},$$

we will assume that we mean to say that "$A$ is the set of all real numbers less than 3." That is, if we write $A = \{x : x < 3\}$, we understand this to mean $A = \{x \in \mathbb{R} : x < 3\}$. In the case when we want to use a specific set of numbers, we will indicate that as we did above, for example, in $A = \{x \in \mathbb{N} : x < 3\}$.

## The Real Line and Interval Notation

Suppose that we draw a line (affectionately known as the "real line"), then plot a point anywhere on that line, then map the number zero to that point (called the "origin"), as shown in **Figure 1**. Secondly, decide on a unit distance and map the number 1 to that point, again shown in **Figure 1**.

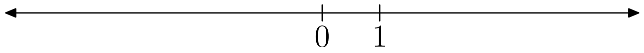

**Figure 1.** Establishing the origin and a unit length on the real line.

Now that we've established a unit distance, every real number corresponds to a point on the real line. Vice-versa, every point on the real line corresponds to a real number. This defines a one-to-one correspondence between the real numbers in $\mathbb{R}$ and the points on the real line. In this manner, the point on the line and the real number can be thought of as synonymous. **Figure 2** shows several real numbers plotted on the real line.

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[14] The notation ± is shorthand for "plus or minus". For example, the sets $\{\pm 1, \pm 2\}$ and $\{-2, -1, 1, 2\}$ are identical.

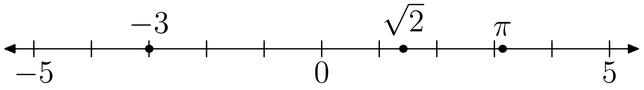


**Figure 2.** Sample numbers on the real line.

Now, suppose that we're asked to shade all real numbers in the set $\{x : x > 3\}$. Because this requires that we shade every real number that is greater than 3 (to the right of 3), we use the shading shown in **Figure 3** to represent the set $\{x : x > 3\}$.

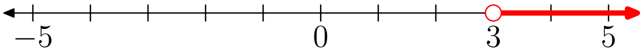


**Figure 3.** Shading all real numbers greater than 3.

Although technically correct, the image in **Figure 3** contains more information than is really needed. The picture is acceptable, but crowded. The really important information is the fact that the shading starts at 3, then moves to the right. Also, because 3 is not in the set $\{x : x > 3\}$, that is, 3 is not greater than 3, we do not shade the point corresponding to the real number 3. Note that we've indicated this fact with an "empty" circle at 3 on the real line.

Thus, when shading the set $\{x : x > 3\}$ on the real line, we need only label the endpoint at 3, use an "empty" circle at 3, and shade all the real numbers to the right of 3, as shown in **Figure 4**.

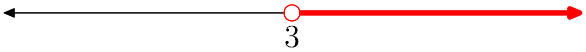


**Figure 4.** Shading all real numbers greater than 3. The endpoint is the only information that needs to be labeled. It is not necessary to show any other tickpoints and/or labels.

Because we're shading all numbers from 3 to positive infinity in **Figure 4**, we'll use the following *interval notation* to represent this "interval" of numbers (everything between 3 and positive infinity).

$$(3, \infty) = \{x : x > 3\}$$

Similarly, **Table 1** lists the set-builder and interval notations, as well as shading of the sets on the real line, for several situations, including the one just discussed.

There are several points of emphasis regarding the intervals in **Table 1**.

1. When we want to emphasize that we are not including a point on the real line, we use an "empty circle." Conversely, a "filled circle" means that we are including the point on the real line. Thus, the real lines in the first two rows of **Table 1** do not include the number 3, but the real lines in the last two rows in **Table 1** do include the number 3.
2. The use of a parenthesis in interval notation means that we are not including that endpoint in the interval. Thus, the parenthesis use in $(-\infty, 3)$ in the second row of **Table 1** means that we are not including the number 3 in the interval.

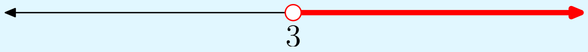
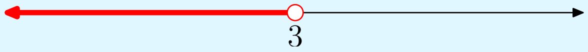
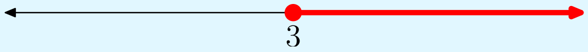
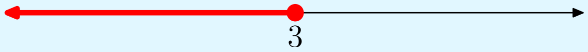
| **Number line** | **Set-builder notation** | **Interval notation** |
|---|---|---|
|  | $\{x : x > 3\}$ | $(3, \infty)$ |
|  | $\{x : x < 3\}$ | $(-\infty, 3)$ |
|  | $\{x : x \geq 3\}$ | $[3, \infty)$ |
|  | $\{x : x \leq 3\}$ | $(-\infty, 3]$ |

**Table 1.** Number lines, set-builder notation, and interval notation.

3. The use of a bracket in interval notation means that we are including the bracketed number in the interval. Thus, the bracket used in $[3, \infty)$, as seen in the third row of **Table 1**, means that we are including the number 3 in the interval.
4. The use of $\infty$ in $(3, \infty)$ in row one of **Table 1** means that we are including every real number greater than 3. The use of $-\infty$ in $(-\infty, 3]$ means that we are including every real number less than or equal to 3. As $-\infty$ and $\infty$ are not actual numbers, it makes no sense to include them with a bracket. Consequently, you must always use a parenthesis with $-\infty$ or $\infty$.

## *Union and Intersection*

The *intersection* of two sets $A$ and $B$ is defined as follows.

**Definition 3.** *The intersection of the sets $A$ and $B$ is the set of all objects that are in $A$* **and** *in $B$. In symbols, we write*

$$A \cap B = \{x : x \in A \text{ and } x \in B\}. \tag{4}$$

In order to understand this definition, it's absolutely crucial that we understand the meaning of the word "and." The word "and" is a conjunction, used between statements $P$ and $Q$, as in "It is raining today and my best friend is the Lone Ranger." In order to determine the truth or falsehood of this statement, you must first examine the truth or falsehood of the statements $P$ and $Q$ on each side of the word "and."

The only way that the speaker is telling the truth is if both statements $P$ and $Q$ are true. In other words, the statement "It is raining today and my best friend is the Lone Ranger" is true if and only if the statement "It is raining today" is true *and* the statement "my best friend is the Lone Ranger" is also true. Logicians like to make up a construct called a truth table, like the one shown in **Table 2**.

Points in **Table 2** to consider:

| $P$ | $Q$ | $P$ **and** $Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

**Table 2.** Truth table for the conjunction "and."

- In the first row (after the header row) of **Table 2**, if statements $P$ and $Q$ are both true (indicated with a T), then the statement "$P$ and $Q$" is also true.
- In the remaining rows of **Table 2**, one or the other of statements $P$ or $Q$ are false (indicated with an F), so the statement "$P$ and $Q$" is also false.

Therefore, the statement "$P$ and $Q$" is true if and only if $P$ is true and $Q$ is true.

▶ **Example 5.** *If $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 5, 7, 8, 11\}$, find the intersection of $A$ and $B$.*

As a reminder, the *intersection* of $A$ and $B$ is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Thus, we are looking for the objects that are in $A$ **and** in $B$. The only objects that are in $A$ and in $B$ (remember, both statements "in $A$" and "in $B$" must be true) are 5 and 7, so we write:

$$A \cap B = \{5, 7\}.$$

Mathematicians and logicians both use a visual aid called a *Venn Diagram* to represent sets. John Venn was an English mathematician who devised this visualization of logical relationships. Consider the ellipse $A$ in **Figure 5**. Everything inside the boundary of this ellipse constitutes the set $A = \{1, 3, 5, 7, 9\}$. That's why you see these numbers inside the boundary of this ellipse.

Consider the ellipse $B$ in **Figure 5**. Everything inside the boundary of this ellipse constitutes the set $B = \{2, 5, 7, 8, 11\}$. That's why you see these numbers inside the boundary of this ellipse.

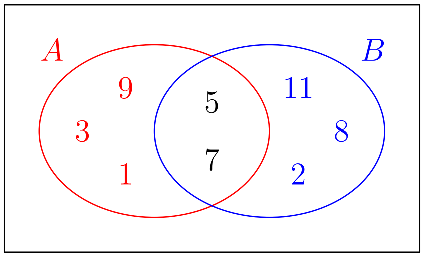


**Figure 5.** Venn Diagram

Now, note that only two numbers, 5 and 7, are contained within the boundaries of both $A$ and $B$. These are the numbers that are in the intersection of the sets $A$ and $B$.

◆

The shaded region in **Figure 6** is the area that belongs to both of the sets $A$ and $B$. Note how this shaded region is aptly named "the intersection of the sets $A$ and $B$." This is the region that is in common to the sets $A$ and $B$, the region where the sets $A$ and $B$ overlap or "intersect."

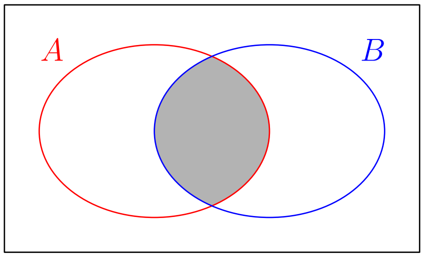


**Figure 6.** The shaded region is the intersection of the sets $A$ and $B$. That is, the shaded region is $A \cap B$.

This leads to the following important piece of advice.

**Tip 6.** *When asked to find the intersection of two sets $A$ and $B$, look to see where the sets intersect or overlap. That is, look to see the elements that are in both sets $A$ and $B$.*

Let's move on to the definition of the *union* of two sets $A$ and $B$.

**Definition 7.** *The union of the sets $A$ and $B$ is the set of all objects that are in $A$* **or** *in $B$. In symbols, we write*

$$A \cup B = \{x : x \in A \text{ or } x \in B\}. \tag{8}$$

In order to understand this definition, it's critical that we understand the meaning of the word "or." The word "or" is a disjunction, used between statements $P$ and $Q$, as in "It is raining today or my best friend is the Lone Ranger." In order to determine the truth or falsehood of this statement, you must first examine the truth or falsehood of the statements $P$ and $Q$ on each side of the word "or."

The speaker is telling the truth if either statement $P$ is true or statement $Q$ is true. In other words, the statement "It is raining today or my best friend is the Lone Ranger" is true if and only if the statement "It is raining today" is true *or* the statement "my best friend is the Lone Ranger" is true. Logicians like to make up a construct called a truth table, like the one shown in **Table 3**.

| $P$ | $Q$ | $P$ **or** $Q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

**Table 3.** Truth table for the disjunction "or."

Points in **Table 3** to consider:

- In the last row of **Table 3**, both statements $P$ and $Q$ are false (indicated with an F), so the statement $P$ or $Q$ is also false.
- In the first three rows (after the header row) of **Table 3**, either statement $P$ is true or statement $Q$ is true (indicated with a T), so the statement $P$ or $Q$ is also true.

Therefore, the statement "$P$ or $Q$" is true if and only if either statement, $P$ or $Q$, is true.

▶ **Example 9.** *If* $A = \{1, 3, 5, 7, 9\}$ *and* $B = \{2, 5, 7, 8, 11\}$, *find the union of* $A$ *and* $B$.

As a reminder, the *union* of $A$ and $B$ is

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Thus, an object is in the union of $A$ and $B$ if and only if it is in either set. The numbers that are in either set are the numbers

$$A \cup B = \{1, 2, 3, 5, 7, 8, 9, 11\}.$$

If we look again at the Venn Diagram in **Figure 5**, we see that this union $A \cup B = \{1, 2, 3, 5, 7, 8, 9, 11\}$ lists every number that is in either set in **Figure 5**.

◇

Thus, the shaded region in **Figure 7** is the union of sets $A$ and $B$. Note how this region is well-named, as that's what you're actually doing, taking the "union" of the two sets $A$ and $B$. That is, the union contains all elements that belong to either $A$ or $B$. Less formally, the union is a way of combining everything that occurs in either set.

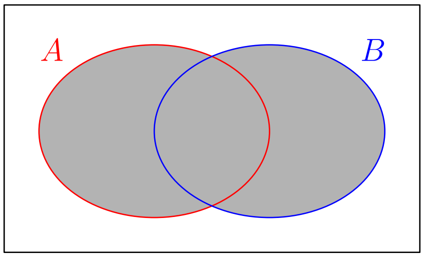


**Figure 7.** The shaded region is the union of sets $A$ and $B$. That is, the shaded region is $A \cup B$.

This leads to the following important piece of advice.

**Tip 10.** *When asked to find the union of two sets $A$ and $B$, in your answer, include everything from both sets.*

## *Simple Compound Inequalities*

Let's apply what we've learned to find the unions and/or intersections of intervals of real numbers. The easiest approach is through a series of examples. Let's begin.

▶ **Example 11.** *On the real line, sketch the set of real numbers in the set $\{x : x < 3 \text{ or } x < 5\}$. Use interval notation to describe your final answer.*

First, let's sketch two sets, $\{x : x < 3\}$ and $\{x : x < 5\}$, on separate real lines, one atop the other as shown in **Figure 8**.

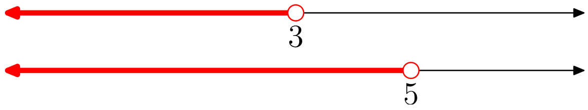


**Figure 8.** Sketch each set separately.

Now, to sketch the solution, note the word "or" in the set $\{x : x < 3 \text{ or } x < 5\}$. Thus, we need to take the union of the two shaded real lines in **Figure 8**. That is, we need to shade everything that is shaded on either of the two number lines. Of course, this would be everything less than 5, as shown in **Figure 9**.

**Figure 9.** The final solution is the union of the two shaded sets in **Figure 8**.

Thus, the final solution is $\{x : x < 5\}$, which in interval notation, is $(-\infty, 5)$.

◇

Let's look at another example.

▶ **Example 12.** *On the real line, sketch the set of real numbers in the set $\{x : x < 3 \text{ and } x < 5\}$. Use interval notation to describe your final answer.*

In **Example 11**, you were asked to shade the set $\{x : x < 3 \text{ or } x < 5\}$ on the real line. In this example, we're asked to sketch the set $\{x : x < 3 \text{ and } x < 5\}$. Note that the set-builder notations are identical except for one change, the "or" of **Example 11** has been replaced with the word "and."

Again, sketch two sets, $\{x : x < 3\}$ and $\{x : x < 5\}$, on separate real lines, one atop the other as shown in **Figure 10**.

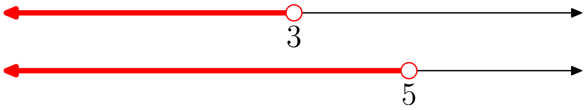


**Figure 10.** Sketch each set separately.

Now, to sketch the solution, note the word "and" in the set $\{x : x < 3 \text{ and } x < 5\}$. Thus, we need to take the intersection of the two shaded real lines in **Figure 10**. That is, we need to shade everything that is common to the two number lines. Of course, this would be everything less than 3, as shown in **Figure 11**.

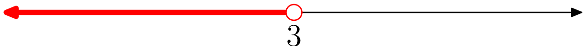


**Figure 11.** The final solution is the intersection of the two shaded sets in **Figure 10**.

Thus, the final solution is $\{x : x < 3\}$, which in interval notation, is $(-\infty, 3)$.

◆

**Warning 13.** *If you answer "or" when the answer requires "and," or vice-versa, you have not made a minor mistake. Indeed, this is a huge error, as demonstrated in* **Example 11** *and* **Example 12**.

Before attempting another example, we pause to define a bit of notation that will be extremely important in our upcoming work.

**Definition 14.** *The notation*

$$a < x < b$$

*is interpreted to mean*

$$x > a \text{ and } x < b.$$

Alternatively, we could have said that $a < x < b$ is identical to saying "$a < x$ and $x < b$," but saying "$a < x$" is the same as saying "$x > a$." We prefer to say "$x > a$ and $x < b$," and will use this order throughout our work, but the form "$a < x$ and $x < b$" is equally valid.

The really key point to make here is the fact that the statement $a < x < b$ is an "and" statement. If it is used properly, it's a good way to describe the numbers that lie between $a$ and $b$.

Let's look at an example.

▶ **Example 15.** *On the real line, sketch the set of real numbers in the set* $\{x : 3 < x < 5\}$. *Use interval notation to describe your answer.*

First, let's write what's meant by the notation $\{x : 3 < x < 5\}$. By definition, this set is the same as the set

$$\{x : x > 3 \text{ and } x < 5\}.$$

Thus, the first step is to sketch the sets $\{x : x > 3\}$ and $\{x : x < 5\}$ on individual real lines, stacked one atop the other, as shown in **Figure 12**.

**Figure 12.** Sketch each set separately.

Now, to sketch the solution, note the word "and" in the set $\{x : x > 3 \text{ and } x < 5\}$. Thus, we need to take the intersection of the two lines in **Figure 12**. That is, we need to shade the numbers on the real line that are in common to the two lines shown in **Figure 12**. The numbers 3 and 5 are not shaded in both sets in **Figure 12**, so they will not be shaded in our final solution. However, all real numbers between 3 and 5 are shaded in both sets in **Figure 12**, so these numbers will be shaded in the final solution shown in **Figure 13**.

**Figure 13.** The final solution is the intersection of the two shaded sets in **Figure 12**.

In a most natural way, the interval notation for the shaded solution in **Figure 13** is $(3, 5)$. That is,

$$(3, 5) = \{x : 3 < x < 5\}$$

◆

Similarly, here are the set-builder and interval notations, as well as shading of the sets on the real line, for several situations, including the one just discussed.

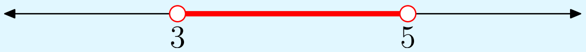
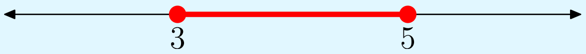
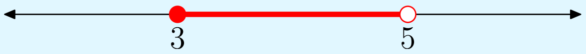
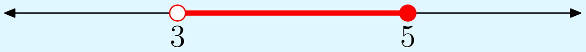
| **Number line** | **Set-builder notation** | **Interval notation** |
|---|---|---|
|  | $\{x : 3 < x < 5\}$ | $(3, 5)$ |
|  | $\{x : 3 \le x \le 5\}$ | $[3, 5]$ |
|  | $\{x : 3 \le x < 5\}$ | $[3, 5)$ |
|  | $\{x : 3 < x \le 5\}$ | $(3, 5]$ |

**Table 4.** Number lines, set-builder notation, and interval notation.

There are several points of emphasis regarding the intervals in **Table 4**.

1. When we want to emphasize that we are not including a point on the real line, we use an "empty circle." Conversely, a "filled circle" means that we are including the point on the real line. Thus, the interval in the first row of **Table 4** does not include the endpoints at 3 and 5, but the interval in the second row of **Table 4** does include the endpoints at 3 and 5.
2. The use of a parenthesis in interval notation means that we are not including that endpoint in the interval. Thus, the parentheses used in $(3, 5)$ in the first row of **Table 4** means that we are not including the numbers 3 and 5 in that interval.
3. The use of a bracket in interval notation means that we are including the bracketed number in the interval. Thus, the brackets used in $[3, 5]$, as seen in the second row of **Table 4**, means that we are including the numbers 3 and 5 in the interval.
4. Finally, note that some of our intervals are "open" on one end but "closed" (filled) on the other end, such as those in rows 3 and 4 of **Table 4**.

**Definition 16.** *Some terminology:*

- *The interval* $(3, 5)$ *is open at each end. Therefore, we call the interval* $(3, 5)$ *an* **open** *interval.*
- *The interval* $[3, 5]$ *is closed (filled) at each end. Therefore, we call the interval* $[3, 5]$ *a* **closed** *interval.*
- *The intervals* $(3, 5]$ *and* $[3, 5)$ *are neither open nor closed.*

Let's look at another example.

▶ **Example 17.** *On the real line, sketch the set of all real numbers in the set* $\{x : x > 3 \text{ or } x < 5\}$*. Use interval notation to describe your answer.*

Note that the only difference between this current example and the set shaded in **Example 15** is the fact that we've replaced the word "and" in $\{x : x > 3 \text{ and } x < 5\}$ with the word "or" in $\{x : x > 3 \text{ or } x < 5\}$. But, as we've seen before, this can make a world of difference.

Thus, the first step is to sketch the sets $\{x : x > 3\}$ and $\{x : x < 5\}$ on individual real lines, stacked one atop the other, as shown in **Figure 14**.

**Figure 14.** Sketch each set separately.

Now, to sketch the solution, note the word "or" in the set $\{x : x > 3 \text{ or } x < 5\}$. Thus, we need to take the union of the two lines in **Figure 14**. That is, we need to shade the numbers on the real line that are shaded on either of the two lines shown in **Figure 14**. However, this means that we will have to shade every number on the line, as shown in **Figure 15**. You'll note no labels for 3 and 5 on the real line in **Figure 15**, as there are no endpoints in this solution. The endpoints, if you will, are at negative and positive infinity.

**Figure 15.** The final solution is the union of the two shaded sets in **Figure 14**.

Thus, in a most natural way, the interval notation for the shaded solution in **Figure 15** is $(-\infty, \infty)$.

◆

Let's look at another example.

▶ **Example 18.** *On the real line, sketch the set of all real numbers in the set* $\{x : x < -1 \text{ or } x > 3\}$*. Use interval notation to describe your answer.*

The first step is to sketch the sets $\{x : x < -1\}$ and $\{x : x > 3\}$ on separate real lines, stacked one atop the other, as shown in **Figure 16**.

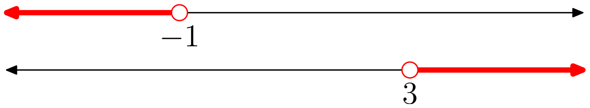


**Figure 16.** Sketch each set separately.

To sketch the solution, note the word "or" in the set $\{x : x < -1 \text{ or } x > 3\}$. Thus, we need to take the union of the two shaded real lines in **Figure 16**. That is, we need to shade the numbers on the real line that are shaded on either real line in **Figure 16**. Thus, every number smaller than $-1$ is shaded, as well as every number greater than 3. The result is shown in **Figure 17**.

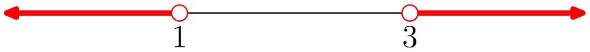


**Figure 17.** The final solution is the union of the shaded real lines in **Figure 16**.

◆

Here is an important tip.

**Tip 19.** *If you wish to use interval notation correctly, follow one simple rule: Always sweep your eyes from left to right describing what you see shaded on the real line.*

If we follow this advice, as we sweep our eyes from left to right across the real line shaded in **Figure 17**, we see that numbers are shaded from negative infinity to $-1$, and from 3 to positive infinity. Thus, in a most natural way, the interval notation for the shaded solution set in **Figure 17** is

$$(-\infty, -1) \cup (3, \infty).$$

There are several important points to make here:

- Note how we used the union symbol $\cup$ to join the two intervals in $(-\infty, -1) \cup (3, \infty)$ in a natural manner.
- The union symbol is used between sets of numbers, while the word "or" is used between statements about numbers. It is incorrect to exchange the roles of the union symbol and the word "or." Thus, writing $\{x : x < -1 \cup x > 3\}$ is incorrect, as it would also be to write $(-\infty, -1)$ or $(3, \infty)$.

We reinforce earlier discussion about the difference between "filled" and "open" circles, brackets, and parentheses in **Table 5**, where we include several comparisons of interval and set-builder notation, including the current solution to **Example 18**.

| Number line | Set-builder notation | Interval notation |
|---|---|---|
|  | $\{x : x < -1 \text{ or } x > 3\}$ | $(-\infty, -1) \cup (3, \infty)$ |
|  | $\{x : x \leq -1 \text{ or } x \geq 3\}$ | $(-\infty, -1] \cup [3, \infty)$ |
|  | $\{x : x \leq -1 \text{ or } x > 3\}$ | $(-\infty, -1] \cup (3, \infty)$ |
|  | $\{x : x < -1 \text{ or } x \geq 3\}$ | $(-\infty, -1) \cup [3, \infty)$ |

**Table 5.** Number lines, set-builder notation, and interval notation.

Again, we reinforce the following points.

- Note how sweeping your eyes from left to right, describing what is shaded on the real line, insures that you write the interval notation in the correct order.
- A bracket is equivalent to a filled dot and includes the endpoint, while a parenthesis is equivalent to an open dot and does not include the endpoint.

Let's do one last example that should forever cement the notion that there is a huge difference between the words "and" and "or."

▶ **Example 20.** *On the real line, sketch the set of all real numbers in the set $\{x : x < -1 \text{ and } x > 3\}$. Describe your solution.*

First and foremost, note that the only difference between this example and **Example 18** is the fact that we changed the "or" in $\{x : x < -1 \text{ or } x > 3\}$ to an "and" in $\{x : x < -1 \text{ and } x > 3\}$. The preliminary sketches are identical to those in **Figure 16**.

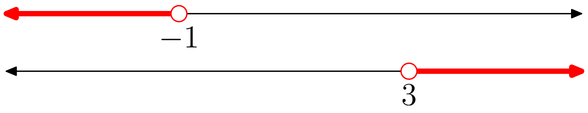


**Figure 18.** Sketch $\{x : x < -1\}$ and $\{x : x > 3\}$ on separate real lines.

Now, note the word "and" in $\{x : x < -1 \text{ and } x > 3\}$. Thus, we need to take the intersection of the shaded real lines in **Figure 18**. That is, we need to shade on a single real line all of the numbers that are shaded on both real lines in **Figure 18**. However, there are no points shaded in common on the real lines in **Figure 18**, so the solution set is empty, as shown in **Figure 19**.

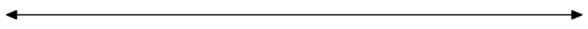


**Figure 19.** The solution is empty so we leave the real line blank.

Pretty impressive! The last two examples clearly demonstrate that if you interchange the roles of "and" and "or," you have not made a minor mistake. Indeed, you've changed the whole meaning of the problem. So, be careful with your "ands" and "ors."

## 1.3 Exercises

Perform each of the following tasks in **Exercises 1-4**.

i. Write out in words the meaning of the symbols which are written in set-builder notation.
ii. Write some of the elements of this set.
iii. Draw a real line and plot some of the points that are in this set.

**1.** $A = \{x \in \mathbb{N} : x > 10\}$

**2.** $B = \{x \in \mathbb{N} : x \geq 10\}$

**3.** $C = \{x \in \mathbb{Z} : x \leq 2\}$

**4.** $D = \{x \in \mathbb{Z} : x > -3\}$

In **Exercises 5-8**, use the sets $A$, $B$, $C$, and $D$ that were defined in **Exercises 1-4**. Describe the following sets using set notation, and draw the corresponding Venn Diagram.

**5.** $A \cap B$

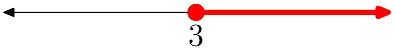
**6.** $A \cup B$

**7.** $A \cup C$.

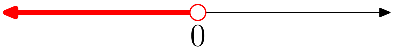
**8.** $C \cap D$.

In **Exercises 9-16**, use both interval and set notation to describe the interval shown on the graph.

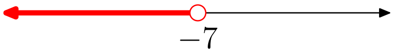
**9.**



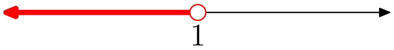
**10.**



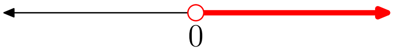
**11.**



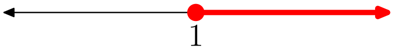
**12.**



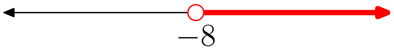
**13.**



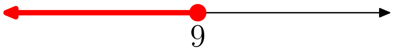
**14.**



**15.**



**16.**



In **Exercises 17-24**, sketch the graph of the given interval.

**17.** $[2, 5)$

**18.** $(-3, 1]$

**19.** $[1, \infty)$

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[15] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

**20.** $(-\infty, 2)$

**21.** $\{x :\ -4 < x < 1\}$

**22.** $\{x :\ 1 \leq x \leq 5\}$

**23.** $\{x :\ x < -2\}$

**24.** $\{x :\ x \geq -1\}$

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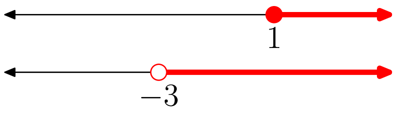
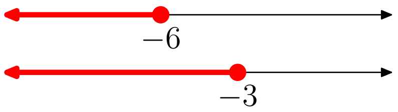
In **Exercises 25-32**, use both interval and set notation to describe the intersection of the two intervals shown on the graph. Also, sketch the graph of the intersection on the real number line.

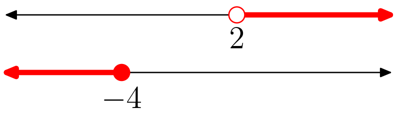
**25.**



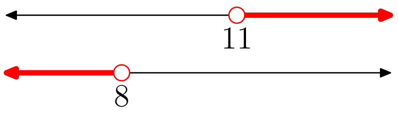
**26.**



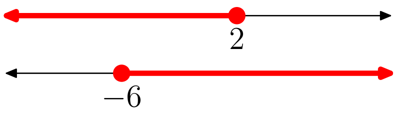
**27.**



**28.**

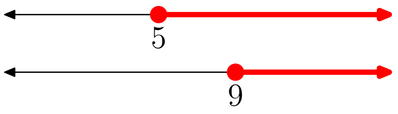


**29.**

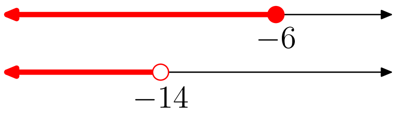


**30.**

**31.**



**32.**



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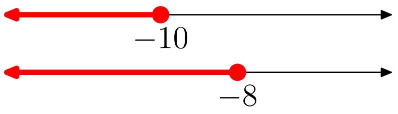
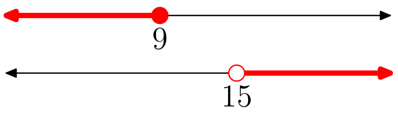
In **Exercises 33-40**, use both interval and set notation to describe the union of the two intervals shown on the graph. Also, sketch the graph of the union on the real number line.

**33.**

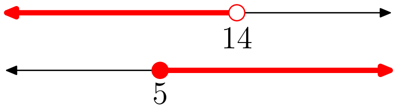


**34.**

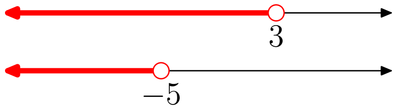
**35.**



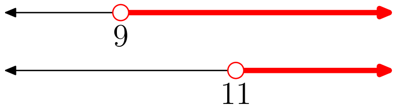
**36.**



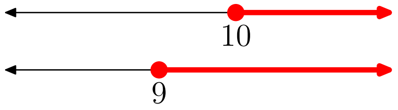
**37.**



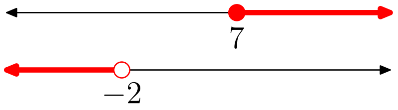
**38.**



**39.**



**40.**



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In **Exercises 41-56**, use interval notation to describe the given set. Also, sketch the graph of the set on the real number line.

**41.** $\{x : x \geq -6 \text{ and } x > -5\}$

**42.** $\{x : x \leq 6 \text{ and } x \geq 4\}$

**43.** $\{x : x \geq -1 \text{ or } x < 3\}$

**44.** $\{x : x > -7 \text{ and } x > -4\}$

**45.** $\{x : x \geq -1 \text{ or } x > 6\}$

**46.** $\{x : x \geq 7 \text{ or } x < -2\}$

**47.** $\{x : x \geq 6 \text{ or } x > -3\}$

**48.** $\{x : x \leq 1 \text{ or } x > 0\}$

**49.** $\{x : x < 2 \text{ and } x < -7\}$

**50.** $\{x : x \leq -3 \text{ and } x < -5\}$

**51.** $\{x : x \leq -3 \text{ or } x \geq 4\}$

**52.** $\{x : x < 11 \text{ or } x \leq 8\}$

**53.** $\{x : x \geq 5 \text{ and } x \leq 1\}$

**54.** $\{x : x < 5 \text{ or } x < 10\}$

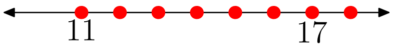
**55.** $\{x : x \leq 5 \text{ and } x \geq -1\}$

**56.** $\{x : x > -3 \text{ and } x < -6\}$

## 1.3 Answers

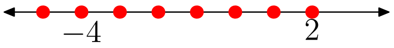
**1.**

i. $A$ is the set of all $x$ in the natural numbers such that $x$ is greater than 10.
ii. $A = \{11, 12, 13, 14, \ldots\}$
iii.

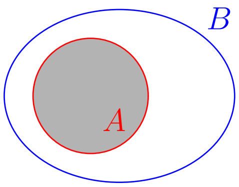


**3.**

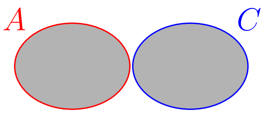
i. $C$ is the set of all $x$ in the set of integers such that $x$ is less than or equal to 2.
ii. $C = \{\ldots, -4, -3, -2, -1, 0, 1, 2\}$
iii.



**5.** $A \cap B = \{x \in \mathbb{N} : x > 10\} = \{11, 12, 13, \ldots\}$



**7.** $A \cup C = \{x \in \mathbb{Z} : x \leq 2 \text{ or } x > 10\} = \{\ldots, -3, -2 -1, 0, 1, 2, 11, 12, 13 \ldots\}$



**9.** $[3, \infty) = \{x : x \geq 3\}$

**11.** $(-\infty, -7) = \{x : x < -7\}$

**13.** $(0, \infty) = \{x : x > 0\}$

**15.** $(-8, \infty) = \{x : x > -8\}$

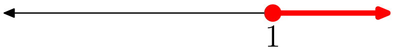
**17.**

**19.**

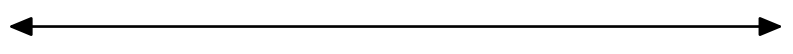
**21.**

**23.**

**25.** $[1, \infty) = \{x : x \geq 1\}$

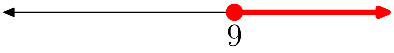


**27.** no intersection

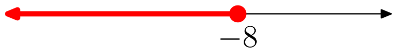


**29.** $[-6, 2] = \{x : -6 \leq x \leq 2\}$

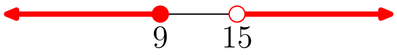
**31.** $[9, \infty) = \{x : x \geq 9\}$



**33.** $(-\infty, -8] = \{x : x \le -8\}$



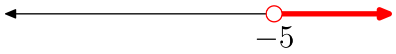
**35.** $(-\infty, 9] \cup (15, \infty)$
$=\{x : x \le 9 \text{ or } x > 15\}$



**37.** $(-\infty, 3) = \{x : x < 3\}$

**39.** $[9, \infty) = \{x : x \ge 9\}$

**41.** $(-5, \infty)$



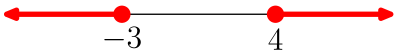
**43.** $(-\infty, \infty)$

**45.** $[-1, \infty)$

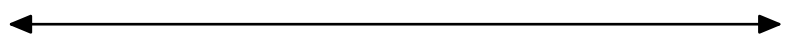
**47.** $(-3, \infty)$

**49.** $(-\infty, -7)$

**51.** $(-\infty, -3] \cup [4, \infty)$



**53.** the set is empty



**55.** $[-1, 5]$

# 1.4 Compound Inequalities

This section discusses a technique that is used to solve *compound inequalities*, which is a phrase that usually refers to a pair of inequalities connected either by the word “and” or the word “or.” Before we begin with the advanced work of solving these inequalities, let’s first spend a word or two (for purposes of review) discussing the solution of simple linear inequalities.

## *Simple Linear Inequalities*

As in solving equations, you may add or subtract the same amount from both sides of an inequality.

**Property 1.** Let $a$ and $b$ be real numbers with $a < b$. If $c$ is any real number, then

$$a + c < b + c$$

and

$$a - c < b - c.$$

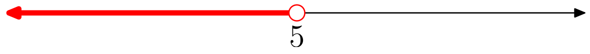
This utility is equally valid if you replace the “less than” symbol with $>$, $\leq$, or $\geq$.

► **Example 2.** *Solve the inequality $x + 3 < 8$ for $x$.*

Subtract 3 from both sides of the inequality and simplify.

$$\begin{aligned} x + 3 &< 8 \\ x + 3 - 3 &< 8 - 3 \\ x &< 5 \end{aligned}$$

Thus, all real numbers less than 5 are solutions of the inequality. It is traditional to sketch the solution set of inequalities on a number line.



We can describe the solution set using set-builder and interval notation. The solution is

$$(-\infty, 5) = \{x : x < 5\}.$$

◆

An important concept is the idea of *equivalent inequalities*.

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**Equivalent Inequalities.** Two inequalities are said to be **equivalent** if and only if they have the same solution set.

Note that this definition is similar to the definition of equivalent equations. That is, two inequalities are equivalent if all of the solutions of the first inequality are also solutions of the second inequality, and vice-versa.

Thus, in **Example 2**, subtracting three from both sides of the original inequality produced an equivalent inequality. That is, the inequalities $x+3 < 8$ and $x < 5$ have the same solution set, namely, all real numbers that are less than 5. It is no coincidence that the tools in **Property 1** produce equivalent inequalities. Whenever you add or subtract the same amount from both sides of an inequality, the resulting inequality is equivalent to the original (they have the same solution set).

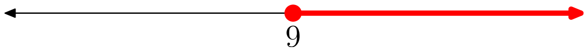
Let's look at another example.

▶ **Example 3.** *Solve the inequality* $x - 5 \geq 4$ *for* $x$.

Add 5 to both sides of the inequality and simplify.

$$\begin{aligned} x - 5 &\geq 4 \\ x - 5 + 5 &\geq 4 + 5 \\ x &\geq 9 \end{aligned}$$

Shade the solution on a number line.



In set-builder and interval notation, the solution is

$$[9, \infty) = \{x : x \geq 9\}$$

You can also multiply or divide both sides by the same *positive* number.

**Property 4.** Let $a$ and $b$ be real numbers with $a < b$. If $c$ is a real **positive** number, then

$$ac < bc$$

and

$$\frac{a}{c} < \frac{b}{c}.$$

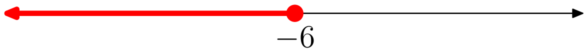
Again, this utility is equally valid if you replace the "less than" symbol by $>$, $\leq$, or $\geq$. The tools in **Property 4** always produce equivalent inequalities.

▶ **Example 5.** *Solve the inequality* $3x \leq -18$ *for* $x$

Divide both sides of the inequality by 3 and simplify.

$$\begin{aligned} 3x &\leq -18 \\ \frac{3x}{3} &\leq \frac{-18}{3} \\ x &\leq -6 \end{aligned}$$

Sketch the solution on a number line.



In set-builder and interval notation, the solution is

$$(-\infty, -6] = \{x : x \leq -6\}.$$

◇

Thus far, there is seemingly no difference between the technique employed for solving inequalities and that used to solve equations. However, there is one important exception. Consider for a moment the true statement

$$-2 < 6. \tag{6}$$

If you multiply both sides of (**6**) by 3, you still have a true statement; i.e.,

$$-6 < 18$$

But if you multiply both sides of (**6**) by −3, you need to "reverse the inequality symbol" to maintain a true statement; i.e.,

$$6 > -18.$$

This discussion leads to the following property.

**Property 7.** Let $a$ and $b$ be real numbers with $a < b$. If $c$ is any real **negative** number, then

$$ac > bc$$

and

$$\frac{a}{c} > \frac{b}{c}.$$

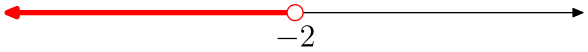
Note that you "reverse the inequality symbol" when you multiply or divide both sides of an inequality by a negative number. Again, this utility is equally valid if you replace the "less than" symbol by $>$, $\leq$, or $\geq$. The tools in **Property 7** always produce equivalent inequalities.

▶ **Example 8.** *Solve the inequality $-5x > 10$ for $x$.*

Divide both sides of the inequality by $-5$ and reverse the inequality symbol. Simplify.

$$\begin{aligned} -5x &> 10 \\ \frac{-5x}{-5} &< \frac{10}{-5} \\ x &< -2 \end{aligned}$$

Sketch the solution on a number line.



In set-builder and interval notation, the solution is

$$(-\infty, -2) = \{x : x < -2\}.$$

## Compound Inequalities

We now turn our attention to the business of solving *compound* inequalities. In the previous section, we studied the subtleties of “and” and “or,” intersection and union, and looked at some simple compound inequalities. In this section, we build on those fundamentals and turn our attention to more intricate examples.

In this case, the best way of learning is by doing. Let’s start with an example.

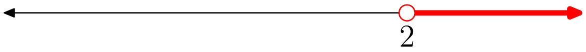
▶ **Example 9.** *Solve the following compound inequality for $x$.*

$$3 - 2x < -1 \quad \text{or} \quad 3 - 2x > 1 \tag{10}$$

First, solve each of the inequalities independently. With the first inequality, add $-3$ to both sides of the inequality, then divide by $-2$, reversing the inequality sign.

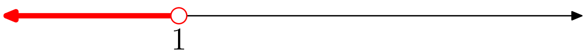
$$\begin{aligned} 3 - 2x &< -1 \\ -2x &< -4 \\ x &> 2 \end{aligned}$$

Shade the solution on a number line.



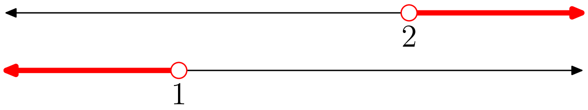
The exact same sequence of operations can be used to solve the second inequality.

$$\begin{aligned} 3 - 2x &> 1 \\ -2x &> -2 \\ x &< 1 \end{aligned}$$



Although you solve each side of the inequality independently, you will want to arrange your work as follows, stacking the number line solution for the first inequality above that of the second inequality.

$$\begin{aligned} 3-2x &< -1 &\quad \text{or} \quad&& 3-2x &> 1 \\ -2x &< -4 &&& -2x &> -2 \\ x &> 2 &&& x &< 1 \end{aligned}$$



Because the inequalities are connected with the word "or," we need to take the union of these two number lines. That is, you want to shade every number from either set on a single number line, as shown in **Figure 1**

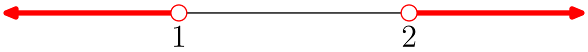


**Figure 1.** The solution of the compound inequality $3-2x<-1$ or $3-2x>1$.

The solution, in interval and set-builder notation, is

$$(-\infty,1)\cup(2,\infty)=\{x:\ x<1 \text{ or } x>2\}.$$

◆

Let's look at another example.

▶ **Example 11.** *Solve the following compound inequality for $x$.*

$$-1<3-2x<1 \tag{12}$$

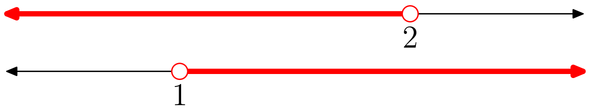
Recall that $a<x<b$ is identical to the statement $x>a$ and $x<b$. Thus, we can write the compound inequality $-1<3-2x<1$ in the form

$$3-2x>-1 \quad \text{and} \quad 3-2x<1. \tag{13}$$

Solve each inequality independently, arranging your work as follows.

$$\begin{aligned} 3-2x &> -1 &\quad \text{and} \quad&& 3-2x &< 1 \qquad (14)\\ -2x &> -4 &&& -2x &< -2 \qquad (15)\\ x &< 2 &&& x &> 1 \end{aligned}$$

Shade the solution of each inequality on separate real lines, one atop the other.



Note the word "and" in our final statement $x < 2$ and $x > 1$. Thus, we must find the intersection of the two shaded solutions. These are the numbers that fall between 1 and 2, as shaded in **Figure 2**.

**Figure 2.** The solution of the compound inequality $-1 < 3 - 2x < 1$.

The solution, in both interval and set-builder notation, is

$$(1, 2) = \{x : 1 < x < 2\}.$$

Note that we used the compact form of the compound inequality in our answer. We could just as well have used

$$(1, 2) = \{x : x > 1 \text{ and } x < 2\}.$$

Both forms of set-builder notation are equally valid. You may use either one, but you must understand both.

**Alternative approach**. You might have noted that in solving the second inequality in (**14**), you found yourself repeating the identical operations used to solve the first inequality. That is, you subtracted 3 from both sides of the inequality, then divided both sides of the inequality by $-2$, reversing the inequality sign.

This repetition is annoying and suggests a possible shortcut in this particular situation. Instead of splitting the compound inequality (**12**) in two parts (as in (**13**)), let's keep the inequality together, as in

$$-1 < 3 - 2x < 1. \tag{16}$$

Now, here are the rules for working with this form.

**Property 17.** When working with a compound inequality having the form

$$a < x < b, \tag{18}$$

you may add (or subtract) the same amount to (from) all three parts of the inequality, as in

$$a + c < x + c < b + c \tag{19}$$

or

$$a - c < x - c < b - c. \tag{20}$$

You may also multiply all three parts by the same **positive** number $c > 0$, as in

$$ca < cx < cb. \tag{21}$$

However, if you multiply all three parts by the same **negative** number $c < 0$, then don't forget to reverse the inequality signs, as in

$$ca > cx > cb. \tag{22}$$

The rules for division are identical to the multiplication rules. If $c > 0$ (positive), then

$$\frac{a}{c} < \frac{x}{c} < \frac{b}{c}. \tag{23}$$

If $c < 0$ (negative), then reverse the inequality signs when you divide.

$$\frac{a}{c} > \frac{x}{c} > \frac{b}{c} \tag{24}$$

Each of the tools in **Property 17** always produce equivalent inequalities.

So, let's return to the compound inequality (**16**) and subtract 3 from all three members of the inequality.

$$\begin{aligned} -1 &< 3 - 2x < 1 \\ -1 - 3 &< 3 - 2x - 3 < 1 - 3 \\ -4 &< -2x < -2 \end{aligned}$$

Next, divide all three members by $-2$, reversing the inequality signs as you do so.

$$\begin{gathered} -4 < -2x < -2 \\ \frac{-4}{-2} > \frac{-2x}{-2} > \frac{-2}{-2} \\ 2 > x > 1 \end{gathered}$$

It is conventional to change the order of this last inequality. By reading the inequality from right to left, we get

$$1 < x < 2,$$

which describes the real numbers that are greater than 1 and less than 2. The solution is drawn on the following real line.

**Figure 3.** The solution of the compound inequality $-1 < 3 - 2x < 1$.

Note that this is identical to the solution set on the real line in **Figure 2**. Note also that this second alternative method is more efficient, particularly if you do a bit of work in your head. Consider the following sequence where we subtract three from all three members, then divide all three members by $-2$, reversing the inequality signs, then finally read the inequality in the opposite direction.

$$\begin{aligned} -1 < 3 - 2x < 1 \\ -4 < -2x < -2 \\ 2 > x > 1 \\ 1 < x < 2 \end{aligned}$$

◆

Let's look at another example.

▶ **Example 25.** *Solve the following compound inequality for $x$.*

$$-1 < x - \frac{x+1}{2} \le 2 \qquad (26)$$

First, let's multiply all three members by 2, in order to clear the fractions.

$$\begin{aligned} 2(-1) < 2\left(x - \frac{x+1}{2}\right) \le 2(2) \\ -2 < 2(x) - 2\left(\frac{x+1}{2}\right) \le 4 \end{aligned}$$

Cancel. Note the use of parentheses, which is crucial when a minus sign is involved.

$$\begin{aligned} -2 < 2x - \cancel{2}\left(\frac{x+1}{\cancel{2}}\right) \le 4 \\ -2 < 2x - (x+1) \le 4 \end{aligned}$$

Distribute the minus sign and simplify.

$$\begin{aligned} -2 < 2x - x - 1 \le 4 \\ -2 < x - 1 \le 4 \end{aligned}$$

Add 1 to all three members.

$$-1 < x \le 5$$

This solution describes the real numbers that are greater than -1 and less than 5, including 5. That is, the real numbers that fall between -1 and 5, including 5, shaded on the real line in **Figure 4**.

**Figure 4.** The solution set of $-1 < x - (x+1)/2 \leq 2$.

The answer, described in both interval and set-builder notation, follows.

$$(-1, 5] = \{x : \, -1 < x \leq 5\}$$

Let's look at another example.

▶ **Example 27.** *Solve the following compound inequality for $x$.*

$$x \leq 2x - 3 \leq 5$$

Suppose that we try to isolate $x$ as we did in **Example 25**. Perhaps we would try adding $-x$ to all three members.

$$\begin{gathered} x \leq 2x - 3 \leq 5 \\ x - x \leq 2x - 3 - x \leq 5 - x \\ 0 \leq x - 3 \leq 5 - x \end{gathered}$$

Well, that didn't help much, just transferring the problem with $x$ to the other end of the inequality. Similar attempts will not help in isolating $x$. So, what do we do?

The solution is we split the inequality (with the word "and," of course).

$$x \leq 2x - 3 \quad \text{and} \quad 2x - 3 \leq 5$$

We can solve the first inequality by subtracting $2x$ from both sides of the inequality, then multiplying both sides by $-1$, reversing the inequality in the process.

$$\begin{aligned} x &\leq 2x - 3 \\ -x &\leq -3 \\ x &\geq 3 \end{aligned}$$

To solve the second inequality, add 3 to both sides, then divide both sides by 2.

$$\begin{aligned} 2x - 3 &\leq 5 \\ 2x &\leq 8 \\ x &\leq 4 \end{aligned}$$

Of course, you'll probably want to arrange your work as follows.

$$\begin{array}{rcr} x \leq 2x - 3 & \quad \text{and} \quad & 2x - 3 \leq 5 \\ -\,x \leq -3 & & 2x \leq 8 \\ x \geq 3 & & x \leq 4 \end{array}$$

Thus, we need to shade on a number line all real numbers that are greater than or equal to 3 and less than or equal to 4, as shown in **Figure 5**.

**Figure 5.** When shading the solution of $x \le 2x - 3 \le 5$, we “fill-in” the endpoints.

The solution, described in both interval and set-builder notation, is

$$[3, 4] = \{x : 3 \le x \le 4\}.$$

◆

## 1.4 Exercises

In **Exercises 1-12**, solve the inequality. Express your answer in both interval and set notations, and shade the solution on a number line.

**1.** $-8x - 3 \leq -16x - 1$

**2.** $6x - 6 > 3x + 3$

**3.** $-12x + 5 \leq -3x - 4$

**4.** $7x + 3 \leq -2x - 8$

**5.** $-11x - 9 < -3x + 1$

**6.** $4x - 8 \geq -4x - 5$

**7.** $4x - 5 > 5x - 7$

**8.** $-14x + 4 > -6x + 8$

**9.** $2x - 1 > 7x + 2$

**10.** $-3x - 2 > -4x - 9$

**11.** $-3x + 3 < -11x - 3$

**12.** $6x + 3 < 8x + 8$

In **Exercises 13-50**, solve the compound inequality. Express your answer in both interval and set notations, and shade the solution on a number line.

**13.** $2x - 1 < 4$ or $7x + 1 \geq -4$

**14.** $-8x + 9 < -3$ and $-7x + 1 > 3$

**15.** $-6x - 4 < -4$ and $-3x + 7 \geq -5$

**16.** $-3x + 3 \leq 8$ and $-3x - 6 > -6$

**17.** $8x + 5 \leq -1$ and $4x - 2 > -1$

**18.** $-x - 1 < 7$ and $-6x - 9 \geq 8$

**19.** $-3x + 8 \leq -5$ or $-2x - 4 \geq -3$

**20.** $-6x - 7 < -3$ and $-8x \geq 3$

**21.** $9x - 9 \leq 9$ and $5x > -1$

**22.** $-7x + 3 < -3$ or $-8x \geq 2$

**23.** $3x - 5 < 4$ and $-x + 9 > 3$

**24.** $-8x - 6 < 5$ or $4x - 1 \geq 3$

**25.** $9x + 3 \leq -5$ or $-2x - 4 \geq 9$

**26.** $-7x + 6 < -4$ or $-7x - 5 > 7$

**27.** $4x - 2 \leq 2$ or $3x - 9 \geq 3$

**28.** $-5x + 5 < -4$ or $-5x - 5 \geq -5$

**29.** $5x + 1 < -6$ and $3x + 9 > -4$

**30.** $7x + 2 < -5$ or $6x - 9 \geq -7$

**31.** $-7x - 7 < -2$ and $3x \geq 3$

**32.** $4x + 1 < 0$ or $8x + 6 > 9$

**33.** $7x + 8 < -3$ and $8x + 3 \geq -9$

**34.** $3x < 2$ and $-7x - 8 \geq 3$

**35.** $-5x + 2 \leq -2$ and $-6x + 2 \geq 3$

**36.** $4x - 1 \leq 8$ or $3x - 9 > 0$

**37.** $2x - 5 \leq 1$ and $4x + 7 > 7$

**38.** $3x + 1 < 0$ or $5x + 5 > -8$

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[17] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

**39.** $-8x+7 \le 9 \quad \text{or} \quad -5x+6 > -2$

**40.** $x-6 \le -5 \quad \text{and} \quad 6x-2 > -3$

**41.** $-4x-8 < 4 \quad \text{or} \quad -4x+2 > 3$

**42.** $9x-5 < 2 \quad \text{or} \quad -8x-5 \ge -6$

**43.** $-9x-5 \le -3 \quad \text{or} \quad x+1 > 3$

**44.** $-5x-3 \le 6 \quad \text{and} \quad 2x-1 \ge 6$

**45.** $-1 \le -7x-3 \le 2$

**46.** $0 < 5x-5 < 9$

**47.** $5 < 9x-3 \le 6$

**48.** $-6 < 7x+3 \le 2$

**49.** $-2 < -7x+6 < 6$

**50.** $-9 < -2x+5 \le 1$

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In **Exercises 51-62**, solve the given inequality for $x$. Graph the solution set on a number line, then use interval and set-builder notation to describe the solution set.

**51.** $-\dfrac{1}{3} < \dfrac{x}{2} + \dfrac{1}{4} < \dfrac{1}{3}$

**52.** $-\dfrac{1}{5} < \dfrac{x}{2} - \dfrac{1}{4} < \dfrac{1}{5}$

**53.** $-\dfrac{1}{2} < \dfrac{1}{3} - \dfrac{x}{2} < \dfrac{1}{2}$

**54.** $-\dfrac{2}{3} \le \dfrac{1}{2} - \dfrac{x}{5} \le \dfrac{2}{3}$

**55.** $-1 < x - \dfrac{x+1}{5} < 2$

**56.** $-2 < x - \dfrac{2x-1}{3} < 4$

**57.** $-2 < \dfrac{x+1}{2} - \dfrac{x+1}{3} \le 2$

**58.** $-3 < \dfrac{x-1}{3} - \dfrac{2x-1}{5} \le 2$

**59.** $x < 4-x < 5$

**60.** $-x < 2x+3 \le 7$

**61.** $-x < x+5 \le 11$

**62.** $-2x < 3-x \le 8$

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**63.** Aeron has arranged for a demonstration of "How to make a Comet" by Professor O'Commel. The wise professor has asked Aeron to make sure the auditorium stays between 15 and 20 degrees Celsius (C). Aeron knows the thermostat is in Fahrenheit (F) and he also knows that the conversion formula between the two temperature scales is $C = (5/9)(F-32)$.

**a)** Setting up the compound inequality for the requested temperature range in Celsius, we get $15 \le C \le 20$. Using the conversion formula above, set up the corresponding compound inequality in Fahrenheit.

**b)** Solve the compound inequality in part (a) for F. Write your answer in set notation.

**c)** What are the possible temperatures (integers only) that Aeron can set the thermostat to in Fahrenheit?

## *1.4 Answers*

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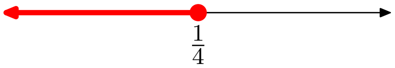
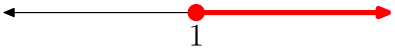
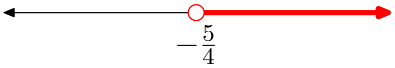
**1.** $(-\infty, \frac{1}{4}] = \{x|x \leq \frac{1}{4}\}$



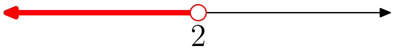
**3.** $[1, \infty) = \{x|x \geq 1\}$



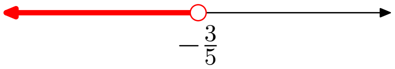
**5.** $(-\frac{5}{4}, \infty) = \{x|x > -\frac{5}{4}\}$



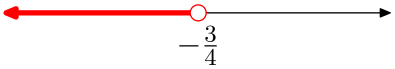
**7.** $(-\infty, 2) = \{x|x < 2\}$



**9.** $(-\infty, -\frac{3}{5}) = \{x|x < -\frac{3}{5}\}$



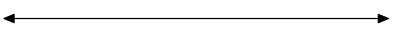
**11.** $(-\infty, -\frac{3}{4}) = \{x|x < -\frac{3}{4}\}$



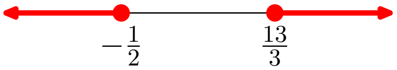
**13.** $(-\infty, \infty) = \{\text{all real numbers}\}$

**15.** $(0, 4] = \{x|0 < x \leq 4\}$

**17.** no solution

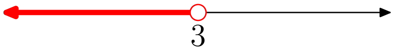


**19.** $(-\infty, -\frac{1}{2}] \bigcup [\frac{13}{3}, \infty)$
$=\{x|x \leq -\frac{1}{2}$ or $x \geq \frac{13}{3}\}$

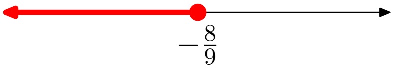


**21.** $(-\frac{1}{5}, 2] = \{x| -\frac{1}{5} < x \leq 2\}$

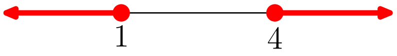
**23.** $(-\infty, 3) = \{x|x < 3\}$



**25.** $(-\infty, -\frac{8}{9}] = \{x|x \leq -\frac{8}{9}\}$

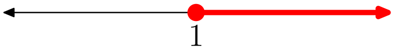


**27.** $(-\infty, 1] \bigcup [4, \infty) = \{x|x \leq 1$ or $x \geq 4\}$

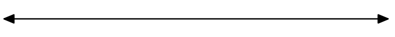


**29.** $(-\frac{13}{3}, -\frac{7}{5}) = \{x| -\frac{13}{3} < x < -\frac{7}{5}\}$

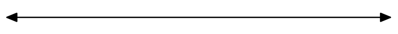
**31.** $[1, \infty) = \{x|x \geq 1\}$



**33.** no solution



**35.** no solution

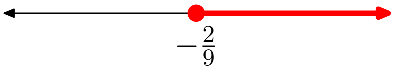


**37.** $(0,3] = \{x|0 < x \leq 3\}$

**39.** $(-\infty,\infty) = \{\text{all real numbers}\}$

**41.** $(-\infty,\infty) = \{\text{all real numbers}\}$

**43.** $[-\frac{2}{9},\infty) = \{x|x \geq -\frac{2}{9}\}$



**45.** $[-\frac{5}{7},-\frac{2}{7}] = \{x| -\frac{5}{7} \leq x \leq -\frac{2}{7}\}$

**47.** $(\frac{8}{9},1] = \{x|\frac{8}{9} < x \leq 1\}$

**49.** $(0,\frac{8}{7}) = \{x|0 < x < \frac{8}{7}\}$

**51.** $(-7/6,1/6) = \{x : -7/6 < x < 1/6\}$

**53.** $(-1/3,5/3) = \{x : -1/3 < x < 5/3\}$

**55.** $(-1,11/4) = \{x : -1 < x < 11/4\}$

**57.** $(-13,11] = \{x : -13 < x \leq 11\}$

**59.** $(-1,2) = \{x : -1 < x < 2\}$

**61.** $(-5/2,6] = \{x : -5/2 < x \leq 6\}$

**63.**

**a)** $15 \leq \frac{5}{9}(F-32) \leq 20$

**b)** $\{F : 59 \leq F \leq 68\}$

**c)** $\{59,\ 60,\ 61,\ 62,\ 63,\ 64,\ 65,\ 66,\ 67,\ 68\}$

## 1.5 Index

# 2 Functions

The concept of a function is a unifying theme in the study of mathematics and it has a rich and storied history. The word "function" was first coined by Gottfried Wilhelm Leibniz (c. 1694) (one of the co-founders of calculus with Sir Isaac Newton). Leibniz's concept of function was relegated to how geometrical properties of a curve (e.g., subtangents and subnormals) depended on the shape of the curve. Johann Bernoulli (c. 1718) described a function of a variable as a quantity that is constructed from that variable and some constants. Indeed, even Leonard Euler (1707-1793), who was a former student of Bernoulli, described the dependence of one variable on another through the means of an analytical expression. In his *Introductio in Analysin Infinitorum* (Introduction to infinite analyses) (1748), Euler states

> *The nature of the curve, provided it is continuous, is expressed through the quality of the function $y$, that is, the rule of formation whereby the value of $y$ is obtained from the composition of constants and the variable $x$.*

Euler equated the word function with an analytic equation describing the relationship between the independent and dependent variables. This is not the modern definition of a function, but it is precisely how many of today's students think about the concept of a function; i.e., a function is an equation.

Euler's definition of function did not change much until mathematicians began studying the equation of the vibrating string, an equation known as the *wave equation*. Jean Baptiste Fourier (1768-1830), in his classic work on heat transfer, claimed that any function could be expressed as an infinite series of trigonometric functions. It turned out that he was wrong, and it was up to Johann Peter Gustav Lejeune Dirichlet (1805-1859) to set sufficient conditions on functions to correct Fourier's error. In order to do that, Dirichlet had to separate the concept of function from its dependence on an analytic expression. Dirichlet's definition of a function closely mirrors the modern day definition.

## Table of Contents

# 2.1 Introduction to Functions

Our development of the function concept is a modern one, but quite quick, particularly in light of the fact that today's definition took over 300 years to reach its present state. We begin with the definition of a relation.

## *Relations*

We use the notation $(2, 4)$ to denote what is called an *ordered pair*. If you think of the positions taken by the ordered pairs $(4, 2)$ and $(2, 4)$ in the coordinate plane (see **Figure 1**), then it is immediately apparent why order is important. The ordered pair $(4, 2)$ is simply not the same as the ordered pair $(2, 4)$.

The first element of an ordered pair is called its *abscissa*. The second element of an ordered pair is called its *ordinate*. Thus, for example, the abscissa of $(4, 2)$ is 4, while the ordinate of $(4, 2)$ is 2.

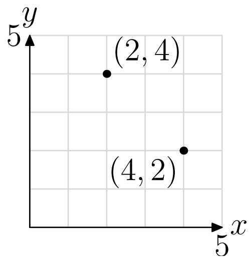


**Figure 1.**

**Definition 1.** *A collection of ordered pairs is called a* **relation**.

For example, the collection of ordered pairs

$$R = \{(0, 1), (0, 2), (3, 4)\} \tag{2}$$

is a relation.

**Definition 3.** *The* **domain** *of a relation is the collection of all abscissas of each ordered pair.*

Thus, the domain of the relation $R$ in (**2**) is

$$\text{Domain} = \{0, 3\}.$$

Note that we list each abscissa only once.

**Definition 4.** *The* **range** *of a relation is the collection of all ordinates of each ordered pair.*

Thus, the range of the relation $R$ in (**2**) is

$$\text{Range} = \{1, 2, 4\}.$$

Let's look at an example.

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[1] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

▶ **Example 5.** *Consider the relation $T$ defined by*

$$T = \{(1,2),(3,2),(4,5)\}. \tag{6}$$

*What are the domain and range of this relation?*

The domain is the collection of abscissas of each ordered pair. Hence, the domain of $T$ is

$$\text{Domain} = \{1,3,4\}.$$

The range is the collection of ordinates of each ordered pair. Hence, the range of $T$ is

$$\text{Range} = \{2,5\}.$$

Note that we list each ordinate in the range only once.

◆

In **Example 5**, the relation is described by listing the ordered pairs. This is not the only way that one can describe a relation. For example, a graph certainly represents a collection of ordered pairs.

▶ **Example 7.** *Consider the graph of the relation $S$ shown in* **Figure 2**.

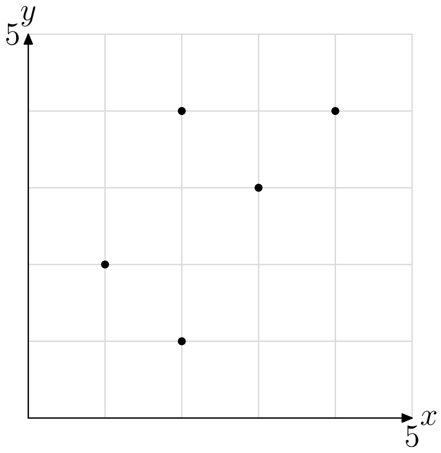


**Figure 2.** The graph of a relation.

*What are the domain and range of the relation $S$?*

There are five ordered pairs (points) plotted in **Figure 2**. They are

$$S = \{(1,2),(2,1),(2,4),(3,3),(4,4)\}.$$

Therefore, the relation $S$ has Domain $= \{1,2,3,4\}$ and Range $= \{1,2,3,4\}$. In the case of the range, note how we've sorted the ordinates of each ordered pair in ascending order, taking care not to list any ordinate more than once.

◆

## *Functions*

A function is a very special type of relation. We begin with a formal definition.

**Definition 8.** *A relation is a* function *if and only if each object in its domain is paired with one and only one object in its range.*

This is not an easy definition, so let's take our time and consider a few examples. Consider, if you will, the relation $R$ in (**2**), repeated here again for convenience.

$$R = \{(0,1),(0,2),(3,4)\}$$

The domain is $\{0,3\}$ and the range is $\{1,2,4\}$. Note that the number 0 in the domain of $R$ is paired with two numbers from the range, namely, 1 and 2. Therefore, $R$ is **not** a function.

There is a construct, called a *mapping diagram*, which can be helpful in determining whether a relation is a function. To craft a mapping diagram, first list the domain on the left, then the range on the right, then use arrows to indicate the ordered pairs in your relation, as shown in **Figure 3**.

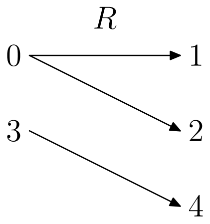


**Figure 3.** A mapping diagram for $R$.

It's clear from the mapping diagram in **Figure 3** that the number 0 in the domain is being paired (mapped) with two different range objects, namely, 1 and 2. Thus, $R$ is **not** a function.

Let's look at another example.

▶ **Example 9.** *Is the relation described in* **Example 5** *a function?*

First, let's repeat the listing of the relation $T$ here for convenience.

$$T = \{(1,2),(3,2),(4,5)\}$$

Next, construct a mapping diagram for the relation $T$. List the domain on the left, the range on the right, then use arrows to indicate the pairings, as shown in **Figure 4**.

From the mapping diagram in **Figure 4**, we can see that each domain object on the left is paired (mapped) with exactly one range object on the right. Hence, the relation $T$ **is** a function.

◆

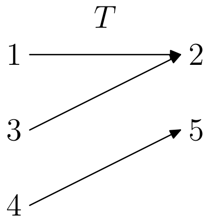


**Figure 4.** A mapping diagram for $T$.

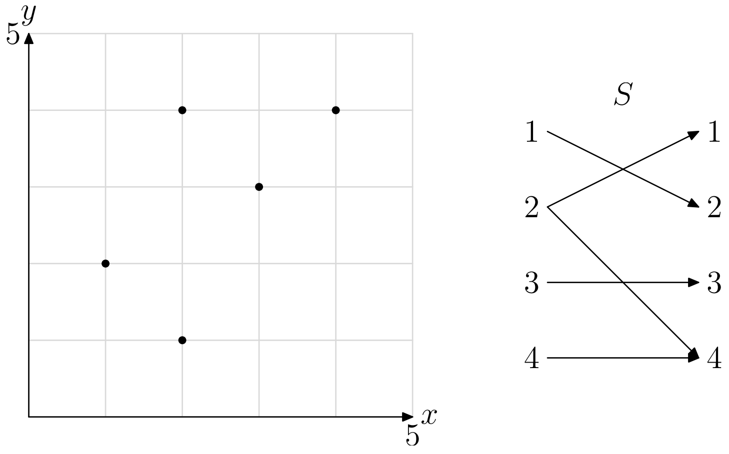
Let's look at another example.

▶ **Example 10.** *Is the relation of* **Example 7**, *pictured in* **Figure 2**, *a function?*

First, we repeat the graph of the relation from **Example 7** here for convenience. This is shown in **Figure 5**(a). Next, we list the ordered pairs of the relation $S$.

$$S = \{(1,2),(2,1),(2,4),(3,3),(4,4)\}$$

Then we create a mapping diagram by first listing the domain on the left, the range on the right, then using arrows to indicate the pairings, as shown in **Figure 5**(b).



(a) (b)

**Figure 5.** A graph of the relation $S$ and its corresponding mapping diagram

Each object in the domain of $S$ gets mapped to exactly one range object with one exception. The domain object 2 is paired with two range objects, namely, 1 and 4. Consequently, $S$ is **not** a function.

◇

This is a good point to summarize what we've learned about functions thus far.

**Summary 11.** *A function consists of three parts:*

1. *a set of objects which mathematicians call the* **domain***,*
2. *a second set of objects which mathematicians call the* **range***,*
3. *and a* **rule** *that describes how to assign a unique range object to each object in the domain.*

The rule can take many forms. For example, we can use sets of ordered pairs, graphs, and mapping diagrams to describe the function. In the sections that follow, we will explore other ways of describing a function, for example, through the use of equations and simple word descriptions.

## Function Notation

We've used the word "mapping" several times in the previous examples. This is not a word to be taken lightly; it is an important concept. In the case of the mapping diagram in **Figure 5**(b), we would say that the number 1 in the domain of $S$ is "mapped" (or "sent") to the number 2 in the range of $S$.

There are a number of different notations we could use to indicate that the number 1 in the domain is "mapped" or "sent" to the number 2 in the range. One possible notation is

$$S : 1 \longrightarrow 2,$$

which we would read as follows: "The relation $S$ maps (sends) 1 to 2." In a similar vein, we see in **Figure 5**(b) that the domain objects 3 and 4 are mapped (sent) to the range objects 3 and 4, respectively. In symbols, we would write

$$\begin{aligned} S &: 3 \longrightarrow 3, \text{ and} \\ S &: 4 \longrightarrow 4. \end{aligned}$$

A difficulty arises when we examine what happens to the domain object 2. There are two possibilities, either

$$S : 2 \longrightarrow 1,$$

or

$$S : 2 \longrightarrow 4.$$

Which should we choose? The 1? Or the 4? Thus, $S$ is not well-defined and is not a function, since we don't know which range object to pair with the domain object 1.

The idea of mapping gives us an alternative way to describe a function. We could say that a function is a rule that assigns a unique object in its range to each object in its domain. Take for example, the function that maps each real number to its square. If we name the function $f$, then $f$ maps 5 to 25, 6 to 36, $-7$ to 49, and so on. In symbols, we would write

$$f : 5 \longrightarrow 25, \qquad f : 6 \longrightarrow 36, \qquad \text{and} \qquad f : -7 \longrightarrow 49.$$

In general, we could write

$$f : x \longrightarrow x^2.$$

Note that each real number $x$ gets mapped to a unique number in the range of $f$, namely, $x^2$. Consequently, the function $f$ is well defined. We've succeeded in writing a rule that completely defines the function $f$.

As another example, let's define a function that takes a real number, doubles it, then adds 3. If we name the function $g$, then $g$ would take the number 7, double it, then add 3. That is,

$$g : 7 \longrightarrow 2(7) + 3$$

Simplifying, $g : 7 \longrightarrow 17$. Similarly, $g$ would take the number 9, double it, then add 3. That is,

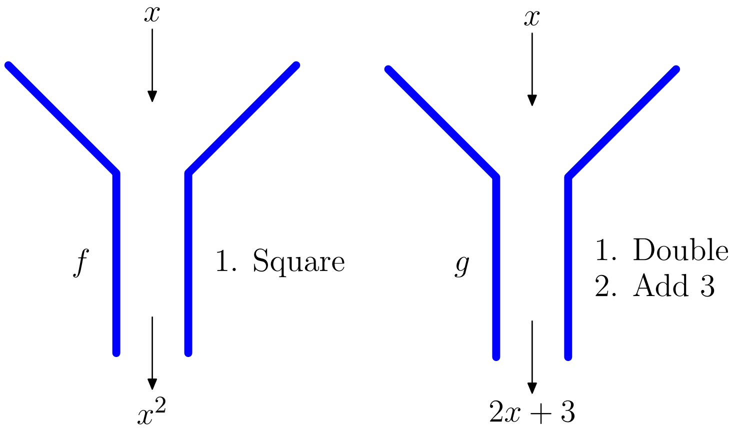
$$g : 9 \longrightarrow 2(9) + 3$$

Simplifying, $g : 9 \longrightarrow 21$. In general, $g$ takes a real number $x$, doubles it, then adds three. In symbols, we would write

$$g : x \longrightarrow 2x + 3.$$

Notice that each real number $x$ is mapped by $g$ to a unique number in its range. Therefore, we've again defined a rule that completely defines the function $g$.

It is helpful to think of a function as a machine. The machine receives input, processes it according to some rule, then outputs a result. Something goes in (input), then something comes out (output). In the case of the function described by the rule $f : x \longrightarrow x^2$, the "$f$-machine" receives input $x$, then applies its "square rule" to the input and outputs $x^2$, as shown in **Figure 6**(a). In the case of the function described by the rule $g : x \longrightarrow 2x + 3$, the "$g$-machine" receives input $x$, then applies the rules "double," then "add 3," in that order, then outputs $2x + 3$, as shown in **Figure 6**(b).



(a) The $f$-machine. (b) The $g$-machine.

**Figure 6.** Function machines.

Let's look at another example.

▶ **Example 12.** *Suppose that $f$ is defined by the following rule. For each real number $x$,*

$$f : x \longrightarrow x^2 - 2x - 3.$$

*Where does $f$ map the number $-3$? Is $f$ a function?*

We substitute $-3$ for $x$ in the rule for $f$ and obtain

$$f : -3 \longrightarrow (-3)^2 - 2(-3) - 3.$$

Simplifying,

$$f : -3 \longrightarrow 9 + 6 - 3,$$

or,

$$f : -3 \longrightarrow 12.$$

Thus, $f$ maps (sends) the number $-3$ to the number 12. It should be clear that each real number $x$ will be mapped (sent) to a unique real number, as defined by the rule $f : x \longrightarrow x^2 - 2x - 3$. Therefore, $f$ is a function.

◆

Let's look at another example.

▶ **Example 13.** *Suppose that $g$ is defined by the following rule. For each real number $x$ that is greater than or equal to zero,*

$$g : x \longrightarrow \pm\sqrt{x}.$$

*Where does $g$ map the number 4? Is $g$ a function?*

Again, we substitute 4 for $x$ in the rule for $g$ and obtain

$$g : 4 \longrightarrow \pm\sqrt{4}.$$

Simplifying,

$$g : 4 \longrightarrow \pm 2.$$

Thus, $g$ maps (sends) the number 4 to **two** different objects in its range, namely, 2 and $-2$. Consequently, $g$ is not well-defined and is **not** a function.

◆

Let's look at another example.

▶ **Example 14.** *Suppose that we have functions $f$ and $g$, defined by*

$$f : x \longrightarrow x^4 + 11 \qquad \text{and} \qquad g : x \longrightarrow (x + 2)^2.$$

*Where does $g$ send 5?*

In this example, we see a clear advantage of function notation. Because our functions have distinct names, we can simply reference the name of the function we want our readers to use. In this case, we are asked where the function $g$ sends the number 5, so we substitute 5 for $x$ in

$$g : x \longrightarrow (x+2)^2.$$

That is,

$$g : 5 \longrightarrow (5+2)^2.$$

Simplifying, $g : 5 \longrightarrow 49$.

◆

## Modern Notation

Function notation is relatively new, with some of the earliest symbolism first occurring in the 17th century. In a letter to Leibniz (1698), Johann Bernoulli wrote "For denoting any function of a variable quantity $x$, I rather prefer to use the capital letter having the same name $X$ or the Greek $\xi$, for it appears at once of what variable it is a function; this relieves the memory."

Mathematicians are fond of the notation

$$f : x \longrightarrow x^2 - 2x,$$

because it conveys a sense of what a function does; namely, it "maps" or "sends" the number $x$ to the number $x^2 - 2x$. This is what functions do, they pair each object in their domain with a unique object in their range. Equivalently, functions "send" each object in their domain to a unique object in their range.

However, in common computational situations, the "arrow" notation can be a bit clumsy, so mathematicians tend to favor a slightly different notation. Instead of writing

$$f : x \longrightarrow x^2 - 2x,$$

mathematicians tend to favor the notation

$$f(x) = x^2 - 2x.$$

It is important to understand from the outset that these two different notations are equivalent; they represent the same function $f$, one that pairs each real number $x$ in its domain with the real number $x^2 - 2x$ in its range.

The first notation, $f : x \longrightarrow x^2 - 2x$, conveys the sense that the function $f$ is a mapping. If we read this notation aloud, we should pronounce it as "$f$ sends (or maps) $x$ to $x^2 - 2x$." The second notation, $f(x) = x^2 - 2x$, is pronounced "$f$ of $x$ equals $x^2 - 2x$."

**Warning 15.** *The phrase "$f$ of $x$" is unfortunate, as our readers might recall being trained from a very early age to pair the word "of" with the operation of multiplication. For example, 1/2 of 12 is 6, as in $1/2 \times 12 = 6$. However, in the context of function notation, even though $f(x)$ is read aloud as "$f$ of $x$," it does* **not** *mean "$f$ times $x$." Indeed, if we remind ourselves that the notation $f(x) = x^2 - 2x$ is equivalent to the notation $f : x \longrightarrow x^2 - 2x$, then even though we might say "$f$ of $x$," we should be thinking "$f$ sends $x$" or "$f$ maps $x$." We should* **not** *be thinking "$f$ times $x$."*

Now, let's see how each of these notations operates on the number 5. In the first case, using the "arrow" notation,

$$f : x \longrightarrow x^2 - 2x.$$

To find where $f$ sends 5, we substitute 5 for $x$ as follows.

$$f : 5 \longrightarrow (5)^2 - 2(5).$$

Simplifying, $f : 5 \longrightarrow 15$. Now, because both notations are equivalent, to compute $f(5)$, we again substitute 5 for $x$ in

$$f(x) = x^2 - 2x.$$

Thus,

$$f(5) = (5)^2 - 2(5).$$

Simplifying, $f(5) = 15$. This result is read aloud as "$f$ of 5 equals 15," but we want to be thinking "$f$ sends 5 to 15."

Let's look at examples that use this modern notation.

► **Example 16.** *Given $f(x) = x^3 + 3x^2 - 5$, determine $f(-2)$.*

Simply substitute $-2$ for $x$. That is,

$$\begin{aligned} f(-2) &= (-2)^3 + 3(-2)^2 - 5 \\ &= -8 + 3(4) - 5 \\ &= -8 + 12 - 5 \\ &= -1. \end{aligned}$$

Thus, $f(-2) = -1$. Again, even though this is pronounced "$f$ of $-2$ equals $-1$," we still should be thinking "$f$ sends $-2$ to $-1$."

◆

▶ **Example 17.** *Given*

$$f(x) = \frac{x+3}{2x-5},$$

*determine* $f(6)$.

Simply substitute 6 for $x$. That is,

$$\begin{aligned} f(6) &= \frac{6+3}{2(6)-5} \\ &= \frac{9}{12-5} \\ &= \frac{9}{7}. \end{aligned}$$

Thus, $f(6) = 9/7$. Again, even though this is pronounced "$f$ of 6 equals 9/7," we should still be thinking "$f$ sends 6 to 9/7."

◆

▶ **Example 18.** *Given* $f(x) = 5x - 3$, *determine* $f(a+2)$.

If we're thinking in terms of mapping notation, then

$$f : x \longrightarrow 5x - 3.$$

Think of this mapping as a "machine." Whatever we put into the machine, it first is multiplied by 5, then 3 is subtracted from the result, as shown in **Figure 7**. For example, if we put a 4 into the machine, then the function rule requires that we multiply input 4 by 5, then subtract 3 from the result. That is,

$$f : 4 \longrightarrow 5(4) - 3.$$

Simplifying, $f : 4 \longrightarrow 17$.

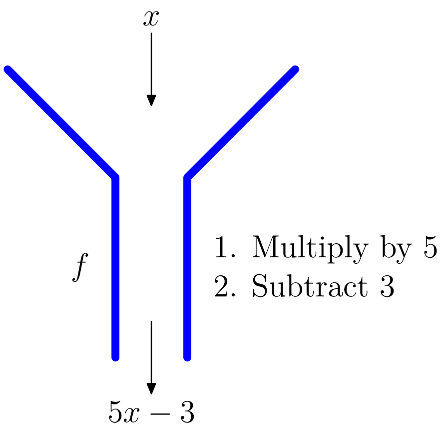


**Figure 7.** The multiply by 5 then subtract 3 machine.

Similarly, if we put an $a+2$ into the machine, then the function rule requires that we multiply the input $a+2$ by 5, then subtract 3 from the result. That is,

$$f: a+2 \longrightarrow 5(a+2)-3.$$

Using modern function notation, we would write

$$f(a+2) = 5(a+2)-3.$$

Note that this is again a simple substitution, where we replace each occurrence of $x$ in the formula $f(x) = 5x-3$ with the expression $a+2$. Finally, use the distributive property to first multiply by 5, then subtract 3.

$$\begin{aligned} f(a+2) &= 5a+10-3 \\ &= 5a+7. \end{aligned}$$

◇

We will often need to substitute the result of one function evaluation into a second function for evaluation. Let's look at an example.

► **Example 19.** *Given two functions defined by $f(x) = 3x+2$ and $g(x) = 5-4x$, find $f(g(2))$.*

The nested parentheses in the expression $f(g(2))$ work in the same manner that they do with nested expressions. The rule is to work the innermost grouping symbols first, proceeding outward as you work. We'll first evaluate $g(2)$, then evaluate $f$ at the result.

We begin. First, evaluate $g(2)$ by substituting 2 for $x$ in the defining equation $g(x) = 5-4x$. Note that $g(2) = 5-4(2)$, then simplify.

$$f(g(2)) = f(5-4(2)) = f(5-8) = f(-3)$$

To complete the evaluation, we substitute $-3$ for $x$ in the defining equation $f(x) = 3x+2$, then simplify.

$$f(-3) = 3(-3)+2 = -9+2 = -7.$$

Hence, $f(g(2)) = -7$.

It is conventional to arrange the work in one contiguous block, as follows.

$$\begin{aligned} f(g(2)) &= f(5-4(2)) \\ &= f(-3) \\ &= 3(-3)+2 \\ &= -7 \end{aligned}$$

You can shorten the task even further if you are willing to do the function substitution and simplification in your head. First, evaluate $g$ at 2, then $f$ at the result.

$$f(g(2)) = f(-3) = -7$$

◆

Let's look at another example of this unique way of combining functions.

▶ **Example 20.** *Given $f(x) = 5x + 2$ and $g(x) = 3 - 2x$, evaluate $g(f(a))$ and simplify the result.*

We work the inner function evaluation in the expression $g(f(a))$ first. Thus, to evaluate $f(a)$, we substitute $a$ for $x$ in the definition $f(x) = 5x + 2$ to get

$$g(f(a)) = g(5a + 2).$$

Now we need to evaluate $g(5a + 2)$. To do this, we substitute $5a + 2$ for $x$ in the definition $g(x) = 3 - 2x$ to get

$$g(5a + 2) = 3 - 2(5a + 2).$$

We can expand this last result and simplify. Thus,

$$g(f(a)) = 3 - 10a - 4 = -10a - 1.$$

Again, it is conventional to arrange the work in one continuous block, as follows.

$$\begin{aligned} g(f(a)) &= g(5a + 2) \\ &= 3 - 2(5a + 2) \\ &= 3 - 10a - 4 \\ &= -10a - 1 \end{aligned}$$

Hence, $g(f(a)) = -10a - 1$.

◆

## *Extracting the Domain of a Function*

We've seen that the domain of a relation or function is the set of all the first coordinates of its ordered pairs. However, if a functional relationship is defined by an equation such as $f(x) = 3x - 4$, then it is not practical to list all ordered pairs defined by this relationship. For any real $x$-value, you get an ordered pair. For example, if $x = 5$, then $f(5) = 3(5) - 4 = 11$, leading to the ordered pair $(5, f(5))$ or $(5, 11)$. As you can see, the number of such ordered pairs is infinite. For each new $x$-value, we get another function value and another ordered pair.

Therefore, it is easier to turn our attention to the values of $x$ that yield real number responses in the equation $f(x) = 3x - 4$. This leads to the following key idea.

**Definition 21.** *If a function is defined by an equation, then the domain of the function is the set of "permissible $x$-values," the values that produce a real number response defined by the equation.*

We sometimes like to say that the domain of a function is the set of "OK $x$-values to use in the equation." For example, if we define a function with the rule $f(x) = 3x - 4$, it is immediately apparent that we can use any value we want for $x$ in the rule $f(x) = 3x - 4$. Thus, the domain of $f$ is all real numbers. We can write that the domain $D = \mathbb{R}$, or we can use interval notation and write that the domain $D = (-\infty, \infty)$.

It is not the case that $x$ can be any real number in the function defined by the rule $f(x) = \sqrt{x}$. It is not possible to take the square root of a negative number.[2] Therefore, $x$ must either be zero or a positive real number. In set-builder notation, we can describe the domain with $D = \{x : x \geq 0\}$. In interval notation, we write $D = [0, \infty)$.

We must also be aware of the fact that we cannot divide by zero. If we define a function with the rule $f(x) = x/(x - 3)$, we immediately see that $x = 3$ will put a zero in the denominator. Division by zero is not defined. Therefore, 3 is not in the domain of $f$. No other $x$-value will cause a problem. The domain of $f$ is best described with set-builder notation as $D = \{x : x \neq 3\}$.

## Functions Without Formulae

In the previous section, we defined functions by means of a formula, for example, as in

$$f(x) = \frac{x + 3}{2 - 3x}.$$

Euler would be pleased with this definition, for as we have said previously, Euler thought of functions as analytic expressions.

However, it really isn't necessary to provide an expression or formula to define a function. There are other forms we can use to express a functional relationship: a graph, a table, or even a narrative description. The only thing that is really important is the requirement that the function be well-defined, and by "well-defined," we mean that each object in the function's domain is paired with one and only one object in its range.

As an example, let's look at a special function $\pi$ on the natural numbers,[3] which returns the number of primes less than or equal to a given natural number. For example, the primes less than or equal to the number 23 are 2, 3, 5, 7, 11, 13, 17, 19, and 23, nine numbers in all. Therefore, the number of primes less than or equal to 23 is nine. In symbols, we would write

$$\pi(23) = 9.$$

---

[2] The square of a real number is either zero or a positive real number. It is not possible to square a real number and get a negative result. Therefore, there is no real square root of a negative number.

[3] The use of $\pi$ in this context is unfortunate and apt to confuse. Readers are more likely to associate the symbol $\pi$ with the formulae for finding the area and circumference of a circle, with approximate value $\pi \approx 3.14159\ldots$. As John Derbyshire states in *Prime Obsession*, "The Greek alphabet has only 24 letters and by the time mathematicians got round to giving this function a symbol (the person responsible in this case is Edmund Landau, in 1909), all 24 had been pretty much used up and they had to start recycling them." In short, the symbol is standard, so we'll just have to live with it.

Note the absence of a formula in the definition of this function. Indeed, the definition is descriptive in nature, so we might write

$$\pi(n) = \text{number of primes less than or equal to } n.$$

The important thing is not how we define this special function $\pi$, but the fact that it is well-defined; that is, for each natural number $n$, there are a fixed number of primes less than or equal to $n$. Thus, each natural number in the domain of $\pi$ is paired with one and only one number in its range.

Now, just because our function doesn't provide an expression for calculating the number of primes less than or equal to a given natural number $n$, it doesn't stop mathematicians from seeking such a formula. Euclid of Alexandria (325-265 BC), a Greek mathematician, proved that the number of primes is infinite, but it was the German mathematician and scientist, Johann Carl Friedrich Gauss (1777-1855), who first proposed that the number of primes less than or equal to $n$ can be approximated by the formula

$$\pi(n) \approx \frac{n}{\ln n},$$

where $\ln n$ is the "natural logarithm" of $n$ (to be explained in Chapter 9). This approximation gets better and better with larger and larger values of $n$. The formula was refined by Gauss, who did not provide a proof, and the problem became known as the *Prime Number Theorem*. It was not until 1896 that Jacques Salomon Hadamard (1865-1963) and Charles Jean Gustave Nicolas Baron de la Vallee Poussin (1866-1962), working independently, provided a proof of the *Prime Number Theorem*.

## 2.1 Exercises

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In **Exercises 1-6**, state the domain and range of the given relation.

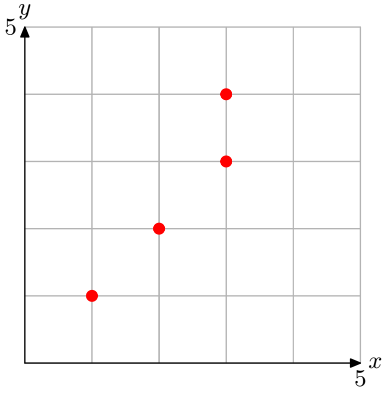
**1.** $R = \{(1, 3), (2, 4), (3, 4)\}$

**2.** $R = \{(1, 3), (2, 4), (2, 5)\}$

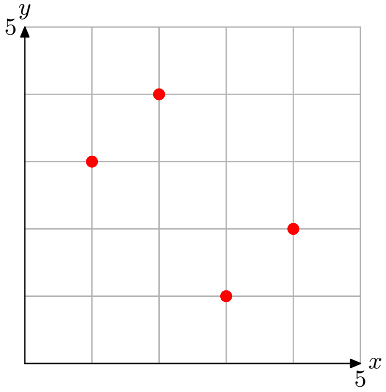
**3.** $R = \{(1, 4), (2, 5), (2, 6)\}$

**4.** $R = \{(1, 5), (2, 4), (3, 6)\}$

**5.**



**6.**



In **Exercises 7-12**, create a mapping diagram for the given relation and state whether or not it is a function.

**7.** The relation in **Exercise 1**.

**8.** The relation in **Exercise 2**.

**9.** The relation in **Exercise 3**.

**10.** The relation in **Exercise 4**.

**11.** The relation in **Exercise 5**.

**12.** The relation in **Exercise 6**.

**13.** Given that $g$ takes a real number and doubles it, then $g : x \longrightarrow ?$.

**14.** Given that $f$ takes a real number and divides it by 3, then $f : x \longrightarrow ?$.

**15.** Given that $g$ takes a real number and adds 3 to it, then $g : x \longrightarrow ?$.

**16.** Given that $h$ takes a real number and subtracts 4 from it, then $h : x \longrightarrow ?$.

**17.** Given that $g$ takes a real number, doubles it, then adds 5, then $g : x \longrightarrow ?$.

**18.** Given that $h$ takes a real number, subtracts 3 from it, then divides the result by 4, then $h : x \longrightarrow ?$.

Given that the function $f$ is defined by the rule $f : x \longrightarrow 3x - 5$, determine where the input number is mapped in **Exercises 19-22**.

**19.** $f : 3 \longrightarrow ?$

---

[4] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

**20.** $f : -5 \longrightarrow ?$

**21.** $f : a \longrightarrow ?$

**22.** $f : 2a + 3 \longrightarrow ?$

---

Given that the function $f$ is defined by the rule $f : x \longrightarrow 4 - 5x$, determine where the input number is mapped in **Exercises 23-26**.

**23.** $f : 2 \longrightarrow ?$

**24.** $f : -3 \longrightarrow ?$

**25.** $f : a \longrightarrow ?$

**26.** $f : 2a + 11 \longrightarrow ?$

---

Given that the function $f$ is defined by the rule $f : x \longrightarrow x^2 - 4x - 6$, determine where the input number is mapped in **Exercises 27-30**.

**27.** $f : 1 \longrightarrow ?$

**28.** $f : -2 \longrightarrow ?$

**29.** $f : -1 \longrightarrow ?$

**30.** $f : a \longrightarrow ?$

---

Given that the function $f$ is defined by the rule $f : x \longrightarrow 3x - 9$, determine where the input number is mapped in **Exercises 31-34**.

**31.** $f : a \longrightarrow ?$

**32.** $f : a + 1 \longrightarrow ?$

**33.** $f : 2a - 5 \longrightarrow ?$

**34.** $f : a + h \longrightarrow ?$

---

Given that the functions $f$ and $g$ are defined by the rules $f : x \longrightarrow 2x + 3$ and $g : x \longrightarrow 4 - x$, determine where the input number is mapped in **Exercises 35-38**.

**35.** $f : 2 \longrightarrow ?$

**36.** $g : 2 \longrightarrow ?$

**37.** $f : a + 1 \longrightarrow ?$

**38.** $g : a - 3 \longrightarrow ?$

---

**39.** Given that $g$ takes a real number and triples it, then $g(x) = ?$.

**40.** Given that $f$ takes a real number and divides it by 5, then $f(x) = ?$.

**41.** Given that $g$ takes a real number and subtracts it from 10, then $g(x) = ?$.

**42.** Given that $f$ takes a real number, multiplies it by 5 and then adds 4 to the result, then $f(x) = ?$.

**43.** Given that $f$ takes a real number, doubles it, then subtracts the result from 11, then $f(x) = ?$.

**44.** Given that $h$ takes a real number, doubles it, adds 5, then takes the square root of the result, then $h(x) = ?$.

---

In **Exercises 45-54**, evaluate the given function at the given value $b$.

**45.** $f(x) = 12x + 2$ for $b = 6$.

**46.** $f(x) = -11x - 4$ for $b = -3$.

**47.** $f(x) = -9x - 1$ for $b = -5$.

**48.** $f(x) = 11x + 4$ for $b = -4$.

**49.** $f(x) = 4$ for $b = -12$.

**50.** $f(x) = 7$ for $b = -7$.

**51.** $f(x) = 0$ for $b = -7$.

**52.** $f(x) = 12x + 8$ for $b = -3$.

**53.** $f(x) = -9x + 3$ for $b = -1$.

**54.** $f(x) = 6x - 3$ for $b = 3$.

In **Exercises 55-58**, given that the function $f$ is defined by the rule $f(x) = 2x + 7$, determine where the input number is mapped.

**55.** $f(a) = ?$

**56.** $f(a + 1) = ?$

**57.** $f(3a - 2) = ?$

**58.** $f(a + h) = ?$

In **Exercises 59-62**, given that the function $g$ is defined by the rule $g(x) = 3 - 2x$, determine where the input number is mapped.

**59.** $g(a) = ?$

**60.** $g(a + 3) = ?$

**61.** $g(2 - 5a) = ?$

**62.** $g(a + h) = ?$

Given that the functions $f$ and $g$ are defined by the rules $f(x) = 1 - x$ and $g(x) = 2x + 13$, determine where the input number is mapped in **Exercises 63-66**.

**63.** $f(a) = ?$

**64.** $g(a) = ?$

**65.** $f(a + 3) = ?$

**66.** $g(4 - a) = ?$

Given that the functions $f$ and $g$ are defined by the rules $f(x) = 3x + 4$ and $g(x) = 2x - 5$, determine where the input number is mapped in **Exercises 67-70**.

**67.** $f(g(2)) = ?$

**68.** $g(f(2)) = ?$

**69.** $f(g(a)) = ?$

**70.** $g(f(a)) = ?$

Given that the functions $f$ and $g$ are defined by the rules $f(x) = 2x - 9$ and $g(x) = 11$, determine where the input number is mapped in **Exercises 71-74**.

**71.** $f(g(2)) = ?$

**72.** $g(f(2)) = ?$

**73.** $f(g(a)) = ?$

**74.** $g(f(a)) = ?$

Use set-builder notation to describe the domain of each of the functions defined in **Exercises 75-78**.

**75.** $f(x) = \dfrac{93}{x + 98}$

**76.** $f(x) = \dfrac{54}{x + 65}$

**77.** $f(x) = -\dfrac{87}{x - 88}$

**78.** $f(x) = -\dfrac{30}{x - 52}$

---

Use set-builder and interval notation to describe the domain of the functions defined in **Exercises 79-82**.

**79.** $f(x) = \sqrt{x + 69}$

**80.** $f(x) = \sqrt{x + 62}$

**81.** $f(x) = \sqrt{x - 81}$

**82.** $f(x) = \sqrt{x - 98}$

---

Two integers are said to be *relatively prime* if their greatest common divisor is 1. For example, the greatest common divisor of 6 and 35 is 1, so 6 and 35 are relatively prime. On the other hand, the greatest common divisor of 14 and 21 is **not** 1 (it is 7), so 14 and 21 are **not** relatively prime. The *Euler $\phi$-function* is defined as follows:

- If $n = 1$, then $\phi(n) = 1$.
- If $n > 1$, then $\phi(n)$ is the number of positive integers less than $n$ that are relatively prime to $n$. In **Exercises 83-84**, evaluate the Euler $\phi$-function at the given input.

**83.** $\phi(12)$

**84.** $\phi(36)$

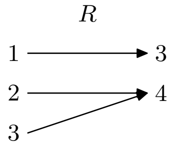
## 2.1 Answers

**1.** Domain $= \{1, 2, 3\}$, Range $= \{3, 4\}$

**3.** Domain $= \{1, 2\}$, Range $= \{4, 5, 6\}$

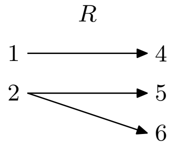
**5.** Domain $= \{1, 2, 3\}$, Range $= \{1, 2, 3, 4\}$

**7.**



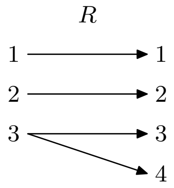
Function.

**9.**



Not a function.

**11.**



Not a function.

**13.** $g : x \longrightarrow 2x$

**15.** $g : x \longrightarrow x + 3$

**17.** $g : x \longrightarrow 2x + 5$

**19.** $f : 3 \longrightarrow 4$

**21.** $f : a \longrightarrow 3a - 5$

**23.** $f : 2 \longrightarrow -6$

**25.** $f : a \longrightarrow 4 - 5a$

**27.** $f : 1 \longrightarrow -9$

**29.** $f : -1 \longrightarrow -1$

**31.** $f : a \longrightarrow 3a - 9$

**33.** $f : 2a - 5 \longrightarrow 6a - 24$

**35.** $f : 2 \longrightarrow 7$

**37.** $f : a + 1 \longrightarrow 2a + 5$

**39.** $g(x) = 3x$

**41.** $g(x) = 10 - x$

**43.** $f(x) = 11 - 2x$

**45.** 74

**47.** 44

**49.** 4

**51.** 0

**53.** 12

**55.** $f(a) = 2a + 7$

**57.** $f(3a - 2) = 6a + 3$

**59.** $g(a) = 3 - 2a$

**61.** $g(2 - 5a) = 10a - 1$

**63.** $f(a) = 1 - a$

**65.** $f(a + 3) = -a - 2$

**67.** $f(g(2)) = 1$

**69.** $f(g(a)) = 6a - 11$

**71.** $f(g(2)) = 13$

**73.** $f(g(a)) = 13$

**75.** Domain $= \{x : x \neq -98\}$

**77.** Domain $= \{x : x \neq 88\}$

**79.** Domain $= [-69, \infty) = \{x : x \geq -69\}$

**81.** Domain $= [81, \infty) = \{x : x \geq 81\}$

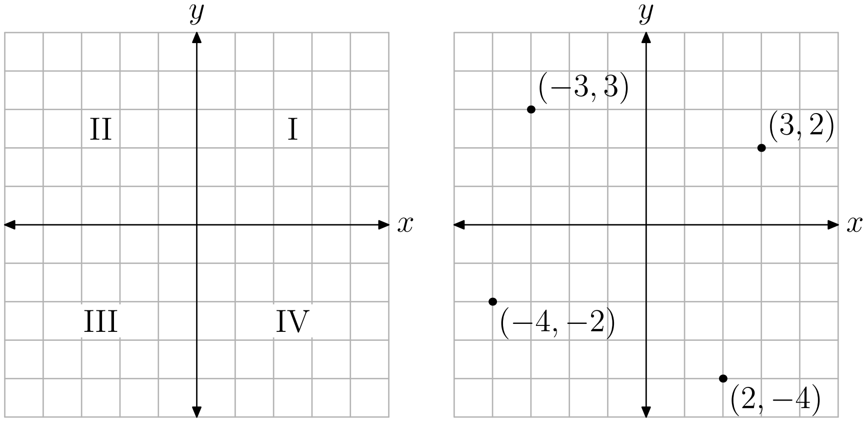
**83.** $\phi(12) = 4$

## 2.2 The Graph of a Function

Rene Descartes (1596-1650) was a French philosopher and mathematician who is well known for the famous phrase "cogito ergo sum" (I think, therefore I am), which appears in his *Discours de la methode pour bien conduire sa raison, et chercher la verite dans les sciences* (Discourse on the Method of Rightly Conducting the Reason, and Seeking Truth in the Sciences). In that same treatise, Descartes introduces his coordinate system, a method for representing points in the plane via pairs of real numbers. Indeed, the Cartesian plane of modern day is so named in honor of Rene Descartes, who some call the "Father of Modern Mathematics."

Descartes' work, which forever linked geometry and algebra, was continued in an appendix to *Discourse on Method*, entitled *La Geometrie*, which some consider the beginning of modern mathematics. Certainly both Newton and Leibniz, in developing the Calculus, built upon the foundation provided in this work by Descartes.

A Cartesian Coordinate System consists of a pair of axes, usually drawn at right angles to one another in the plane, one horizontal (labeled $x$) and one vertical (labeled $y$), as shown in the **Figure 1**. The quadrants are numbered I, II, III, and IV, in counterclockwise order, and samples of ordered pairs of the form $(x, y)$ are shown in each quadrant of the Cartesian coordinate system in **Figure 1**.



Numbering the quadrants.

To the right and up is positive, left and down is negative.

**Figure 1.** The Cartesian coordinate system.

Now, suppose that we have a relation

$$R = \{(1, 2), (3, 1), (3, 4), (4, 3)\}.$$

Recall that *relation* is the name given to a collection of ordered pairs. In **Figure 2**(b) we've plotted each of the ordered pairs in the relation $R$. This is called the *graph* of the relation $R$.

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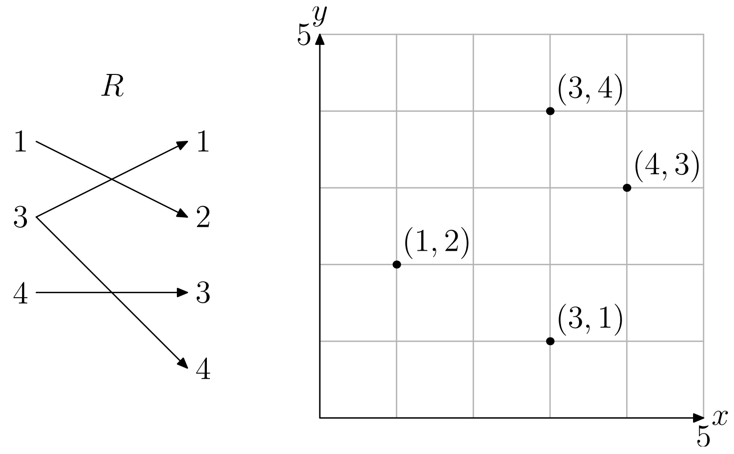
[5] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

**Definition 1.** *The graph of a relation is the collection of all ordered pairs of the relation. These are usually represented as points in a Cartesian coordinate system.*



(a) (b)

**Figure 2.** A mapping diagram and its graph.

In **Figure 2**(a), we've created a mapping diagram of the ordered pairs. Note that the domain object 3 is paired with two range elements, namely 1 and 4. Hence the relation $R$ is **not** a function. It is interesting to note that there are two points in the graph of $R$ in **Figure 2**(b) that have the same first coordinate, namely $(3, 1)$ and $(3, 4)$. This is a signal that the graph of the relation $R$ is not a function. In the next section we will discuss the *Vertical Line Test*, which will use this dual use of the first coordinate to determine when a relation is a not a function.

## Creating the Graph of a Function

Some texts will speak of the graph of an equation, such as "Draw the graph of the equation $y = x^2$." This instruction raises a number of difficulties.

- First, the instruction provides no direction to the reader; that is, what does the instruction mean? It's not very helpful.

- Secondly, the instruction is incorrect. You don't draw the graphs of equations. Rather, you draw the graphs of relations and/or functions. A graph is just another way of representing a function, a relation that pairs each element in its domain with exactly one element in its range.

So, what is the proper instruction? First, we will provide the formal definition of the graph of a function, then we will break it down by means of examples.

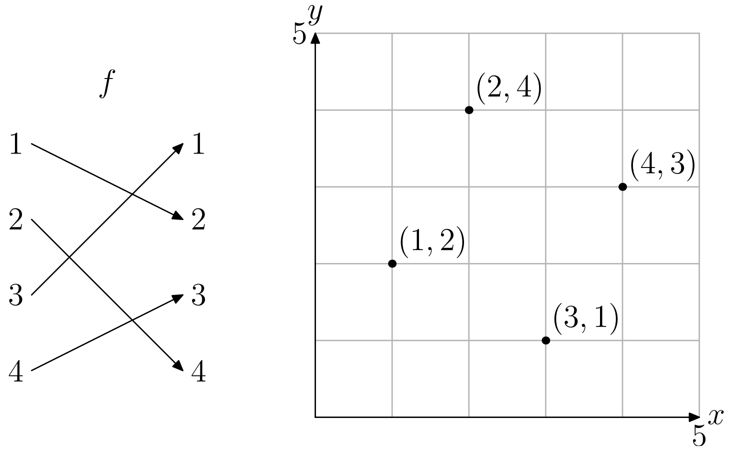
**Definition 2.** *The graph of a function is the collection of all ordered pairs of the function. These are usually represented as points in a Cartesian coordinate system.*

As an example, consider the function

$$f = \{(1,2),(2,4),(3,1),(4,3)\}. \tag{3}$$

Readers will note that each object in the domain is paired with one and only one object in the range, as seen in the mapping diagram of **Figure 3**(a).

Thus, we have two representations of the function $f$, the collection of ordered pairs (**3**), and the mapping diagram of in **Figure 3**(a). A third representation of the function $f$ is the graph of the ordered pairs of the function, shown in the Cartesian plane in **Figure 3**(b).



(a) (b)

**Figure 3.** A mapping diagram and its graph.

When the function is represented by an equation or formula, then we adjust our definition of its graph somewhat.

**Definition 4.** *The graph of $f$ is the set of all ordered pairs $(x, f(x))$ so that $x$ is in the domain of $f$. In symbols,*

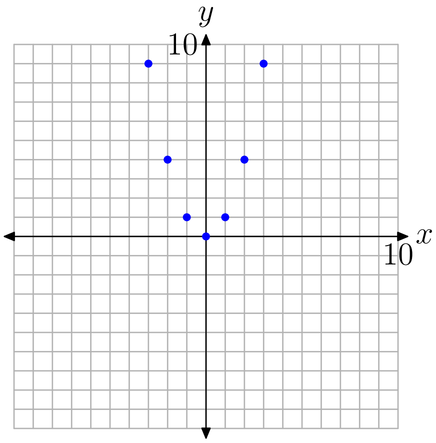
$$\text{Graph of } f = \{(x, f(x)) : x \text{ is in the domain of } f.\}.$$

This last definition is most easily explained by example. So, let's define a function $f$ that maps any real number $x$ to the real number $x^2$; that is, let $f(x) = x^2$. Now, according to **Definition 4**, the graph of $f$ is the set of all points $(x, f(x))$, such that $x$ is in the domain of $f$.

The way is now clear. We begin by creating a table of points $(x, f(x))$, where $x$ is in the domain of the function $f$ defined by $f(x) = x^2$. The choice of $x$ is both subjective and experimental, so we begin by choosing integer values of $x$ between $-3$ and 3. We then evaluate the function at each of these $x$-values (e.g., $f(-3) = (-3)^2 = 9$). The results are shown in the table in **Figure 4**(a). We then plot the points in our table in the Cartesian plane as shown in **Figure 4**(b).

| $x$ | $f(x) = x^2$ | $(x, f(x))$ |
|---|---|---|
| $-3$ | 9 | $(-3, 9)$ |
| $-2$ | 4 | $(-2, 4)$ |
| $-1$ | 1 | $(-1, 1)$ |
| 0 | 0 | $(0, 0)$ |
| 1 | 1 | $(1, 1)$ |
| 2 | 4 | $(2, 4)$ |
| 3 | 9 | $(3, 9)$ |

(a)



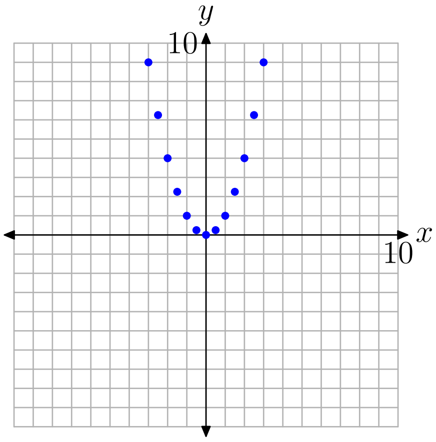
(b)

**Figure 4.** Plotting pairs satisfying the functional relationship defined by the equation $f(x) = x^2$.

Although this is a good start, the graph in **Figure 4**(b) is far from complete. Definition **4** requires that we plot the ordered pairs $(x, f(x))$ for **every** value of $x$ that is in the domain of $f$. We've only plotted seven such points, so we're not done. Let's add more points to the graph of $f$. We'll evaluate the function at each of the $x$-values shown in the table in **Figure 5**(a), then plot the additional pairs $(x, f(x))$ from the table in the Cartesian plane, as shown in **Figure 5**(b).

| $x$ | $f(x) = x^2$ | $(x, f(x))$ |
|---|---|---|
| $-5/2$ | $25/4$ | $(-5/2, 25/4)$ |
| $-3/2$ | $9/4$ | $(-3/2, 9/4)$ |
| $-1/2$ | $1/4$ | $(-1/2, 1/4)$ |
| $1/2$ | $1/4$ | $(1/2, 1/4)$ |
| $3/2$ | $9/4$ | $(3/2, 9/4)$ |
| $5/2$ | $25/4$ | $(5/2, 25/4)$ |

(a)



(b)

**Figure 5.** Plotting additional pairs $(x, f(x))$ defined by the equation $f(x) = x^2$.

We're still not finished, because we've only plotted 13 pairs $(x, f(x))$, such that $f(x) = x^2$. **Definition 4** requires that we plot the ordered pairs $(x, f(x))$ for **every** value of $x$ in the domain of $f$.

However, a pattern is certainly establishing itself, as seen in **Figure 5**(b). At some point, we need to "make a leap of faith," and plot **all** ordered pairs $(x, f(x))$, such that $x$ is in the domain of $f$. This is done in **Figure 6**.

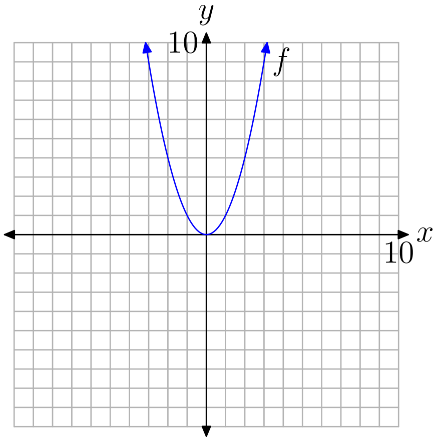


**Figure 6.** Plotting all pairs $(x, f(x))$ so that $x$ is in the domain of $f$.

There are several important points we need to make about the final result in **Figure 6**.

- When we draw a smooth curve, such as that shown in **Figure 6**, it is important to understand that this is a simply a shortcut for plotting all pairs $(x, f(x))$, where $f(x) = x^2$ and $x$ is in the domain of $f$.[6]
- It is important to understand that we are **NOT** "connecting the dots," neither with a ruler nor with curved segments. Rather, the curve in **Figure 6** is the result of plotting all of the individual pairs $(x, f(x))$.
- The "arrows" at each end of the curve have an important meaning. Much as the ellipsis at the end of the progression 2, 4, 6, ... mean "et-cetera," the arrows at each end of the curve have a similar meaning. The arrow at the end of the left-half of the curve indicates that the graph continues opening upward and to the left, while the arrow at the end of the right-half of the curve indicates that the graph continues opening upward and to the right.

## *Creating Graphs by Hand*

We're going to look at several basic graphs, which we'll create by employing the strategy used to create the graph of $f(x) = x^2$. First, let's summarize that process.

---

[6] It would take too long to plot the individual pairs "one at a time."

**Summary 5.** *If a function is defined by an equation, you can create the graph of the function as follows.*

1. *Select several values of $x$ in the domain of the function $f$.*
2. *Use the selected values of $x$ to create a table of pairs $(x, f(x))$ that satisfy the equation that defines the function $f$.*
3. *Create a Cartesian coordinate system on a sheet of graph paper. Label and scale each axis, then plot the pairs $(x, f(x))$ from your table on your coordinate system.*
4. *If the plotted pairs $(x, f(x))$ provide enough of a pattern for you to intuit the shape of the graph of $f$, make the "leap of faith" and plot* **all** *pairs that satisfy the equation defining $f$ by drawing a smooth curve on your coordinate system. Of course, this curve should contain all previously plotted pairs.*
5. *If your plotted pairs do not provide enough of a pattern to determine the final shape of the graph of $f$, then add more pairs to your table and plot them on your Cartesian coordinate system. Continue in this manner until you are confident in the shape of the graph of $f$.*

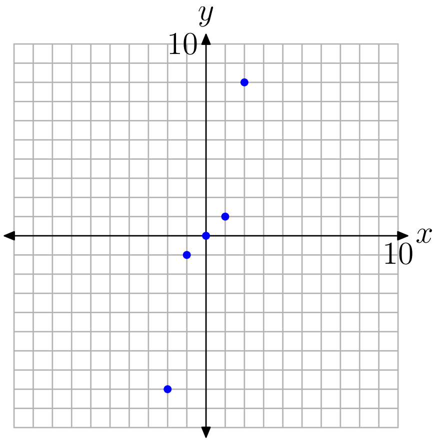
Let's look at an example.

▶ **Example 6.** *Sketch the graph of the function defined by the equation $f(x) = x^3$.*

We'll start with $x$-values $-2$, $-1$, $0$, $1$, and $2$, then use the equation $f(x) = x^3$ to determine pairs $(x, f(x))$ (e.g., $f(-2) = (-2)^3 = -8$). These are listed in the table in **Figure 7**(a). We then plot the points from the table on a Cartesian coordinate system, as shown in **Figure 7**(b).

| $x$ | $f(x) = x^3$ | $(x, f(x))$ |
|---|---|---|
| $-2$ | $-8$ | $(-2, -8)$ |
| $-1$ | $-1$ | $(-1, -1)$ |
| $0$ | $0$ | $(0, 0)$ |
| $1$ | $1$ | $(1, 1)$ |
| $2$ | $8$ | $(2, 8)$ |

(a)



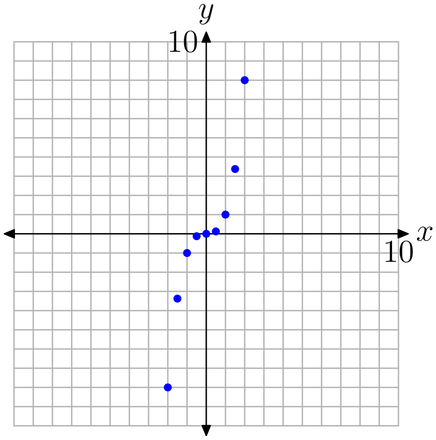
(b)

**Figure 7.** Plotting pairs $(x, f(x))$ defined by the equation $f(x) = x^3$.

We're a bit unsure of the shape of the graph of $f$, so we'll add a few more pairs to our table and plot them. This is shown in **Figures 8**(a) and (b).

| $x$ | $f(x) = x^3$ | $(x, f(x))$ |
|---|---|---|
| $-3/2$ | $-27/8$ | $(-3/2, -27/8)$ |
| $-1/2$ | $-1/8$ | $(-1/2, -1/8)$ |
| $1/2$ | $1/8$ | $(1/2, 1/8)$ |
| $3/2$ | $27/8$ | $(3/2, 27/8)$ |

(a)



(b)

**Figure 8.** Plotting additional pairs $(x, f(x))$ defined by the equation $f(x) = x^3$.

The additional pairs fill in the shape of $f$ in **Figure 8**(b) a bit better than those in **Figure 7**(b), enough so that we're confident enough to make a "leap of faith" and draw the final shape of the graph of $f(x) = x^3$ in **Figure 9**.

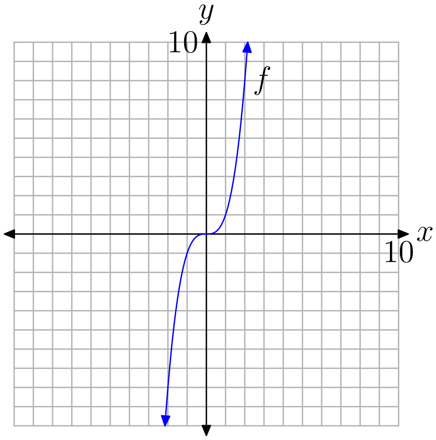


**Figure 9.** The final graph of $f(x) = x^3$.

◆

Let's look at another example.

▶ **Example 7.** *Sketch the graph of* $f(x) = \sqrt{x}$.

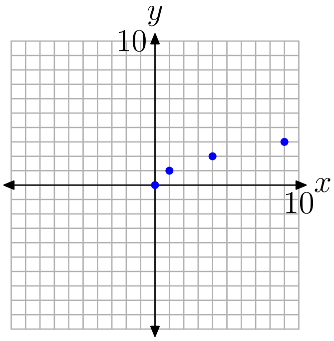
Again, we'll start by selecting several values of $x$ in the domain of $f$. In this case, $f(x) = \sqrt{x}$, and it's not possible to take the square root of a negative number.[7] Also,

---

[7] Whenever you square a real number, the result is either positive or zero. Hence, the square root of a negative number cannot be a real number.

if we're creating a table of pairs by hand, it's good strategy to select known squares. Thus, we'll use $x = 0$, 1, 4, and 9 for starters.

| $x$ | $f(x) = \sqrt{x}$ | $(x, f(x))$ |
|---|---|---|
| 0 | 0 | $(0, 0)$ |
| 1 | 1 | $(1, 1)$ |
| 4 | 2 | $(4, 2)$ |
| 9 | 3 | $(9, 3)$ |

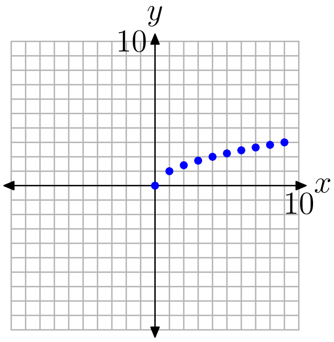


(a) (b)

**Figure 10.** Plotting pairs $(x, f(x))$ defined by the equation $f(x) = \sqrt{x}$.

Some might be ready to make a "leap of faith" based on these initial results. Others might want to use a calculator to compute decimal approximations for additional square roots. The resulting pairs are shown in the table in **Figure 11**(a) and the additional pairs are plotted in **Figure 11**(b).

| $x$ | $f(x) = \sqrt{x}$ | $(x, f(x))$ |
|---|---|---|
| 2 | 1.4 | $(2, 1.4)$ |
| 3 | 1.7 | $(3, 1.7)$ |
| 5 | 2.2 | $(5, 2.2)$ |
| 6 | 2.4 | $(6, 2.4)$ |
| 7 | 2.6 | $(7, 2.6)$ |
| 8 | 2.8 | $(8, 2.8)$ |



(a) (b)

**Figure 11.** Plotting additional pairs $(x, f(x))$ defined by the equation $f(x) = \sqrt{x}$.

The pattern in **Figure 11**(b) is clear enough to make a "leap of faith" and complete the graph as shown in **Figure 12**.

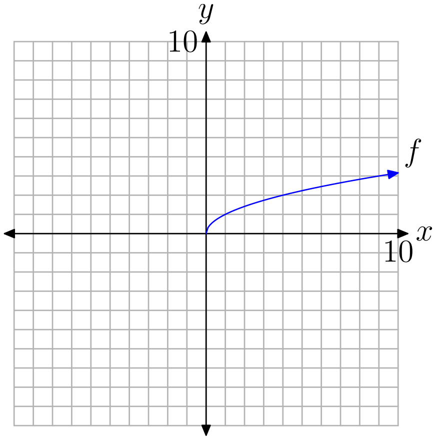


**Figure 12.** The graph of $f$ defined by the equation $f(x) = \sqrt{x}$.

◆

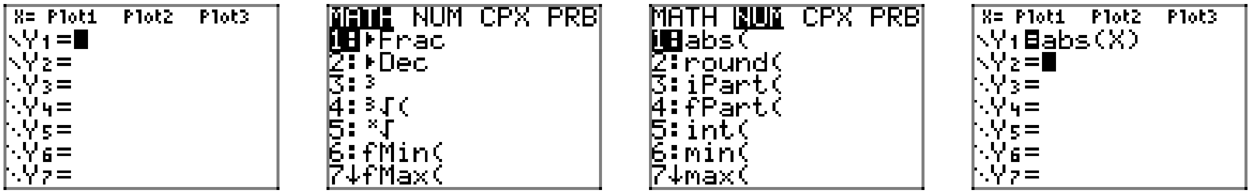
## *Using the Table Feature of the Graphing Calculator*

The `TABLE` feature on your graphing calculator can be of immense help when creating tables of points that satisfy the equation defining the function $f$. Let's look at an example.

▶ **Example 8.** *Sketch the graph of $f(x) = |x|$.*

Enter the function $f(x) = |x|$ in the `Y=` menu as follows.[8]

1. Press the `Y=` button on your calculator. This will open the `Y=` menu as shown in **Figure 13**(a). Use the arrow keys and the `CLEAR` button on your calculator to delete any existing functions.
2. Press the `MATH` button to open the menu shown in **Figure 13**(b).
3. Press the right-arrow on your calculator to select the `NUM` submenu as shown in **Figure 13**(c).
4. Select `1:abs(`, then enter `X` and close the parentheses, as shown in **Figure 13**(d).



(a) (b) (c) (d)

**Figure 13.** Entering $f(x) = |x|$ in the `Y=` menu.

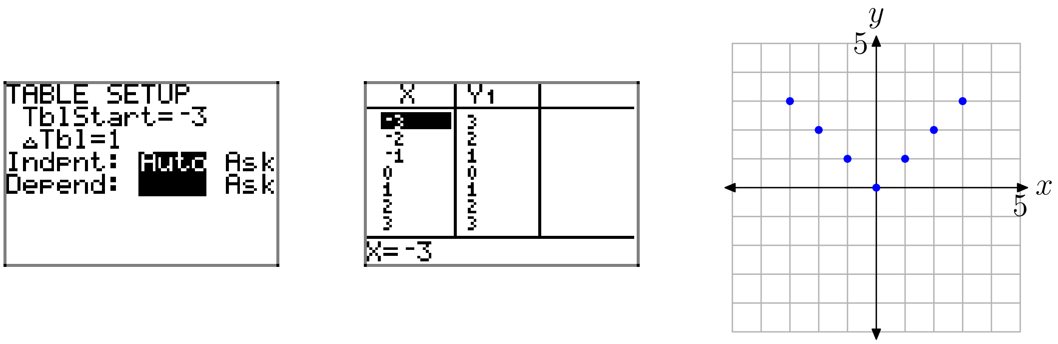
---

[8] Readers will recall that the absolute value function takes a real number and makes it nonnegative. For example, $|-3| = 3$, $|0| = 0$, and $|3| = 3$. We'll have more to say about the absolute value function in Chapter 3.

We will now use the `TABLE` feature of the graphing calculator to help create a table of pairs $(x, f(x))$ satisfying the equation $f(x) = |x|$. Proceed as follows.

1. Select `2nd TBLSET` (i.e., push the `2nd` button followed by `TBLSET`), which is located over the `WINDOW` button. Enter `TblStart=-3`, $\Delta$`Tbl` $= 1$, and set the independent and dependent variables to `Auto` (this is done by highlighting `Auto` and pressing the `Enter` button), as shown in **Figure 14**(a).
2. Press `2nd TABLE`, which is located above the `GRAPH` button, to produce the table of pairs $(x, f(x))$ shown in **Figure 14**(b).

We've plotted the pairs directly from the calculator onto a Cartesian coordinate system on graph paper in **Figure 14**(c).



```
TABLE SETUP
 TblStart=-3
 ΔTbl=1
Indpnt: Auto Ask
Depend:      Ask
```

```
  X   | Y1
  -3  | 3
  -2  | 2
  -1  | 1
  0   | 0
  1   | 1
  2   | 2
  3   | 3
X=-3
```

(a) (b) (c)

**Figure 14.** Creating a table with the `TABLE` feature of the graphing calculator.

Based on what we see in **Figure 14**(c), we're ready to make a "leap of faith" and draw the graph of $f$ shown in **Figure 15**.

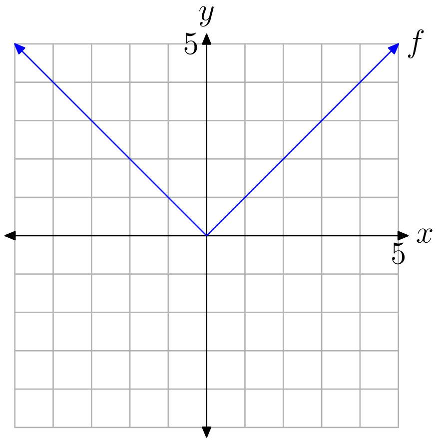
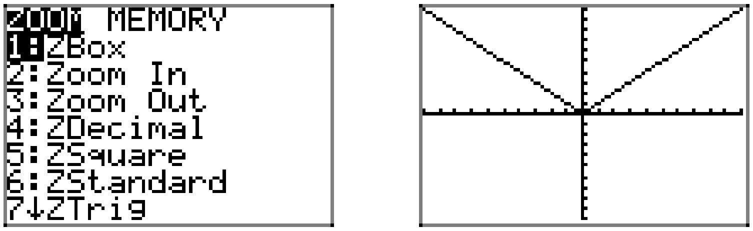


**Figure 15.** The graph of $f$ defined by $f(x) = |x|$.

Alternatively, or as a check, we can have the graphing calculator draw the graph for us. Push the `ZOOM` button, then select `6:ZStandard` (shown in **Figure 16**(a)) to produce the graph shown in **Figure 16**(b).



(a) (b)

**Figure 16.** Creating the graph of $f(x) = |x|$ with the graphing calculator.

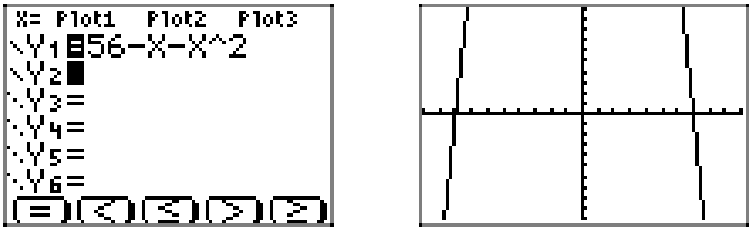
## *Adjusting the Viewing Window*

In **Example 8**, we used the graphing calculator to draw the graph of the function defined by the equation $f(x) = |x|$. For the functions we've encountered thus far, drawing their graphs using the graphing calculator is pretty trivial. Simply enter the equation in the `Y=` menu, then press the `ZOOM` button and select `6:ZStandard`. However, if the graph of a function doesn't fit (or even appear) in the "standard" viewing window, it can be quite challenging to find optimal view settings so that the important features of the graph are visible.

Indeed, as one might not even know what "important" features to look for, setting the viewing window is usually highly subjective and experimental by nature. Let's look at some examples.

▶ **Example 9.** *Use a graphing calculator to sketch the graph of* $f(x) = 56 - x - x^2$. *Experiment with the* `WINDOW` *settings until you feel you have a viewing window that exhibits the important features of the graph.*

First, start by entering the function in the `Y=` menu, as shown in **Figure 17**(a). The caret ˆon the keyboard is used for exponents. Press the `ZOOM` button and select `6:ZStandard` to produce the graph shown in **Figure 17**(b).



(a) (b)

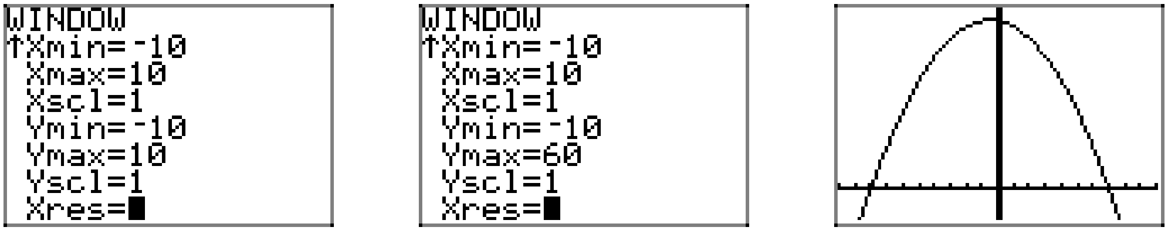
**Figure 17.** The graph of $f(x) = 56 - x - x^2$ in the "standard" viewing window.

As the graph draws, observe that the graph rises from the bottom of the screen, leaves the top of the screen, then returns, falling from the top of the screen and leaving again at the bottom of the screen. This would indicate that there must be some sort of "turning point" that is not visible at the top of the screen.

Press the `WINDOW` button to reveal the "standard viewing window" settings shown in **Figure 18**(a). The following legend explains each of the `WINDOW` parameters in **Figure 18**(a).

$Xmin$ = $x$-value of left edge of viewing window
$Xmax$ = $x$-value of right edge of viewing window
$Xscl$ = $x$-axis tick increment
$Ymin$ = $y$-value of bottom edge of viewing window
$Ymax$ = $y$-value of top edge of viewing window
$Yscl$ = $y$-axis tick increment

It is easy to evaluate the function $f(x) = 56 - x - x^2$ at $x = 0$. Indeed, $f(0) = 56 - 0 - 0^2 = 56$. This indicates that the graph of $f$ must pass through the point $(0, 56)$. This gives us a clue at how we should set the upper bound on our viewing window. Set `Ymax = 60`, as shown in **Figure 18**(b), then press the `GRAPH` button to produce the graph and viewing window shown in **Figure 18**(c).



```
WINDOW
↑Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=█
```

```
WINDOW
↑Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=60
 Yscl=1
 Xres=█
```

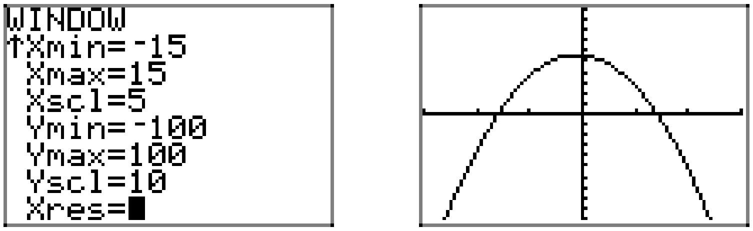
(a) (b) (c)

**Figure 18.** Changing the viewing window.

Although the viewing window in **Figure 18**(c) shows the "turning point" of the graph of $f$, we will make some additional changes to the window settings, as shown in **Figure 19**(a). First, we "widen" the viewing window a bit, setting `Xmin = -15` and `Xmax = 15`, then we set tick marks on the $x$-axis every 5 units with `Xscl = 5`. Next, to create a little room at the top of the screen, we set `Ymax = 100`, then we "balance" this setting with `Ymin = -100`. Finally, we set tick marks on the $y$-axis every 10 units with `Yscl = 10`.

Push the `GRAPH` button to view the effects of these changes to the `WINDOW` parameters in **Figure 19**(b). Note that these settings are *highly subjective*, and what one reader might find quite pleasing will not necessarily find favor with other readers.

However, what is important is the fact that we've captured the "important features" of the graph of $f(x) = 56 - x - x^2$. Note that this is a very controversial statement. If one is just beginning to learn about the graphs of functions, how is one to determine what are the "important features" of the graph? Unfortunately, the answer to this question is, "through experience." Undoubtedly, this is a very frustrating phrase for readers to hear, but at least it's truthful. The more graphs that you draw, the more



```
WINDOW
↑Xmin=-15
 Xmax=15
 Xscl=5
 Ymin=-100
 Ymax=100
 Yscl=10
 Xres=█
```

(a) (b)

**Figure 19.** Improving the `WINDOW` settings.

you will learn how to look for “turning points,” “end-behavior,” “$x$- and $y$-intercepts,” and the like.

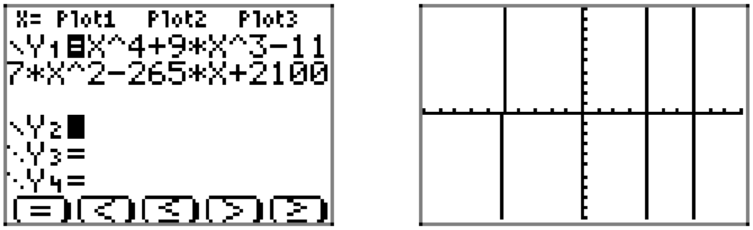
For example, how do we know that the `WINDOW` settings in **Figure 19**(a) determine a viewing window (**Figure 19**(b)) that reveals all “important features” of the graph? The answer at this point is, “we don’t, not without further experiment.” For example, the careful reader might want to try the window settings `Xmin=-50`, `Xmax=50`, `Xscl=10`, `Ymin=-500`, `Ymax=500`, and `Yscl=100` to see if any unexpected behavior crops up.

◆

Let’s look at one last example.

▶ **Example 10.** *Sketch the graph of the function $f$ defined by the equation* $f(x) = x^4 + 9x^3 - 117x^2 - 265x + 2100$.

Load the function into the `Y=` menu (shown in **Figure 20**(a)) and select `6:ZStandard` to produce the graph shown in **Figure 20**(b).



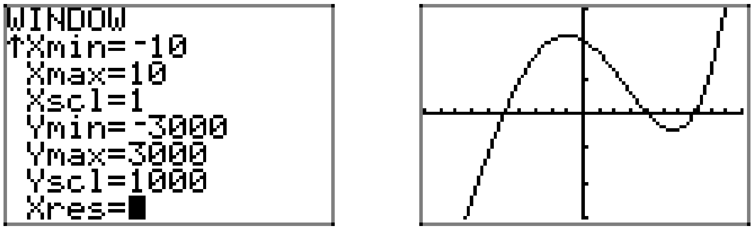
```
Plot1 Plot2 Plot3
\Y1=X^4+9*X^3-11
7*X^2-265*X+2100
\Y2=█
\Y3=
\Y4=
```

(a) (b)

**Figure 20.** Sketching the graph of $f(x) = x^4 + 9x^3 - 117x^2 - 265x + 2100$.

As the graph draws, observe that it rises form the bottom of the viewing window, leaves the top of the viewing window, then returns to fall off the bottom of the viewing window, then returns again and rises off the top of the viewing window.

We notice that $f(0) = 2100$, so we’ll need to set the top of the viewing window to that value or higher. With this thought in mind, we’ll set `Ymax=3000`, then set `Ymin=-3000` for balance, then to avoid a million little tick marks, we’ll set `Yscl=1000`, all shown in **Figure 21**(a). Pressing the `GRAPH` button then produces the image shown in **Figure 21**(b).

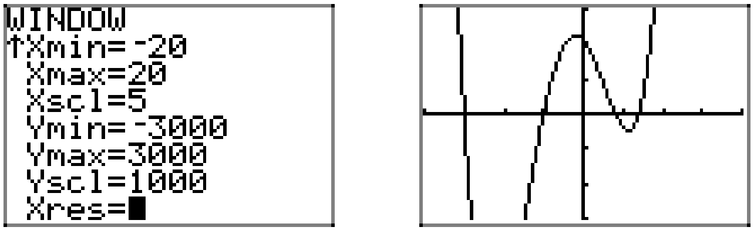


```
WINDOW
↑Xmin=⁻10
 Xmax=10
 Xscl=1
 Ymin=⁻3000
 Ymax=3000
 Yscl=1000
 Xres=█
```

(a) (b)

**Figure 21.** Adjusting the viewing window.

Does it appear that we have all of the “important features” of this graph displayed in our viewing window? Note that we did not experiment very much. Perhaps we should try expanding the window a bit more to see if we have missed any important behavior. With that thought in mind, we set `Xmin=-20`, `Xmax=20`, and to avoid a ton of tick marks, `Xscl=5`, as shown in **Figure 22**(a). Pushing the `GRAPH` button produces the image in **Figure 22**(b).



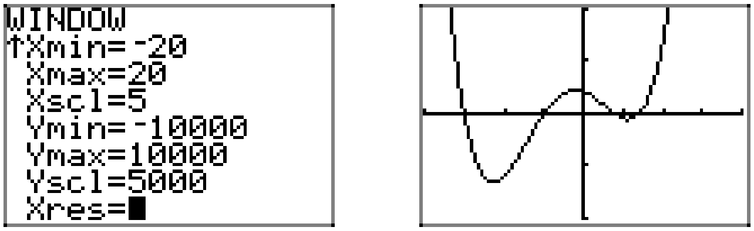
```
WINDOW
↑Xmin=⁻20
 Xmax=20
 Xscl=5
 Ymin=⁻3000
 Ymax=3000
 Yscl=1000
 Xres=█
```

(a) (b)

**Figure 22.** Adjusting the viewing window again reveals behavior not seen.

Note that the viewing window in **Figure 22**(b) reveals behavior not seen in the viewing window of **Figure 21**(b). If we had not experimented further, if we had not expanded the viewing window, we would not have seen this new behavior. This is an important lesson.

Note that one of the “turning points” of the graph in **Figure 22**(b) lies off the bottom of the viewing window. We’ll make one more adjustment to include this important feature. Set `Ymin=-10000`, `Ymax=10000`, and `Yscl=5000`, as shown in **Figure 23**(a), then push the `GRAPH` button to produce the image shown in **Figure 23**(b).



```
WINDOW
↑Xmin=⁻20
 Xmax=20
 Xscl=5
 Ymin=⁻10000
 Ymax=10000
 Yscl=5000
 Xres=█
```

(a) (b)

**Figure 23.** Adjusting the viewing window again reveals behavior not seen.

The graph in **Figure 23**(b) shows all of the "important features" of the graph of $f$, but the careful reader will continue to experiment, expanding the viewing window to ascertain the truth of this statement.

◆

## 2.2 Exercises

Perform each of the following tasks for the functions defined by the equations in **Exercises 1-8**.

i. Set up a table of points that satisfy the given equation. Please place this table of points next to your graph on your graph paper.
ii. Set up a coordinate system on a sheet of graph paper. Label and scale each axis, then plot each of the points from your table on your coordinate system.
iii. If you are confident that you "see" the shape of the graph, make a "leap of faith" and plot **all** pairs that satisfy the given equation by drawing a smooth curve (free-hand) on your coordinate system that contains all previously plotted points (use a ruler only if the graph of the equation is a line). If you are not confident that you "see" the shape of the graph, then add more points to your table, plot them on your coordinate system, and see if this helps. Continue this process until you "see" the shape of the graph and can fill in the rest of the points that satisfy the equation by drawing a smooth curve (or line) on your coordinate system.

**1.** $f(x) = 2x + 1$

**2.** $f(x) = 1 - x$

**3.** $f(x) = 3 - \frac{1}{2}x$

**4.** $f(x) = -1 + \frac{1}{2}x$

**5.** $f(x) = x^2 - 2$

**6.** $f(x) = 4 - x^2$

**7.** $f(x) = \frac{1}{2}x^2 - 6$

**8.** $f(x) = 8 - \frac{1}{2}x^2$

Perform each of the following tasks for the functions **Exercises 9-10**.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.
ii. Use the table feature of your graphing calculator to evaluate the function at the given values of $x$. Record these results in a table next to your coordinate system on your graph paper.
iii. Plot the points in the table on your coordinate system then use them to draw the graph of the given function. Label the graph with its equation.

**9.** $f(x) = \sqrt{x-4}$ at $x = 4, 5, 6, 7, 8, 9$, and 10.

**10.** $f(x) = \sqrt{4-x}$ at $x = -10, -8, -6, -4, -2, 0, 2$, and 4.

In **Exercises 11-14**, the graph of the given function is a *parabola*, a graph that has a "U-shape." A parabola has only one turning point. For each exercise, perform the following tasks.

i. Load the equation into the `Y=` menu of your graphing calculator. Adjust the `WINDOW` parameters so that the "turning point" (actually called the *vertex*) is visible in the viewing window.
ii. Make a reasonable copy of the image in the viewing window on your home-

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[9] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

work paper. Draw all lines with a ruler (including the axes), but draw curves freehand. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graph with its equation.

**11.** $f(x) = x^2 - x - 30$

**12.** $f(x) = 24 - 2x - x^2$

**13.** $f(x) = 11 + 10x - x^2$

**14.** $f(x) = x^2 + 11x - 12$

Each of the equations in **Exercises 15-18** are called "cubic polynomials." Each equation has been carefully chosen so that its graph has exactly two "turning points." For each exercise, perform each of the following tasks.

i. Load the equation into the `Y=` menu of your graphing calculator and adjust the `WINDOW` parameters so that both "turning points" are visible in the viewing window.
ii. Make a reasonable copy of the graph in the viewing window on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax, then label the graph with its equation. *Remember to draw all lines with a ruler.*

**15.** $f(x) = x^3 - 2x^2 - 29x + 30$

**16.** $f(x) = -x^3 + 2x^2 + 19x - 20$

**17.** $f(x) = x^3 + 8x^2 - 53x - 60$

**18.** $f(x) = -x^3 + 16x^2 - 43x - 60$

Perform each of the following tasks for the equations in **Exercises 19-22**.

i. Load the equation into the `Y=` menu. Adjust the `WINDOW` parameters until you think all important behavior ("turning points," etc.) is visible in the viewing window. *Note: This is more difficult than it sounds, particularly when we have no advance notion of what the graph might look like. However, experiment with several settings until you "discover" the settings that exhibit the most important behavior.*
ii. Copy the image on the screen onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graph with its equation.

**19.** $f(x) = 2x^2 - x - 465$

**20.** $f(x) = x^3 - 24x^2 + 65x + 1050$

**21.** $f(x) = x^4 - 2x^3 - 168x^2 + 288x + 3456$

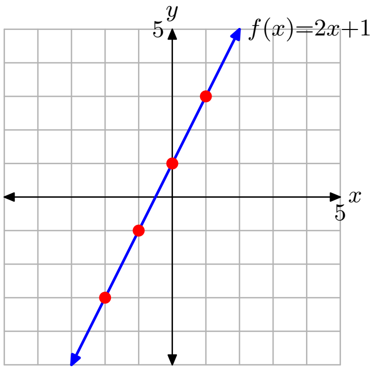
**22.** $f(x) = -x^4 - 3x^3 + 141x^2 + 523x - 660$

## 2.2 *Answers*

---

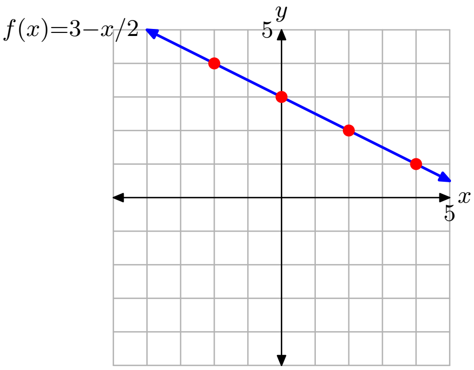
**1.**

| $x$ | $f(x) = 2x + 1$ | $(x, f(x))$ |
|---|---|---|
| $-2$ | $-3$ | $(-2, -3)$ |
| $-1$ | $-1$ | $(-1, -1)$ |
| $0$ | $1$ | $(0, 1)$ |
| $1$ | $3$ | $(1, 3)$ |



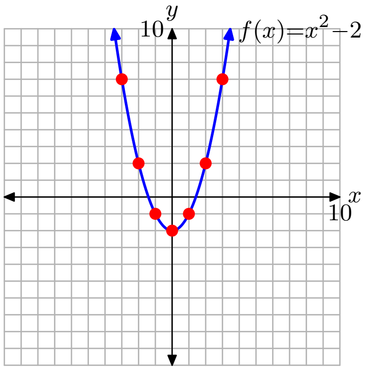
**3.**

| $x$ | $f(x) = 3 - x/2$ | $(x, f(x))$ |
|---|---|---|
| $-2$ | $4$ | $(-2, 4)$ |
| $0$ | $3$ | $(0, 3)$ |
| $2$ | $2$ | $(2, 2)$ |
| $4$ | $1$ | $(4, 1)$ |



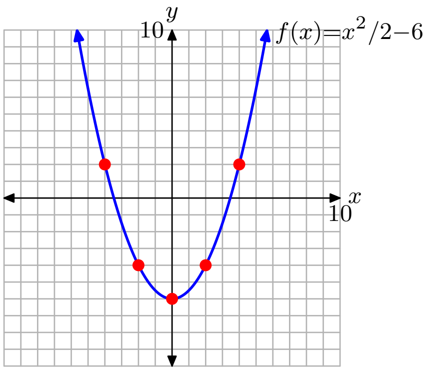
**5.**

| $x$ | $f(x) = x^2 - 2$ | $(x, f(x))$ |
|---|---|---|
| $-3$ | $7$ | $(-3, 7)$ |
| $-2$ | $2$ | $(-2, 2)$ |
| $-1$ | $-1$ | $(-1, -1)$ |
| $0$ | $-2$ | $(0, -2)$ |
| $1$ | $-1$ | $(1, -1)$ |
| $2$ | $2$ | $(2, 2)$ |
| $3$ | $7$ | $(3, 7)$ |

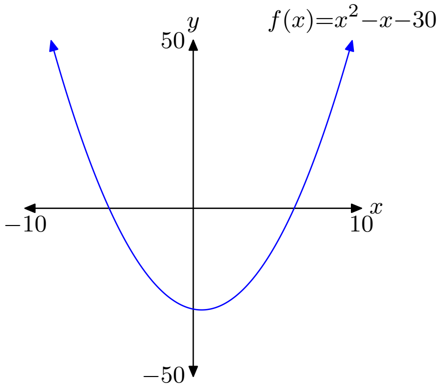


**7.**

| $x$ | $f(x) = x^2/2 - 6$ | $(x, f(x))$ |
|---|---|---|
| $-4$ | $2$ | $(-4, 2)$ |
| $-2$ | $-4$ | $(-2, -4)$ |
| $0$ | $-6$ | $(0, -6)$ |
| $2$ | $-4$ | $(2, -4)$ |
| $4$ | $2$ | $(4, 2)$ |

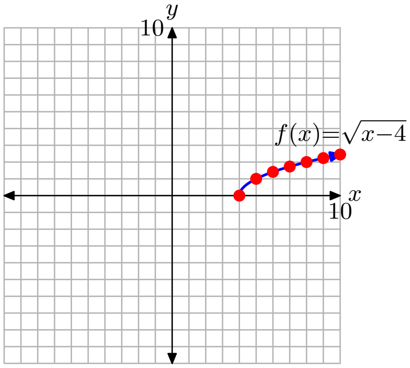


**11.**

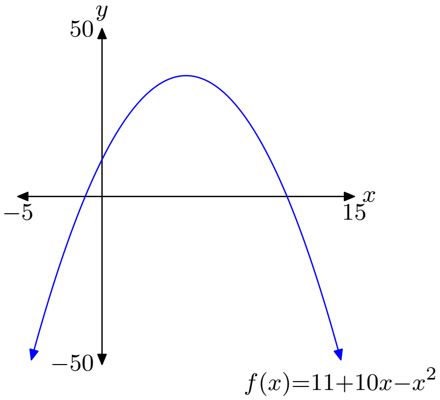


**9.**

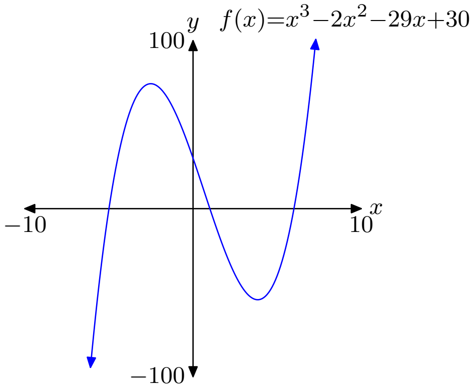
| $x$ | $f(x) = \sqrt{x-4}$ | $(x, f(x))$ |
|---|---|---|
| 4 | 0 | $(4, 0)$ |
| 5 | 1 | $(5, 1)$ |
| 6 | 1.4142 | $(6, 1.4142)$ |
| 7 | 1.7321 | $(7, 1.7321)$ |
| 8 | 2 | $(8, 2)$ |
| 9 | 2.2361 | $(9, 2.2361)$ |
| 10 | 2.4495 | $(10, 2.4495)$ |



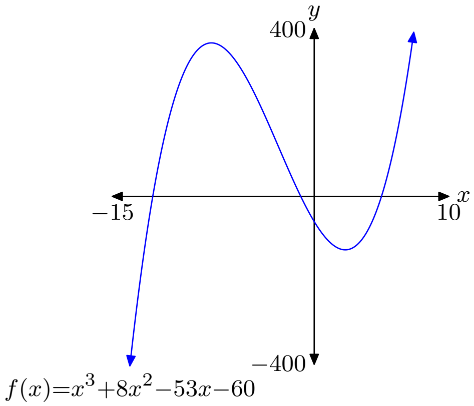
**13.**



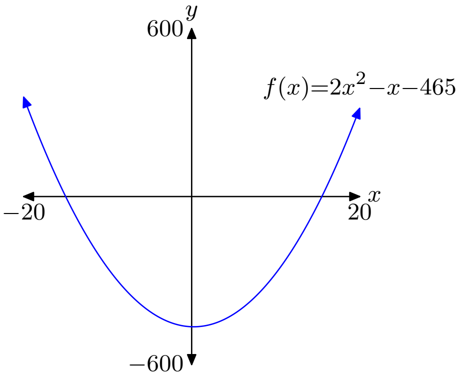
**15.**



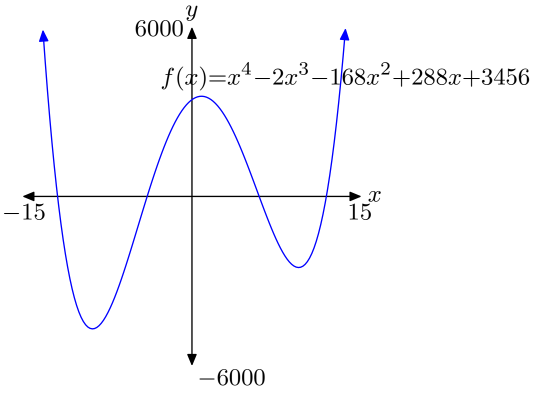
**17.**


y
400
x
−15
10
−400
$f(x)=x^3+8x^2-53x-60$

**19.**


y
600
$f(x)=2x^2-x-465$
x
−20
20
−600

**21.**


y
6000
$f(x)=x^4-2x^3-168x^2+288x+3456$
x
−15
15
−6000

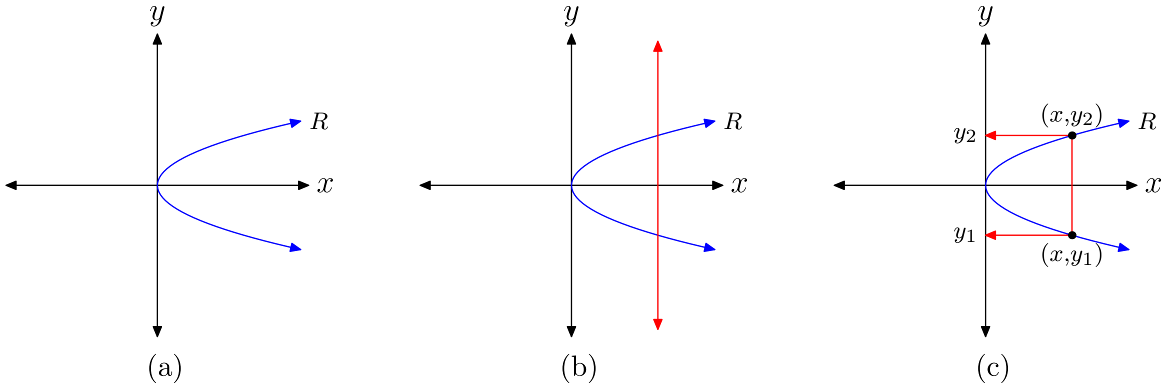
# 2.3 Interpreting the Graph of a Function

In the previous section, we began with a function and then drew the graph of the given function. In this section, we will start with the graph of a function, then make a number of interpretations based on the given graph: function evaluations, the domain and range of the function, and solving equations and inequalities.

## *The Vertical Line Test*

Consider the graph of the relation $R$ shown in **Figure 1**(a). Recall that we earlier defined a relation as a set of ordered pairs. Surely, the graph shown in **Figure 1**(a) is a set of ordered pairs. Indeed, it is an infinite set of ordered pairs, so many that the graph is a solid curve.

In **Figure 1**(b), note that we can draw a vertical line that cuts the graph more than once. In **Figure 1**(b), we've drawn a vertical line that cuts the graph in two places, once at $(x, y_1)$, then again at $(x, y_2)$, as shown in **Figure 1**(c). This means that the domain object $x$ is paired with two different range objects, namely $y_1$ and $y_2$, so relation $R$ is **not** a function.



(a) (b) (c)

**Figure 1.** Explaining the vertical line test for functions.

Recall the definition of a function.

**Definition 1.** *A relation is a* **function** *if and only if each object in its domain is paired with one and only one object in its range.*

Consider the mapping diagram in **Figure 2**, where we've used arrows to indicate the ordered pairs $(x, y_1)$ and $(x, y_2)$ in **Figure 1**(c). Note that $x$, an object in the domain of $R$, is mapped to **two** objects in the range of $R$, namely $y_1$ and $y_2$. Hence, the relation $R$ is **not** a function.

---

[10] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

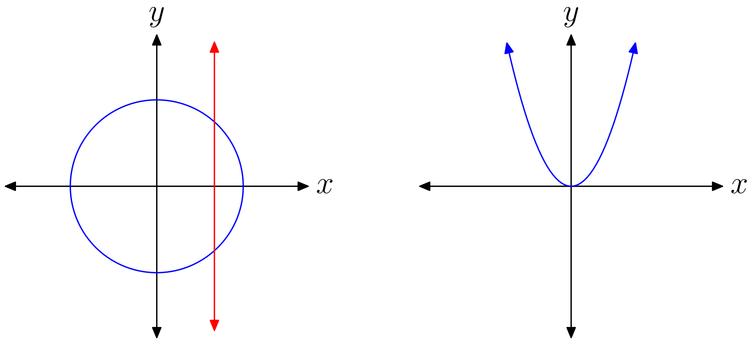
$$\begin{tikzcd} & R & \\ x \arrow[rr] \arrow[rrd] & & y_1 \\ & & y_2 \end{tikzcd}$$

**Figure 2.** A mapping diagram representing the points $(x, y_1)$ and $(x, y_2)$ in **Figure 1**(c).

This discussion leads to the following result, called the *vertical line test* for functions.

**The Vertical Line Test.** If any vertical line cuts the graph of a relation more than once, then the relation is **NOT** a function.

Hence, the circle pictured in **Figure 3**(a) is a relation, but it is **not** the graph of a function. It is possible to cut the graph of the circle more than once with a vertical line, as shown in **Figure 3**(a). On the other hand, the parabola shown in **Figure 3**(b) **is** the graph of a function, because no vertical line will cut the graph more than once.



(a) (b)

**Figure 3.** Use the vertical line test to determine if the graph is the graph of a function.

## *Reading the Graph for Function Values*

We know that the graph of $f$ pictured in **Figure 4** is the graph of a function. We know this because no vertical line will cut the graph of $f$ more than once.

We earlier defined the graph of $f$ as the set of all ordered pairs $(x, f(x))$, so that $x$ is in the domain of $f$. Consequently, if we select a point $P$ on the graph of $f$, as in **Figure 4**(a), we label the point $P(x, f(x))$. However, we can also label this point as $P(x, y)$, as shown in **Figure 4**(b). This leads to a new interpretation of $f(x)$ as the $y$-value of the point $P$. That is, $f(x)$ is the $y$-value that is paired with $x$.[11]

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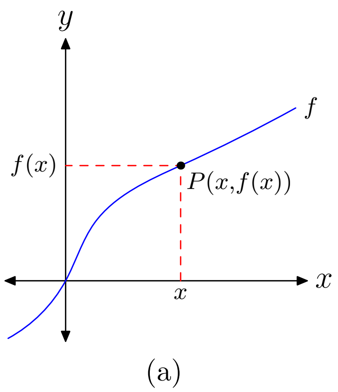
[11] Of course, if the axes were labeled $A$ and $t$, then there would be a similar interpretation based on the variables $A$ and $t$.



(a) (b)

**Figure 4.** Reading the graph of a function.

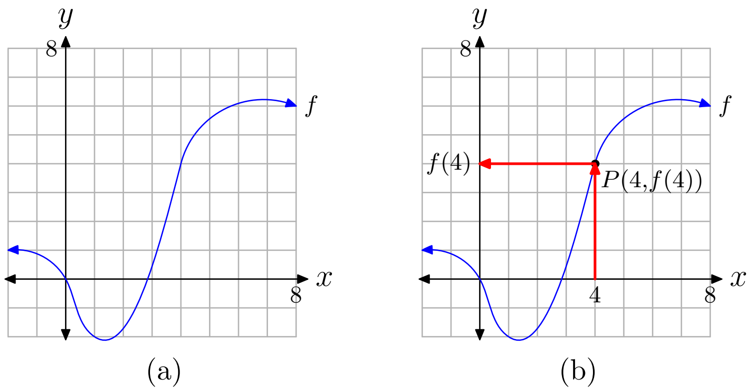
**Definition 2.** *$f(x)$ is the $y$-value that is paired with $x$.*

Two more comments are in order. In **Figure 4**(a), we select a point $P$ on the graph of $f$.

1. To find the $x$-value of the point $P$, we must project the point $P$ onto the $x$-axis.
2. To find $f(x)$, the value of $y$ that is paired with $x$, we must project the point $P$ onto the $y$-axis.

Let's look at an example.

▶ **Example 3.** *Given the graph of $f$ in* **Figure 5***(a), find $f(4)$.*



(a) (b)

**Figure 5.** Finding the value of $f(4)$.

First, note that the graph of $f$ represents a function. No vertical line will cut the graph of $f$ more than once.

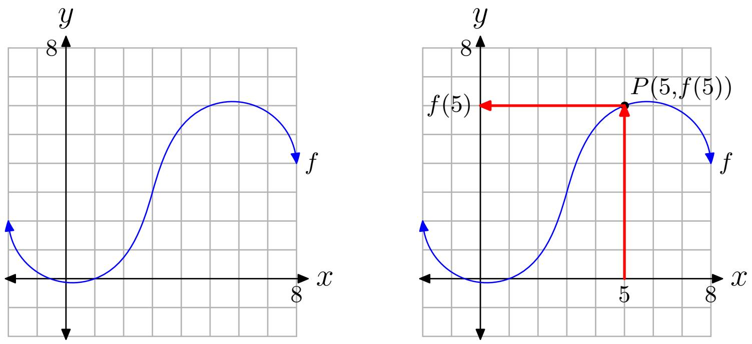
Because $f(4)$ represents the $y$-value that is paired with an $x$-value of 4, we first locate 4 on the $x$-axis, as shown in **Figure 5**(b). We then draw a vertical arrow until we intercept the graph of $f$ at the point $P(4, f(4))$. Finally, we draw a horizontal arrow from the point $P$ until we intercept the $y$-axis. The projection of the point $P$ onto the $y$-axis is the value of $f(4)$.

Because we have a grid that shows a scale on each axis, we can approximate the value of $f(4)$. It would appear that the $y$-value of point $P$ is approximately 4. Thus, $f(4) \approx 4$.

◆

Let's look at another example.

▶ **Example 4.** *Given the graph of $f$ in* **Figure 6***(a), find $f(5)$.*



(a) (b)

**Figure 6.** Finding the value of $f(5)$.

First, note that the graph of $f$ represents a function. No vertical line will cut the graph of $f$ more than once.

Because $f(5)$ represents the $y$-value that is paired with an $x$-value of 5, we first locate 5 on the $x$-axis, as shown in **Figure 6**(b). We then draw a vertical arrow until we intercept the graph of $f$ at the point $P(5, f(5))$. Finally, we draw a horizontal arrow from the point $P$ until we intercept the $y$-axis. The projection of the point $P$ onto the $y$-axis is the value of $f(5)$.

Because we have a grid that shows a scale on each axis, we can approximate the value of $f(5)$. It would appear that the $y$-value of point $P$ is approximately 6. Thus, $f(5) \approx 6$.

◆

Let's reverse the interpretation in another example.

▶ **Example 5.** *Given the graph of $f$ in* **Figure 7***(a), for what value of $x$ does $f(x) = -4$?*

Again, the graph in **Figure 7** passes the vertical line test and represents the graph of a function.

This time, in the equation $f(x) = -4$, we're given a $y$-value equal to $-4$. Consequently, we must reverse the process used in **Example 3** and **Example 4**. We first locate the $y$-value $-4$ on the $y$-axis, then draw a horizontal arrow until we intercept

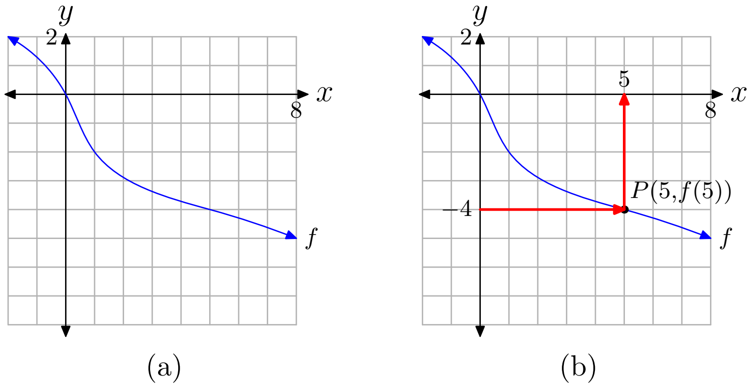


**Figure 7.** Finding $x$ so that $f(x) = -4$.

the graph of $f$ at $P$, as shown in **Figure 7**(b). Finally, we draw a vertical arrow from the point $P$ until we intercept the $x$-axis. The projection of the point $P$ onto the $x$-axis is the solution of $f(x) = -4$.

Because we have a grid that shows a scale on each axis, we can approximate the $x$-value of the point $P$. It seems that $x \approx 5$. Thus, we label the point $P(5, f(5))$, and the solution of $f(x) = -4$ is approximately $x \approx 5$.

This solution can easily be checked by computing $f(5)$. Simply start with 5 on the $x$-axis, then reverse the order of the arrows shown in **Figure 7**(b). You should wind up at $-4$ on the $y$-axis, demonstrating that $f(5) = -4$.

◆

## *The Domain and Range of a Function*

We can use the graph of a function to determine its domain and range. For example, consider the graph of the function shown in **Figure 8**(a).

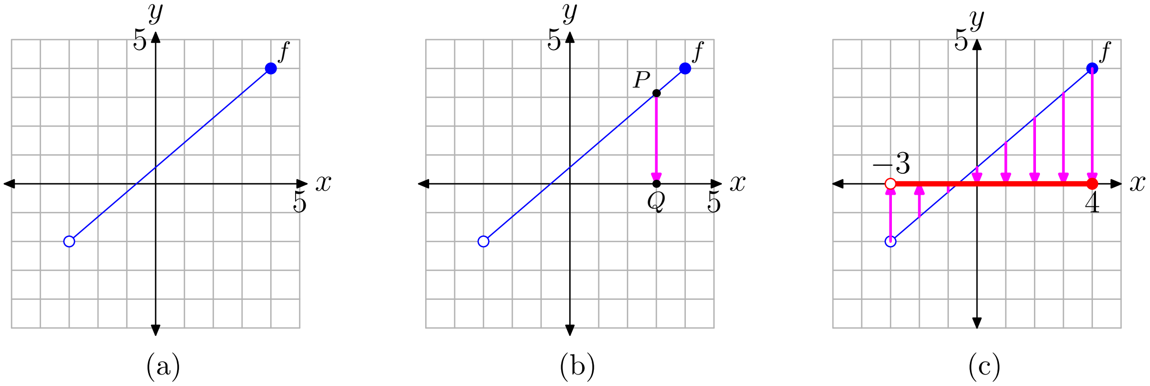


**Figure 8.** Determining the domain of a function from its graph.

Note that no vertical line will cut the graph of $f$ more than once, so the graph of $f$ represents a function.

To determine the domain, we must collect the $x$-values (first coordinates) of every point on the graph of $f$. In **Figure 8**(b), we've selected a point $P$ on the graph of $f$, which we then project onto the $x$-axis. The image of this projection is the point $Q$, and the $x$-value of the point $Q$ is an element in the domain of $f$.

Think of the projection shown in **Figure 8**(b) in the following manner. Imagine a light source above the point $P$. The point $P$ blocks out the light and its shadow falls onto the $x$-axis at the point $Q$. That is, think of point $Q$ as the "shadow" that the point $P$ produces when it is projected vertically onto the $x$-axis.

Now, to find the domain of the function $f$, we must project each point on the graph of $f$ onto the $x$-axis. Here's the question: if we project each point on the graph of $f$ onto the $x$-axis, what part of the $x$-axis will "lie in shadow" when the process is complete? The answer is shown in **Figure 8**(c).

In **Figure 8**(c), note that the "shadow" created by projecting each point on the graph of $f$ onto the $x$-axis is shaded in red (a thicker line if you are viewing this in black and white). This collection of $x$-values is the domain of the function $f$. There are three critical points that we need to make about the "shadow" on the $x$-axis in **Figure 8**(c).

1. All points lying between $x = -3$ and $x = 4$ have been shaded on the $x$-axis in red.
2. The left endpoint of the graph of $f$ is an open circle. This indicates that there is no point plotted at this endpoint. Consequently, there is no point to project onto the $x$-axis, and this explains the open circle at the left end of our "shadow" on the $x$-axis.
3. On the other hand, the right endpoint of the graph of $f$ is a filled endpoint. This indicates that this is a plotted point and part of the graph of $f$. Consequently, when this point is projected onto the $x$-axis, a shadow falls at $x = 4$. This explains the filled endpoint at the right end of our "shadow" on the $x$-axis.

We can describe the $x$-values of the "shadow" on the $x$-axis using set-builder notation.

$$\text{Domain of } f = \{x : \, -3 < x \leq 4\}.$$

Note that we don't include $-3$ in this description because the left end of the shadow on the $x$-axis is an empty circle. Note that we do include 4 in this description because the right end of the shadow on the $x$-axis is a filled circle.

We can also describe the $x$-values of the "shadow" on the $x$-axis using interval notation.

$$\text{Domain of } f = (-3, 4]$$

We remind our readers that the parenthesis on the left means that we are **not** including $-3$, while the bracket on the right means that we **are** including 4.

To find the range of the function, picture again the graph of $f$ shown in **Figure 9**(a). Proceed in a similar manner, only this time project points on the graph of $f$ onto the $y$-axis, as shown in **Figures 9**(b) and (c).

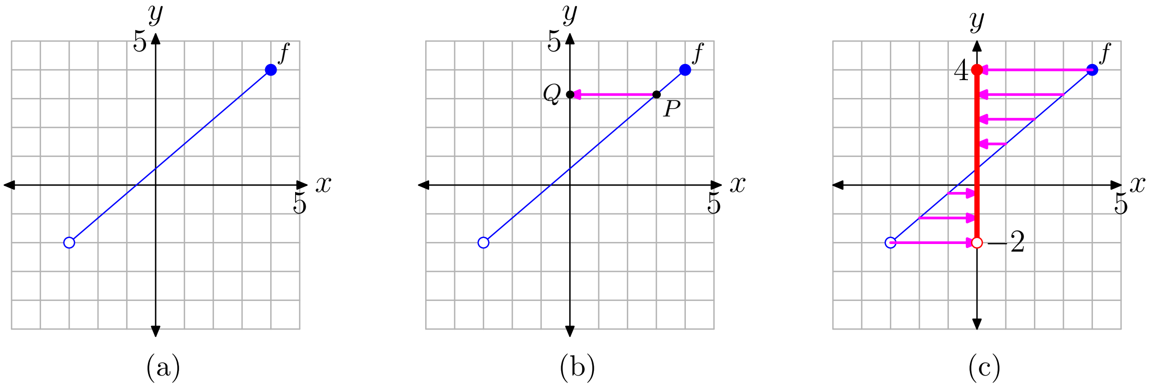


**Figure 9.** Determining the range of a function from its graph.

Note which part of the $y$-axis "lies in shadow" once we've projected all points on the graph of $f$ onto the $y$-axis.

1. All points lying between $y = -2$ and $y = 4$ have been shaded on the $y$-axis in red (a thicker line style if you are viewing this in black and white).
2. The left endpoint of the graph of $f$ is an empty circle, so there is no point to project onto the $y$-axis. Consequently, there is no "shadow" at $y = -2$ on the $y$-axis and the point is left unshaded (an empty circle).
3. The right endpoint of the graph of $f$ is a filled circle, so there is a "shadow" at $y = 4$ on the $y$-axis and this point is shaded (a filled circle).

We can now easily describe the range in both set-builder and interval notation.

$$\text{Range of } f = (-2, 4] = \{y : \ -2 < y \leq 4\}$$

◆

Let's look at another example.

▶ **Example 6.** *Use set-builder and interval notation to describe the domain and range of the function represented by the graph in* **Figure 10***(a).*

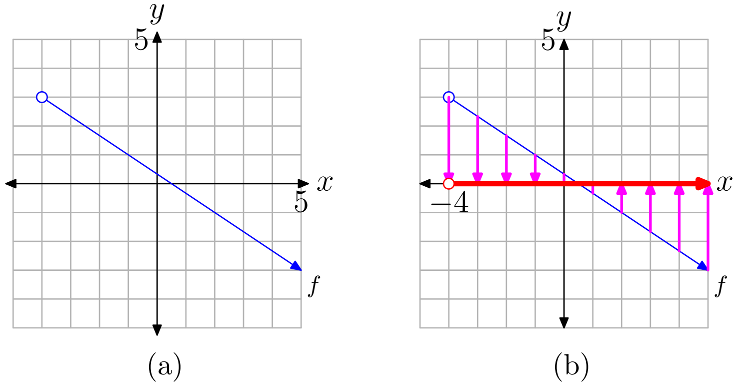


**Figure 10.** Determining the domain from the graph of $f$.

To determine the domain of $f$, project each point on the graph of $f$ onto the $x$-axis. This projection is indicated by the "shadow" on the $x$-axis in **Figure 10**(b). Two important points need to be made about this "shadow" or projection.

1. The left endpoint of the graph of $f$ is empty (indicated by the open circle), so it has no projection onto the $x$-axis. This is indicated by an open circle at the left end (at $x = -4$) of the "shadow" or projection on the $x$-axis.
2. The arrowhead on the right end of the graph of $f$ indicates that the graph of $f$ continues downward and to the right indefinitely. Consequently, the projection onto the $x$-axis is a shadow that moves indefinitely to the right. This is indicated by an arrowhead at the right end of the "shadow" or projection on the $x$-axis.

Consequently, the domain of $f$ is the collection of $x$-values represented by the "shadow" or projection onto the $x$-axis. Note that all $x$-values to the right of $x = -4$ are shaded on the $x$-axis. Consequently,

$$\text{Domain of } f = (-4, \infty) = \{x : x > -4\}.$$

To find the range, we must project each point on the graph of $f$ (redrawn in **Figure 11**(a)) onto the $y$-axis. The projection is indicated by a "shadow" or projection on the $y$-axis, as seen in **Figure 11**(b). Two important points need to be made about this "shadow" or projection.

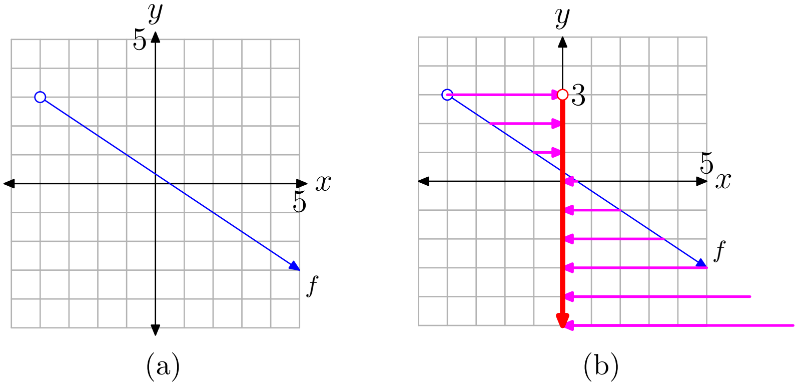


**Figure 11.** Determining the range from the graph of $f$.

1. The left endpoint of the graph of $f$ is empty (indicated by an open circle), so it has no projection onto the $y$-axis. This is indicated by an open circle at the top end (at $y = 3$) of the "shadow" on the $y$-axis.
2. The arrowhead on the right end of the graph of $f$ indicates that the graph of $f$ continues downward and to the right indefinitely. Consequently, the projection of the graph of $f$ onto the $y$-axis is a shadow that moves indefinitely downward. In **Figure 11**(b), note how projections of points on the graph of $f$ not visible in the viewing window come in from the lower right corner and cast "shadows" on the $y$-axis.

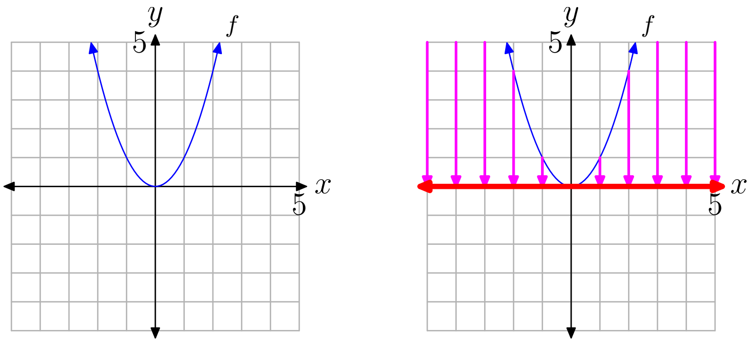
Consequently, the range of $f$ is the collection of $y$-values shaded on the $y$-axis of the coordinate system shown in **Figure 11**(b). Note that all $y$-values lower than $y = 3$ are shaded on the $y$-axis. Thus, the range of $f$ is

$$\text{Range of } f = (-\infty, 3) = \{y : y < 3\}.$$

◆

Let's look at another example.

▶ **Example 7.** *Use set-builder and interval notation to describe the domain and range of the function represented by the graph in* **Figure 12***(a).*



(a) (b)

**Figure 12.** Determining the domain from the graph of $f$.

To determine the domain of $f$, we must project all points on the graph of $f$ onto the $x$-axis. This projection is indicated by the red "shadow" (or thicker line style if you are viewing this in black and white) shown on the $x$-axis in **Figure 12**(b). Two important points need to be made about this "shadow" or projection.

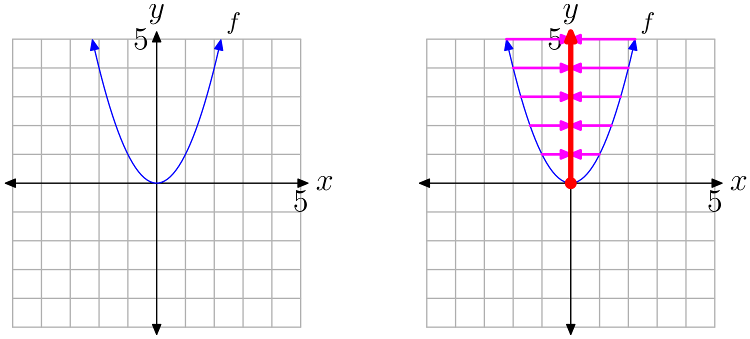
1. The arrow at the end of the left half of the graph of $f$ in **Figure 12**(a) indicates that this half of the graph of $f$ opens indefinitely to the left and upward. Consequently, when the points on the left half of the graph of $f$ are projected onto the $x$-axis, the "shadow" or projection extends indefinitely to the left. Note how points on the graph that fall outside the viewing window come in from the upper left corner and cast "shadows" on the $x$-axis.

2. The arrow at the end of the right half of the graph of $f$ in **Figure 12**(a) indicates that this half of the graph of $f$ opens indefinitely to the right and upward. Consequently, when the points on this half of the graph of $f$ are projected onto the $x$-axis, the "shadow" or projection extends indefinitely to the right.

Consequently, the entire $x$-axis lies in "shadow," making the domain of $f$ to be

$$\text{Domain of } f = (-\infty, \infty) = \{x : x \in \mathbb{R}\}.$$

To determine the range of $f$, we must project all points on the graph of $f$ onto the $y$-axis. This projection is indicated by the red "shadow" (or thicker line if you are viewing this in black and white) shown on the $y$-axis in **Figure 13**(b). Two important points need to made about this "shadow" or projection.



(a) (b)

**Figure 13.** Determining the range from the graph of $f$.

1. The graph of $f$ passes through the origin (the point $(0, 0)$). This is the lowest point on the graph and hence its shadow is the endpoint on the low end of the shaded region on the $y$-axis.
2. The arrows at the end of each half of the graph of $f$ indicate that the graph opens upward indefinitely. Hence, when points on the graph of $f$ are projected onto the $y$-axis, the "shadow" or projection extends upward indefinitely. This is indicated by an arrow on the upper end of the "shadow" on the $y$-axis.

Consequently, all points on the $y$-axis above and including the point at the origin "lie in shadow." Thus, the range of $f$ is

$$\text{Range of } f = [0, \infty) = \{y : y \geq 0\}.$$

◆

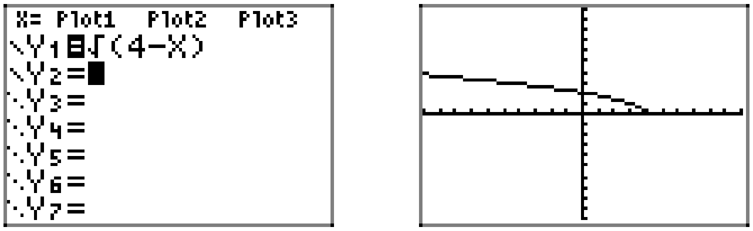
## *Using a Graphing Calculator to Determine Domain and Range*

We've learned how to find the domain and range of a function by looking at its graph. Therefore, if we define a function by means of an expression, such as $f(x) = \sqrt{4 - x}$, then we should be able to capture the domain and range of $f$ from its graph, provided, of course, that we can draw the graph of $f$. We'll find the graphing calculator will be a handy tool for this exercise.

▶ **Example 8.** *Use set-builder and interval notation to describe the domain and range of the function defined by the rule*

$$f(x) = \sqrt{4 - x}. \tag{9}$$

Load the expression defining $f$ into the `Y=` menu, as shown in **Figure 14**(a). Select `6:ZStandard` from the `ZOOM` menu to produce the graph of $f$ shown in **Figure 14**(b).

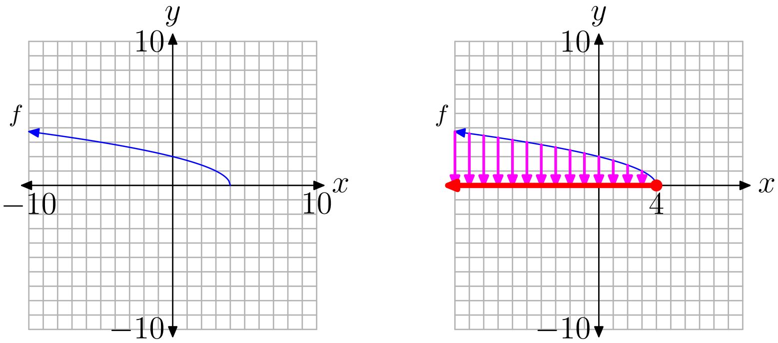


```
X= Plot1  Plot2  Plot3
\Y1=√(4-X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

(a) (b)

**Figure 14.** Sketching the graph of $f(x) = \sqrt{4-x}$.

Copy the image in **Figure 14**(b) onto a sheet of graph paper. Label and scale each axis with the `WINDOW` parameters xmin, xmax, ymin, and ymax, as shown in **Figure 15**(a).



(a) (b)

**Figure 15.** Capturing the domain of $f(x) = \sqrt{4-x}$ from its graph.

Next, project each point on the graph of $f$ onto the $x$-axis, as shown in **Figure 15**(b). Note that we've made two assumptions about the graph of $f$.

1. At the left end of the graph in **Figures 14**(b) and **15**(b), we assume that the graph of $f$ continues upward and to the left indefinitely. Hence, the "shadow" or projection onto the $x$-axis will move indefinitely to the left. This is indicated by attaching an arrowhead to the left-hand end of the region that "lies in shadow" on the $x$-axis, as shown in **Figure 15**(b).
2. We also assume that the right end of the graph ends at the point $(4, 0)$. This accounts for the "filled dot" when this point on the graph of $f$ is projected onto the $x$-axis.

Note that the "shadow" or projection onto the $x$ axis in **Figure 15**(b) includes all values of $x$ less than or equal to 4. Thus, the domain of $f$ is

$$\text{Domain of } f = (-\infty, 4] = \{x : x \leq 4\}.$$

We can intuit this result by considering the expression that defines $f$. That is, consider the rule or definition

$$f(x) = \sqrt{4-x}.$$

Recall that we earlier defined the domain of $f$ as the set of "permissible" $x$-values. In this case, it is impossible to take the square root of a negative number, so we must be careful selecting the $x$-values we use in this rule. Note that $x = 4$ is allowable, as

$$f(0) = \sqrt{4-4} = \sqrt{0} = 0.$$

However, numbers larger than 4 cannot be used in this rule. For example, consider what happens when we attempt to use $x = 5$.

$$f(x) = \sqrt{4-5} = \sqrt{-1}$$

This result is not a real number, so 5 is not in the domain of $f$.

On the other hand, if we try $x$-values that are smaller than 4, such as $x = 3$,

$$f(3) = \sqrt{4-3} = \sqrt{1} = 1.$$

We'll leave it to our readers to test other values of $x$ that are less than 4. They will also produce real answers when they are input into the rule $f(x) = \sqrt{4-x}$. Note that this also verifies our earlier conjecture that the "shadow" or projection shown in **Figure 15**(b) continues indefinitely to the left.

Instead of "guessing and checking," we can speed up the analysis of the domain of $f(x) = \sqrt{4-x}$ by noting that the expression under the radical must not be a negative number. Hence, $4 - x$ must either be greater than or equal to zero. This argument produces an inequality that is easily solved for $x$.

$$\begin{aligned} 4 - x &\geq 0 \\ -x &\geq -4 \\ x &\leq 4 \end{aligned}$$

This last result verifies that the domain of $f$ is all values of $x$ that are less than or equal to 4, which is in complete agreement with the "shadow" or projection onto the $x$-axis shown in **Figure 15**(b).

To determine the range of $f$, we must project each point on the graph of $f$ onto the $y$-axis, as shown in **Figure 16**(b).

Again, we make two assumptions about the graph of $f$.

1. At the left-end of the graph of $f(x) = \sqrt{4-x}$ in **Figures 14**(b) and **16**(b), we assume that the graph of $f$ continues upward and to the left indefinitely. Thus, when points on the graph of $f$ are projected onto the $y$-axis, there will be projections coming from the upper left from points on the graph of $f$ that are not visible in the viewing window selected in **Figure 14**(b). Hence, the "shadow" or projection on the $y$-axis shown in **Figure 16**(b) continues upward indefinitely. This is indicated with a arrowhead at the upper end of the "shadow" on the $y$-axis in **Figure 16**(b).
2. Again, we assume that the right end of the graph of $f$ ends at the point $(4, 0)$. The projection of this point onto the $y$-axis produces the "filled" endpoint at the origin shown in **Figure 16**(b).

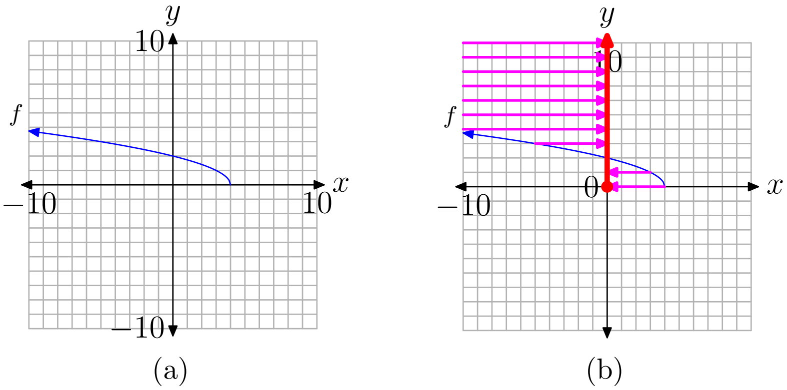


**Figure 16.** Determining the range of $f(x) = \sqrt{4-x}$ from its graph.

Note that the "shadow" or projection onto the $y$-axis in **Figure 16**(b) includes all values of $y$ that are greater than or equal to zero. Hence,

$$\text{Range of } f = [0, \infty) = \{y : y \geq 0\}.$$

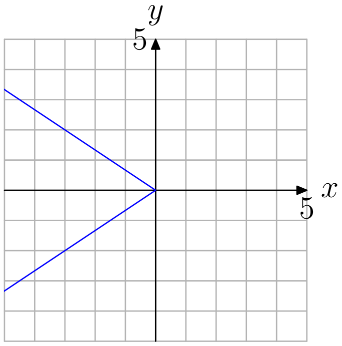
◇

# 2.3 Exercises

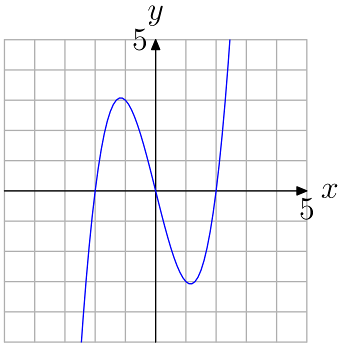
For **Exercises 1-6**, perform each of the following tasks.

i. Make a copy of the graph on a sheet of graph paper and apply the vertical line test.
ii. Write a complete sentence stating whether or not the graph represents a function. Explain the reason for your response.

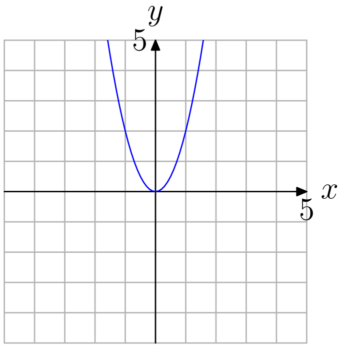
**1.**



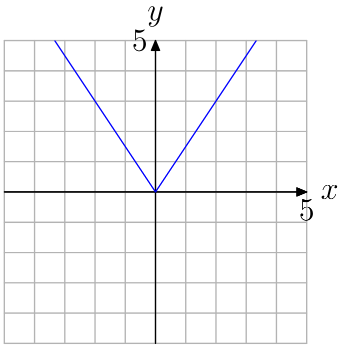
**2.**



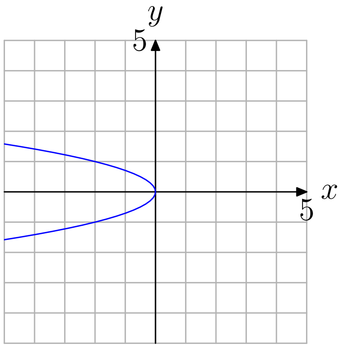
**3.**



**4.**



**5.**



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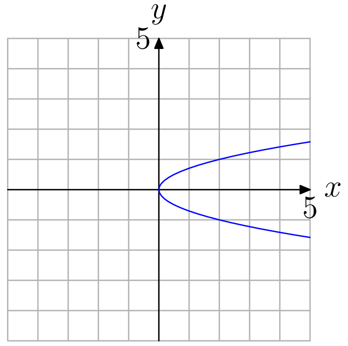
[12] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

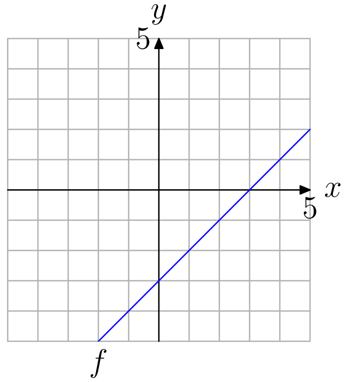
**6.**



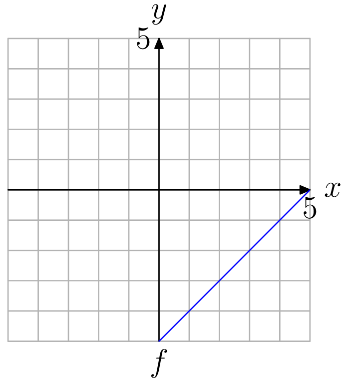
In **Exercises 7-12**, perform each of the following tasks.

i. Make an exact copy of the graph of the function $f$ on a sheet of graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Use the technique of Examples 3 and 4 in the narrative to evaluate the function at the given value. Draw and label the arrows as shown in Figures 4 and 5 in the narrative.

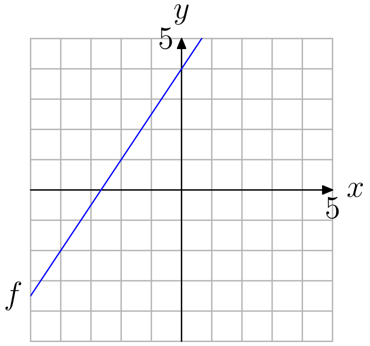
**7.** Use the graph of $f$ to determine $f(2)$.



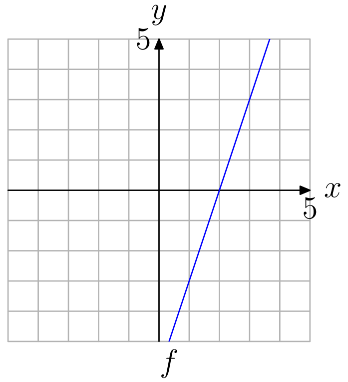
**8.** Use the graph of $f$ to determine $f(3)$.



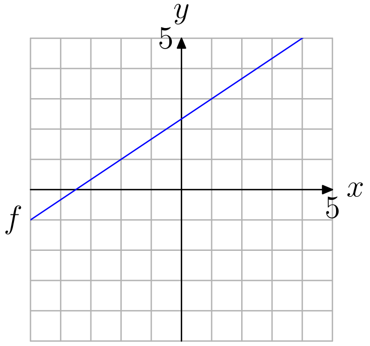
**9.** Use the graph of $f$ to determine $f(-2)$.



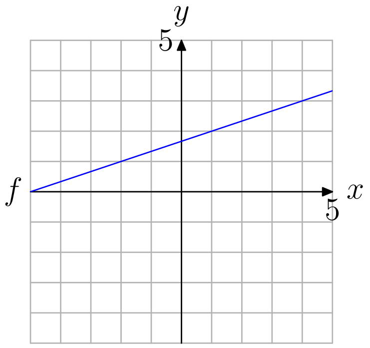
**10.** Use the graph of $f$ to determine $f(1)$.



**11.** Use the graph of $f$ to determine $f(1)$.



**12.** Use the graph of $f$ to determine $f(-2)$.

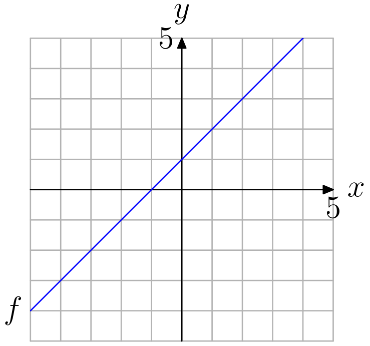


In **Exercises 13-18**, perform each of the following tasks.

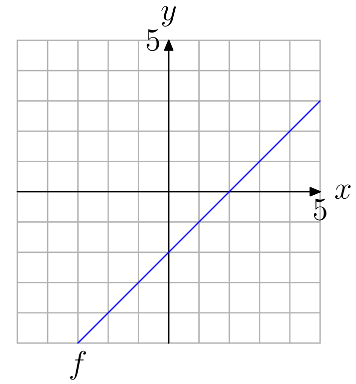
i. Make an exact copy of the graph of the function $f$ on a sheet of graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*

ii. Use the technique of Example 5 in the narrative to find the value of $x$ that maps onto the given value. Draw and label the arrows as shown in Figure 6 in the narrative.

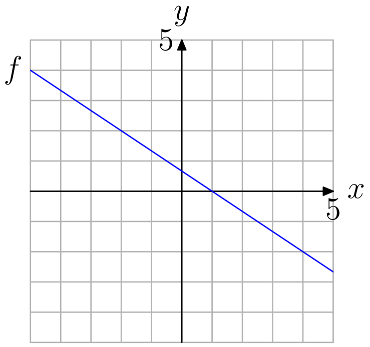
**13.** Use the graph of $f$ to solve the equation $f(x) = -2$.



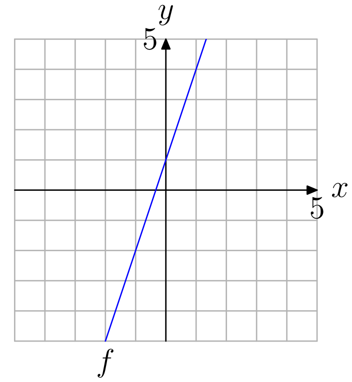
**14.** Use the graph of $f$ to solve the equation $f(x) = 1$.



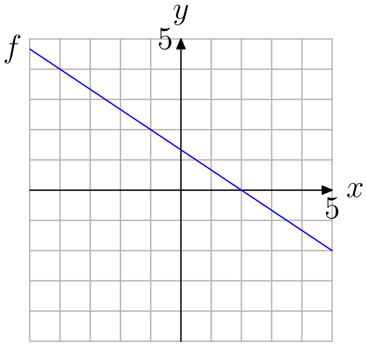
**15.** Use the graph of $f$ to solve the equation $f(x) = 2$.



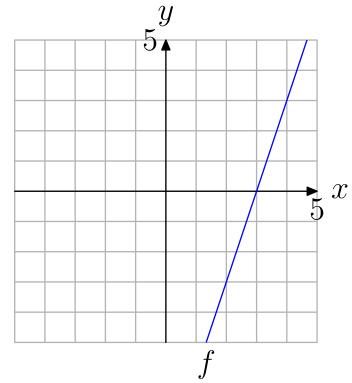
**16.** Use the graph of $f$ to solve the equation $f(x) = -2$.



**17.** Use the graph of $f$ to solve the equation $f(x) = 2$.



**18.** Use the graph of $f$ to solve the equation $f(x) = -3$.



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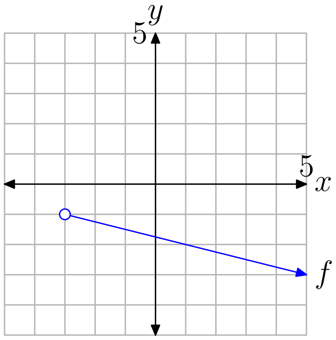
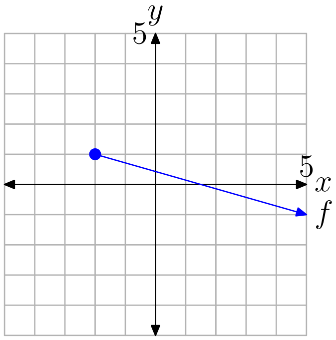
In the **Exercises 19-22**, perform each of the following tasks.

i. Make a copy of the graph of $f$ on a sheet of graph paper. Label and scale each axis.
ii. Using a different colored pen or pencil, project each point on the graph of $f$ onto the $x$-axis. Shade the resulting domain on the $x$-axis.
iii. Use both set-builder and interval notation to describe the domain.

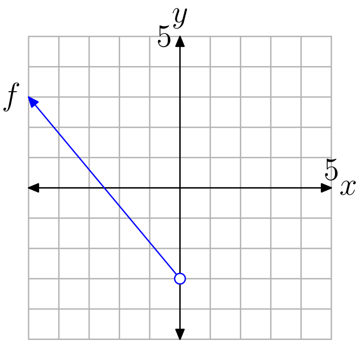
**19.**



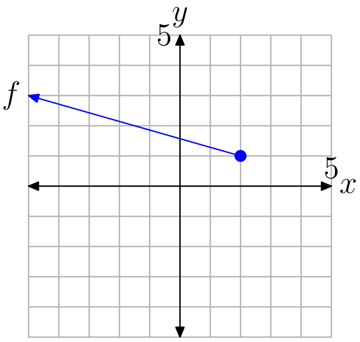
**20.**



**21.**



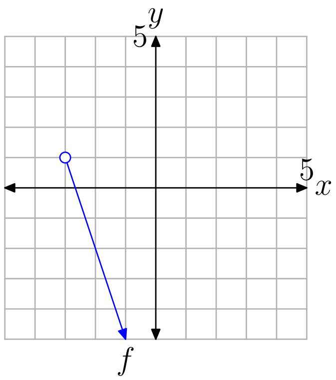
**22.**



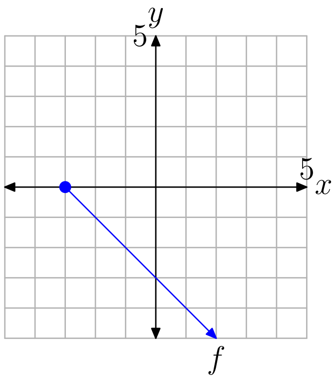
In **Exercises 23-26**, perform each of the following tasks.

i. Make a copy of the graph of $f$ on a sheet of graph paper. Label and scale each axis.
ii. Using a different colored pen or pencil, project each point on the graph of $f$ onto the $y$-axis. Shade the resulting range on the $y$-axis.
iii. Use both set-builder and interval notation to describe the range.

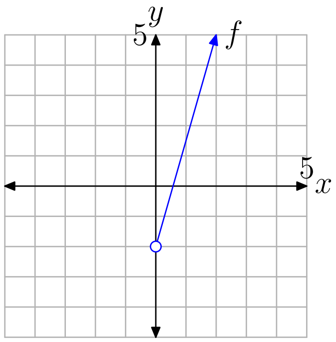
**23.**



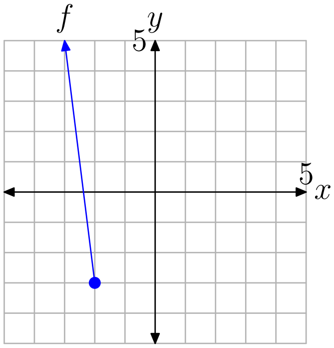
**24.**



**25.**



**26.**



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In **Exercises 27-30**, perform each of the following tasks.

i. Use your graphing calculator to draw the graph of the given function. Make a reasonably accurate copy of the image in your viewing screen on your homework paper. Label and scale each axis with the `WINDOW` parameters xmin, xmax, ymin, and ymax. Label the graph with its equation.

ii. Using a colored pencil, project each point on the graph onto the $x$-axis; i.e., shade the domain on the $x$-axis. Use interval and set-builder notation to describe the domain.

iii. Use a purely algebraic technique, as demonstrated in Example 8 in the narrative, to find the domain. Compare this result with that found in part (ii).

iv. Using a different colored pencil, project each point on the graph onto the $y$-axis; i.e., shade the range on the $y$-axis. Use interval and set-builder notation to describe the range.

**27.** $f(x) = \sqrt{x+5}$.

**28.** $f(x) = \sqrt{5-x}$.

**29.** $f(x) = -\sqrt{4-x}$.

**30.** $f(x) = -\sqrt{x+4}$.

## 2.3 *Answers*

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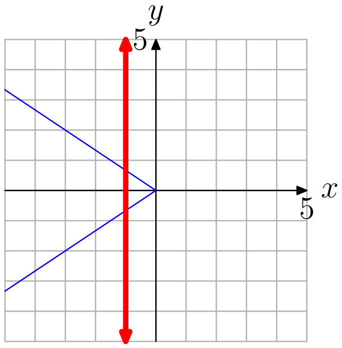
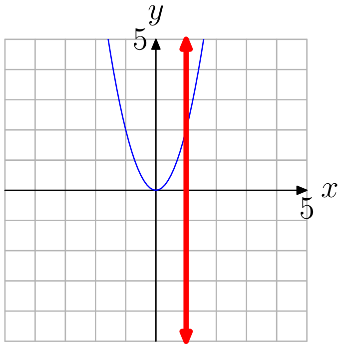
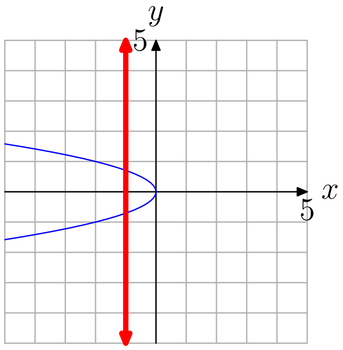
**1.** Note that in the figure below a vertical line cuts the graph more than once. Therefore, the graph does not represent the graph of a function.



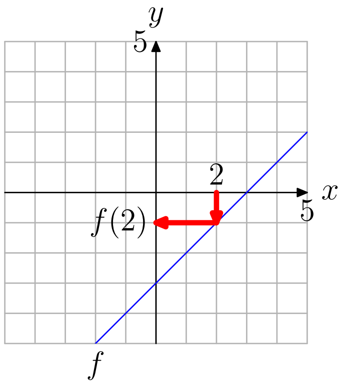
**3.** No vertical line cuts the graph more than once (see figure below). Therefore, the graph represents a function.



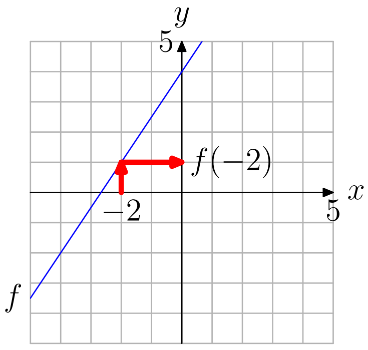
**5.** Note that in the figure below a vertical line cuts the graph more than once. Therefore, the graph does not represent the graph of a function.



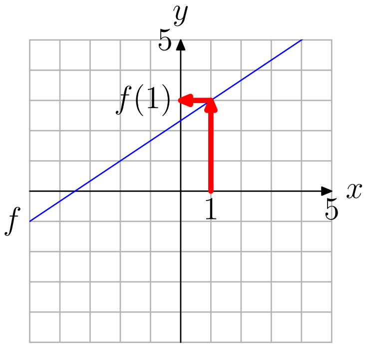
**7.** $f(2) = -1$



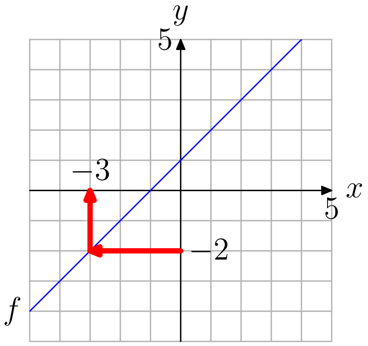
**9.** $f(-2) = 1$



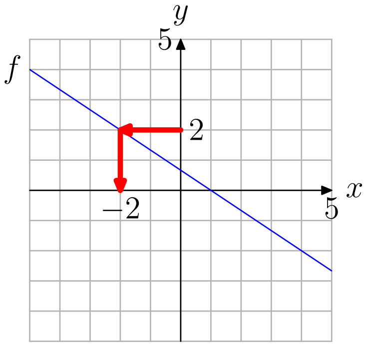
**11.** $f(1) = 3$



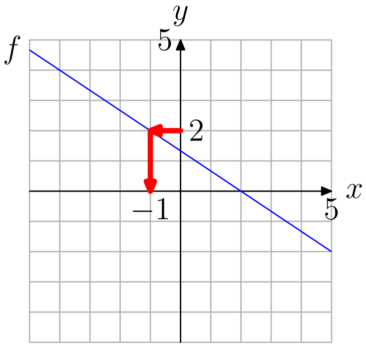
**13.** The solution of $f(x) = -2$ is $x = -3$.



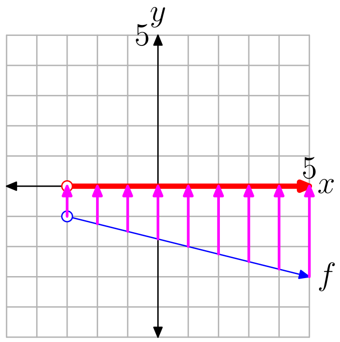
**15.** The solution of $f(x) = 2$ is $x = -2$.



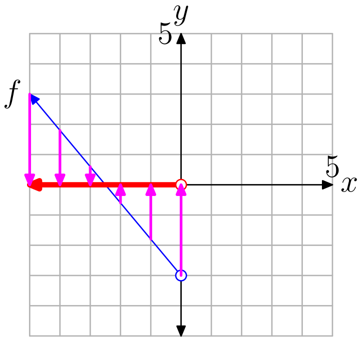
**17.** The solution of $f(x) = 2$ is $x = -1$.



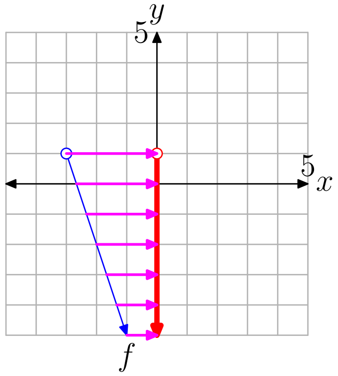
**19.** $\{x : x > -3\} = (-3, \infty)$



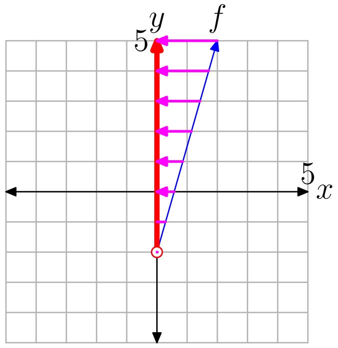
**21.** $\{x : x < 0\} = (-\infty, 0)$



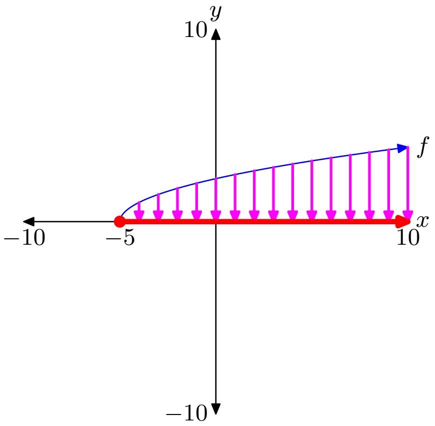
**23.** $\{y : y < 1\} = (-\infty, 1)$



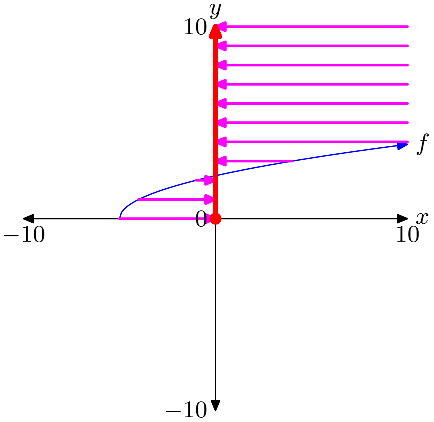
**25.** $\{y : y > -2\} = (-2, \infty)$



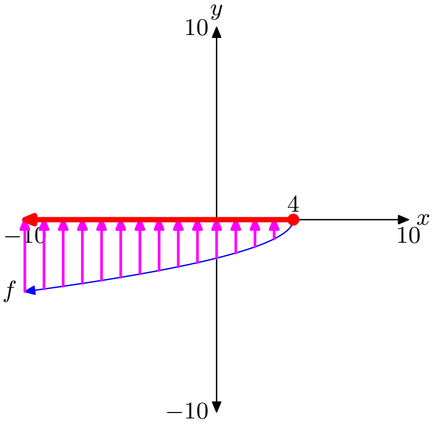
**27.** Domain $= [-5, \infty)$
$= \{x : x \geq -5\}$



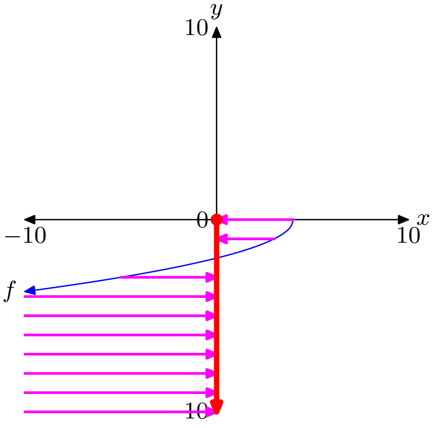
Range $= \{y : y \geq 0\} = [0, \infty)$



**29.** Domain $= (-\infty, 4] = \{x : x \leq 4\}$



Range $= \{y : y \leq 0\} = (-\infty, 0]$



# 2.4 Solving Equations and Inequalities by Graphing

Our emphasis in the chapter has been on functions and the interpretation of their graphs. In this section, we continue in that vein and turn our exploration to the solution of equations and inequalities by graphing. The equations will have the form $f(x) = g(x)$, and the inequalities will have form $f(x) < g(x)$ and/or $f(x) > g(x)$.

You might wonder why we have failed to mention inequalities having the form $f(x) \leq g(x)$ and $f(x) \geq g(x)$. The reason for this omission is the fact that the solution of the inequality $f(x) \leq g(x)$ is simply the union of the solutions of $f(x) = g(x)$ and $f(x) < g(x)$. After all, $\leq$ is pronounced "less than **or** equal." Similar comments are in order for the inequality $f(x) \geq g(x)$.

We will begin by comparing the function values of two functions $f$ and $g$ at various values of $x$ in their domains.

## *Comparing Functions*

Suppose that we evaluate two functions $f$ and $g$ at a particular value of $x$. One of three outcomes is possible. Either

$$f(x) = g(x), \qquad \text{or} \qquad f(x) > g(x), \qquad \text{or} \qquad f(x) < g(x).$$

It's pretty straightforward to compare two function values at a particular value if rules are given for each function.

▶ **Example 1.** *Given $f(x) = x^2$ and $g(x) = 2x+3$, compare the functions at $x = -2$, 0, and 3.*

Simple calculations reveal the relations.

- At $x = -2$,

  $$f(-2) = (-2)^2 = 4 \qquad \text{and} \qquad g(-2) = 2(-2) + 3 = -1,$$

  so clearly, $f(-2) > g(-2)$.
- At $x = 0$,

  $$f(0) = (0)^2 = 0 \qquad \text{and} \qquad g(0) = 2(0) + 3 = 3,$$

  so clearly, $f(0) < g(0)$.
- Finally, at $x = 3$,

  $$f(3) = (3)^2 = 9 \qquad \text{and} \qquad g(3) = 2(3) + 3 = 9,$$

  so clearly, $f(3) = g(3)$.

◆

We can also compare function values at a particular value of $x$ by examining the graphs of the functions. For example, consider the graphs of two functions $f$ and $g$ in **Figure 1**.

---

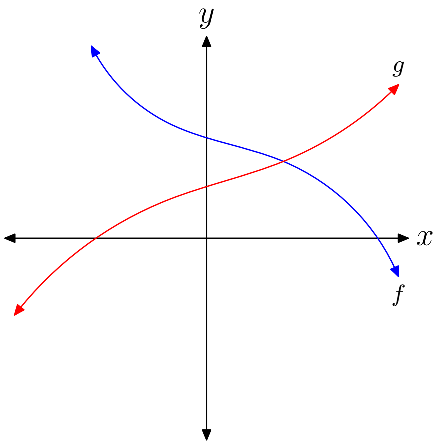
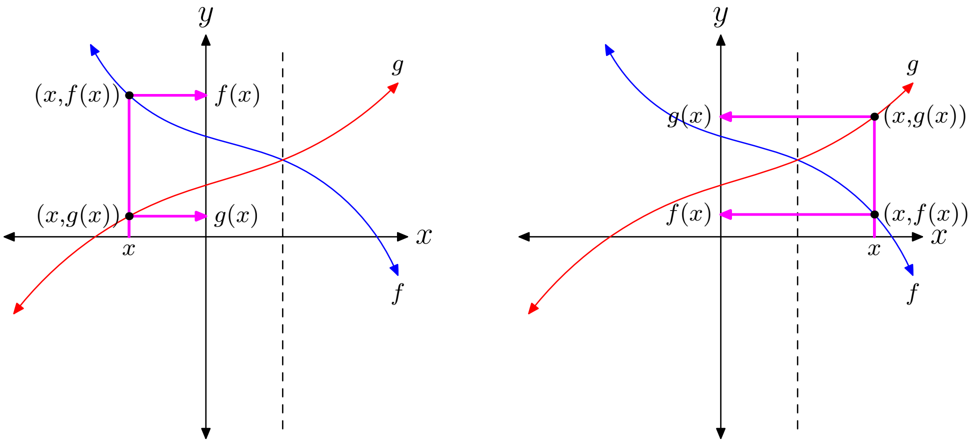


**Figure 1.** Each side of the equation $f(x) = g(x)$ has its own graph.

Next, suppose that we draw a dashed vertical line through the point of intersection of the graphs of $f$ and $g$, then select a value of $x$ that lies to the *left* of the dashed vertical line, as shown in **Figure 2**(a). Because the graph of $f$ lies above the graph of $g$ for all values of $x$ that lie to the left of the dashed vertical line, it will be the case that $f(x) > g(x)$ for all such $x$ (see **Figure 2**(a)).[14]

On the other hand, the graph of $f$ lies *below* the graph of $g$ for all values of $x$ that lie to the *right* of the dashed vertical line. Hence, for all such $x$, it will be the case that $f(x) < g(x)$ (see **Figure 2**(b)).[15]



(a) To the left of the vertical dashed line, the graph of $f$ lies *above* the graph of $g$.

(b) To the right of the vertical dashed line, the graph of $f$ lies *below* the graph of $g$.

**Figure 2.** Comparing $f$ and $g$.

---

[14] When thinking in terms of the vertical direction, "greater than" is equivalent to saying "above."

[15] When thinking in terms of the vertical direction, "less than" is equivalent to saying "below."

Finally, if we select the $x$-value of the point of intersection of the graphs of $f$ and $g$, then for this value of $x$, it is the case that $f(x)$ and $g(x)$ are equal; that is, $f(x) = g(x)$ (see **Figure 3**).

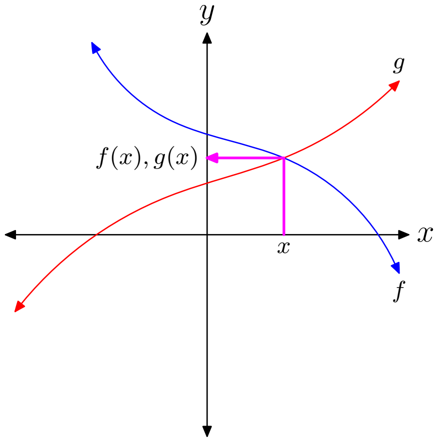


**Figure 3.** The function values $f(x)$ and $g(x)$ are equal where the graphs of $f$ and $g$ intersect.

Let's summarize our findings.

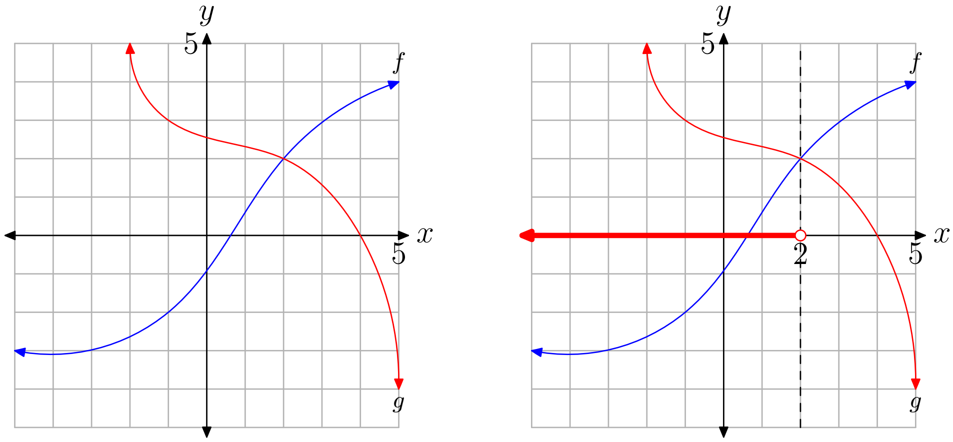
**Summary 2.**

- *The solution of the equation $f(x) = g(x)$ is the set of all $x$ for which the graphs of $f$ and $g$ intersect.*
- *The solution of the inequality $f(x) < g(x)$ is the set of all $x$ for which the graph of $f$ lies* below *the graph of $g$.*
- *The solution of the inequality $f(x) > g(x)$ is the set of all $x$ for which the graph of $f$ lies* above *the graph of $g$.*

Let's look at an example.

▶ **Example 3.** *Given the graphs of $f$ and $g$ in* **Figure 4***(a), use both set-builder and interval notation to describe the solution of the inequality $f(x) < g(x)$. Then find the solutions of the inequality $f(x) > g(x)$ and the equation $f(x) = g(x)$ in a similar fashion.*

To find the solution of $f(x) < g(x)$, we must locate where the graph of $f$ lies below the graph of $g$. We draw a dashed vertical line through the point of intersection of the graphs of $f$ and $g$ (see **Figure 4**(b)), then note that the graph of $f$ lies *below* the graph of $g$ to the *left* of this dashed line. Consequently, the solution of the inequality $f(x) < g(x)$ is the collection of all $x$ that lie to the left of the dashed line. This set is shaded in red (or in a thicker line style if viewing in black and white) on the $x$-axis in **Figure 4**(b).



(a) The graphs of $f$ and $g$. (b) The solution of $f(x) < g(x)$.

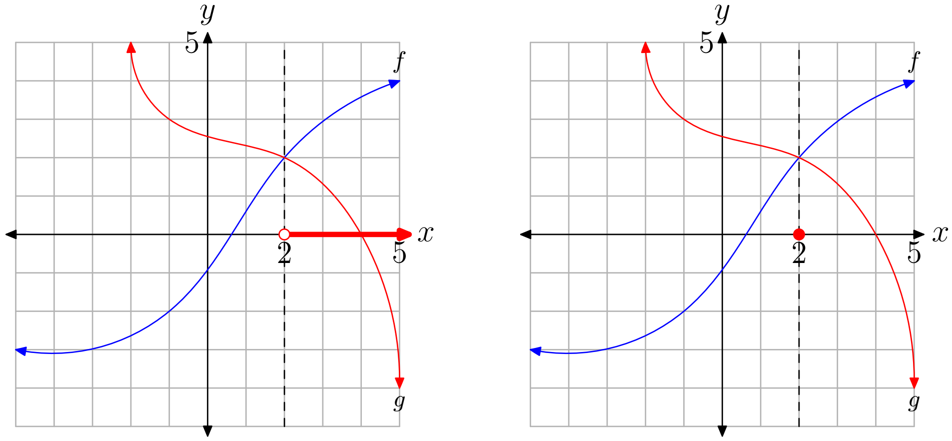
**Figure 4.** Comparing $f$ and $g$.

Note that the shaded points on the $x$-axis have $x$-values less than 2. Hence, the solution of $f(x) < g(x)$ is

$$(-\infty, 2) = \{x : x < 2\}.$$

In like manner, the solution of $f(x) > g(x)$ is found by noting where the graph of $f$ lies *above* the graph of $g$ and shading the corresponding $x$-values on the $x$-axis (see **Figure 5**(a)). The solution of $f(x) > g(x)$ is $(2, \infty)$, or alternatively, $\{x : x > 2\}$.

To find the solution of $f(x) = g(x)$, note where the graph of $f$ intersects the graph of $g$, then shade the $x$-value of this point of intersection on the $x$-axis (see **Figure 5**(b)). Therefore, the solution of $f(x) = g(x)$ is $\{x : x = 2\}$. This is not an interval, so it is not appropriate to describe this solution with interval notation.



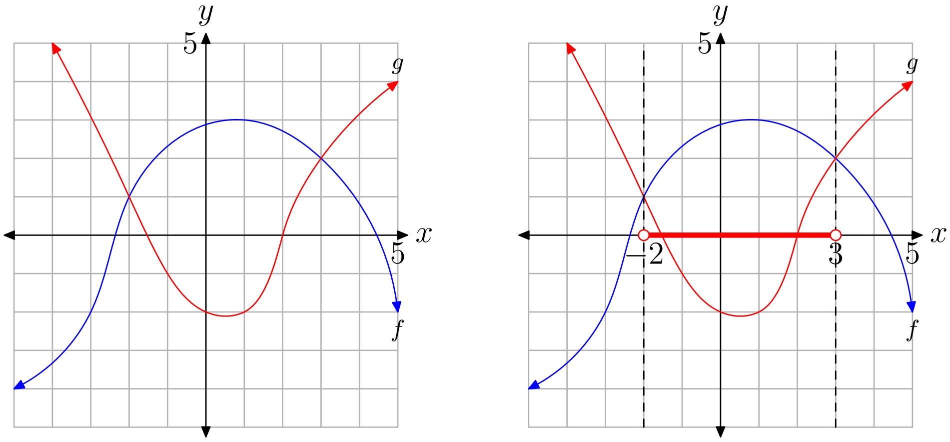
(a) The solution of $f(x) > g(x)$. (b) The solution of $f(x) = g(x)$.

**Figure 5.** Further comparisons.

◆

Let's look at another example.

▶ **Example 4.** *Given the graphs of $f$ and $g$ in* **Figure 6***(a), use both set-builder and interval notation to describe the solution of the inequality $f(x) > g(x)$. Then find the solutions of the inequality $f(x) < g(x)$ and the equation $f(x) = g(x)$ in a similar fashion.*



(a) The graphs of $f$ and $g$ (b) The solution of $f(x) > g(x)$.

**Figure 6.** Comparing $f$ and $g$.

To determine the solution of $f(x) > g(x)$, we must locate where the graph of $f$ lies *above* the graph of $g$. Draw dashed vertical lines through the points of intersection of the graphs of $f$ and $g$ (see **Figure 6**(b)), then note that the graph of $f$ lies above the graph of $g$ between the dashed vertical lines just drawn. Consequently, the solution of the inequality $f(x) > g(x)$ is the collection of all $x$ that lie between the dashed vertical lines. We have shaded this collection on the $x$-axis in red (or with a thicker line style for those viewing in black and white) in **Figure 6**(b).

Note that the points shaded on the $x$-axis in **Figure 6**(b) have $x$-values between $-2$ and 3. Consequently, the solution of $f(x) > g(x)$ is

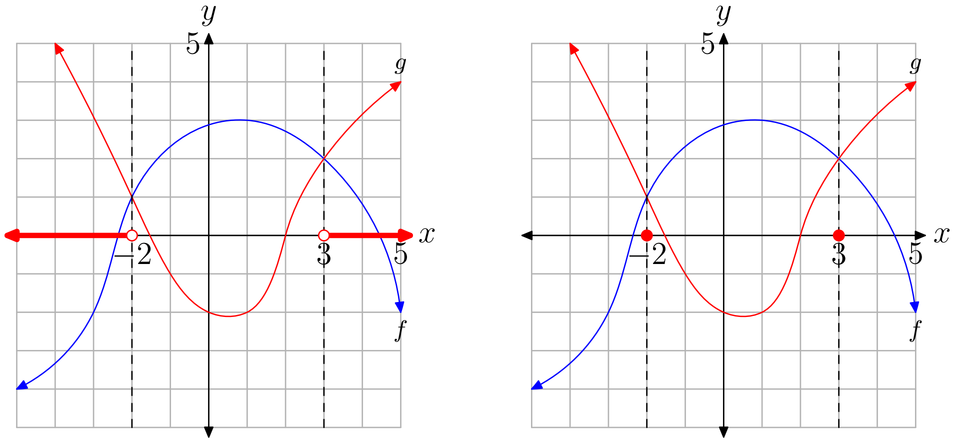
$$(-2, 3) = \{x : -2 < x < 3\}.$$

In like manner, the solution of $f(x) < g(x)$ is found by noting where the graph of $f$ lies *below* the graph of $g$ and shading the corresponding $x$-values on the $x$-axis (see **Figure 7**(a)). Thus, the solution of $f(x) < g(x)$ is

$$(-\infty, -2) \cup (3, \infty) = \{x : x < -2 \text{ or } x > 3\}.$$

To find the solution of $f(x) = g(x)$, note where the graph of $f$ intersects the graph of $g$, and shade the $x$-value of each point of intersection on the $x$-axis (see **Figure 7**(b)). Therefore, the solution of $f(x) = g(x)$ is $\{x : x = -2 \text{ or } x = 3\}$. Because this solution set is not an interval, it would be inappropriate to describe it with interval notation.

◆



(a) The solution of $f(x) < g(x)$. (b) The solution of $f(x) = g(x)$.

**Figure 7.** Further comparisons.

## *Solving Equations and Inequalities with the Graphing Calculator*

We now know that the solution of $f(x) = g(x)$ is the set of all $x$ for which the graphs of $f$ and $g$ intersect. Therefore, the graphing calculator becomes an indispensable tool when solving equations.

▶ **Example 5.** *Use a graphing calculator to solve the equation*

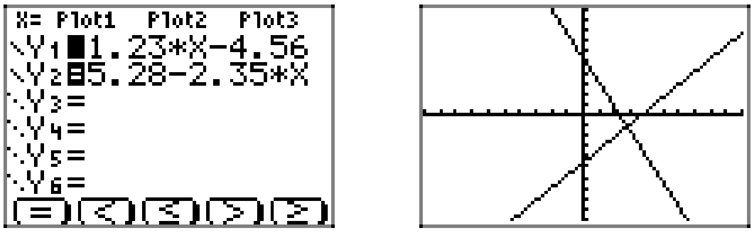
$$1.23x - 4.56 = 5.28 - 2.35x. \tag{6}$$

Note that **equation (6)** has the form $f(x) = g(x)$, where

$$f(x) = 1.23x - 4.56 \qquad \text{and} \qquad g(x) = 5.28 - 2.35x.$$

Thus, our approach will be to draw the graphs of $f$ and $g$, then find the $x$-value of the point of intersection.

First, load $f(x) = 1.23x - 4.56$ into `Y1` and $g(x) = 5.28 - 2.35x$ into `Y2` in the `Y=` menu of your graphing calculator (see **Figure 8**(a)). Select `6:ZStandard` in the `ZOOM` menu to produce the graphs in **Figure 8**(b).



```
\Y1■1.23*X-4.56
\Y2=5.28-2.35*X
\Y3=
\Y4=
\Y5=
\Y6=
```

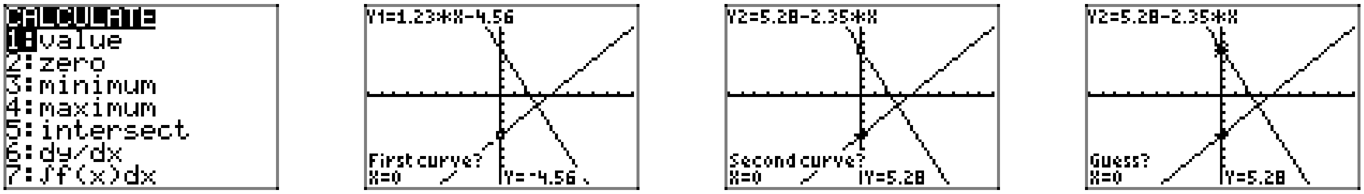
(a) (b)

**Figure 8.** Sketching the graphs of $f(x) = 1.23x - 4.56$ and $g(x) = 5.28 - 2.35x$.

The solution of **equation (6)** is the $x$-value of the point of intersection of the graphs of $f$ and $g$ in **Figure 8**(b). We will use the `intersect` utility in the `CALC` menu on the graphing calculator to determine the coordinates of the point of intersection.

We proceed as follows:

- Select `2nd CALC` (push the `2nd` button, followed by the `TRACE` button), which opens the menu shown in **Figure 9**(a).
- Select `5:intersect`. The calculator responds by placing the cursor on one of the graphs, then asks if you want to use the selected curve. You respond in the affirmative by pressing the `ENTER` key on the calculator.
- The calculator responds by placing the cursor on the second graph, then asks if you want to use the selected curve. Respond in the affirmative by pressing the `ENTER` key.
- The calculator responds by asking you to make a guess. In this case, there are only two graphs on the calculator, so any guess is appropriate.[16] Simply press the `ENTER` key to use the current position of the cursor as your guess.



```
Y1=1.23*X-4.56
First curve?
X=0          Y=-4.56
```

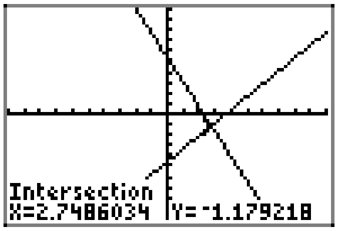
```
Y2=5.28-2.35*X
Second curve?
X=0          Y=5.28
```

```
Y2=5.28-2.35*X
Guess?
X=0          Y=5.28
```

(a) (b) (c) (d)

**Figure 9.** Using the `intersect` utility.

The result of this sequence of steps is shown in **Figure 10**. The coordinates of the point of intersection are approximately $(2.7486034, -1.179218)$. The $x$-value of this point of intersection is the solution of **equation (6)**. That is, the solution of $1.23x - 4.56 = 5.28 - 2.35x$ is approximately $x \approx 2.7486034$.[17]



```
Intersection
X=2.7486034  Y=-1.179218
```

**Figure 10.** The coordinates of the point of intersection.

---

[16] We will see in the case where there are two points of intersection, that the guess becomes more important.

[17] It is important to remember that every time you pick up your calculator, you are only approximating a solution.

[18] Please use a ruler to draw all lines.

**Summary 7. Guidelines**. *You'll need to discuss expectations with your teacher, but we expect our students to summarize their results as follows.*

1. *Set up a coordinate system.*[18] *Label and scale each axis with* `xmin`, `xmax`, `ymin`, *and* `ymax`.
2. *Copy the image in your viewing window onto your coordinate system. Label each graph with its equation.*
3. *Draw a dashed vertical line through the point of intersection.*
4. *Shade and label the solution of the equation on the $x$-axis.*

The result of following this standard is shown in **Figure 11**.

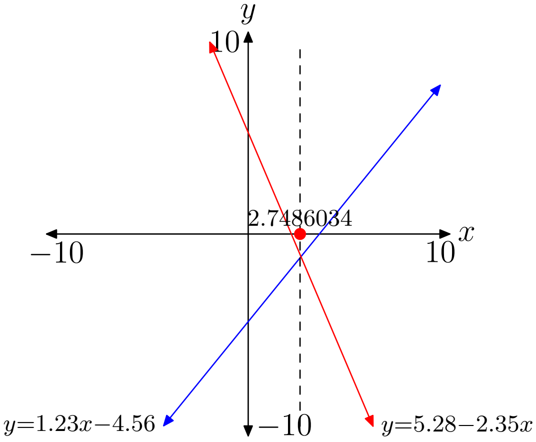


**Figure 11.** Summarizing the solution of **equation (6)**.

◆

Let's look at another example.

▶ **Example 8.** *Use set-builder and interval notation to describe the solution of the inequality*

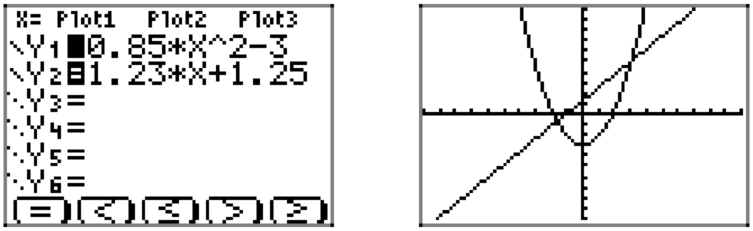
$$0.85x^2 - 3 \geq 1.23x + 1.25. \qquad (9)$$

Note that the inequality (**9**) has the form $f(x) \geq g(x)$, where

$$f(x) = 0.85x^2 - 3 \qquad \text{and} \qquad g(x) = 1.23x + 1.25.$$

Load $f(x) = 0.85x^2 - 3$ and $g(x) = 1.23x + 1.25$ into `Y1` and `Y2` in the `Y=` menu, respectively, as shown in **Figure 12**(a). Select `6:ZStandard` from the `ZOOM` menu to produce the graphs shown in **Figure 12**(b).

To find the points of intersection of the graphs of $f$ and $g$, we follow the same sequence of steps as we did in **Example 5** up to the point where the calculator asks you to make a guess (i.e., `2nd CALC`, `5:intersect`, First curve `ENTER`, Second curve `ENTER`). Because there are two points of intersection, when the calculator asks you to

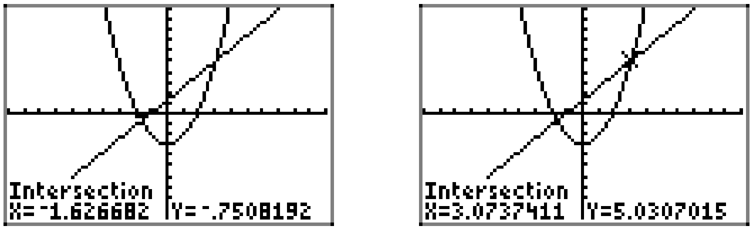


```
X= Plot1  Plot2  Plot3
\Y1■0.85*X^2-3
\Y2■1.23*X+1.25
\Y3=
\Y4=
\Y5=
\Y6=
```

(a) (b)

**Figure 12.** The graphs of
$f(x) = 0.85x^2 - 3$ and $g(x) = 1.23x + 1.25$.

make a guess, you must move your cursor (with the arrow keys) so that it is closer to the point of intersection you wish to find than it is to the other point of intersection. Using this technique produces the two points of intersection found in **Figures 13**(a) and (b).



```
Intersection
X=-1.626682  Y=-.7508192
```

```
Intersection
X=3.0737411  Y=5.0307015
```

(a) (b)

**Figure 13.** The points of
intersection of the graphs of $f$ and $g$.

The approximate coordinates of the first point of intersection are $(-1.626682, -0.7508192)$. The second point of intersection has approximate coordinates $(3.0737411, 5.0307015)$.

It is important to remember that every time you pick up your calculator, you are only getting an approximation. It is possible that you will get a slightly different result for the points of intersection. For example, you might get $(-1.626685, -0.7508187)$ for your point of intersection. Based on the position of the cursor when you marked the curves and made your guess, you can get slightly different approximations. Note that this second solution is very nearly the same as the one we found, differing only in the last few decimal places, and is perfectly acceptable as an answer.

We now summarize our results by creating a coordinate system, labeling the axes, and scaling the axes with the values of the window parameters `xmin`, `xmax`, `ymin`, and `ymax`. We copy the image in our viewing window onto this coordinate system, labeling each graph with its equation. We then draw dashed vertical lines through each point of intersection, as shown in **Figure 14**.

We are solving the inequality $0.85x^2 - 3 \geq 1.23x + 1.25$. The solution will be the union of the solutions of $0.85x^2 - 3 > 1.23x + 1.25$ and $0.85x^2 - 3 = 1.23x + 1.25$.

- To solve $0.85x^2 - 3 > 1.23x + 1.25$, we note where the graph of $y = 0.85x^2 - 3$ lies *above* the graph of $y = 1.23x + 1.25$ and shade the corresponding $x$-values

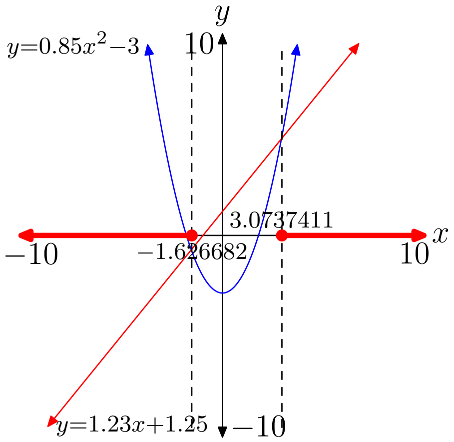


**Figure 14.** Summarizing the solution of $0.85x^2 - 3 \geq 1.23x + 1.25$.

on the $x$-axis. In this case, the graph of $y = 0.85x^2 - 3$ lies *above* the graph of $y = 1.23x + 1.25$ for values of $x$ that lie outside of our dashed vertical lines.

- To solve $0.85x^2 - 3 = 1.23x + 1.25$, we note where the graph of $y = 0.85x^2 - 3$ *intersects* the graph of $y = 1.23x + 1.25$ and shade the corresponding $x$-values on the $x$-axis. This is why the points at $x \approx -1.626682$ and $x \approx 3.0737411$ are "filled."

Thus, all values of $x$ that are either less than or equal to $-1.626682$ or greater than or equal to $3.0737411$ are solutions. That is, the solution of inequality $0.85x^2 - 3 > 1.23x + 1.25$ is approximately

$$(-\infty, -1.626682] \cup [3.0737411, \infty) = \{x : \ x \leq -1.626682 \text{ or } x \geq 3.0737411\}.$$

◆

## Comparing Functions with Zero

When we evaluate a function $f$ at a particular value of $x$, only one of three outcomes is possible. Either

$$f(x) = 0, \quad \text{or} \quad f(x) > 0, \quad \text{or} \quad f(x) < 0.$$

That is, either $f(x)$ equals zero, or $f(x)$ is positive, or $f(x)$ is negative. There are no other possibilities.

We could start fresh, taking a completely new approach, or we can build on what we already know. We choose the latter approach. Suppose that we are asked to compare $f(x)$ with zero? Is it equal to zero, is it greater than zero, or is it smaller than zero?

We set $g(x) = 0$. Now, if we want to compare the function $f$ with zero, we need only compare $f$ with $g$, which we already know how to do. To find where $f(x) = g(x)$, we note where the graphs of $f$ and $g$ intersect, to find where $f(x) > g(x)$, we note where the graph of $f$ lies above the graph of $g$, and finally, to find where $f(x) < g(x)$, we simply note where the graph of $f$ lies below the graph of $g$.

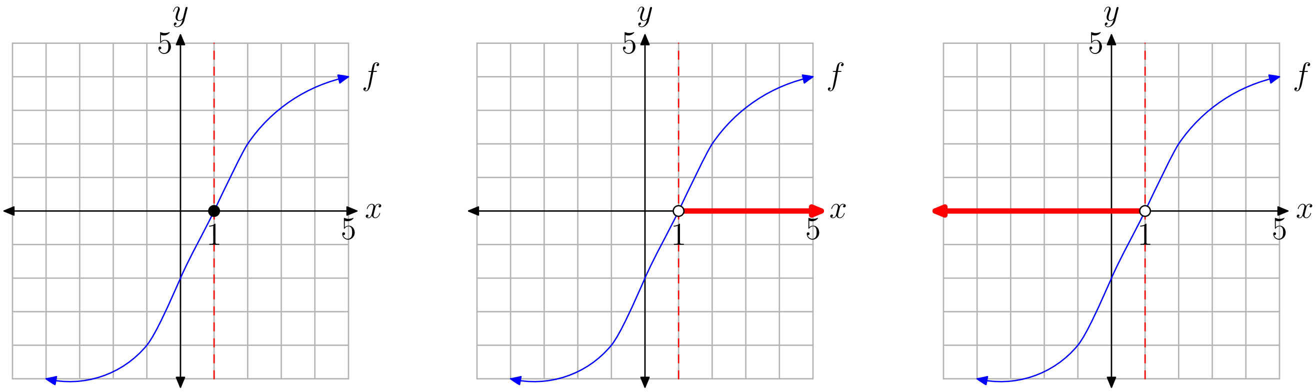
However, the graph of $g(x) = 0$ is a horizontal line coincident with the $x$-axis. Indeed, $g(x) = 0$ is the equation of the $x$-axis. This argument leads to the following key results.

**Summary 10.**

- *The solution of $f(x) = 0$ is the set of all $x$ for which the graph of $f$ intersects the $x$-axis.*
- *The solution of $f(x) > 0$ is the set of all $x$ for which the graph of $f$ lies strictly above the $x$-axis.*
- *The solution of $f(x) < 0$ is the set of all $x$ for which the graph of $f$ lies strictly below the $x$-axis.*

For example:

- To find the solution of $f(x) = 0$ in **Figure 15**(a), we simply note where the graph of $f$ crosses the $x$-axis in **Figure 15**(a). Thus, the solution of $f(x) = 0$ is $x = 1$.
- To find the solution of $f(x) > 0$ in **Figure 15**(b), we simply note where the graph of $f$ lies above the $x$-axis in **Figure 15**(b), which is to the right of the vertical dashed line through $x = 1$. Thus, the solution of $f(x) > 0$ is $(1, \infty) = \{x : x > 1\}$.
- To find the solution of $f(x) < 0$ in **Figure 15**(c), we simply note where the graph of $f$ lies below the $x$-axis in **Figure 15**(c), which is to the left of the vertical dashed line at $x = 1$. Thus, the solution of $f(x) < 0$ is $(-\infty, 1) = \{x : x < 1\}$.



(a) The solution of $f(x) = 0$ (b) The solution of $f(x) > 0$ (c) The solution of $f(x) < 0$

**Figure 15.** Comparing the function $f$ with zero.

We next define some important terminology.

**Definition 11.** *If $f(a) = 0$, then $a$ is called a* **zero** *of the function $f$. The graph of $f$ will intercept the $x$-axis at $(a, 0)$, a point called the* **x-intercept** *of the graph of $f$.*

Your calculator has a utility that will help you to find the zeros of a function.

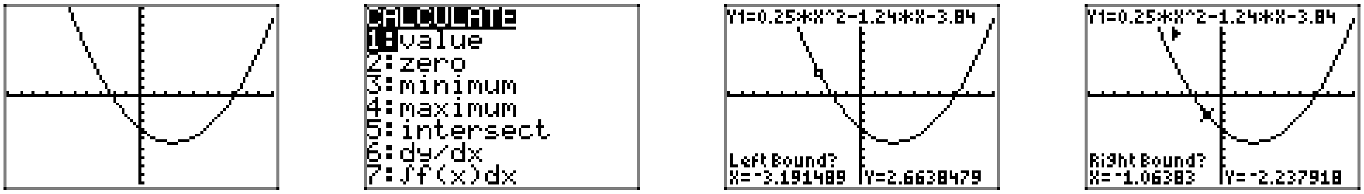
▶ **Example 12.** *Use a graphing calculator to solve the inequality*

$$0.25x^2 - 1.24x - 3.84 \leq 0.$$

Note that this inequality has the form $f(x) \leq 0$, where $f(x) = 0.25x^2 - 1.24x - 3.84$. Our strategy will be to draw the graph of $f$, then determine where the graph of $f$ lies *below or on* the $x$-axis.

We proceed as follows:

- First, load the function $f(x) = 0.25x^2 - 1.24x - 3.84$ into the `Y1` in the `Y=` menu of your calculator. Select `6:ZStandard` from the `ZOOM` menu to produce the image in **Figure 16**(a).
- Press `2nd CALC` to open the menu shown in **Figure 16**(b), then select `2:zero` to start the utility that will find a zero of the function (an $x$-intercept of the graph).
- The calculator asks for a "Left Bound," so use your arrow keys to move the cursor slightly to the left of the leftmost $x$-intercept of the graph, as shown in **Figure 16**(c). Press `ENTER` to record this "Left Bound."
- The calculator then asks for a "Right Bound," so use your arrow keys to move the cursor slightly to the right of the $x$-intercept, as shown in **Figure 16**(d). Press `ENTER` to record this "Right Bound."



```
Y1=0.25*X^2-1.24*X-3.84
Left Bound?
X=-3.191489   Y=2.6638479
```

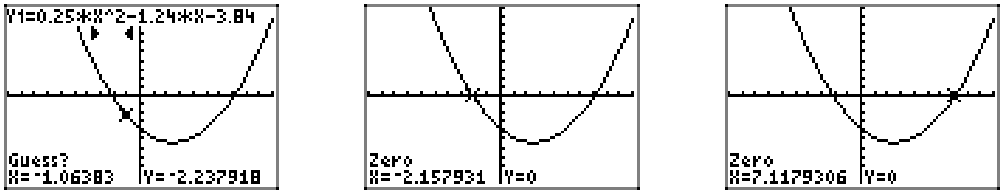
```
Y1=0.25*X^2-1.24*X-3.84
Right Bound?
X=-1.06383   Y=-2.237918
```

(a) (b) (c) (d)

**Figure 16.** Finding a zero or $x$-intercept with the calculator.

- The calculator responds by marking the left and right bounds on the screen, as shown in **Figure 17**(a), then asks you to make a reasonable starting guess for the zero or $x$-intercept. You may use the arrow keys to move your cursor to any point, so long as the cursor remains between the left- and right-bound marks on the viewing window. We usually just leave the cursor where it is and press the `ENTER` to record this guess. We suggest you do that as well.
- The calculator responds by finding the coordinates of the $x$-intercept, as shown in **Figure 17**(b). Note that the $x$-coordinate of the $x$-intercept is approximately $-2.157931$.
- Repeat the procedure to find the coordinates of the rightmost $x$-intercept. The result is shown in **Figure 17**(c). Note that the $x$-coordinate of the intercept is approximately 7.1179306.

The final step is the interpretation of results and recording of our solution on our homework paper. Referring to the **Summary 7** Guidelines, we come up with the graph shown in **Figure 18**.



(a) (b) (c)

**Figure 17.** Finding a zero or $x$-intercept with the calculator.

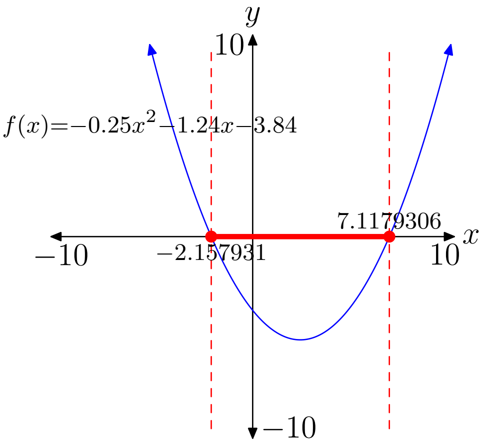


**Figure 18.** The solution of $0.25x^2 - 1.24x - 3,84 \leq 0$.

Several comments are in order. Noting that $f(x) = 0.25x^2 - 1.24x - 3.84$, we note:

1. The solutions of $f(x) = 0$ are the points where the graph crosses the $x$-axis. That's why the points $(-2.157931, 0)$ and $(7.1179306, 0)$ are shaded and filled in **Figure 18**.
2. The solutions of $f(x) < 0$ are those values of $x$ for which the graph of $f$ falls strictly below the $x$-axis. This occurs for all values of $x$ between $-2.157931$ and $7.1179306$. These points are also shaded on the $x$-axis in **Figure 18**.
3. Finally, the solution of $f(x) \leq 0$ is the union of these two shadings, which we describe in interval and set-builder notation as follows:

$$[-2.157931, 7.1179306] = \{x : -2.157931 \leq x \leq 7.1179306\}$$

◆

## 2.4 Exercises

---

In **Exercises 1-6**, you are given the definition of two functions $f$ and $g$. Compare the functions, as in Example 1 of the narrative, at the given values of $x$.

**1.** $f(x) = x + 2$, $g(x) = 4 - x$ at $x = -3$, 1, and 2.

**2.** $f(x) = 2x - 3$, $g(x) = 3 - x$ at $x = -4$, 2, and 5.

**3.** $f(x) = 3 - x$, $g(x) = x + 9$ at $x = -4$, $-3$, and $-2$.

**4.** $f(x) = x^2$, $g(x) = 4x + 5$ at $x = -2$, 1, and 6.

**5.** $f(x) = x^2$, $g(x) = -3x - 2$ at $x = -3$, $-1$, and 0.

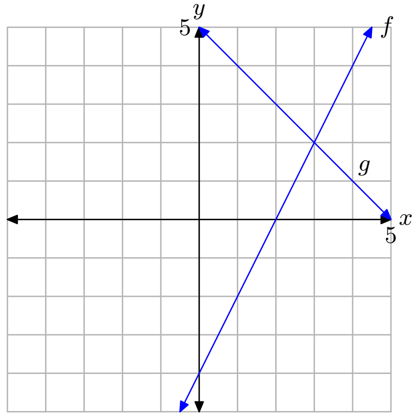
**6.** $f(x) = |x|$, $g(x) = 4 - x$ at $x = 1$, 2, and 3.

---

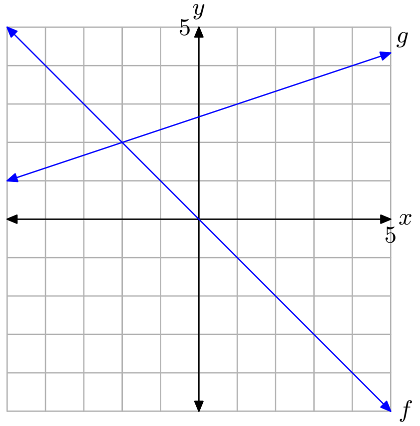
In **Exercises 7-12**, perform each of the following tasks. *Remember to use a ruler to draw all lines.*

i. Make an accurate copy of the image on graph paper (label each equation, label and scale each axis), drop a dashed vertical line through the point of intersection, then label and shade the solution of $f(x) = g(x)$ on the $x$-axis.
ii. Make a second copy of the image on graph paper, drop a dashed, vertical line through the point of intersection, then label and shade the solution of $f(x) > g(x)$ on the $x$-axis. Use set-builder and interval notation to describe your solution set.
iii. Make a third copy of the image on graph paper, drop a dashed, vertical line through the point of intersection, then label and shade the solution of $f(x) < g(x)$ on the $x$-axis. Use set-builder and interval notation to describe your solution set.

**7.**



**8.**



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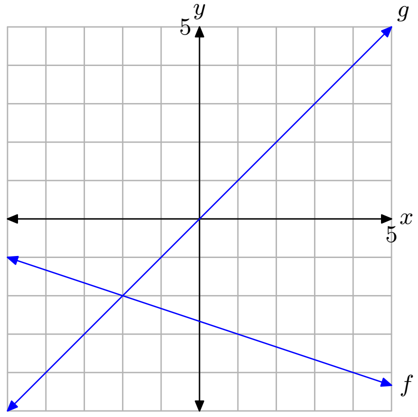
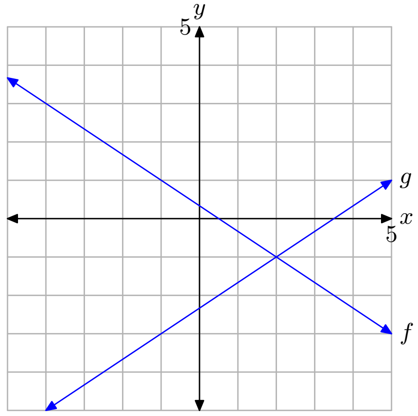
[19] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

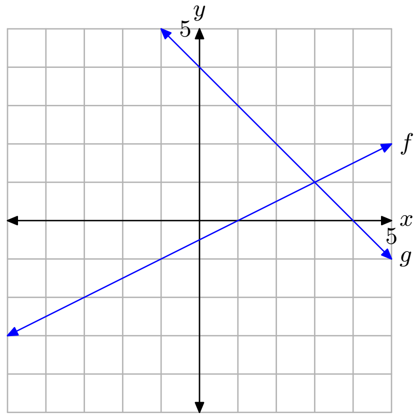
**9.**



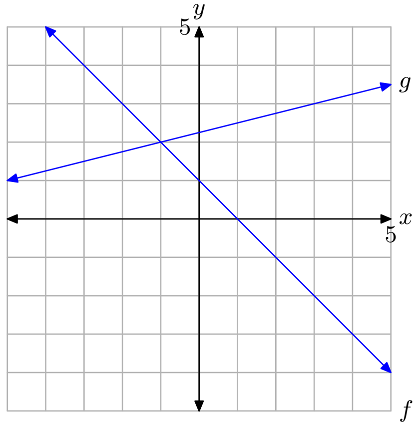
**10.**



**11.**



**12.**



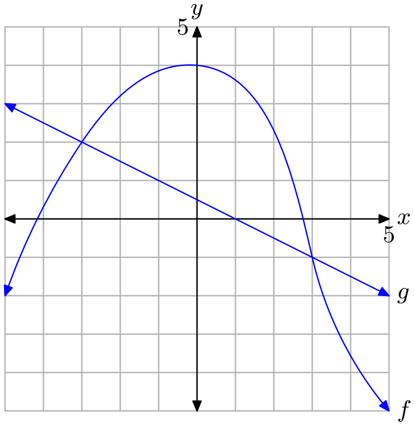
---

In **Exercises 13-16**, perform each of the following tasks. *Remember to use a ruler to draw all lines.*

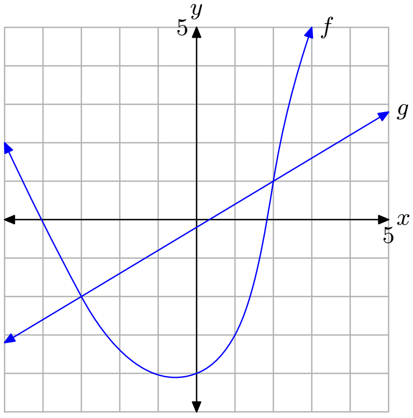
i. Make an accurate copy of the image on graph paper, drop dashed, vertical lines through the points of intersection, then label and shade the solution of $f(x) \geq g(x)$ on the $x$-axis. Use set-builder and interval notation to describe your solution set.

ii. Make a second copy of the image on graph paper, drop dashed, vertical lines through the points of intersection, then label and shade the solution of $f(x) < g(x)$ on the $x$-axis. Use set-builder and interval notation to describe your solution set.

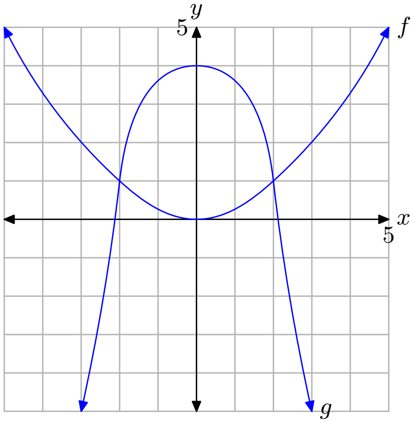
**13.**



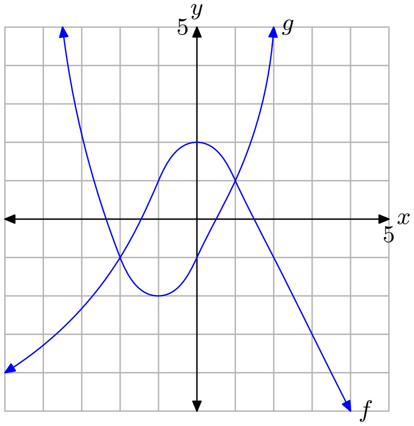
**14.**



**15.**



**16.**



---

In **Exercises 17-20**, perform each of the following tasks. *Remember to use a ruler to draw all lines.*

- i. Load each side of the equation into the `Y=` menu of your calculator. Adjust the `WINDOW` parameters so that the point of intersection of the graphs is visible in the viewing window. Use the `intersect` utility in the `CALC` menu of your calculator to determine the x-coordinate of the point of intersection.
- ii. Make an accurate copy of the image in your viewing window on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax, and label each graph with its equation.
- iii. Draw a dashed, vertical line through the point of intersection. Shade and label the solution of the equation on the $x$-axis.

**17.** $1.23x - 4.56 = 3.46 - 2.3x$

**18.** $2.23x - 1.56 = 5.46 - 3.3x$

**19.** $5.46 - 1.3x = 2.2x - 5.66$

**20.** $2.46 - 1.4x = 1.2x - 2.66$

---

In **Exercises 21-26**, perform each of the following tasks. *Remember to use a ruler to draw all lines.*

i. Load each side of the inequality into the `Y=` menu of your calculator. Adjust the `WINDOW` parameters so that the point(s) of intersection of the graphs is visible in the viewing window. Use the `intersect` utility in the `CALC` menu of your calculator to determine the coordinates of the point(s) of intersection.
ii. Make an accurate copy of the image in your viewing window on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax, and label each graph with its equation.
iii. Draw a dashed, vertical line through the point(s) of intersection. Shade and label the solution of the inequality on the $x$-axis. Use both set-builder and interval notation to describe the solution set.

**21.** $1.6x + 1.23 \geq -2.3x - 4.2$

**22.** $1.24x + 5.6 < 1.2 - 0.52x$

**23.** $0.15x - 0.23 > 8.2 - 0.6x$

**24.** $-1.23x - 9.76 \leq 1.44x + 22.8$

**25.** $0.5x^2 - 5 < 1.23 - 0.75x$

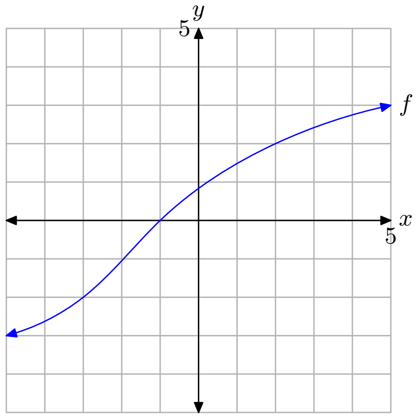
**26.** $4 - 0.5x^2 \leq 0.72x - 1.34$

---

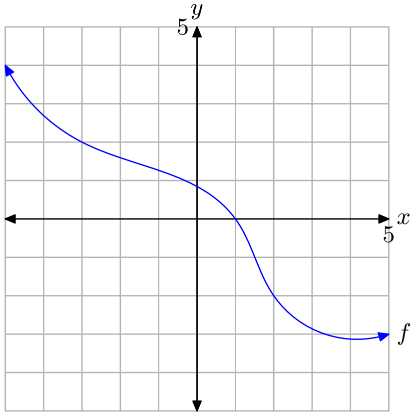
In **Exercises 27-30**, perform each of the following tasks. *Remember to use a ruler to draw all lines.*

i. Make an accurate copy of the image on graph paper (label the graph with the letter $f$ and label and scale each axis), drop a dashed vertical line through the $x$-intercept of the graph of $f$, then label and shade the solution of $f(x) = 0$ on the $x$-axis. Use set-builder notation to describe your solution.
ii. Make a second copy of the image on graph paper, drop a dashed, vertical line through the $x$-intercept of the graph of $f$, then label and shade the solution of $f(x) > 0$ on the $x$-axis. Use set-builder and interval notation to describe your solution set.
iii. Make a third copy of the image on graph paper, drop a dashed, vertical line through the $x$-intercept of the graph of $f$, then label and shade the solution of $f(x) < 0$ on the $x$-axis. Use set-builder and interval notation to describe your solution set.

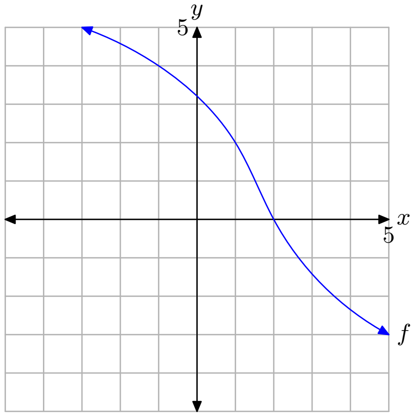
**27.**



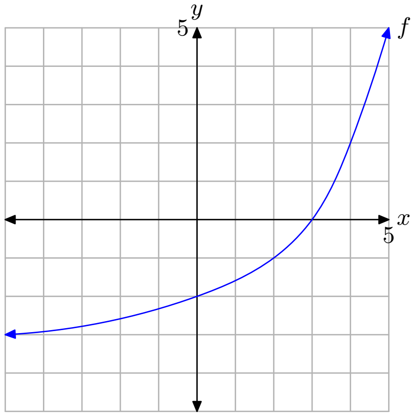
**28.**



**29.**



**30.**

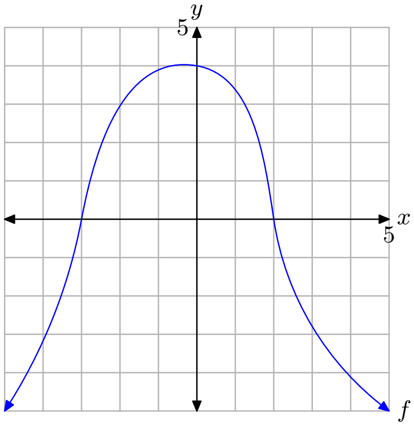


In **Exercises 31-34**, perform each of the following tasks. *Remember to use a ruler to draw all lines.*

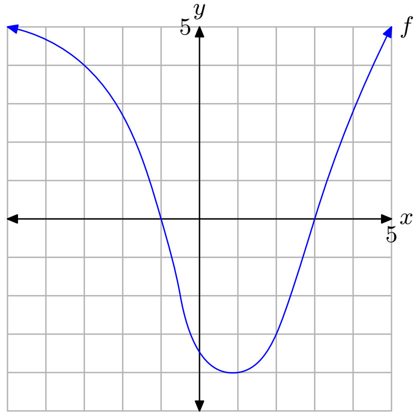
i. Make an accurate copy of the image on graph paper, drop dashed, vertical lines through the $x$-intercepts, then label and shade the solution of $f(x) \geq 0$ on the $x$-axis. Use set-builder and interval notation to describe your solution set.

ii. Make a second copy of the image on graph paper, drop dashed, vertical lines through the $x$-intercepts, then label and shade the solution of $f(x) < 0$ on the $x$-axis. Use set-builder and interval notation to describe your solution set.

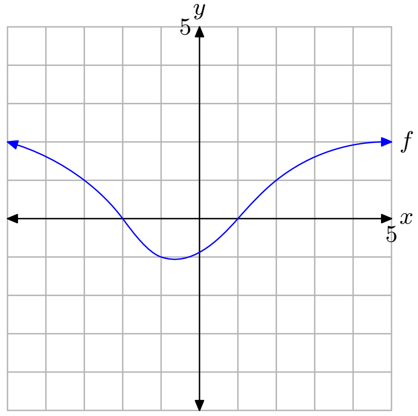
**31.**



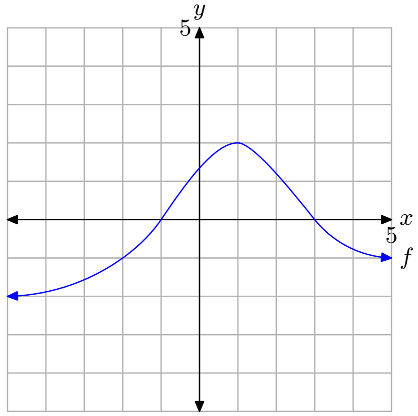
**32.**



**33.**



**34.**



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In **Exercises 35-38**, perform each of the following tasks. *Remember to use a ruler to draw all lines.*

i. Load the given function $f$ into the `Y=` menu of your calculator. Adjust the `WINDOW` parameters so that the $x$-intercept(s) of the graph of $f$ is visible in the viewing window. Use the `zero` utility in the `CALC` menu of your calculator to determine the coordinates of the $x$-intercept(s) of the graph of $f$.

ii. Make an accurate copy of the image in your viewing window on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax, and label the graph with its equation.

iii. Draw a dashed, vertical line through the $x$-intercept(s). Shade and label the solution of the inequality $f(x) > 0$ on the $x$-axis. Use both set-builder and interval notation to describe the solution set.

**35.** $f(x) = -1.25x + 3.58$

**36.** $f(x) = 1.34x - 4.52$

**37.** $f(x) = 1.25x^2 + 4x - 5.9125$

**38.** $f(x) = -1.32x^2 - 3.96x + 5.9532$

---

In **Exercises 39-42**, perform each of the following tasks. *Remember to use a ruler to draw all lines.*

i. Load the given function $f$ into the `Y=` menu of your calculator. Adjust the `WINDOW` parameters so that the $x$-intercept(s) of the graph of $f$ is visible in the viewing window. Use the `zero` utility in the `CALC` menu of your calculator to determine the coordinates of the $x$-intercept(s) of the graph of $f$.

ii. Make an accurate copy of the image

in your viewing window on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax, and label the graph with its equation.

iii. Draw a dashed, vertical line through the $x$-intercept(s). Shade and label the solution of the inequality $f(x) \leq 0$ on the $x$-axis. Use both set-builder and interval notation to describe the solution set.

**39.** $f(x) = -1.45x - 5.6$

**40.** $f(x) = 1.35x + 8.6$

**41.** $f(x) = -1.11x^2 - 5.9940x + 1.2432$

**42.** $f(x) = 1.22x^2 - 6.3440x + 1.3176$

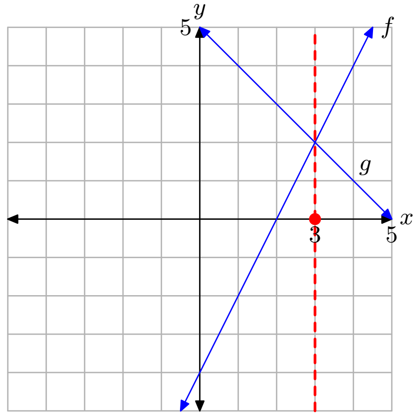
## 2.4 *Answers*

**1.** $f(-3) < g(-3)$, $f(1) = g(1)$, and $f(2) > g(2)$.

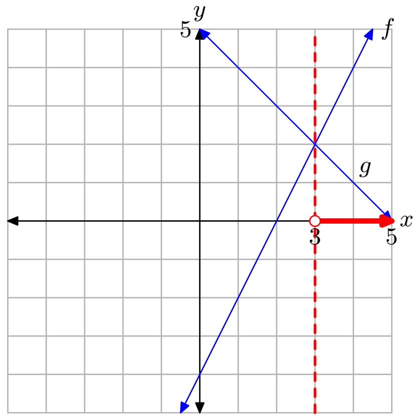
**3.** $f(-4) > g(-4)$, $f(-3) = g(-3)$, and $f(-2) < g(-2)$.

**5.** $f(-3) > g(-3)$, $f(-1) = g(-1)$, and $f(0) > g(0)$.

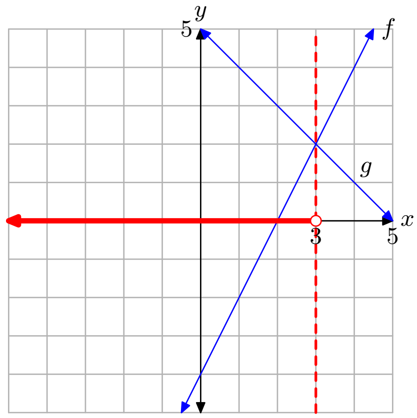
**7.** The solution of $f(x) = g(x)$ is $x = 3$.



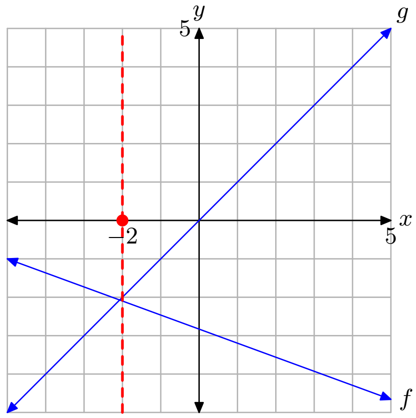
The solution of $f(x) > g(x)$ is $(3, \infty) = \{x : x > 3\}$.



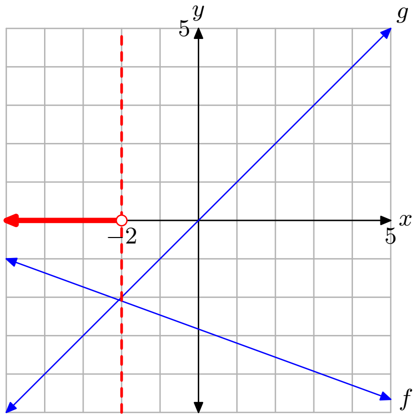
The solution of $f(x) < g(x)$ is $(-\infty, 3) = \{x : x < 3\}$.



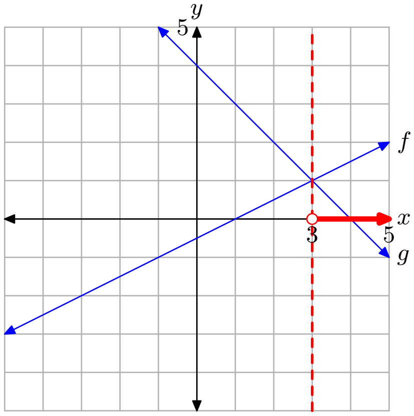
**9.** The solution of $f(x) = g(x)$ is $x = -2$.



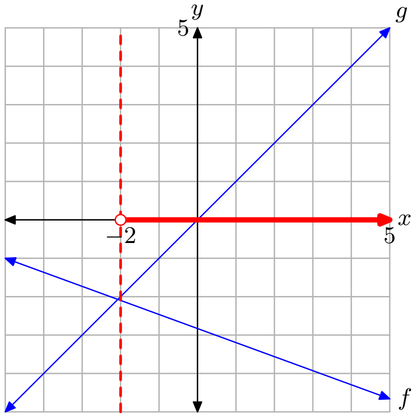
The solution of $f(x) > g(x)$ is $(-\infty, -2) = \{x : x < -2\}$.



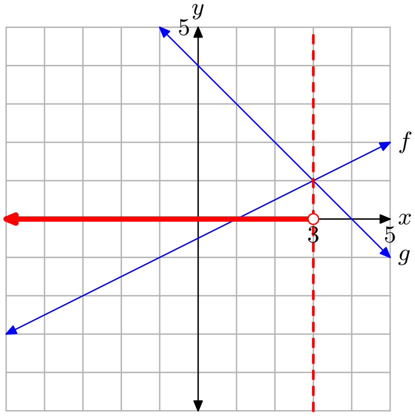
The solution of $f(x) > g(x)$ is $(3, \infty) = \{x : x > 3\}$.



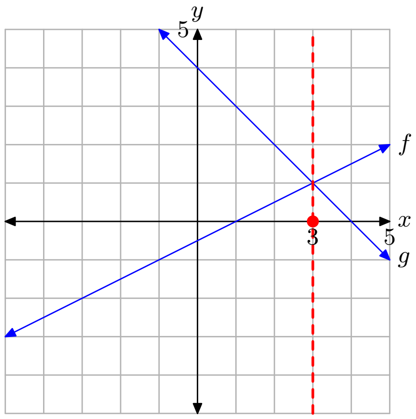
The solution of $f(x) < g(x)$ is $(-2, \infty) = \{x : x > -2\}$.



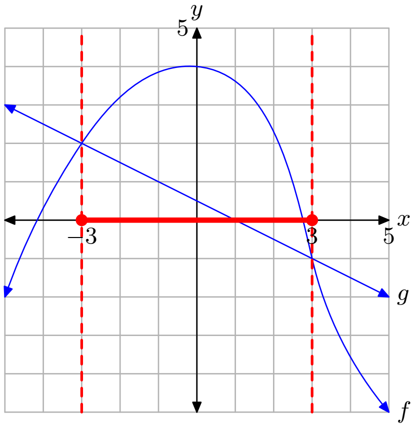
The solution of $f(x) < g(x)$ is $(-\infty, 3) = \{x : x < 3\}$.



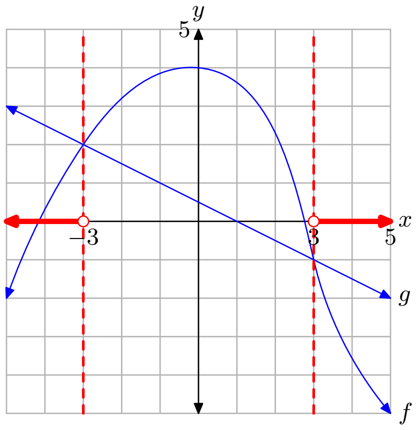
**11.** The solution of $f(x) = g(x)$ is $x = 3$.



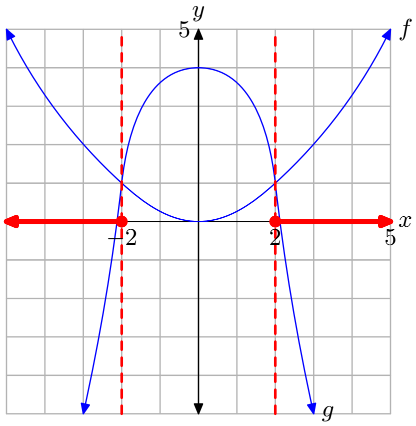
**13.** The solution of $f(x) \geq g(x)$ is $[-3, 3] = \{x : -3 \leq x \leq 3\}$.



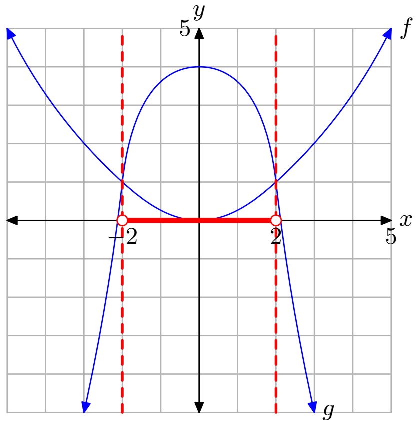
The solution of $f(x) < g(x)$ is
$(-\infty, -3) \cup (3, \infty)$
$=\{x : x < -3 \text{ or } x > 3\}$.



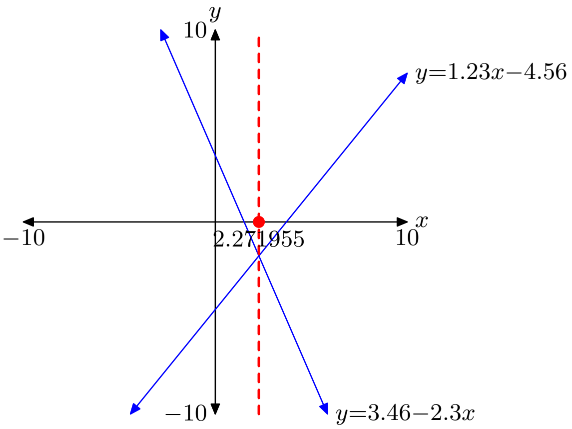
**15.** The solution of $f(x) \geq g(x)$ is
$(-\infty, -2] \cup [2, \infty)$
$=\{x : x \leq -2 \text{ or } x \geq 2\}$.



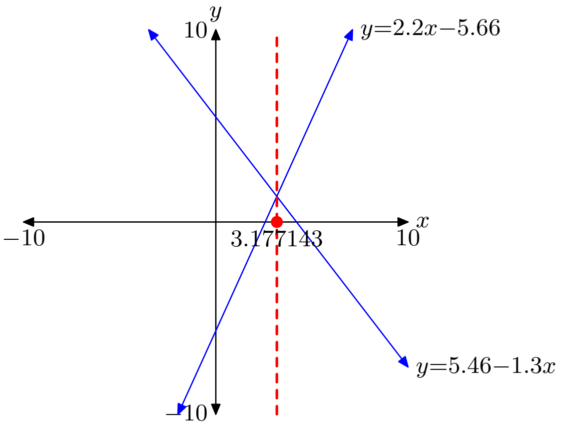
The solution of $f(x) < g(x)$ is $(-2, 2) = \{x : -2 < x < 2\}$.



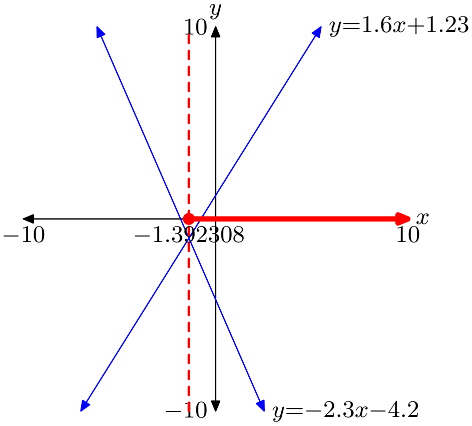
**17.** $x = 2.271955$



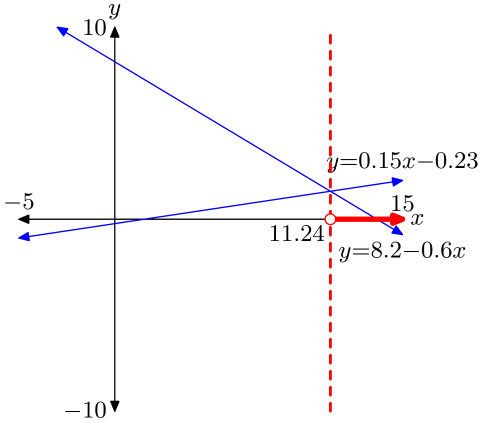
**19.** $x = 3.177143$



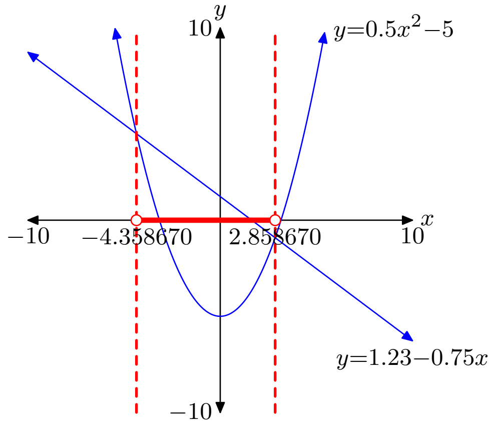
**21.** $[-1.392308, \infty) = \{x : x \geq -1.392308\}$



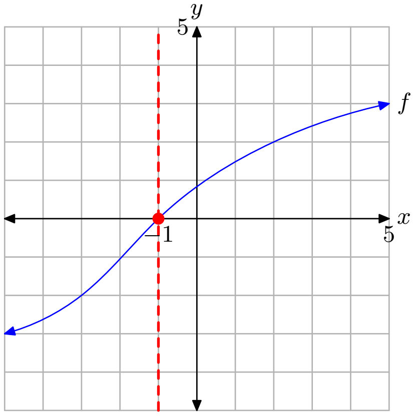
**23.** $(11.24, \infty) = \{x : x > 11.24\}$



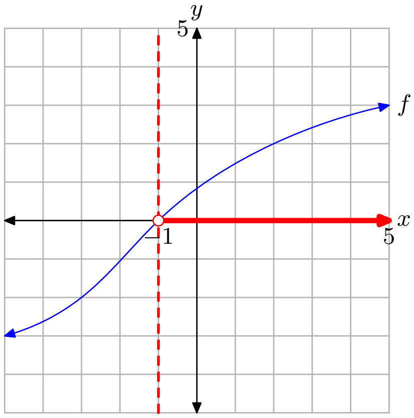
**25.** $(-4.358670, 2.858670)$
$= \{x : -4.358670 < x < 2.858670\}$



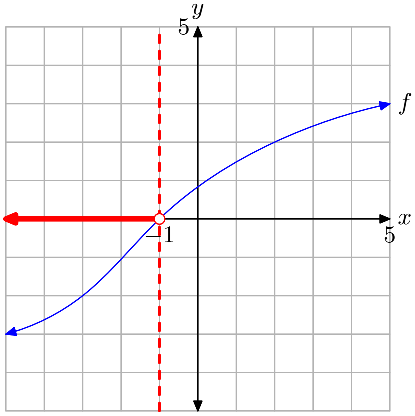
**27.** The solution of $f(x) = 0$ is $x = -1$.



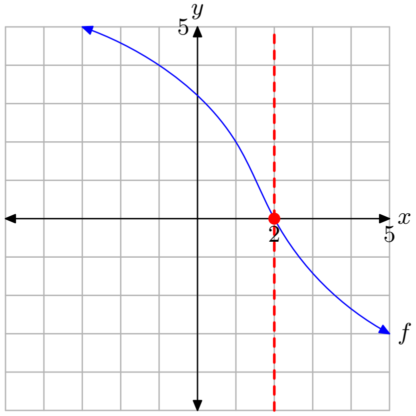
The solution of $f(x) > 0$ is $(-1, \infty) = \{x : x > -1\}$.



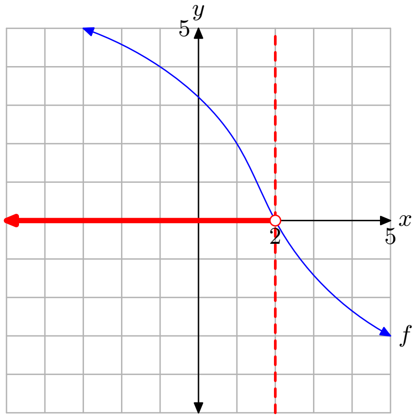
The solution of $f(x) < 0$ is $(-\infty, -1) = \{x : x < -1\}$



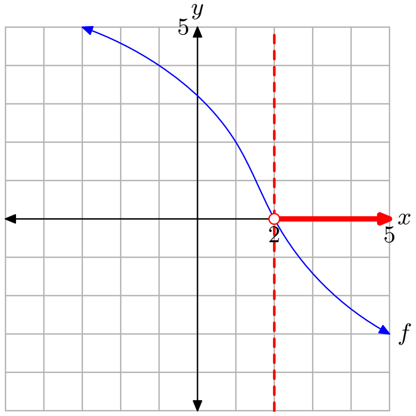
**29.** The solution of $f(x) = 0$ is $x = 2$.



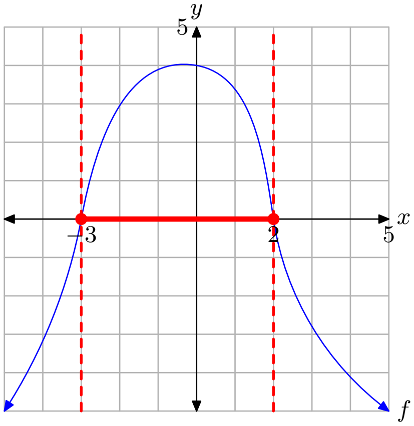
The solution of $f(x) > 0$ is $(-\infty, 2) = \{x : x < 2\}$.



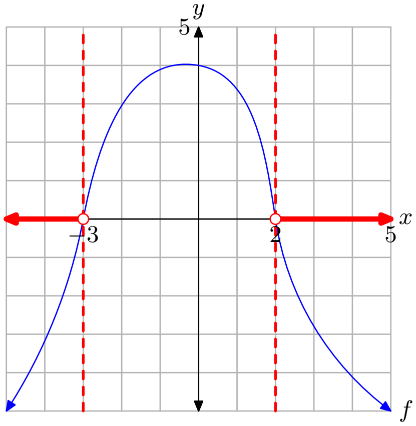
The solution of $f(x) < 0$ is $(2, \infty) = \{x : x > 2\}$



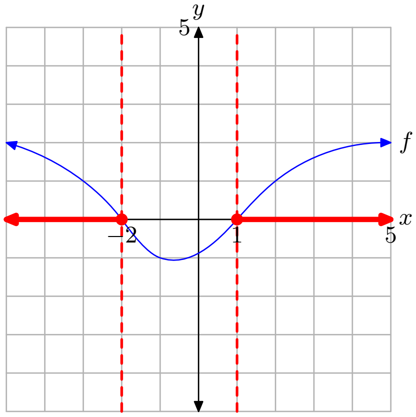
**31.** The solution of $f(x) \geq 0$ is $[-3, 2] = \{x : -3 \leq x \leq 2\}$.



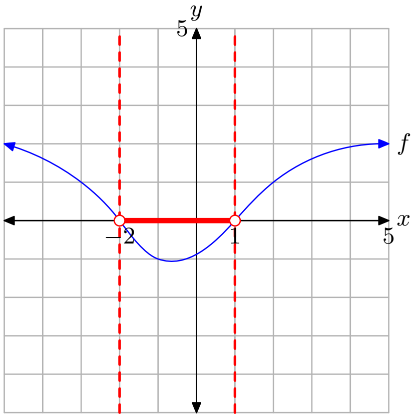
The solution of $f(x) < 0$ is $(-\infty, -3) \cup (2, \infty) = \{x : x < -3 \text{ or } x > 2\}$.



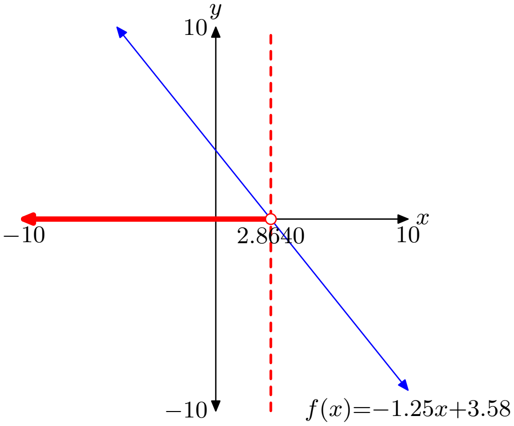
**33.** The solution of $f(x) \geq 0$ is $(-\infty, -2] \cup [1, \infty) = \{x : x \leq -2 \text{ or } x \geq 1\}$.



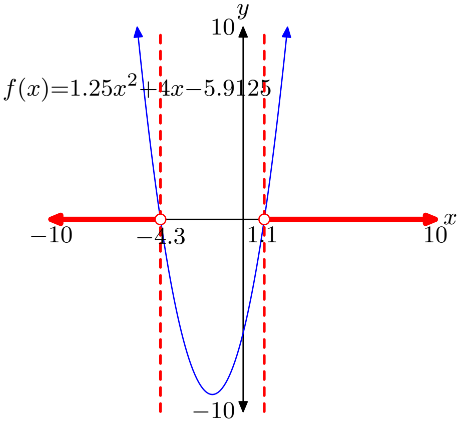
The solution of $f(x) < 0$ is $(-2, 1) = \{x : -2 < x < 1\}$.



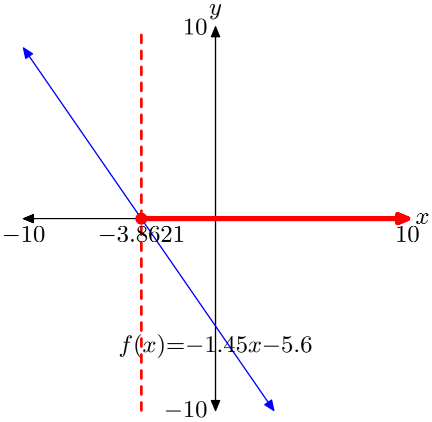
**35.** $(-\infty, 2.8640) = \{x : x < 2.8640\}$



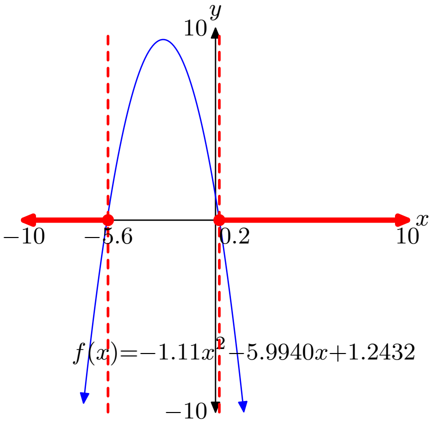
**37.** $(-\infty, -4.3) \cup (1.1, \infty) = \{x : x < -4.3 \text{ or } x > 1.1\}$



**39.** $[-3.8621, \infty) = \{x : x \geq -3.8621\}$



**41.** $(-\infty, -5.6] \cup [0.2, \infty) = \{x : x \leq -5.6 \text{ or } x \geq 0.2\}$

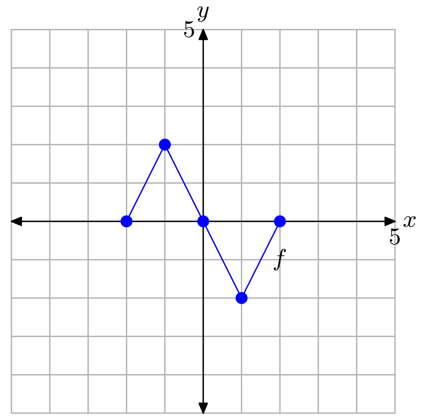


## 2.5 Vertical Transformations

In this section we study the art of transformations: scalings, reflections, and translations. We will restrict our attention to transformations in the vertical or $y$-direction. Our goal is to apply certain transformations to the equation of a function, then ask what effect it has on the graph of the function.

We begin our task with an example that requires that we read the graph of a function to capture several key points that lie on the graph of the function.

▶ **Example 1.** *Consider the graph of $f$ presented in* **Figure 1***(a). Use the graph of $f$ to complete the table in* **Figure 1***(b).*



| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-2$ | | |
| $-1$ | | |
| $0$ | | |
| $1$ | | |
| $2$ | | |

*(a) The graph of $f$.* *(b) The table.*

**Figure 1.** Reading key values from the graph of $f$.

To compute $f(-1)$, we would locate $-1$ on the $x$-axis, draw a vertical arrow to the graph of $f$, then a horizontal arrow to the $y$-axis, as shown in **Figure 2**(a). The $y$-value of this final destination is the value of $f(-1)$. That is, $f(-1) = 2$. This allows us to complete one entry in the table, as shown in **Figure 2**(b). Continue in this manner to complete all of the entries in the table. The result is shown in **Figure 2**(c).

◆

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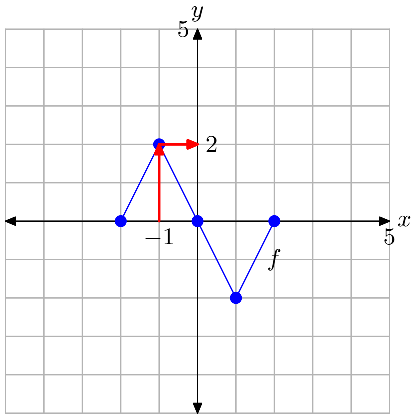


(a) The graph of $f$.

| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-2$ | | |
| $-1$ | 2 | $(-1, 2)$ |
| 0 | | |
| 1 | | |
| 2 | | |

(b) Recording $f(-1) = 2$.

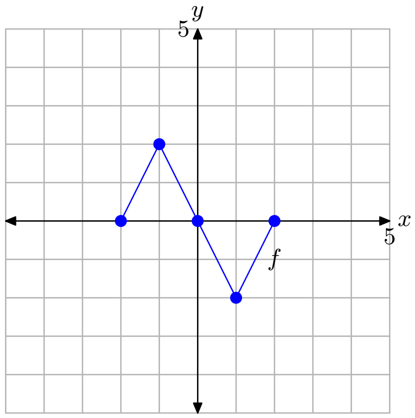
| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-2$ | 0 | $(-2, 0)$ |
| $-1$ | 2 | $(-1, 2)$ |
| 0 | 0 | $(0, 0)$ |
| 1 | $-2$ | $(1, -2)$ |
| 2 | 0 | $(2, 0)$ |

(c) Completed table.

**Figure 2.** Recording coordinates of points on the graph of $f$ in the tables.

## Vertical Scaling

In the narrative that follows, we will have repeated need of the graph in **Figure 2**(a) and the table in **Figure 2**(c). They characterize the basic function that will be the starting point for the concepts of scaling, reflection, and translation that we develop in this section. Consequently, let's place them side-by-side for emphasis in **Figure 3**.



(a)

| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-2$ | 0 | $(-2, 0)$ |
| $-1$ | 2 | $(-1, 2)$ |
| 0 | 0 | $(0, 0)$ |
| 1 | $-2$ | $(1, -2)$ |
| 2 | 0 | $(2, 0)$ |

(b)

**Figure 3.** The original graph of $f$ and a table of key points on the graph of $f$.

We are now going to scale the graph of $f$ in the vertical direction.

▶ **Example 2.** *If $y = f(x)$ has the graph shown in* **Figure 3**(a)*, sketch the graph of $y = 2f(x)$.*

What do we do when we meet a graph whose shape we are unsure of? The answer to this question is we plot some points that satisfy the equation in order to get an idea of the shape of the graph. With that thought in mind, let's evaluate the function $y = 2f(x)$ at $x = -2$.

The letter $f$ refers to the original function shown in **Figure 3**(a) and the table in **Figure 3**(b) contains the values of that function at the given values of $x$. Thus, in computing $y = 2f(-2)$, the first step is to look up the value of $f(-2)$ in the table in **Figure 3**(b). There we find that $f(-2) = 0$. Thus, we can write

$$y = 2f(-2) = 2(0) = 0.$$

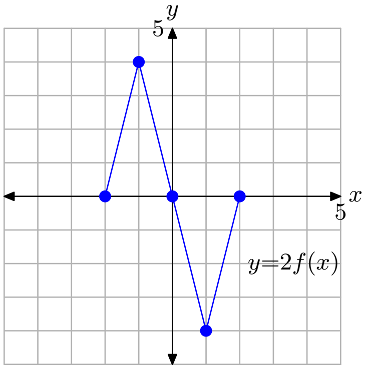
In similar fashion, let's evaluate the function $y = 2f(x)$ at $x = -1$. First, look up the value of $f(-1)$ in the table in **Figure 3**(b). There we find that $f(-1) = 2$. Thus, we can write

$$y = 2f(-1) = 2(2) = 4.$$

We finish by evaluating the function $y = 2f(x)$ at $x = 0$, 1, and 2. Each time you need to evaluate the function $f$ at a number, take the result from the table or graph in **Figure 3**. What follows are the evaluations of $y = 2f(x)$ at $x = -2$, $-1$, 0, 1, and 2.

$$\begin{aligned}
y &= 2f(-2) = 2(0) = 0\\
y &= 2f(-1) = 2(2) = 4\\
y &= 2f(0) = 2(0) = 0\\
y &= 2f(1) = 2(-2) = -4\\
y &= 2f(2) = 2(0) = 0
\end{aligned}$$

We can arrange these results in a table shown in **Figure 4**(b), then plot them in the figure shown in **Figure 4**(a).



(a)

| $x$ | $y = 2f(x)$ | $(x, 2f(x))$ |
|---|---|---|
| $-2$ | 0 | $(-2, 0)$ |
| $-1$ | 4 | $(-1, 4)$ |
| 0 | 0 | $(0, 0)$ |
| 1 | $-4$ | $(1, -4)$ |
| 2 | 0 | $(2, 0)$ |

(b)

**Figure 4.** The points in the table are points on the graph of $y = 2f(x)$.

At this point, there are a number of comparisons you can make.

1. Compare the data in the tables in **Figure 3**(b) and **Figure 4**(b). Note that the $x$-values are identical. In both tables, $x = -2$, $-1$, 0, 1, and 2. However, note

that each $y$-value in the table in **Figure 4**(b) is precisely *double* the corresponding $y$-value in the table in **Figure 3**(b).

2. Compare the graphs in **Figure 3**(a) and **Figure 4**(a). Note that the $y$-value of each point in the graph of $y = 2f(x)$ in **Figure 4**(a) is precisely *double* the $y$-value of the corresponding point in **Figure 3**(a).

Note the result. The graph of $y = 2f(x)$ has been *stretched* vertically (away from the $x$-axis) , both positively and negatively, by a factor of 2.

◆

Let's look at another example.

▶ **Example 3.** *If $y = f(x)$ has the graph shown in* **Figure 3***(a), sketch the graph of $y = (1/2)f(x)$.*

Let's begin by evaluating the function $y = (1/2)f(x)$ at $x = -2$. First, look up the value of $f(-2)$ in the table in **Figure 3**(b). There we find that $f(-2) = 0$. Thus, we can write

$$y = (1/2)f(-2) = (1/2)(0) = 0.$$

In similar fashion, let's evaluate the function $y = (1/2)f(x)$ at $x = -1$. First, look up the value of $f(-1)$ in the table in **Figure 3**(b). There we find that $f(-1) = 2$. Thus, we can write

$$y = (1/2)f(-1) = (1/2)(2) = 1.$$

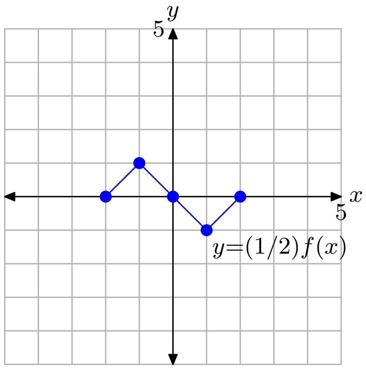
Continuing in this manner, we can evaluate the function $y = (1/2)f(x)$ at $x = 0$, 1, and 2.

$$\begin{aligned} y &= (1/2)f(0) = (1/2)(0) = 0 \\ y &= (1/2)f(1) = (1/2)(-2) = -1 \\ y &= (1/2)f(2) = (1/2)(0) = 0 \end{aligned}$$

The results are recorded in the table in **Figure 5**(b). Rather than double each value of $y$ as did the function $y = 2f(x)$ in **Example 2**, this function $y = (1/2)f(x)$ *halves* each value of $y$. The graph of $y = (1/2)f(x)$ and a table of key points on the graph are presented in **Figures 5**(a) and (b), respectively.

Again, there are a number of comparisons.

1. Compare the data in the tables in **Figure 5**(b) and **Figure 3**(b). Note that the $x$-values are identical. In both tables $x = -2$, $-1$, 0, 1, and 2. However, note that each $y$-value in the table in **Figure 5**(b) is precisely *half* the corresponding $y$-value in the table in **Figure 3**(b).
2. When you compare the graph of $y = (1/2)f(x)$ in **Figure 5**(a) with the original graph of $y = f(x)$ in **Figure 3**(a), note that each point on the graph of $y = (1/2)f(x)$ has a $y$-value that is precisely *half* of the corresponding $y$-value on the original graph of $y = f(x)$ in **Figure 3**(a).



(a)

| $x$ | $y=(1/2)f(x)$ | $(x,(1/2)f(x))$ |
|---|---|---|
| $-2$ | 0 | $(-2,0)$ |
| $-1$ | 1 | $(-1,1)$ |
| 0 | 0 | $(0,0)$ |
| 1 | $-1$ | $(1,-1)$ |
| 2 | 0 | $(2,0)$ |

(b)

**Figure 5.** The points in the table are points on the graph of $y = (1/2)f(x)$.

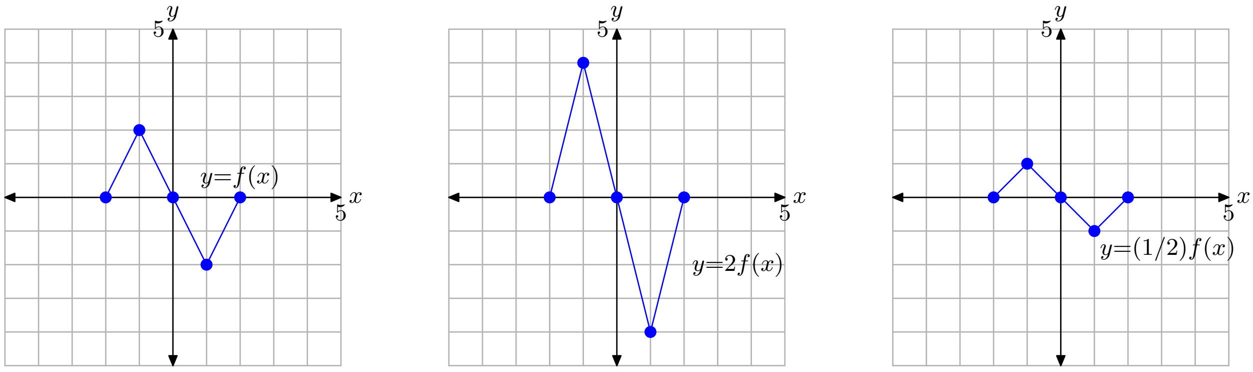
Note the result. The graph of $f$ has been *compressed* vertically (toward the $x$-axis), both positively and negatively, by a factor of 2.

◆

Let's summarize our findings.

**A Visual Summary — Vertical Scaling**. Consider the images in **Figure 6**.

- In **Figure 6**(a), we see pictured the graph of the original function $y = f(x)$.
- In **Figure 6**(b), note that each key point on the graph of $y = 2f(x)$ has a $y$-value that is precisely double the $y$-value of the corresponding point on the graph of $y = f(x)$ in **Figure 6**(a).
- In **Figure 6**(c), note that each key point on the graph of $y = (1/2)f(x)$ has a $y$-value that is precisely half the $y$-value of the corresponding point on the graph of $y = f(x)$ in **Figure 6**(a).
- Note that the $x$-value of each transformed point remains the same.



(a) $y = f(x)$ (b) $y = 2f(x)$ (c) $y = (1/2)f(x)$

**Figure 6.** The graph of $y = 2f(x)$ stretches vertically (away from the $x$-axis) by a factor of 2. The graph of $y = (1/2)f(x)$ compresses vertically (toward the $x$-axis) by a factor of 2.

The visual summary in **Figure 6** makes sketching the graphs of $y = 2f(x)$ and $y = (1/2)f(x)$ an easy task.

- Given the graph of $y = f(x)$, to sketch the graph of $y = 2f(x)$, simply take each point on the graph of $y = f(x)$ and double its $y$-value, keeping the same $x$-value.
- Given the graph of $y = f(x)$, to sketch the graph of $y = (1/2)f(x)$, simply take each point on the graph of $y = f(x)$ and halve its $y$-value, keeping the same $x$-value.

Follow the same procedures for other scaling factors. For example, in the case of $y = 3f(x)$, take each point on the graph of $y = f(x)$ and multiply its $y$-value by 3, keeping the same $x$-value. On the other hand, to draw the graph of $y = (1/3)f(x)$, take each point on the graph of $f$ and multiply its $y$-value by $1/3$, keeping the same $y$-value.

In general, we can state the following.

**Summary 4.** *Suppose we are given the graph of $y = f(x)$.*

- *If $a > 1$, then the graph of $y = af(x)$ is* stretched *vertically (away from the $x$-axis), both positively and negatively, by a factor of $a$.*
- *If $0 < a < 1$, then the graph of $y = af(x)$ is* compressed *vertically (toward the $x$-axis), both positively and negatively, by a factor of $1/a$.*

The second item in Summary **4** warrants a word of explanation. Compare the general form $y = af(x)$ with the function of **Example 3**, $y = (1/2)f(x)$. In this case, $a = 1/2$, so

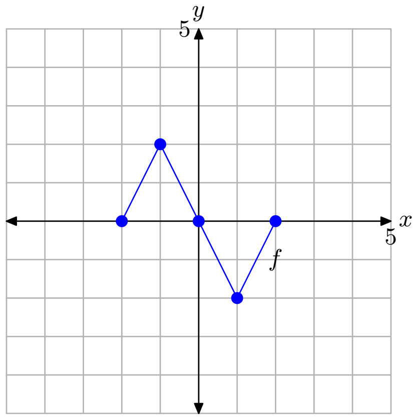
$$\frac{1}{a} = \frac{1}{1/2} = 1 \times 2 = 2.$$

The second item says that when $0 < a < 1$, the graph of $y = af(x)$ is compressed vertically by a factor of $1/a$. Indeed, this is exactly what happens in the case of $y = (1/2)f(x)$, which is compressed by a factor of $1/(1/2)$, or 2.

## *Vertical Reflections*

For convenience, we begin by repeating the original graph of $y = f(x)$ and its accompanying data.

We are now going to reflect the graph in the vertical direction (across the $x$-axis).



(a)

| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-2$ | $0$ | $(-2, 0)$ |
| $-1$ | $2$ | $(-1, 2)$ |
| $0$ | $0$ | $(0, 0)$ |
| $1$ | $-2$ | $(1, -2)$ |
| $2$ | $0$ | $(2, 0)$ |

(b)

**Figure 7.** The original graph of $f$ and a table of key points on the graph of $f$.

▶ **Example 5.** *If $y = f(x)$ has the graph shown in **Figure 7**(a), sketch the graph of $y = -f(x)$.*

To set up a table of points in preparation for the plot of $y = -f(x)$, we'll use exactly the same values of $x$ that you see in the table in **Figure 7**(b), namely $x = -2$, $-1$, $0$, $1$, and $2$.

To evaluate $y = -f(x)$ at the first value of $x$, namely $x = -2$, we make the following calculation,

$$y = -f(-2) = -(0) = 0,$$

where we've used the fact that $f(-2) = 0$ from the table in **Figure 7**(b). In similar fashion, we evaluate $y = -f(x)$ at each of the remaining values of $x$, namely $x = -1$, $0$, $1$, and $2$.

$$\begin{aligned} y &= -f(-1) = -(2) = -2 \\ y &= -f(0) = -(0) = 0 \\ y &= -f(1) = -(-2) = 2 \\ y &= -f(2) = -(0) = 0 \end{aligned}$$

We assemble these points in the table in **Figure 8**(b) and plot them in **Figure 8**(a).

Note that the graph of $y = -f(x)$ in **Figure 8**(a) is a reflection of the graph of $y = f(x)$ in **Figure 7**(a) across the $x$-axis.[21]

◇

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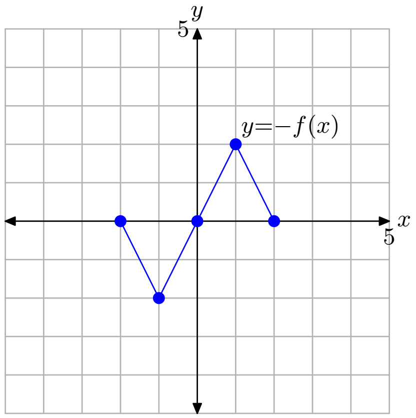
[21] Be sure to note that this is a reflection of the graph of $y = f(x)$ across the $x$-axis. Note that a reflection of the graph of $y = f(x)$ across the $y$-axis gives the same result, but that's not what we've done here. We'll address reflections across the $y$-axis in the next section.



(a)

| $x$ | $y=-f(x)$ | $(x,-f(x))$ |
|---|---|---|
| $-2$ | $0$ | $(-2,0)$ |
| $-1$ | $-2$ | $(-1,-2)$ |
| $0$ | $0$ | $(0,0)$ |
| $1$ | $2$ | $(1,2)$ |
| $2$ | $0$ | $(2,0)$ |

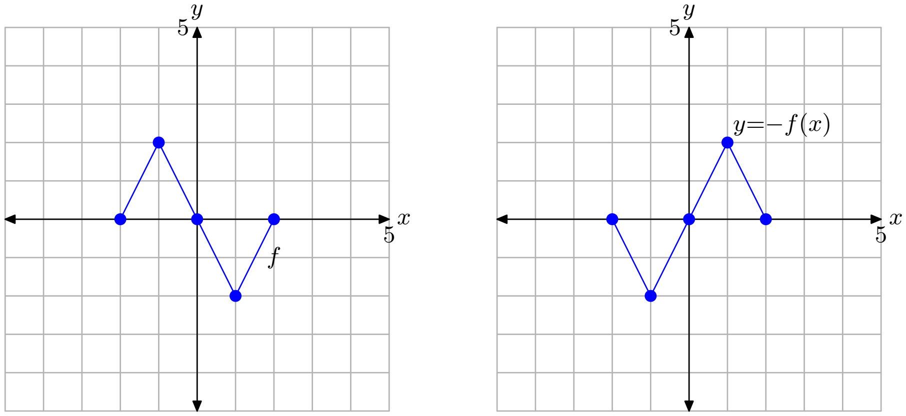
(b)

**Figure 8.** The graph of $y=-f(x)$ and a table of key points on the graph.

Let's summarize what we've learned about vertical reflections.

**A Visual Summary — Vertical Reflections.** Consider the images in **Figure 9**.

- In **Figure 9**(a), we see pictured the original graph of $y=f(x)$.
- In **Figure 9**(b), the graph of $y=-f(x)$ is a reflection of the graph of $y=f(x)$ across the $x$-axis.



(a) $y=f(x)$ (b) $y=-f(x)$

**Figure 9.** The graph of $y=-f(x)$ is a reflection of the graph of $y=f(x)$ across the $x$-axis.

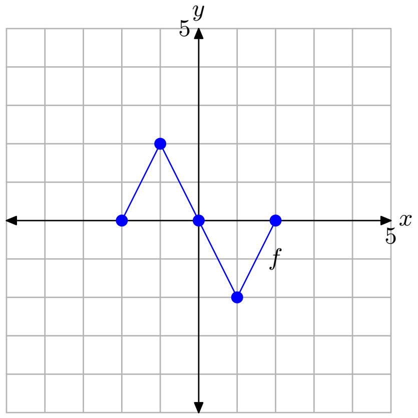
Thus, given the graph of $y=f(x)$, it is a simple task to draw the graph of $y=-f(x)$.

- To draw the graph of $y=-f(x)$, take each point on the graph of $y=f(x)$ and reflect it across the $x$-axis, keeping the $x$-value the same, but negating the $y$-value.

## *Vertical Translations*

Translations are perhaps the easiest transformation of all. A translation is a “shift” or a “slide.” Pretend, for a moment, that you’ve placed a transparent sheet of thin plastic over a sheet of graph paper. You’ve drawn a Cartesian coordinate system on your graph paper, but you’ve plotted your graph on the transparent sheet of plastic. Now, “shift” or “slide” the transparency over your graph paper in a constant direction without rotating the transparency. This is what we mean by a “translation.” In this section, we will focus strictly on vertical translations.

For convenience, we begin by repeating the original graph of $y = f(x)$ and its accompanying data in **Figure 10**(a) and( b), respectively. We will now translate this graph in the vertical direction.



| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-2$ | $0$ | $(-2, 0)$ |
| $-1$ | $2$ | $(-1, 2)$ |
| $0$ | $0$ | $(0, 0)$ |
| $1$ | $-2$ | $(1, -2)$ |
| $2$ | $0$ | $(2, 0)$ |

(a) (b)

**Figure 10.** The original graph of $f$ and a table of key points on the graph of $f$.

▶ **Example 6.** *If $y = f(x)$ has the graph shown in* **Figure 10***(a), sketch the graph of $y = f(x) + 1$.*

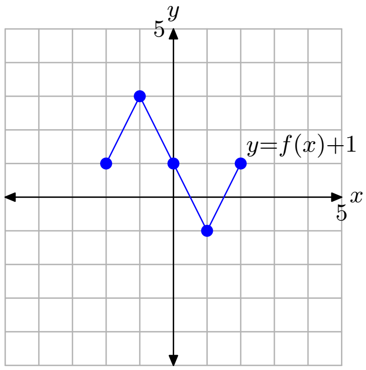
We will evaluate $y = f(x)+1$ at the same values shown in the table in **Figure 10**(b), namely $x = -2$, $-1$, $0$, $1$, and $2$. To evaluate $y = f(x)+1$ at the first value of $x$, namely $x = -2$, we make the following calculation

$$y = f(-2) + 1 = 0 + 1 = 1,$$

where we’ve used that fact that $f(-2) = 0$ from the table in **Figure 10**(b). In similar fashion, we can evaluate $y = f(x) + 1$ at each of the remaining values of $x$, namely $x = -1$, $0$, $1$, and $2$.

$$\begin{aligned} y &= f(-1) + 1 = 2 + 1 = 3 \\ y &= f(0) + 1 = 0 + 1 = 1 \\ y &= f(1) + 1 = -2 + 1 = -1 \\ y &= f(2) + 1 = 0 + 1 = 1 \end{aligned}$$

We assemble these points in the table in **Figure 11**(b) and plot them in **Figure 11**(a).



(a)

| $x$ | $y = f(x) + 1$ | $(x, f(x) + 1)$ |
|---|---|---|
| $-2$ | $1$ | $(-2, 1)$ |
| $-1$ | $3$ | $(-1, 3)$ |
| $0$ | $1$ | $(0, 1)$ |
| $1$ | $-1$ | $(1, -1)$ |
| $2$ | $1$ | $(2, 1)$ |

(b)

**Figure 11.** The graph of $y = f(x) + 1$ and a table of key points on the graph.

When you compare the entries in the table in **Figure 11**(b) with the original values in the table in **Figure 10**(b), you'll note that the $x$-values in each table are identical, but the $y$-values in the table in **Figure 11**(b) are all increased by 1. This makes sense, because these are the $y$-values of the points associated with the function $y = f(x) + 1$. Of course, all the $y$-values should be 1 larger than the $y$-values associated with the original equation $y = f(x)$.

Note the result. The graph of $y = f(x) + 1$ in **Figure 11**(a), when compared with the graph of $y = f(x)$ in **Figure 10**(a), is shifted 1 unit upwards.

◆

Let's look at another example.

▶ **Example 7.** *If $y = f(x)$ has the graph shown in* **Figure 10***(a), sketch the graph of $y = f(x) - 2$.*

Evaluate the function $y = f(x) - 2$ at each value of $x$ in the table in **Figure 10**(b). At $x = -2$,

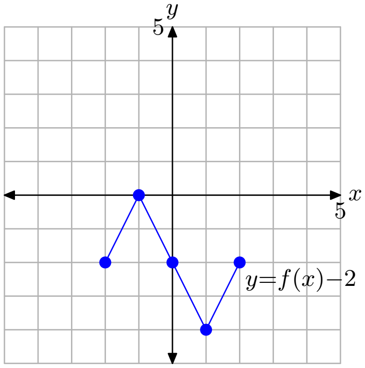
$$y = f(-2) - 2 = 0 - 2 = -2.$$

In similar fashion, evaluate $y = f(x) - 2$ at each remaining $x$-value in the table in **Figure 10**(b).

$$\begin{aligned} y &= f(-1) - 2 = 2 - 2 = 0 \\ y &= f(0) - 2 = 0 - 2 = -2 \\ y &= f(1) - 2 = -2 - 2 = -4 \\ y &= f(2) - 2 = 0 - 2 = -2 \end{aligned}$$

We assemble these points in the table in **Figure 12**(b) and plot them in **Figure 12**(a).

When you compare the entries in the table in **Figure 12**(b) with the original values in the table in **Figure 10**(b), you'll note that the $x$-values in each table are identical, but the $y$-values in the table in **Figure 12**(b) are all decremented by 2. This makes



| $x$ | $y = f(x) - 2$ | $(x, f(x) - 2)$ |
|---|---|---|
| $-2$ | $-2$ | $(-2, -2)$ |
| $-1$ | $0$ | $(-1, 0)$ |
| $0$ | $-2$ | $(0, -2)$ |
| $1$ | $-4$ | $(1, -4)$ |
| $2$ | $-2$ | $(2, -2)$ |

(a) (b)

**Figure 12.** The graph of $y = f(x) - 2$ and a table of key points on the graph.

sense, because these are the $y$-values of the points associated with the function $y = f(x) - 2$. Of course, all the $y$-values should be 2 less than the $y$-values associated with the original equation $y = f(x)$.

Note the result. The graph of $y = f(x) - 2$ in **Figure 12**(a), when compared with the graph of $y = f(x)$ in **Figure 10**(a), is shifted *downward* 2 units.

◆

Let's summarize what we've learned about vertical translations.

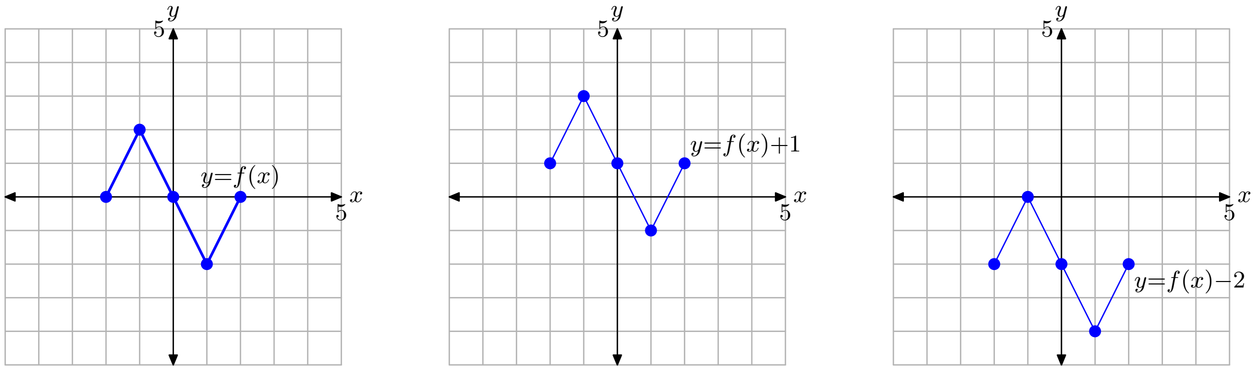
**A Visual Summary — Vertical Translations (Shifts)**. Consider the images in **Figure 13**.

- In **Figure 13**(a), we see pictured the graph of the original function $y = f(x)$.
- In **Figure 13**(b), note that each key point on the graph of $y = f(x) + 1$ has a $y$-value that is precisely 1 unit larger than the $y$-value of the corresponding point on the graph of $y = f(x)$ in in **Figure 13**(a).
- In **Figure 13**(c), note that each key point on the graph of $y = f(x) - 2$ has a $y$-value that is precisely 2 units smaller than the $y$-value of the corresponding point on the graph of $y = f(x)$ in in **Figure 13**(a).
- Note that the $x$-value of each transformed point remains the same.

The visual summary in **Figure 13** makes sketching the graphs of $y = f(x) + 1$ and $y = f(x) - 2$ an easy task.

- Given the graph of $y = f(x)$, to sketch the graph of $y = f(x) + 1$, simply take each point on the graph of $y = f(x)$ and move it upwards 1 unit, keeping the same $x$-value.

- Given the graph of $y = f(x)$, to sketch the graph of $y = f(x) - 2$, simply take each point on the graph of $y = f(x)$ and move it downwards 2 units, keeping the same $x$-value.



(a) $y = f(x)$ (b) $y = f(x) + 1$ (c) $y = f(x) - 2$

**Figure 13.** The graph of $y = f(x) + 1$ is formed by shifting (vertically) the graph of $y = f(x)$ upward 1 unit. The graph of $y = f(x) - 2$ is formed by shifting (vertically) the graph of $y = f(x)$ downward 2 units.

In general, we can state the following.

**Summary 8.** *Suppose that we are given the graph of $y = f(x)$ and suppose that $c$ is any positive real number.*

- *The graph of $y = f(x)+c$ is shifted $c$ units upward from the graph of $y = f(x)$.*
- *The graph of $y = f(x) - c$ is shifted $c$ units downward from the graph of $y = f(x)$.*

## *Composing Transformations*

Sometimes we will want to perform one transformation, then take the result of the first transformation and apply a second transformation. Let's look at an example.

▶ **Example 9.** *Consider the graph of $y = f(x)$ presented in* **Figure 14**.

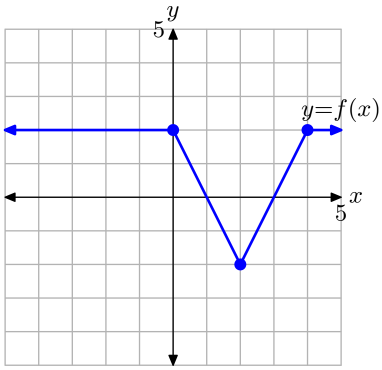


**Figure 14.** The graph of $y = f(x)$ that will be transformed in **Example 9**.

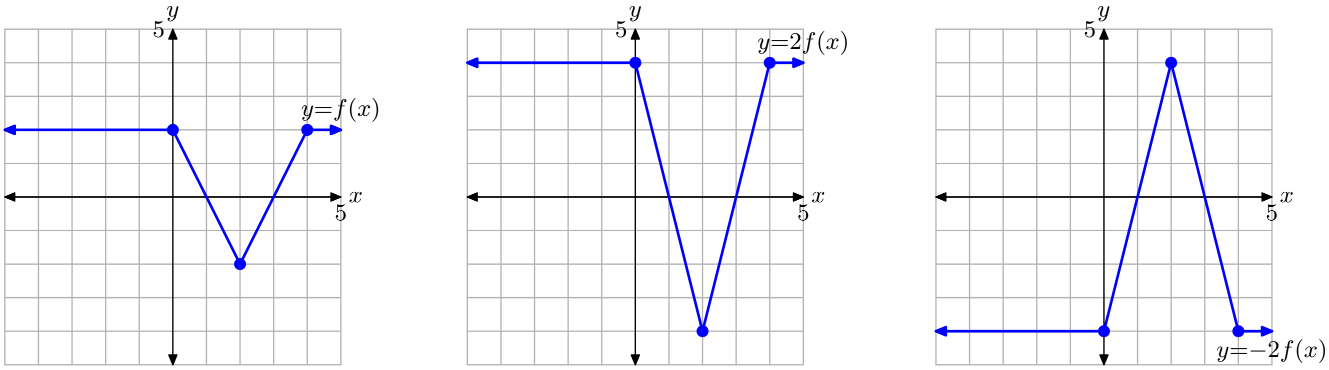
*Use the concepts discussed in the Visual Summaries to sketch the graph of $y = -2f(x)$ without creating and referring to a table of points.*

Note that the equation $y = -2f(x)$ can be formed by a sequence of two transformations.

1. First, scale the original function $y = f(x)$ to obtain the equation $y = 2f(x)$.
2. Second, negate the resulting function $y = 2f(x)$ to obtain the equation $y = -2f(x)$.

Thus, the graph of $y = -2f(x)$ can be formed as follows:

1. Start with the graph of $y = f(x)$ and double the $y$-value of each point on the graph of $y = f(x)$, keeping the same $x$-value. The result is the graph of $y = 2f(x)$ shown in **Figure 15**(b).
2. Next, negate the $y$-value of each point on the graph of $y = 2f(x)$, keeping the same $x$-value. The result is the graph of $y = -2f(x)$ in **Figure 15**(c).



(a) $y = f(x)$ (b) $y = 2f(x)$ (c) $y = -2f(x)$

**Figure 15.** Transforming the graph of $y = f(x)$ with a sequence of two transformations.

It is interesting to note that you will get the same result if you negate first, then scale the result. We will leave it to our readers to check that this is true.

◆

Let's look at one final example.

▶ **Example 10.** *Consider the graph of $y = f(x)$ presented in* **Figure 16**. *Use the concepts discussed in the Visual Summaries to sketch the graph of $y = -f(x)+2$ without creating and referring to a table of points.*

Note that the equation $y = -f(x) + 2$ can be formed by a sequence of two transformations.

1. First, negate the original function $y = f(x)$ to obtain the equation $y = -f(x)$.
2. Second, add 2 to the resulting function $y = -f(x)$ to obtain the equation $y = -f(x) + 2$.

Thus, the graph of $y = -f(x) + 2$ can be formed as follows.

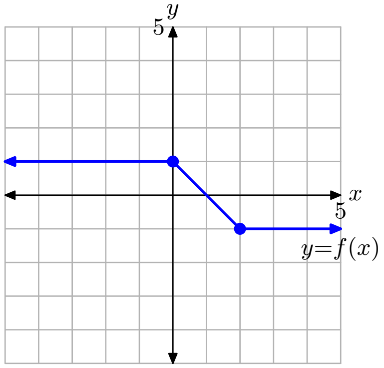
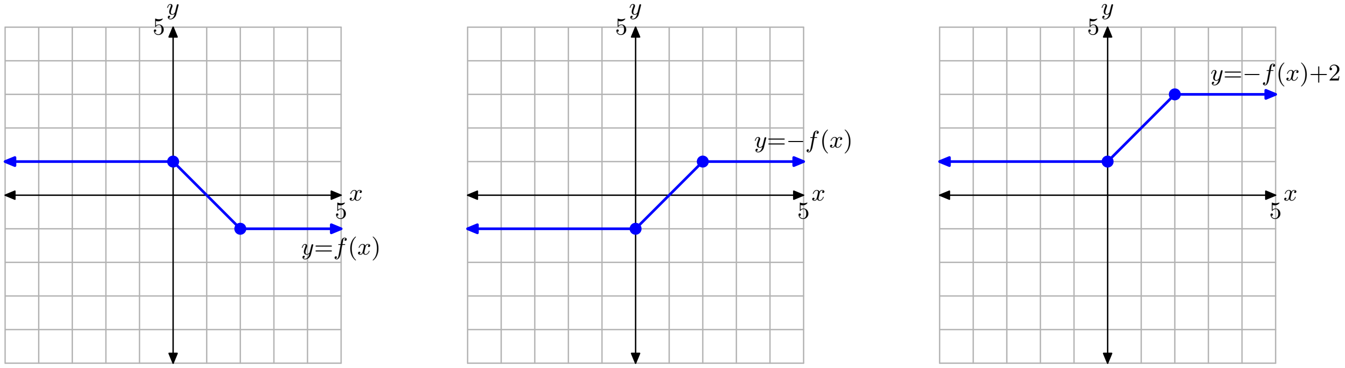


**Figure 16.** The graph of $y = f(x)$ that will be transformed in **Example 10**.

1. First, start with the graph of $y = f(x)$ in **Figure 17**(a) and negate the $y$-value of each point to produce the graph of $y = -f(x)$ **Figure 17**(b).
2. Next, add 2 to the $y$-value of each point on the graph of $y = -f(x)$ in **Figure 17**(b) to produce the graph of $y = -f(x) + 2$ in **Figure 17**(c).



(a) $y = f(x)$ (b) $y = -f(x)$ (c) $y = -f(x) + 2$

**Figure 17.** Transforming the graph of $y = f(x)$, first reflecting across the $x$-axis, then shifting 2 units upward to obtain the graph of $y = -f(x) + 2$.

In **Example 9**, where we started with the graph of $y = f(x)$ and then graphed $y = 2f(x)$, the order of the transformations did not matter. Scale by 2, then negate, or negate and scale by 2, you get the same result (readers should verify this claim). However, in this example, the order in which the transformations are applied **does** matter. To see this, let's do the following:

1. Add 2 to shift the graph of $y = f(x)$ in **Figure 18**(a) two units upward to obtain the graph of $y = f(x) + 2$ in **Figure 18**(b).
2. Negate the $y$-value of each point on the graph of $y = f(x) + 2$ in **Figure 18**(b) to obtain the graph of $y = -(f(x) + 2)$ in **Figure 18**(c). Note that we must negate the **entire** $y$-value. Hence the parentheses.

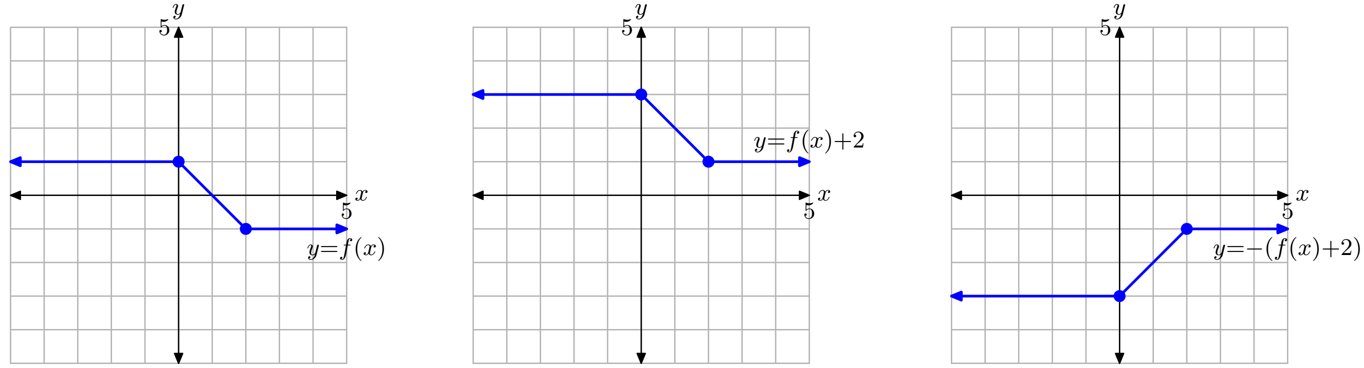
Unfortunately, the graph of $y = -(f(x)+2)$ in **Figure 18**(c) is not the same as the graph of $y = -f(x)+2$ in **Figure 17**(c). But of course, this makes complete sense, as the equations (in the case of **Figure 18**(c))

$$y = -(f(x) + 2) = -f(x) - 2$$

and (in the case of **Figure 17**(c))

$$y = -f(x) + 2 \tag{11}$$

are also not the same.



(a) $y = f(x)$ (b) $y = f(x) + 2$ (c) $y = -(f(x) + 2)$

**Figure 18.** Transforming the graph of $y = f(x)$, shifting 2 units upward to obtain the graph of $y = f(x) + 2$, then reflecting across the $x$-axis to obtain the graph of $y = -(f(x) + 2)$.

Therefore, care must be taken when applying more than one transformation. Here is a good rule of thumb to live by.

**Do Vertical Scalings and Reflections First, then Vertical Translations.** When performing a sequence of vertical transformations, it is usually easier (less confusing) to apply vertical scalings and reflections before vertical translations.

However, as long as you perform the transformations correctly, you should obtain the correct result. In **Example 10**, if you want to sketch the graph of $y = -f(x)+2$ by doing the translation first, the correct way to proceed is as follows (though somewhat counterintuitive):

1. First, shift the graph of $y = f(x)$ downward 2 units to obtain the graph of $y = f(x) - 2$.
2. Second, reflect the graph of $y = f(x) - 2$ across the $x$-axis to obtain the graph of $y = -(f(x) - 2)$. Again, note the use of parentheses as we negate the **entire** $y$-value.

Finally, note that

$$y = -(f(x) - 2) = -f(x) + 2.$$

We will leave it to our readers to show that this sequence produces the correct result, a graph identical to the correct answer shown in **Figure 17**(c).

◆

## *Summary*

In this section we've seen how a handful of transformations greatly enhance our graphing capability. We end this section by listing the transformations presented in this section and their effects on the graph of a function.

**Vertical Transformations**. Suppose we are given the graph of $y = f(x)$.

- If $a > 1$, then the graph of $y = af(x)$ is *stretched* vertically (away from the $x$-axis), both positively and negatively, by a factor of $a$.
- If $0 < a < 1$, then the graph of $y = af(x)$ is *compressed* vertically (toward the $x$-axis), both positively and negatively, by a factor of $1/a$.
- The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ across the $x$-axis.
- If $c > 0$, then the graph of $y = f(x) + c$ is shifted $c$ units upward from the graph of $y = f(x)$.
- If $c > 0$, then the graph of $y = f(x) - c$ is shifted $c$ units downward from the graph of $y = f(x)$.

## 2.5 Exercises

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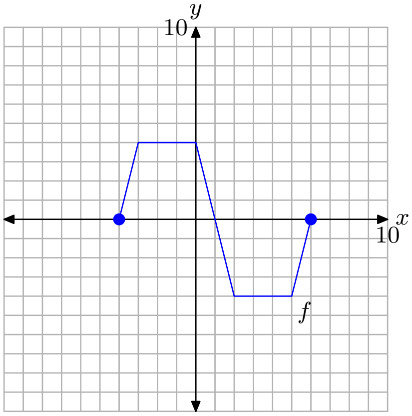
Pictured below is the graph of a function $f$.



The table that follows evaluates the function $f$ in the plot at key values of $x$. Notice the horizontal format, where the first point in the table is the ordered pair $(-4, 0)$.

| $x$ | $-4$ | $-3$ | $0$ | $2$ | $5$ | $6$ |
|---|---|---|---|---|---|---|
| $f(x)$ | 0 | 4 | 4 | $-4$ | $-4$ | 0 |

Use the graph and the table to complete each of following tasks for **Exercises 1-10**.

i. Set up a coordinate system on graph paper. Label and scale each axis, then copy and label the original graph of $f$ onto your coordinate system. *Remember to draw all lines with a ruler.*
ii. Use the original table to help complete the table for the given function in the exercise.
iii. Using a different colored pencil, plot the data from your completed table on the *same* coordinate system as the original graph of $f$. Use these points to help complete the graph of the given function in the exercise, then label this graph with its equation given in the exercise.

**1.** $y = 2f(x)$.

| $x$ | $-4$ | $-3$ | $0$ | $2$ | $5$ | $6$ |
|---|---|---|---|---|---|---|
| $y$ |  |  |  |  |  |  |

**2.** $y = (1/2)f(x)$.

| $x$ | $-4$ | $-3$ | $0$ | $2$ | $5$ | $6$ |
|---|---|---|---|---|---|---|
| $y$ |  |  |  |  |  |  |

**3.** $y = -f(x)$.

| $x$ | $-4$ | $-3$ | $0$ | $2$ | $5$ | $6$ |
|---|---|---|---|---|---|---|
| $y$ |  |  |  |  |  |  |

**4.** $y = f(x) - 2$.

| $x$ | $-4$ | $-3$ | $0$ | $2$ | $5$ | $6$ |
|---|---|---|---|---|---|---|
| $y$ |  |  |  |  |  |  |

**5.** $y = f(x) + 4$.

| $x$ | $-4$ | $-3$ | $0$ | $2$ | $5$ | $6$ |
|---|---|---|---|---|---|---|
| $y$ |  |  |  |  |  |  |

**6.** $y = -2f(x)$.

| $x$ | $-4$ | $-3$ | $0$ | $2$ | $5$ | $6$ |
|---|---|---|---|---|---|---|
| $y$ |  |  |  |  |  |  |

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**7.** $y = (-1/2)f(x)$.

| $x$ | $-4$ | $-3$ | $0$ | $2$ | $5$ | $6$ |
|---|---|---|---|---|---|---|
| $y$ | | | | | | |

**8.** $y = -f(x) + 3$.

| $x$ | $-4$ | $-3$ | $0$ | $2$ | $5$ | $6$ |
|---|---|---|---|---|---|---|
| $y$ | | | | | | |

**9.** $y = -f(x) - 2$.

| $x$ | $-4$ | $-3$ | $0$ | $2$ | $5$ | $6$ |
|---|---|---|---|---|---|---|
| $y$ | | | | | | |

**10.** $y = (-1/2)f(x) + 3$.

| $x$ | $-4$ | $-3$ | $0$ | $2$ | $5$ | $6$ |
|---|---|---|---|---|---|---|
| $y$ | | | | | | |

**11.** Use your graphing calculator to draw the graph of $y = \sqrt{x}$. Then, draw the graph of $y = -\sqrt{x}$. In your own words, explain what you learned from this exercise.

**12.** Use your graphing calculator to draw the graph of $y = |x|$. Then, draw the graph of $y = -|x|$. In your own words, explain what you learned from this exercise.

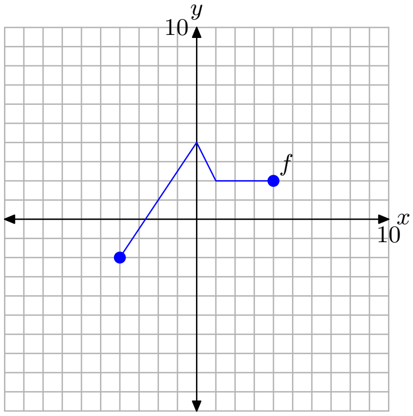
**13.** Use your graphing calculator to draw the graph of $y = x^2$. Then, in succession, draw the graphs of $y = x^2 - 2$, $y = x^2 - 4$, and $y = x^2 - 6$. In your own words, explain what you learned from this exercise.

**14.** Use your graphing calculator to draw the graph of $y = x^2$. Then, in succession, draw the graphs of $y = x^2 + 2$, $y = x^2 + 4$, and $y = x^2 + 6$. In your own words, explain what you learned from this exercise.

**15.** Use your graphing calculator to draw the graph of $y = |x|$. Then, in succession, draw the graphs of $y = 2|x|$, $y = 3|x|$, and $y = 4|x|$. In your own words, explain what you learned from this exercise.

**16.** Use your graphing calculator to draw the graph of $y = |x|$. Then, in succession, draw the graphs of $y = (1/2)|x|$, $y = (1/3)|x|$, and $y = (1/4)|x|$. In your own words, explain what you learned from this exercise.

Pictured below is the graph of a function $f$. In **Exercises 17-22**, use this graph to perform each of the following tasks.



i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Make an exact copy of the graph of $f$ on your coordinate system. *Remember to draw all lines with a ruler.*

ii. In the narrative, a shadow box at the end of the section summarizes the concepts and technique of vertical scaling, vertical reflection, and vertical translation. Use the shortcut ideas presented in this summary shadow box to draw the graphs of the functions that follow **without** using tables.

iii. Use a different colored pencil to draw the graph of the function given in the exercise. Label this graph with its equation. Be sure that key points are accurately plotted. In each exercise, please plot exactly two plots per coordinate system, the graph of original function $f$ and the graph of the function in the exercise.

**17.** $y = (1/2)f(x)$.

**18.** $y = 2f(x)$.

**19.** $y = -f(x)$.

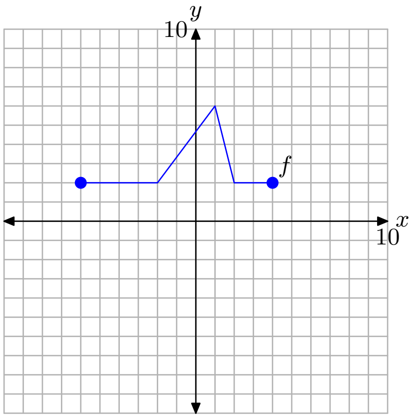
**20.** $y = f(x) - 1$.

**21.** $y = f(x) + 3$.

**22.** $y = f(x) - 4$.

---

Pictured below is the graph of a function $f$. In **Exercises 23-28**, use this graph to perform each of the following tasks.



i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Make an exact copy of the graph of $f$ on your coordinate system. *Remember to draw all lines with a ruler.*

ii. In the narrative, a shadow box at the end of the section summarizes the concepts and technique of vertical scaling, vertical reflection, and vertical translation. Use the shortcut ideas presented in this summary shadow box to draw the graphs of the functions that follow **without** using tables.

iii. Use a different colored pencil to draw the graph of the function given in the exercise. Label this graph with its equation. Be sure that key points are accurately plotted. In each exercise, please plot exactly two plots per coordinate system, the graph of original function $f$ and the graph of the function in the exercise.

**23.** $y = 2f(x)$.

**24.** $y = (1/2)f(x)$.

**25.** $y = -f(x)$.

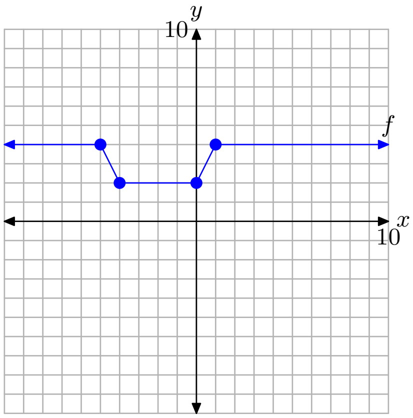
**26.** $y = f(x) + 3$.

**27.** $y = f(x) - 2$.

**28.** $y = f(x) - 1$.

---

Pictured below is the graph of a function $f$. In **Exercises 29-34**, use this graph to perform each of the following tasks.



i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Make an exact copy of the graph of $f$ on your coordinate system. *Remember to draw all lines with a ruler.*

ii. In the narrative, a shadow box at the end of the section summarizes the concepts and technique of vertical scaling, vertical reflection, and vertical translation. Use the shortcut ideas presented in this summary shadow box to draw the graphs of the functions that follow **without** using tables.

iii. Use a different colored pencil to draw the graph of the function given in the exercise. Label this graph with its equation. Be sure that key points are accurately plotted. In each exercise, please plot exactly two plots per coordinate system, the graph of original function $f$ and the graph of the function in the exercise.

**29.** $y = (-1/2)f(x)$.

**30.** $y = -2f(x)$.

**31.** $y = -f(x) + 2$.

**32.** $y = -f(x) - 3$.

**33.** $y = 2f(x) - 3$.

**34.** $y = (-1/2)f(x) + 1$.

## 2.5 *Answers*

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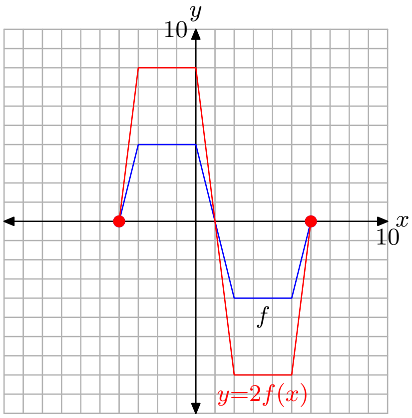
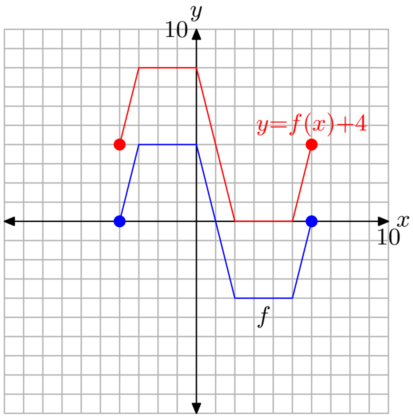
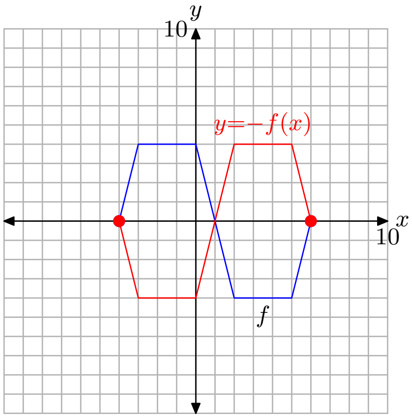
**1.**



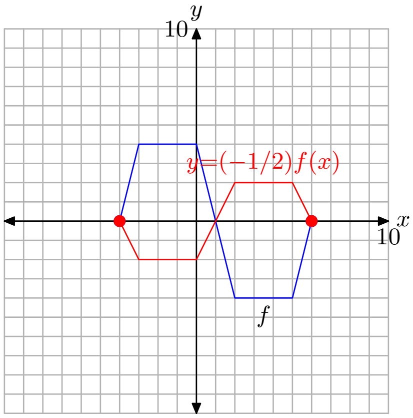
**5.**



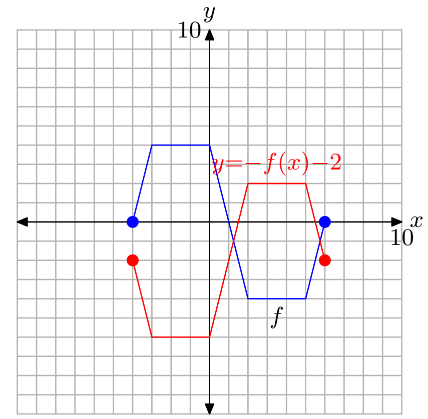
**3.**



**7.**



**9.**

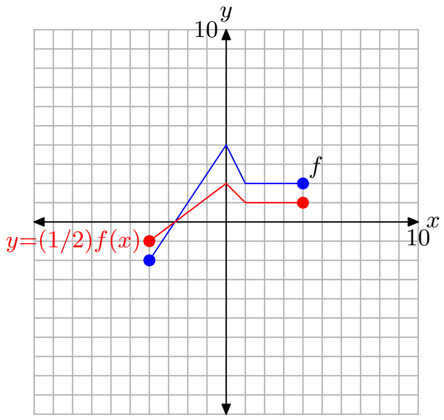


**11.** Mutiplying by $-1$, as in $y = -\sqrt{x}$, reflects the graph across the $x$-axis.

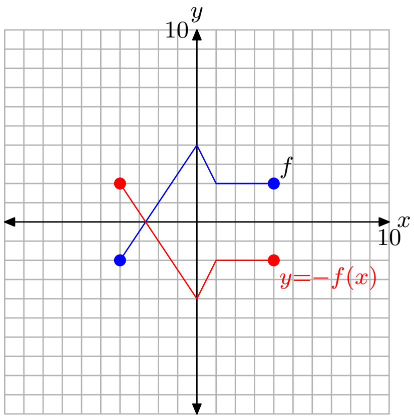
**13.** Subtracting $c$, where $c > 0$, moves the graph $c$ units downward.

**15.** Multiply by a scalar $a$, such that $a$ is larger than 1, stretches the graph vertically by a factor of $a$.

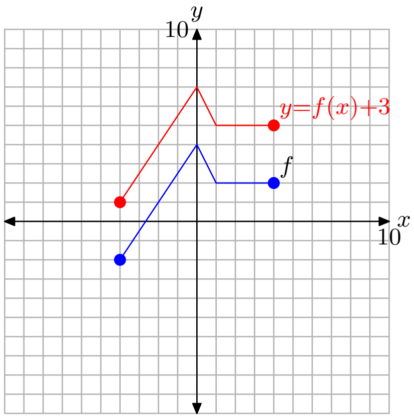
**17.**



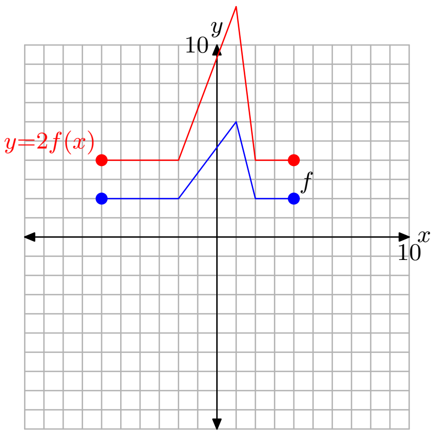
**19.**



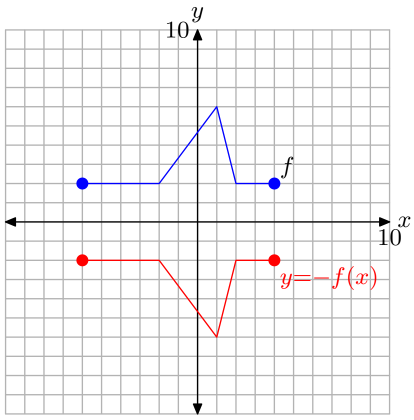
**21.**



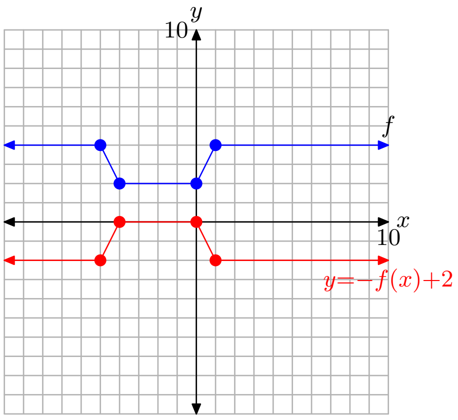
**23.**



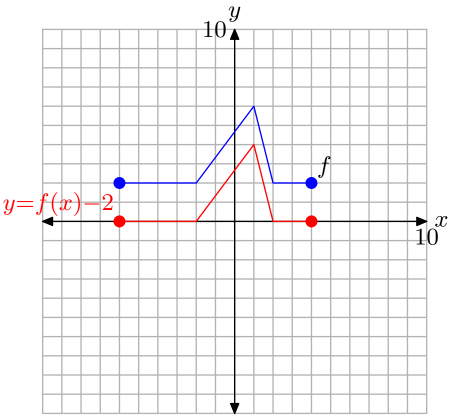
**25.**


y
10
x
10
f
y=−f(x)

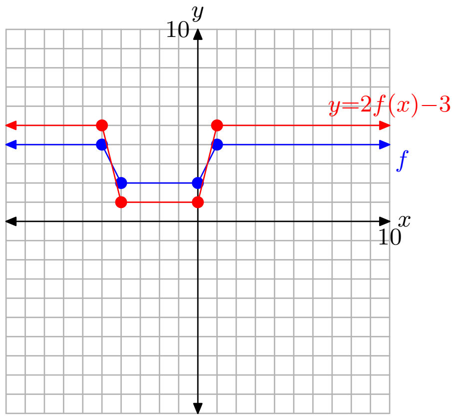
**31.**


y
10
x
10
f
y=−f(x)+2

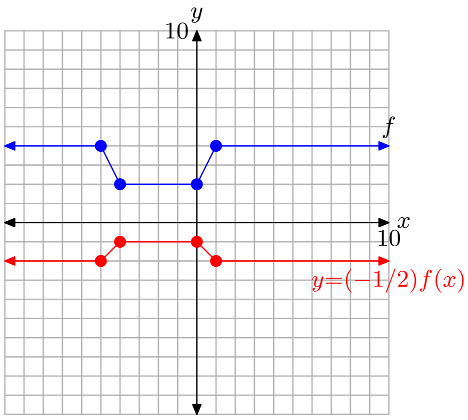
**27.**


y
10
x
10
f
y=f(x)−2

**33.**


y
10
x
10
y=2f(x)−3
f

**29.**

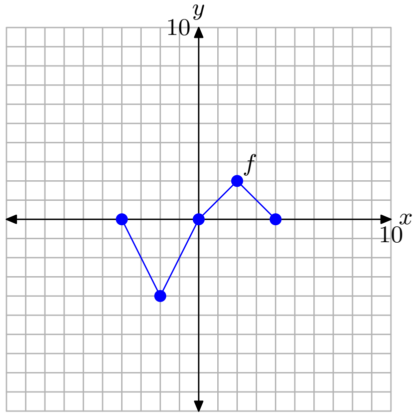

y
10
x
10
f
y=(−1/2)f(x)

# 2.6 Horizontal Transformations

In the previous section, we introduced the concept of transformations. We made a change to the basic equation $y = f(x)$, such as $y = af(x)$, $y = -f(x)$, $y = f(x) - c$, or $y = f(x) + c$, then studied how these changes affected the shape of the graph of $y = f(x)$. In that section, we concentrated strictly on transformations that applied in th vertical direction. In this section, we will study transformations that will affect the shape of the graph in the horizontal direction.

We begin our task with an example that requires that we read the graph of a function to capture several key points that lie on the graph of the function.

▶ **Example 1.** *Consider the graph of f presented in Figure 1(a). Use the graph of f to complete the table in Figure 1(b).*



| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-4$ | | |
| $-2$ | | |
| $0$ | | |
| $2$ | | |
| $4$ | | |

*(a) The graph of $f$.* *(b) The table.*

**Figure 1.** Reading key values from the graph of $f$.

To compute $f(-2)$, for example, we would first locate $-2$ on the $x$-axis, draw a vertical arrow to the graph of $f$, then a horizontal arrow to the $y$-axis, as shown in **Figure 2**(a). The $y$-value of this final destination is the value of $f(-2)$. That is, $f(-2) = -4$. This allows us to complete one entry in the table, as shown in **Figure 2**(b). Continue in this manner to complete all of the entries in the table. The result is shown in **Figure 2**(c).

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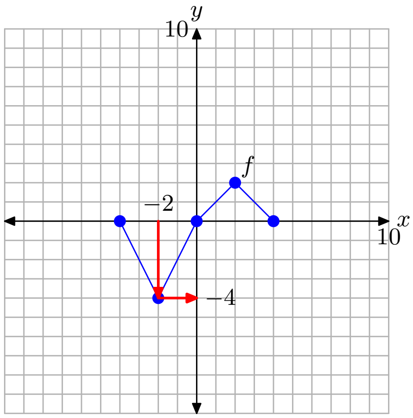


| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-4$ | | |
| $-2$ | $-4$ | $(-2, -4)$ |
| $0$ | | |
| $2$ | | |
| $4$ | | |

| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-4$ | $0$ | $(-4, 0)$ |
| $-2$ | $-4$ | $(-2, -4)$ |
| $0$ | $0$ | $(0, 0)$ |
| $2$ | $2$ | $(2, 2)$ |
| $4$ | $0$ | $(4, 0)$ |

(a) The graph of $f$.

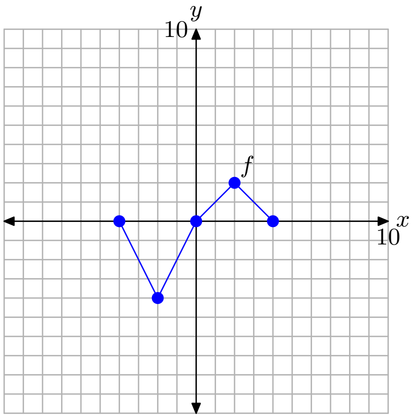
(b) Recording $f(-2) = -4$.

(c) Completed table.

**Figure 2.** Recording coordinates of points on the graph of $f$ in the tables.

## Horizontal Scaling

In the narrative that follows, we will have repeated need of the graph in **Figure 2**(a) and the table in **Figure 2**(c). They characterize the basic function that will be the starting point for the concepts of scaling, reflection, and translation that we develop in this section. Consequently, let's place them side-by-side for emphasis in **Figure 3**.



| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-4$ | $0$ | $(-4, 0)$ |
| $-2$ | $-4$ | $(-2, -4)$ |
| $0$ | $0$ | $(0, 0)$ |
| $2$ | $2$ | $(2, 2)$ |
| $4$ | $0$ | $(4, 0)$ |

(a)

(b)

**Figure 3.** The original graph of $f$ and a table of key points on the graph of $f$

We are now going to scale the graph of $f$ in the horizontal direction.

▶ **Example 2.** *If $y = f(x)$ has the graph shown in* **Figure 3***(a), sketch the graph of $y = f(2x)$.*

In the previous section, we investigated the graph of $y = 2f(x)$. The number 2 was *outside* the function notation and as a result we stretched the graph of $y = f(x)$ vertically by a factor of 2. However, note that the 2 is now *inside* the function notation

$y = f(2x)$. Intuition would demand that this might have something to do with scaling in the $x$-direction (horizontal direction), but how?

Again, when we're unsure of the shape of the graph, we rely on plotting a table of points. We begin by picking these $x$-values: $x = -2$, $-1$, 0, 1, and 2. Note that these are precisely half of each of the $x$-values presented in the table in **Figure 3**(b). We will now evaluate the function $y = f(2x)$ at each of these $x$-values. For example, to compute $y = f(2x)$ at $x = -2$, we first insert $x = -2$ for $x$ to obtain

$$y = f(2(-2)) = f(-4).$$

To complete the computation, we must now evaluate $f(-4)$. However, this result is recorded in the table in **Figure 3**(b). There we find that $f(-4) = 0$, and we can complete the computation started above.

$$y = f(2(-2)) = f(-4) = 0$$

In similar fashion, to evaluate the function $y = f(2x)$ at $x = -1$, first substitute $x = -1$ in $y = f(2x)$ to obtain

$$y = f(2(-1)) = f(-2).$$

Now, note that $f(-2)$ is the next recorded value in the table in **Figure 3**(b). There we find that $f(-2) = -4$, so we can complete the computation started above.

$$y = f(2(-1)) = f(-2) = -4$$

At this point, you might see why we chose $x$-values: $-2$, $-1$, 0, 1, and 2. These are precisely half of the $x$-values in the table of original values for the function $y = f(x)$ in **Figure 3**(b). When the values $-2$, $-1$, 0, 1, and 2 are substituted into the function $y = f(2x)$, they are first doubled before we go to look up the function value in the table in **Figure 3**(b).

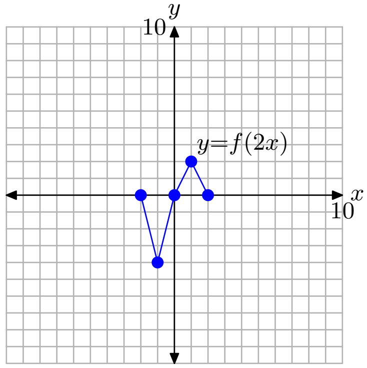
Continuing in this manner, we evaluate the function $y = f(2x)$ at the remaining values of $x$, namely, 0, 1, and 2.

$$\begin{aligned} y &= f(2(0)) = f(0) = 0, \\ y &= f(2(1)) = f(2) = 2, \\ y &= f(2(2)) = f(4) = 0 \end{aligned}$$

We enter these values into the table in **Figure 4**(b) and plot them to determine the graph of $y = f(2x)$ in **Figure 4**(a).

At this point, there are a number of comparisons you can make.

1. Compare the data in the table in **Figure 4**(b) with the original function data in the table in **Figure 3**(b). Note that the y-values in each table are identical. However, note that each $x$-value in the table of **Figure 4**(b) is precisely half of the corresponding $x$-value in the table of **Figure 3**(b).
2. Compare the graph of $y = f(2x)$ in **Figure 4**(a) with the original graph of $y = f(x)$ in **Figure 3**(a). Note that each $x$-value at each point on the graph of $y = f(2x)$ in



(a)

| $x$ | $y=f(2x)$ | $(x,f(2x))$ |
|---|---|---|
| $-2$ | $0$ | $(-2,0)$ |
| $-1$ | $-4$ | $(-1,-4)$ |
| $0$ | $0$ | $(0,0)$ |
| $1$ | $2$ | $(1,2)$ |
| $2$ | $0$ | $(2,0)$ |

(b)

**Figure 4.** The points in the table are points on the graph of $y = f(2x)$.

**Figure 4**(a) is precisely half the $x$-value of the corresponding point on the graph of $y = f(x)$ in **Figure 3**(a).

Note the result. The graph of $y = f(2x)$ is compressed horizontally (toward the $y$-axis), both positively and negatively, by a factor of 2. Note that this is exactly the opposite of what you might expect by intuition, but a careful examination of the data in the tables in Figures **3**(b) and **4**(b) will explain why.

◇

Let's look at another example.

► **Example 3.** *If $y = f(x)$ has the graph shown in* **Figure 3***(a), sketch the graph of $y = f((1/2)x)$.*

Rather than doubling each value of $x$ at the start, this function first halves each value of $x$. Thus, we will want to evaluate the function $y = f((1/2)x)$ at $x = -8$, $-4$, 0, 4, and 8. For example, to evaluate the function $y = f((1/2)x)$ at $x = -8$, first substitute $x = -8$ to obtain

$$y = f((1/2)(-8)) = f(-4).$$

Now, look up this value in the table in **Figure 3**(b) and note that $f(-4) = 0$. Thus, we can complete the computation as follows.

$$y = f((1/2)(-8)) = f(-4) = 0$$

Similarly, to evaluate the function $y = f((1/2)x)$ at $x = -4$, first substitute $x = -4$ to obtain

$$y = f((1/2)(-4)) = f(-2).$$

Now, look up this value in the table in **Figure 3**(b) and note that $f(-2) = -4$. Thus, we can complete the computation as follows.

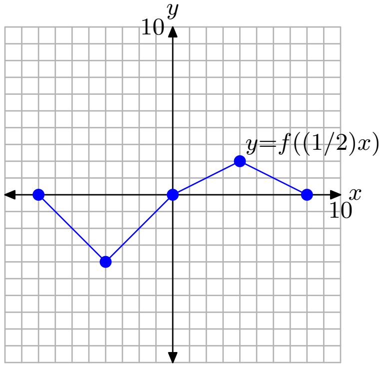
$$y = f((1/2)(-4)) = f(-2) = -4$$

At this point, you will see why we chose $x$-values: $-8$, $-4$, 0, 4, and 8. These values are precisely double the $x$-values in the table of original values for the function $y = f(x)$ in **Figure 3**(b). When the values $-8$, $-4$, 0, 4, and 8 are substituted into the function $y = f((1/2)x)$, they are first halved before we go to look up the function value in the table in **Figure 3**(b). This halving leads to the values $-4$, $-2$, 0, 2, and 4, which are precisely the values available in the table in **Figure 3**(b).

We make similar computations at the remaining values of $x$, namely $x = 0$, 4, and 8.

$$\begin{aligned} y &= f((1/2)(0)) = f(0) = 0 \\ y &= f((1/2)(4)) = f(2) = 2 \\ y &= f((1/2)(8)) = f(4) = 0 \end{aligned}$$

Hopefully, these computations explain our choice of $x$-values above. Each of these results is recorded in the table in **Figure 5**(b) and plotted on the graph shown in **Figure 5**(a).



(a)

| $x$ | $y = f((1/2)x)$ | $(x, f((1/2)x))$ |
|---|---|---|
| $-8$ | 0 | $(-8, 0)$ |
| $-4$ | $-4$ | $(-4, -4)$ |
| 0 | 0 | $(0, 0)$ |
| 4 | 2 | $(4, 2)$ |
| 8 | 0 | $(8, 0)$ |

(b)

**Figure 5.** The points in the table are points on the graph of $y = f((1/2)x)$.

Again, note that the $y$-values in the table in **Figure 5**(b) are identical to the $y$-values in the table in **Figure 3**(b). However, each $x$-value in the table in **Figure 5**(b) is precisely double the corresponding $x$-value in the table in **Figure 3**(b).

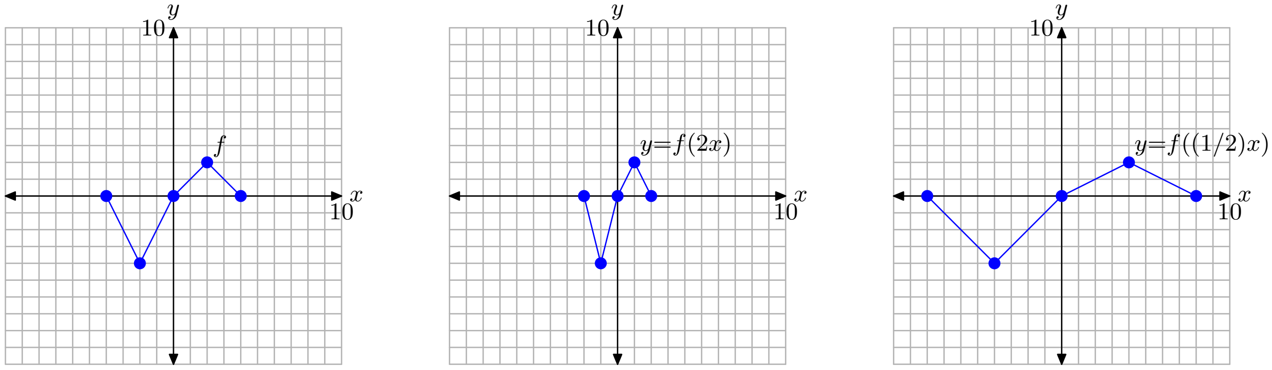
This doubling of the $x$-values is apparent in the graph of $y = f((1/2)x)$ shown in **Figure 5**(a), where the graph is stretched by a factor of 2 horizontally (away from the $y$-axis), both positively and negatively. Note that this is exactly the opposite of what you might expect by intuition, but a careful examination of the data in the tables in Figures **3**(b) and **5**(b) will explain why.

◆

Let's summarize our findings.

**A Visual Summary — Horizontal Scaling**. Consider the images in **Figure 6**.

- In **Figure 6**(a), we see pictured the graph of the original function $y = f(x)$.
- In **Figure 6**(b), note that each key point on the graph of $y = f(2x)$ has an $x$-value that is precisely half the $x$-value of the corresponding point on the graph of $y = f(x)$ in **Figure 6**(a).
- In **Figure 6**(c), note that each key point on the graph of $y = f((1/2)x)$ has an $x$-value that is twice the $x$-value of the corresponding point on the graph of $y = f(x)$ in **Figure 6**(a).
- Note that the $y$-value of each transformed point remains the same.



(a) $y = f(x)$ (b) $y = f(2x)$ (c) $y = f((1/2)x)$

**Figure 6.** The graph of $y = f(2x)$ compresses horizontally (toward the $y$-axis) by a factor of 2. The graph of $y = f((1/2)x)$ stretches horizontally (away from the $y$-axis) by a factor of 2.

The visual summary in **Figure 6** makes sketching the graphs of $y = f(2x)$ and $y = f(1/2)x)$ an easy task.

- Given the graph of $y = f(x)$, to sketch the graph of $y = f(2x)$, simply take each point on the graph of $y = f(x)$ and cut its $x$-value in half, keeping the same $y$-value.
- Given the graph of $y = f(x)$, to sketch the graph of $y = f((1/2)x)$, simply take each point on the graph of $y = f(x)$ and double its $x$-value, keeping the same $y$-value.

Follow the same procedures for other scaling factors. For example, in the case of $y = f(3x)$, take each point on the graph of $y = f(x)$ and divide its $x$-value by 3, keeping the same $y$-value. On the other hand, to draw the graph of $y = f((1/3)x)$, take each point on the graph of $f$ and multiply its $x$-value by 3, keeping the same $y$-value.

In general, we can state the following.

**Summary 4.** *Suppose we are given the graph of $y = f(x)$.*

- *If $a > 1$, the graph of $y = f(ax)$ compresses horizontally (toward the $y$-axis) , both positively and negatively, by a factor of $a$.*
- *If $0 < a < 1$, the graph of $y = f(ax)$ stretches horizontally (away from the $y$-axis), both positively and negatively, by a factor of $1/a$.*

In the case of the first item in **Summary 4**, when we compare the general form $y = f(ax)$ with $y = f(2x)$, we see that $a = 2$. In this case, note that $a > 1$ and the graph of $y = f(2x)$ compresses horizontally by a factor of 2 when compared with the graph of $y = f(x)$ (see **Figure 6**(b)).

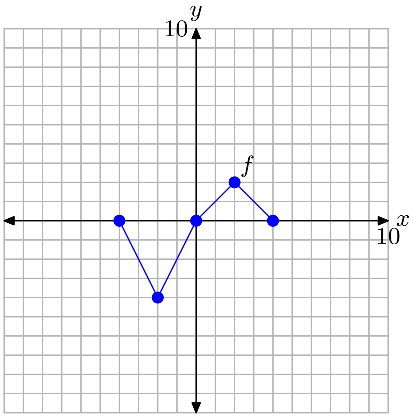
In the case of the second item in **Summary 4**, when we compare the general form $y = f(ax)$ with the equation $y = f((1/2)x)$, we see that $a = 1/2$, so

$$\frac{1}{a} = \frac{1}{1/2} = 2.$$

The second item in **Summary 4** says that when $0 < a < 1$, the graph of $y = f(ax)$ stretches horizontally by a factor of $1/a$. Indeed, this is exactly what happened in the case of $y = f((1/2)x)$, which stretched in the horizontal direction by a factor of $1/(1/2)$, or 2 (see **Figure 6**(c)).

## *Horizontal Reflections*

For convenience, we begin by repeating the original graph of $y = f(x)$ and its accompanying data in **Figure 7**. We are now going to reflect the graph of $y = f(x)$ in the horizontal direction (across the $y$-axis).



(a)

| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-4$ | 0 | $(-4, 0)$ |
| $-2$ | $-4$ | $(-2, -4)$ |
| 0 | 0 | $(0, 0)$ |
| 2 | 2 | $(2, 2)$ |
| 4 | 0 | $(4, 0)$ |

(b)

**Figure 7.** The original graph of $f$ and a table of key points on the graph of $f$

▶ **Example 5.** *If $y = f(x)$ has the graph shown in **Figure 7**(a), draw the graph of $y = f(-x)$.*

In the previous section, we were asked to draw the graph of $y = -f(x)$. Note how the minus sign appears on the *outside* of the function. Clearly, the $y$-values of $y = -f(x)$ must be opposite in sign to the $y$-values of $y = f(x)$. That is why the graph of $y = -f(x)$ was a reflection of the graph of $y = f(x)$ across the $x$-axis.

However, in this example, the minus sign is *inside* the function, leaving one to intuit that it is the $x$-values, not the $y$-values, that are being negated. We will choose the following $x$-values: $x = 4$, 2, 0, $-2$, and $-4$. This is a bit deceptive, as it looks like we are choosing the same $x$-values, only in reverse order. This is not the case. We are choosing the negative of each $x$-value in the table in **Figure 7**(b).

To evaluate $y = f(-x)$ at our first $x$-value, namely $x = 4$, we perform the following calculation. First substitute $x = 4$ to obtain

$$y = f(-(4)) = f(-4).$$

Now, look up this value in the table in **Figure 7**(b) and note that $f(-4) = 0$. Thus, we can complete the computation as follows.

$$y = f(-(4)) = f(-4) = 0$$

Similarly, to evaluate the function $y = f(-x)$ at $x = 2$, first substitute $x = 2$ to obtain

$$y = f(-(2)) = f(-2).$$

Now, look up this value in the table in **Figure 7**(b) and note that $f(-2) = -4$. Thus, we can complete the computation as follows.

$$y = f(-(2)) = f(-2) = -4$$

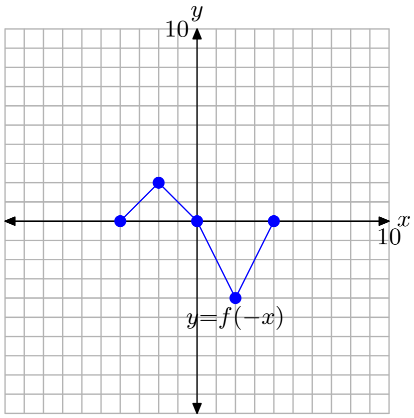
At this point, you will see why we chose $x$-values: 4, 2, 0, $-2$, and $-4$. These values are the negatives of the $x$-values in the table of original values for the function $y = f(x)$ in **Figure 7**(b). When the values 4, 2, 0, $-2$, and $-4$ are substituted into the function $y = f(-x)$, they are first negated before we go to look up the function value in the table in **Figure 7**(b). This negating leads to the values $-4$, $-2$, 0, 2, and 4, which are precisely the values available in the table in **Figure 7**(b).

We make similar computations at the remaining values of $x$, namely $x = 0$, $-2$, and $-4$.

$$\begin{aligned} y &= f(-(0)) = f(0) = 0 \\ y &= f(-(-2)) = f(2) = 2 \\ y &= f(-(-4)) = f(4) = 0 \end{aligned}$$

We organize these points in the table in **Figure 8**(b), then plot them in **Figure 8**(a).

When you compare the entries in the table in **Figure 8**(b) with those in the table in **Figure 7**(b), note that the $y$-values appear in the same order, but the $x$-values of the table in **Figure 7**(b) have been negated in the table in **Figure 8**(b). This means that a former point such as $(-2, -4)$ is transformed to the point $(2, -4)$, which is a reflection of the point $(-2, -4)$ across the $y$-axis.



(a)

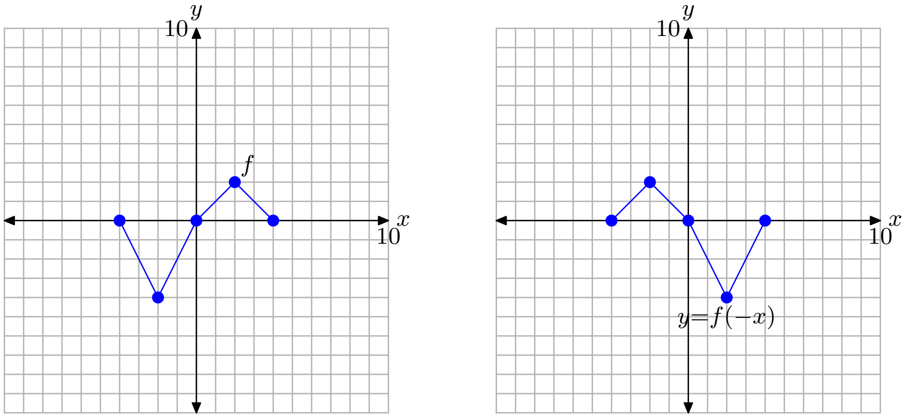
| $x$ | $y = f(-x)$ | $(x, f(-x))$ |
|---|---|---|
| 4 | 0 | $(4, 0)$ |
| 2 | $-4$ | $(2, -4)$ |
| 0 | 0 | $(0, 0)$ |
| $-2$ | 2 | $(-2, 2)$ |
| $-4$ | 0 | $(-4, 0)$ |

(b)

**Figure 8.** The graph of $y = f(-x)$ and a table of key points on the graph.

Thus, to produce the graph of $y = f(-x)$, simply reflect the graph of $y = f(x)$ across the $y$-axis.

Let's summarize what we've learned about horizontal reflections.



(a) $y = f(x)$ (b) $y = f(-x)$

**Figure 9.** The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ across the $y$-axis.

**A Visual Summary — Horizontal Reflections.** Consider the images in **Figure 9**.

- In **Figure 9**(a), we see pictured the original graph of $y = f(x)$.
- In **Figure 9**(b), the graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ across the $y$-axis.

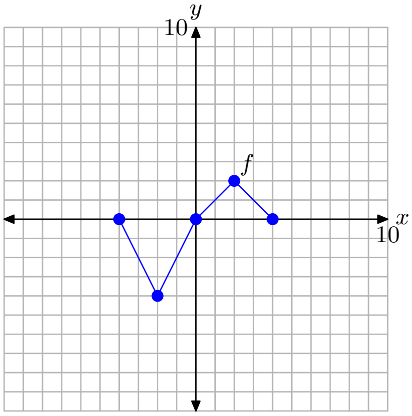
Thus, given the graph of $y = f(x)$, it is a simple task to draw the graph of $y = f(-x)$.

- To draw the graph of $y = f(-x)$, take each point on the graph of $y = f(x)$ and reflect it across the $y$-axis, keeping the $y$-value the same, but negating the $x$-value.

## *Horizontal Translations*

In the previous section, we saw that the graphs of $y = f(x) + c$ and $y = f(x) - c$ were vertical translations of the graph of $y = f(x)$. If $c$ is a positive number, then the graph of $y = f(x) + c$ shifts $c$ units upward while the graph of $y = f(x) - c$ shifts $c$ units downward.

In this section, we will study horizontal translations. For convenience, we begin by repeating the original graph of $y = f(x)$ and its accompanying data in **Figure 10**.



(a)

| $x$ | $f(x)$ | $(x, f(x))$ |
|---|---|---|
| $-4$ | $0$ | $(-4, 0)$ |
| $-2$ | $-4$ | $(-2, -4)$ |
| $0$ | $0$ | $(0, 0)$ |
| $2$ | $2$ | $(2, 2)$ |
| $4$ | $0$ | $(4, 0)$ |

(b)

**Figure 10.** The original graph of $f$ and a table of key points on the graph of $f$

▶ **Example 6.** *If $y = f(x)$ has the graph shown in* **Figure 10***(a), sketch the graph of $y = f(x + 1)$.*

In the previous section, we drew the graph of $y = f(x)+1$. Note that in $y = f(x)+1$, the number 1 is *outside* the function. The result was a graph that was shifted 1 unit upwards in the $y$-direction.

In this case, $y = f(x + 1)$ and the 1 is *inside* the function notation, leading one to intuit that the translation might be in the horizontal direction ($x$-direction). But how?

Again, we will set up a table of points that satisfy the equation $y = f(x + 1)$, then plot them. Because this function requires that we first add 1 to each $x$-value before inserting it into the function, we will choose $x$-values appropriately, namely $x = -5$, $-3$, $-1$, 1, and 3. In a moment, it will be clear why we have chosen these particular values of $x$. Perhaps you already see why?

We need to evaluate the function $y = f(x + 1)$ at each of these chosen values of $x$. To evaluate $y = f(x+1)$ at the first value, namely $x = -5$, we insert $x = -5$ and make the calculation

$$y = f(-5+1) = f(-4).$$

To complete the calculation, we must now evaluate $f(-4)$. However, this result is recorded in the table in **Figure 10**(b). There we find that $f(-4) = 0$, and we can complete the calculation started above.

$$y = f(-5+1) = f(-4) = 0$$

In similar fashion, we can evaluate the function $y = f(x+1)$ at $x = -3$. First, substitute $x = -3$ in $y = f(x+1)$ to obtain

$$y = f(-3+1) = f(-2).$$

To complete the calculation, we must now evaluate $f(-2)$. However, this result is recorded in the table in **Figure 10**(b). There we find that $f(-2) = -4$, and we can complete the calculation started above.

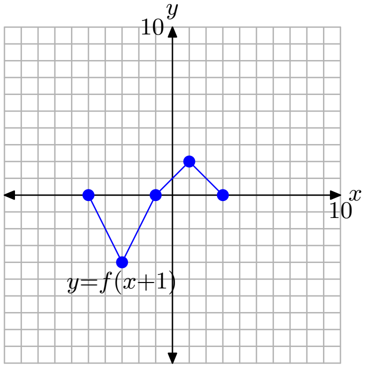
$$y = f(-3+1) = f(-2) = -4$$

At this point, you might see why we chose $x$-values: $-5$, $-3$, $-1$, 1, and 3. These are precisely one less than the $x$-values in the table of original values for the function $y = f(x)$ in **Figure 10**(b). When the values $-5$, $-3$, $-1$, 1, and 3 are substituted into the function $y = f(x+1)$, we first add 1 to each value before we go to look up the function value in the table in **Figure 10**(b). This adding of 1 leads to the values $-4$, $-2$, 0, 2, and 4, which are precisely the values available in the table in **Figure 10**(b).

Continuing in this manner, we evaluate the function $y = f(x+1)$ at the remaining values of $x$, namely, $-1$, 1, and 3.

$$\begin{aligned} y &= f(-1+1) = f(0) = 0 \\ y &= f(1+1) = f(2) = 2 \\ y &= f(3+1) = f(4) = 0 \end{aligned}$$

We assemble these results in the table in **Figure 11**(b) and plot them in **Figure 11**(a).



(a)

| $x$ | $y = f(x+1)$ | $(x, f(x+1))$ |
|---|---|---|
| $-5$ | 0 | $(-5, 0)$ |
| $-3$ | $-4$ | $(-3, -4)$ |
| $-1$ | 0 | $(-1, 0)$ |
| 1 | 2 | $(1, 2)$ |
| 3 | 0 | $(3, 0)$ |

(b)

**Figure 11.** The graph of $y = f(x+1)$ and a table of key points on the graph.

When you compare the points on the graph of $y = f(x+1)$ in the table in **Figure 11**(b) with the original points on the graph of $y = f(x)$ in the table in **Figure 10**(b), note that the $y$-values are identical, but the $x$-values in the table in **Figure 11**(b) are all 1 unit less than the corresponding $x$-values in the table in **Figure 10**(b). It is no wonder that the graph of $y = f(x+1)$ in **Figure 11**(a) is shifted 1 unit to the left of the original graph of $y = f(x)$ in **Figure 10**(a).

Note that this is somewhat counterintuitive, because we're seemingly adding 1 to each $x$-value in $y = f(x+1)$. Why doesn't the graph move one unit to the right? Well, a careful comparison of the $x$-values in the tables in Figures **10**(b) and **11**(b) reveals the answer. In order to use the data in the table in **Figure 10**(b), we must first subtract 1 from each $x$-value to produce the $x$-values in the table in **Figure 11**(b). This is why the graph of $y = f(x+1)$ moves 1 unit to the left instead of 1 unit to the right.

You might also recall that the function $y = f(2x)$ compressed by a factor of 2, which is also the opposite of what intuition might dictate. Similarly, the function $y = f((1/2)x)$ stretches by a factor of 2, which also goes counter to intuition. With these thoughts in mind, it is not surprising that $y = f(x+1)$ shifts one unit to the left. Still, a comparison of the $x$-values in the tables in Figures **10**(b) and **11**(b) provide irrefutable evidence that the shift is 1 unit to the left.

◇

Let's look at another example.

▶ **Example 7.** *If $y = f(x)$ has the graph shown in* **Figure 10***(a), sketch the graph of $y = f(x-2)$.*

Again, we will set up a table of points that satisfy the equation $y = f(x-2)$, then plot them. Because this function requires that we first subtract 2 from each $x$-value before inserting it into the function, we will choose $x$-values: $-2$, 0, 2, 4, and 6. We need to evaluate the function $y = f(x-2)$ at each of these values of $x$.

To evaluate $y = f(x-2)$ at the first value, namely $x = -2$, insert $x = -2$ into the function $y = f(x-2)$ to obtain

$$y = f(-2-2) = f(-4).$$

In the table in **Figure 10**(b), we find that $f(-4) = 0$, which allows us to complete the calculation above.

$$y = f(-2-2) = f(-4) = 0$$

In similar fashion, we evaluate $y = f(x-2)$ at $x = 0$ to obtain

$$y = f(0-2) = f(-2).$$

In the table in **Figure 10**(b), we find that $f(-2) = -4$, which allows us to complete the calculation above.

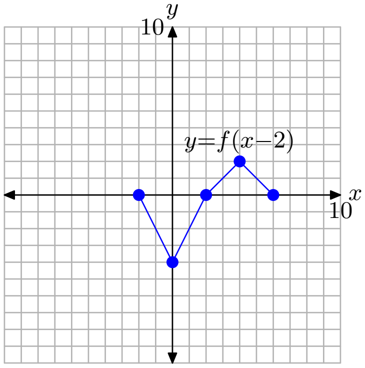
$$y = f(0-2) = f(-2) = -4$$

Hopefully, you see why we chose the $x$-values: $-2$, 0, 2, 4, and 6. These values are 2 larger than the $x$-values in the table of original values for the function $y = f(x)$ in **Figure 10**(b). When the values $-2$, 0, 2, 4, and 6 are substituted into the function $y = f(x - 2)$, we first subtract 2 from each value before we go to look up the function value in the table in **Figure 10**(b). This subtracting of 2 leads to $-4$, $-2$, 0, 2, and 4, precisely the values that are available in the table in **Figure 10**(b).

Continuing in this manner, we evaluate $y = f(x - 2)$ at the remaining values of $x$, namely, $x = 2$, 4, and 6.

$$\begin{aligned} y &= f(2-2) = f(0) = 0 \\ y &= f(4-2) = f(2) = 2 \\ y &= f(6-2) = f(4) = 0 \end{aligned}$$

We assemble these results in the table in **Figure 12**(b) and plot them in **Figure 12**(a).



| $x$ | $y = f(x-2)$ | $(x, f(x-2))$ |
|---|---|---|
| $-2$ | 0 | $(-2, 0)$ |
| 0 | $-4$ | $(0, -4)$ |
| 2 | 0 | $(2, 0)$ |
| 4 | 2 | $(4, 2)$ |
| 6 | 0 | $(6, 0)$ |

(a) (b)

**Figure 12.** The graph of $y = f(x - 2)$ and a table of key points on the graph.

When you compare the points on the graph of $y = f(x-2)$ in the table in **Figure 12**(b) with the original points on the graph of $y = f(x)$ in the table in **Figure 10**(b), note that the $y$-values are identical, but the $x$-values in the table in **Figure 12**(b) are all 2 larger than the corresponding $x$-values in the table in **Figure 10**(b). It is no wonder that the graph of $y = f(x - 2)$ in **Figure 12**(a) is shifted 2 units to the right of the original graph of $y = f(x)$ in **Figure 10**(a).

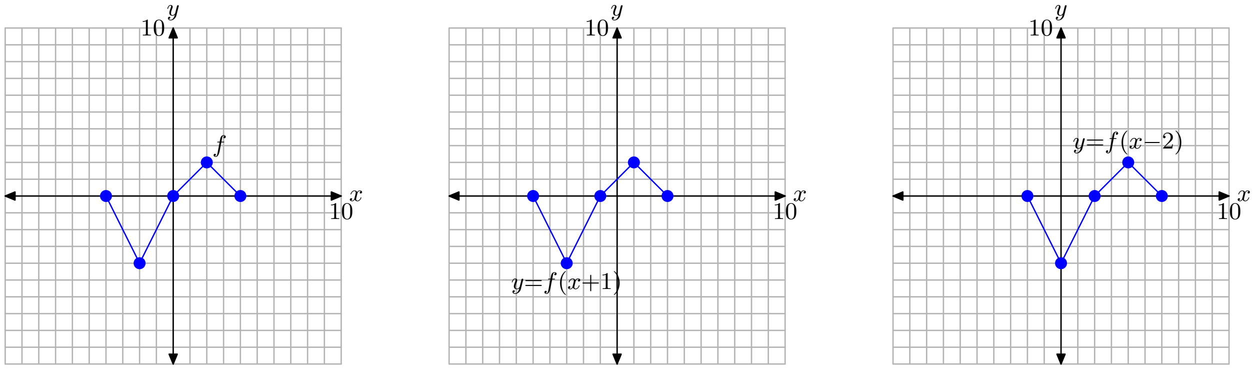
Again, this runs counterintuitive (why doesn't the graph of $y = f(x - 2)$ shift 2 units to the left?), but a comparison of the $x$-values in the tables in Figures **10**(b) and **12**(b) clearly indicates a shift to the right.

◆

Let's summarize what we've learned about horizontal translations.

**Visual Summary — Horizontal Translations (Shifts).** Consider the images in **Figure 13**.

- In **Figure 13**(a), we see pictured the graph of the original function $y = f(x)$.
- In **Figure 13**(b), note that each point on the graph of $y = f(x+1)$ has an $x$-value that is 1 unit less than the $x$-value of the corresponding point on the graph of $y = f(x)$ in **Figure 13**(a).
- In **Figure 13**(c), note that each point on the graph of $y = f(x-2)$ has an $x$-value that is 2 units greater than the $x$-value of the corresponding point on the graph of $y = f(x)$ in **Figure 13**(a).
- Note that the $y$-value of each transformed point remains the same.



(a) $y = f(x)$ (b) $y = f(x+1)$ (c) $y = f(x-2)$

**Figure 13.** The graph of $y = f(x+1)$ is formed by shifting (horizontally) the graph of $y = f(x)$ one unit to the left. The graph of $y = f(x-2)$ is formed by shifting (horizontally) the graph of $y = f(x)$ two units to the right.

The visual summary in **Figure 13** makes sketching the graphs of $y = f(x+1)$ and $y = f(x-2)$ an easy task.

- Given the graph of $y = f(x)$, to sketch the graph of $y = f(x+1)$, simply take each point on the graph of $y = f(x)$ and shift it 1 unit to the left, keeping the same $y$-value.
- Given the graph of $y = f(x)$, to sketch the graph of $y = f(x-2)$, simply take each point on the graph of $y = f(x)$ and shift it 2 units to the right, keeping the same $y$-value.

In general, we can state the following.

**Summary 8.** *Suppose that we are given the graph of $y = f(x)$ and suppose that $c$ is any positive real number.*

- *The graph of $y = f(x+c)$ is shifted $c$ units to the left of the graph of $y = f(x)$.*
- *The graph of $y = f(x-c)$ is shifted $c$ units to the right of the graph of $y = f(x)$.*

When we looked at vertical translations in the previous section, a translation was described by first imagining a graph on a sheet of transparent plastic, then sliding the transparency (without rotating it) over a coordinate system on a sheet of graph paper. Horizontal translations can be thought of in the same way, as sliding the graph on the transparency $c$ units to the left, or $c$ units to the right.

## Extra Practice

In this section, let's take the concepts from the Visual Summaries and put them to work on some final examples.

▶ **Example 9.** *Consider the graph of $f$ in* **Figure 14**.

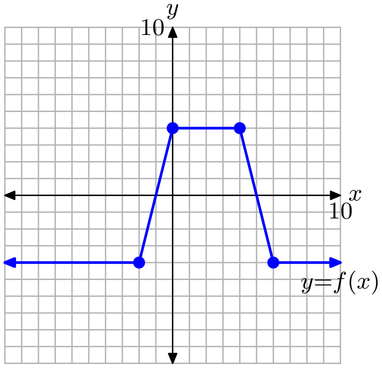
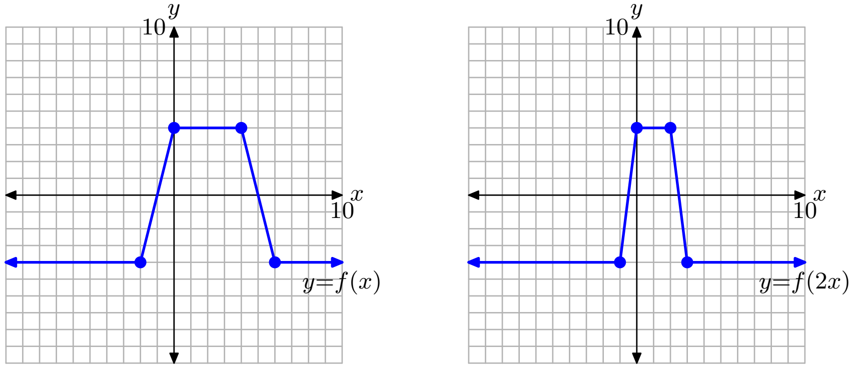


**Figure 14.** The graph of $y = f(x)$ for **Example 9**.

*Use the concepts from the Visual Summaries (scaling, reflection, and translation) to sketch the graphs of $y = f(2x)$, $y = f(-x)$, and $y = f(x+2)$ without creating and referring to tables.*

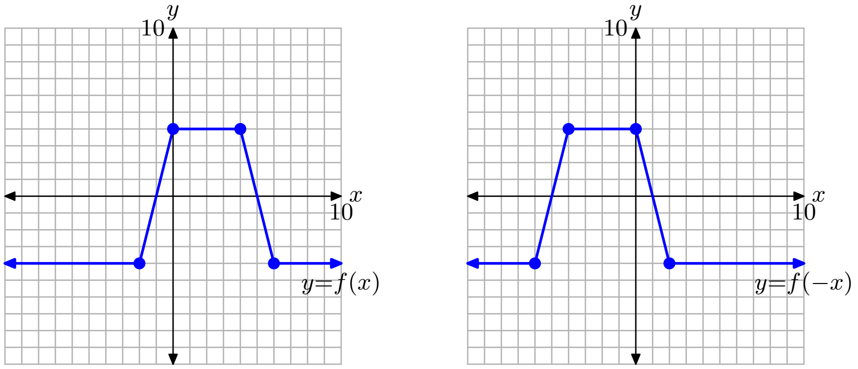
To sketch the graph of $y = f(2x)$, simply take each point on the graph of $y = f(x)$ in **Figure 15**(a) and divide its $x$-value by 2, keeping the same $y$-value. The result is shown in **Figure 15**(b).



(a) $y = f(x)$ (b) $y = f(2x)$

**Figure 15.** Compress the graph of $y = f(x)$ by a factor of 2 to produce the graph of $y = f(2x)$.

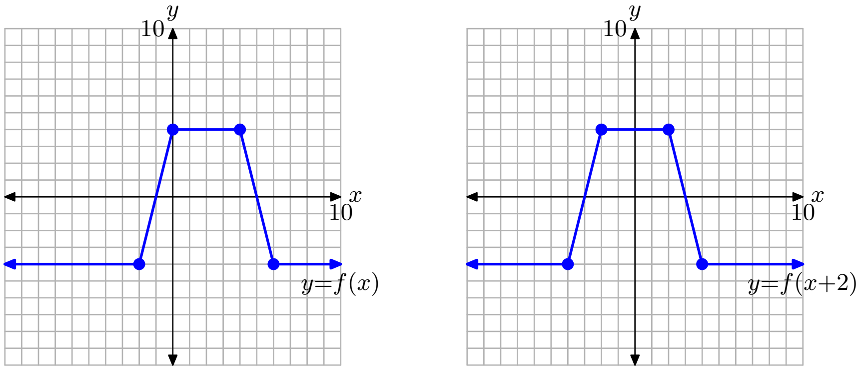
To sketch the graph of $y = f(-x)$, simply take each point on the graph of $y = f(x)$ in **Figure 16**(a) and negate its $x$-value, keeping the same $y$-value. The result is shown in **Figure 16**(b).



(a) $y = f(x)$ (b) $y = f(-x)$

**Figure 16.** Reflect the graph of $y = f(x)$ across the $y$-axis to produce the graph of $y = f(-x)$.

To sketch the graph of $y = f(x + 2)$, simply take each point on the graph of $y = f(x)$ in **Figure 17**(a) and subtract 2 from its $x$-value, keeping the same $y$-value. The result is shown in **Figure 17**(b).



(a) $y = f(x)$ (b) $y = f(x + 2)$

**Figure 17.** Shift the graph of $y = f(x)$ to the left 2 units to produce the graph of $y = f(x + 2)$.

◆

## *Summary*

In this section we've seen how a handful of transformations greatly enhance our graphing capability. We end this section by listing the transformations presented in this section and their effects on the graph of a function.

**Vertical Transformations**. Suppose we are given the graph of $y = f(x)$.

- If $a > 1$, the graph of $y = f(ax)$ compresses horizontally (toward the $y$-axis) , both positively and negatively, by a factor of $a$.
- If $0 < a < 1$, the graph of $y = f(ax)$ stretches horizontally (away from the $y$-axis), both positively and negatively, by a factor of $1/a$.
- The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ across the $y$-axis.
- If $c > 0$, then the graph of $y = f(x + c)$ is shifted $c$ units to the left of the graph of $y = f(x)$.
- If $c > 0$, then the graph of $y = f(x - c)$ is shifted $c$ units to the right of the graph of $y = f(x)$.

## 2.6 Exercises

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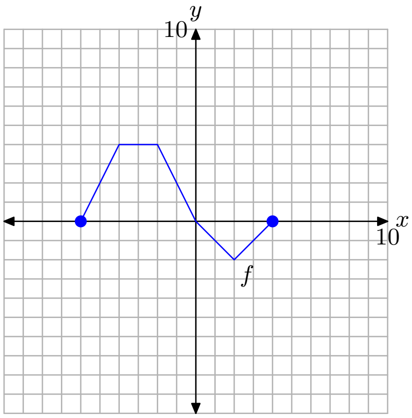
Pictured below is the graph of a function $f$.



The table that follows evaluates the function $f$ in the plot at key values of $x$. Notice the horizontal format, where the first point in the table is the ordered pair $(-6, 0)$.

| $x$ | $-6$ | $-4$ | $-2$ | $0$ | $2$ | $4$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $0$ | $4$ | $4$ | $0$ | $-2$ | $0$ |

Use the graph and the table to complete each of following tasks for **Exercises 1-10**.

i. Set up a coordinate system on graph paper. Label and scale each axis, then copy and label the original graph of $f$ onto your coordinate system. *Remember to draw all lines with a ruler.*
ii. Use the original table to help complete the table for the given function in the exercise.
iii. Using a different colored pencil, plot the data from your completed table on the *same* coordinate system as the original graph of $f$. Use these points to help complete the graph of the given function in the exercise, then label this graph with its equation given in the exercise.

**1.** $y = f(2x)$.

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|---|
| $y$ | | | | | | |

**2.** $y = f((1/2)x)$.

| $x$ | $-12$ | $-8$ | $-4$ | $0$ | $4$ | $8$ |
|---|---|---|---|---|---|---|
| $y$ | | | | | | |

**3.** $y = f(-x)$.

| $x$ | $-4$ | $-2$ | $0$ | $2$ | $4$ | $6$ |
|---|---|---|---|---|---|---|
| $y$ | | | | | | |

**4.** $y = f(x + 3)$.

| $x$ | $-9$ | $-7$ | $-5$ | $-3$ | $-1$ | $1$ |
|---|---|---|---|---|---|---|
| $y$ | | | | | | |

**5.** $y = f(x - 1)$.

| $x$ | $-5$ | $-3$ | $-1$ | $1$ | $3$ | $5$ |
|---|---|---|---|---|---|---|
| $y$ | | | | | | |

**6.** $y = f(-2x)$.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|
| $y$ | | | | | | |

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**7.** $y = f((-1/2)x)$.

| $x$ | $-8$ | $-4$ | $0$ | $4$ | $8$ | $12$ |
|---|---|---|---|---|---|---|
| $y$ | | | | | | |

**8.** $y = f(-x - 2)$.

| $x$ | $-6$ | $-4$ | $-2$ | $0$ | $2$ | $4$ |
|---|---|---|---|---|---|---|
| $y$ | | | | | | |

**9.** $y = f(-x + 1)$.

| $x$ | $-3$ | $-1$ | $1$ | $3$ | $5$ | $7$ |
|---|---|---|---|---|---|---|
| $y$ | | | | | | |

**10.** $y = f(-x/4)$.

| $x$ | $-16$ | $-8$ | $0$ | $8$ | $16$ | $24$ |
|---|---|---|---|---|---|---|
| $y$ | | | | | | |

**11.** Use your graphing calculator to draw the graph of $y = \sqrt{x}$. Then, draw the graph of $y = \sqrt{-x}$. In your own words, explain what you learned from this exercise.

**12.** Use your graphing calculator to draw the graph of $y = |x|$. Then, draw the graph of $y = |-x|$. In your own words, explain what you learned from this exercise.

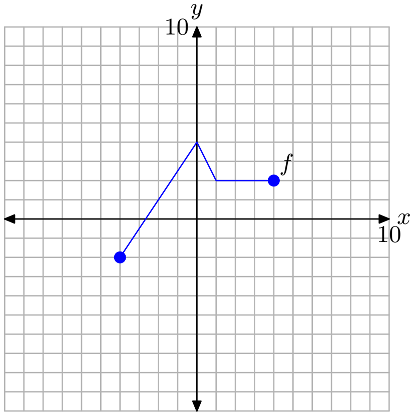
**13.** Use your graphing calculator to draw the graph of $y = x^2$. Then, in succession, draw the graphs of $y = (x - 2)^2$, $y = (x - 4)^2$, and $y = (x - 6)^2$. In your own words, explain what you learned from this exercise.

**14.** Use your graphing calculator to draw the graph of $y = x^2$. Then, in succession, draw the graphs of $y = (x + 2)^2$, $y = (x + 4)^2$, and $y = (x + 6)^2$. In your own words, explain what you learned from this exercise.

**15.** Use your graphing calculator to draw the graph of $y = |x|$. Then, in succession, draw the graphs of $y = |2x|$, $y = |3x|$, and $y = |4x|$. In your own words, explain what you learned from this exercise.

**16.** Use your graphing calculator to draw the graph of $y = |x|$. Then, in succession, draw the graphs of $y = |(1/2)x|$, $y = |(1/3)x|$, and $y = |(1/4)x|$. In your own words, explain what you learned from this exercise.

Pictured below is the graph of a function $f$. In **Exercises 17-22**, use this graph to perform each of the following tasks.



i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Make an exact copy of the graph of $f$ on your coordinate system. *Remember to draw all lines with a ruler.*

ii. In the narrative, a shadow box at the end of the section summarizes the concepts and technique of horizontal scaling, horizontal reflection, and horizontal translation. Use the shortcut ideas presented in this summary shadow box to draw the graphs of the functions that follow **without** using tables.

iii. Use a different colored pencil to draw the graph of the function given in the exercise. Label this graph with its equation. Be sure that key points are accurately plotted. In each exercise, please plot exactly two plots per coordinate system, the graph of original function $f$ and the graph of the function in the exercise.

**17.** $y = f(2x)$.

**18.** $y = f((1/2)x)$.

**19.** $y = f(-x)$.

**20.** $y = f(x-1)$.

**21.** $y = f(x+3)$.

**22.** $y = f(x-2)$.

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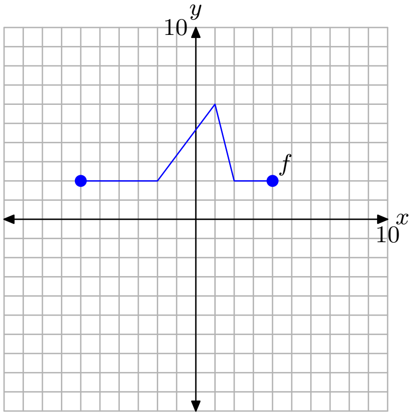
Pictured below is the graph of a function $f$. In **Exercises 23-28**, use this graph to perform each of the following tasks.



i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Make an exact copy of the graph of $f$ on your coordinate system. *Remember to draw all lines with a ruler.*

ii. In the narrative, a shadow box at the end of the section summarizes the concepts and technique of horizontal scaling, horizontal reflection, and horizontal translation. Use the shortcut ideas presented in this summary shadow box to draw the graphs of the functions that follow **without** using tables.

iii. Use a different colored pencil to draw the graph of the function given in the exercise. Label this graph with its equation. Be sure that key points are accurately plotted. In each exercise, please plot exactly two plots per coordinate system, the graph of original function $f$ and the graph of the function in the exercise.

**23.** $y = f(2x)$.

**24.** $y = f((1/2)x)$.

**25.** $y = f(-x)$.

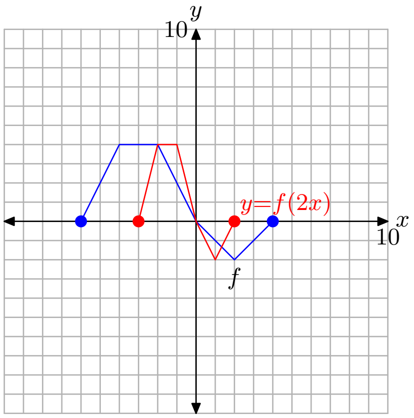
**26.** $y = f(x+3)$.

**27.** $y = f(x-2)$.

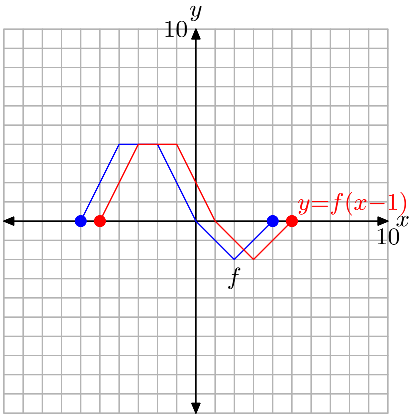
**28.** $y = f(x+1)$.

## 2.6 *Answers*

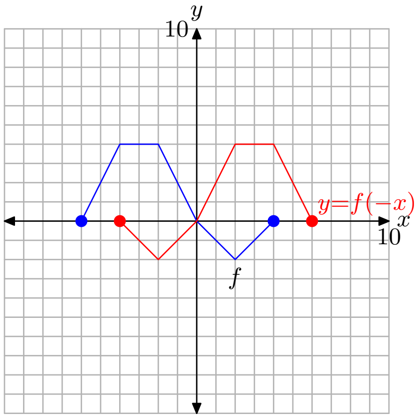
**1.**



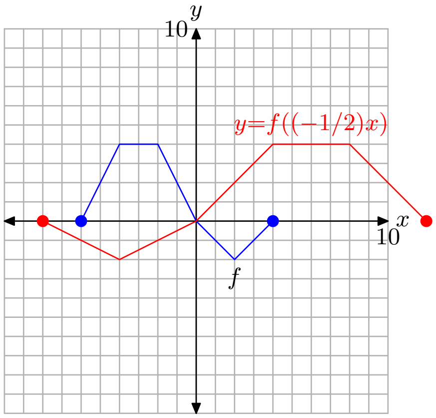
**5.**



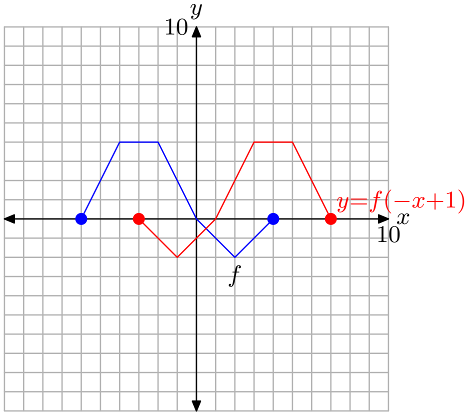
**3.**



**7.**



**9.**

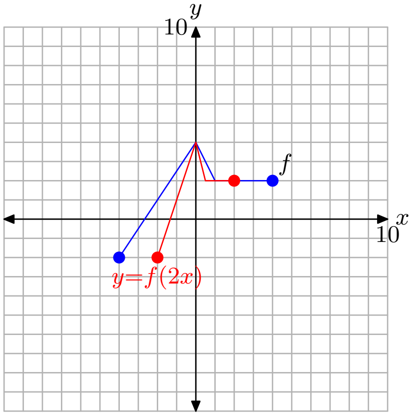


**11.** Mutiplying on the inside by −1, as in $y = \sqrt{-x}$, reflects the graph across the $y$-axis.

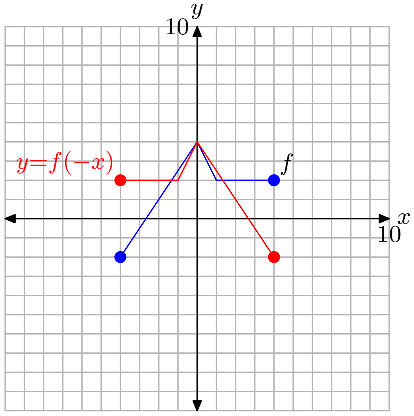
**13.** Replacing $x$ with $x - c$, where $c$ is positive, moves the graph $c$ units to the right.

**15.** Multiplying by a scalar $a$, such that $a$ is larger than 1, compresses the graph horizonally by a factor of $a$.

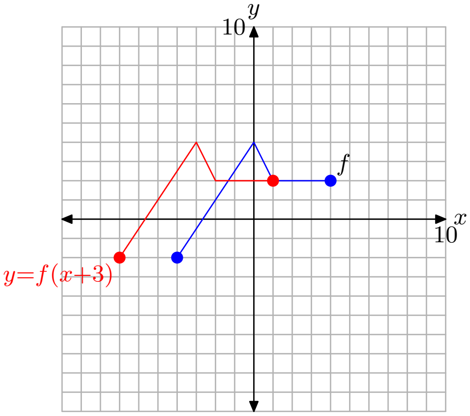
**17.**



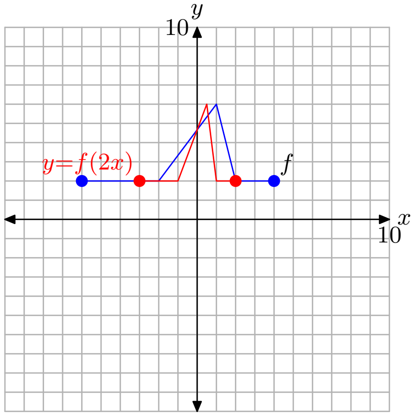
**19.**



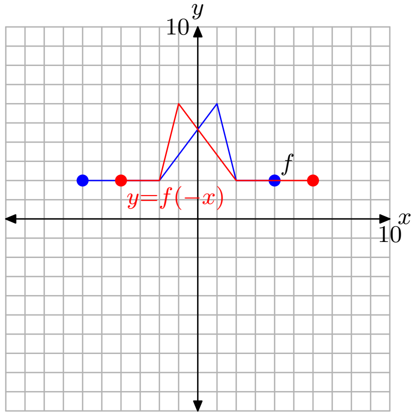
**21.**



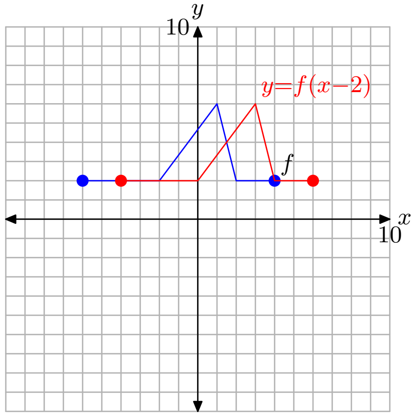
**23.**



**25.**



**27.**



## 2.7 Index

# 3 Linear Functions

In this chapter we will study a class of function called a *linear function*, so named because the graph of a linear function is a line.

We begin our study of linear functions by examining some linear models, where we will present a thorough discussion of the modeling process, including the notion of dependent and independent variables, and representing the data with a graph, properly labeled and scaled. We will learn that if one quantity changes at constant rate with respect to a second quantity, the functional relationship must be linear and the graph will be a line. We will also learn how to develop model equations, then use both the model equation and the graph to make predictions.

We will then present a discussion on slope, making the connection to the constant rates provided in the linear models section previously studied. From there we move to a more formal definition of the slope of a line, a number that controls the "steepness" of the line.

We conclude the chapter with a discussion of the equation of a linear function, using two important forms: the *slope-intercept form* and *point-slope form*. Finally, we will use these forms to determine a "line of best fit" for a variety of data sets.

Welcome to the world of linear models. Let's begin.

## Table of Contents

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# 3.1 Linear Models

Sebastian waves good-bye to his brother, who is talking to a group of his friends approximately 20 feet away. Sebastian then begins to walk away from his brother at a *constant* rate of 4 feet per second. Let's model the distance separating the two brothers as a function of time.

Our first approach will be graphical. We will let the variable $d$ represent the distance (in feet) between the brothers and the variable $t$ represent the amount of time (in seconds) that has passed since Sebastian waved good-bye to his brother. Because the distance separating the brothers *depends* on the amount of time that has passed, we will say that the distance $d$ is the *dependent variable* and the time $t$ is the *independent variable*.

It is somewhat traditional in the modeling process to place the independent variable on the horizontal axis and the dependent variable on the vertical axis. This is not a hard and fast rule, more a matter of personal taste, but we will follow this rule in our example nonetheless. Thus, we will place distance on the vertical axis and time on the horizontal axis, as shown in **Figure 1**. Notice that we've labeled each axis with its variable representation and included the units, an important practice.

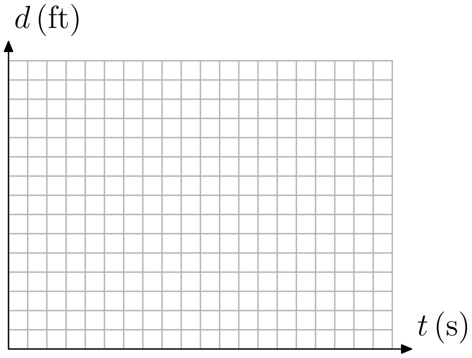

**Figure 1.** Distance depends upon time.

**Warning 1.** *The label on the horizontal axis, t (s), might look like function notation to some readers. This is not the case. Rather, the variable t represents time, and the (s) in parentheses that follows represents seconds, a standard abbreviation in physics. Similar comments are in order for the label d (ft). The variable d represents distance, and the (ft) in parentheses that follows represent feet, another standard abbreviation in physics.*

There are a number of different ways that you can label the axes of your graph with units appropriate for the problem at hand. For example, consider the technique presented in **Figure 2**, where the labels are placed to the left of the vertical axis and

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underneath the horizontal axis. Another difference is the fact that the unit abbreviations in **Figure 1** are spelled out in their entirety in **Figure 2**.

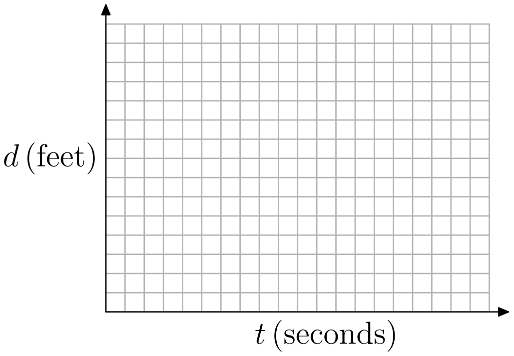


**Figure 2.** Distance depends upon time.

Some instructors prefer that you rotate the distance label on the vertical axis ninety degrees, so that it appears sideways. Others prefer that you label the ends of each axis with the variable, as we have done in **Figure 1**, but spell out the units in their entirety alongside each axis as we've done in **Figure 2**. The list of preferences goes on and on.

**Tip 2.** *It is important to have a conversation with your instructor in order to determine what your instructor's expectations are when it comes to labeling the axes and indicating the units on your graphs.*

We prefer to label the axes as shown in **Figure 1**, and we will try to be consistent to this standard throughout the remainder of the text, though we might stray to alternate forms of labeling from time to time.

We must now *scale* each axis appropriately, a task that is harder than it first seems. A poor choice of scale can make the task ahead more difficult than it needs to be. We will choose a scale for each axis with the following thoughts in mind.

**Guidelines for Scaling Axes.** Here is some good advice to follow when scaling the dependent and independent axes.

1. We want to avoid postage-stamp-sized graphs. A large graph is easier to interpret than one that is cramped in a small corner of our graph paper.
2. It is not necessary to have the same scale on each axis, but once a scale is chosen, you must remain consistent.
3. We want to choose a scale that correlates easily with the given rate.

Sebastian is walking away from his brother at a constant rate of 4 feet per second. Let's let each box on the vertical axis represent 4 feet and every two boxes on the horizontal axis represent 1 second, as shown in **Figure 3**.

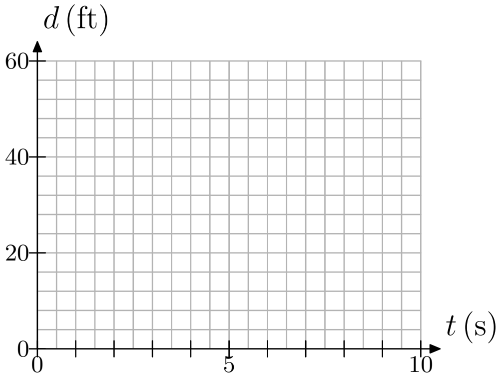
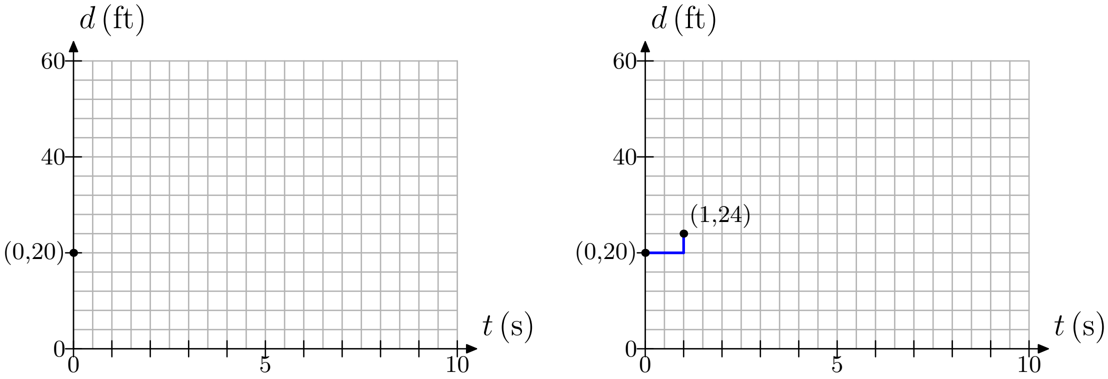


**Figure 3.** Scaling each axis to accommodate the rate.

At time $t = 0$, Sebastian is separated from his brother by a distance of $d = 20$ feet. This corresponds to the point $(t, d) = (0, 20)$ shown in **Figure 4**(a).

Next, Sebastian walks away from his brother at a constant rate of 4 feet per second. This means that for every second of time that elapses, the distance between the brothers increases by 4 feet. Starting at the point $(0, 20)$, move 1 second (two boxes) to the right and 4 feet (1 box) upward to the point $(1, 24)$, as shown in **Figure 4**(b).



(a) Initially 20 feet from his brother. (b) Walking away at 4 ft/s.

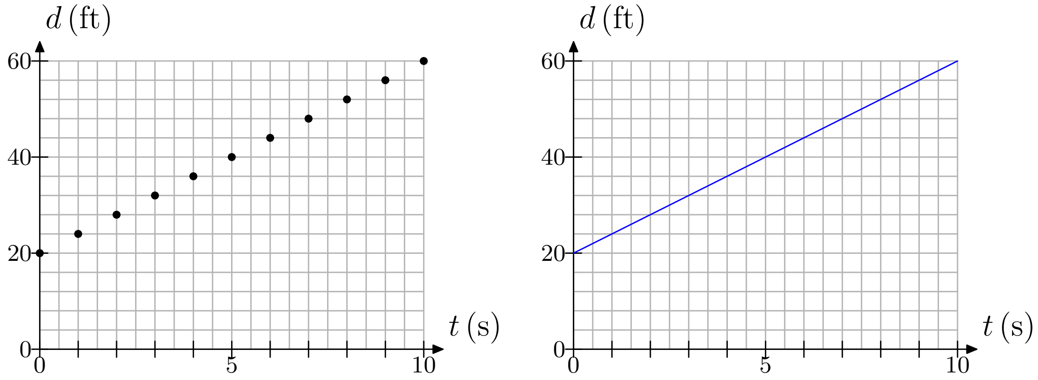
**Figure 4.**

The rate of separation is a *constant* 4 feet per second. So, continue indefinitely in the manner of **Figure 4**(b), moving 1 second (2 boxes) to the right, then 4 feet upward (1 box). This will produce the linear relationship between distance and time suggested in **Figure 5**(a).

If we assume that the distance is a continuous function of time, a legitimate assumption due to the fact that the distance is increasing continuously at a constant rate of 4 feet per second, then we can replace the discrete set of data points in **Figure 5**(a) with the line shown in **Figure 5**(b).

The line in **Figure 5**(b) is a *continuous model.* It can be drawn with a simple stroke of the pencil, without the tip of the pencil ever leaving contact with our graph paper. On the other hand, the set of points in **Figure 5**(a) is a *discrete model.* After plotting a point, our pencil must break contact with our graph paper before plotting the next point. This is the essential difference between a discrete model and a continuous model.

In this case, the continuous model is a more accurate representation of the distance between the brothers. We say this because the distance between them is increasing at a *constant rate* of 4 feet per second, or 2 feet every half second, or 1 foot every quarter second, etc. Shortly, we will show an example where this sort of continuous model is unreasonable.



(a) A discrete model. (b) A continuous model.

**Figure 5.** Constant rate yields a linear relationship.

Now that we've modeled the distance between the brothers with a graph, we can use the graph to make predictions. For example, to determine the distance between the brothers after 8 seconds, locate 8 seconds on the time axis, draw a vertical arrow to the line, then a horizontal arrow to the distance axis, as shown in **Figure 6**.

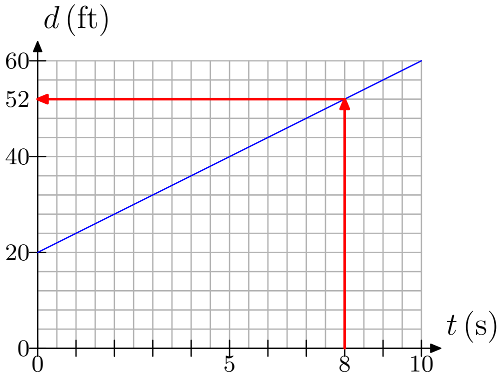


**Figure 6.** Predicting the distance between the brothers after 8 seconds.

However, suppose that we want to determine the distance between the brothers after 2 minutes. As the graph in **Figure 5**(b) only models the distance over the first 10 seconds, we would have to redraw the graph over the first 2 minutes (120 seconds) to determine the answer. We did not plan ahead for this contingency, so perhaps we can model the distance between the brothers in another way, one that will more easily predict the distance between the brothers after an arbitrary amount of time $t$.

To this end, we search for a pattern that describes the distance $d$ between the brothers as a function of time $t$. Because the distance between the brothers is increasing at a rate of 4 feet per second, we note that:

- At $t = 0$ seconds, the distance between the brothers is $d = 20$ feet.
- At $t = 1$ second, the distance between the brothers is $d = 24$ feet.
- At $t = 2$ seconds, the distance between the brothers is $d = 28$ feet.
- At $t = 3$ seconds, the distance between the brothers is $d = 32$ feet.

We summarize these results in **Table 1**(a).

However, you don't want to simplify the distances as we have in **Table 1**(a), because you hide the pattern or the relationship between the distance $d$ and the time $t$. It is more efficient to seek a relationship between distance and time in the following manner. After $t = 1$ second, the distance increases by 1 increment of 4 feet, so $d = 20 + 4(1)$. After $t = 2$ seconds, the distance increases by 2 increments of 4 feet, so $d = 20 + 4(2)$. Continuing in this manner, we have:

- At $t = 3$ seconds, the distance between the brothers is $d = 20 + 4(3)$ feet.
- At $t = 4$ seconds, the distance between the brothers is $d = 20 + 4(4)$ feet.

These results are summarized in **Table 1**(b).

| $t$ | $d$ |
|---|---|
| 0 | 20 |
| 1 | 24 |
| 2 | 28 |
| 3 | 32 |

(a)

| $t$ | $d$ |
|---|---|
| 0 | 20 |
| 1 | $20 + 4(1)$ |
| 2 | $20 + 4(2)$ |
| 3 | $20 + 4(3)$ |

(b)

**Table 1.** Determining a model equation.

Unlike **Table 1**(a), **Table 1**(b) reveals a relationship between distance $d$ and time $t$ that can be described by the equation

$$d = 20 + 4t. \tag{3}$$

The careful reader will check that **equation (3)** reveals the correct distances for $t = 0$, 1, 2, and 3 seconds, as recorded in **Table 1**(a). Two important observations can be made about **equation (3)**.

1. The 20 in $d = 20 + 4t$ is the initial distance between the brothers and corresponds to the point $(0, 20)$ in **Figure 4**(a).
2. The 4 in $N = 20 + 4t$ is the rate at which the distance between the brothers is increasing (4 feet per second).

Moreover, **equation (3)** can be used to predict the distance between the brothers at 2 minutes. First, convert $t = 2$ minutes to $t = 120$ seconds, then substitute this number in our model **equation (3)**.

$$d = 20 + 4(120) = 500.$$

Thus, the distance between the brothers after 2 minutes is $d = 500$ feet.

We can also write the equation $d = 20 + 4t$ using function notation.

$$d(t) = 20 + 4t$$

Then, to find the distance between the brothers at the end of 2 minutes, we would perform the following calculation.

$$\begin{aligned} d(120) &= 20 + 4(120) \\ d(120) &= 500 \end{aligned}$$

Unlike function notation, when the result is written $d = 500$ feet, note how one piece of information is hidden, namely the time. With function notation, we interpret $d(120) = 500$ to mean "the distance between the two brothers after 120 seconds is 500 feet." Note how both the distance and the time are available in the notation $d(120) = 500$.

## *Modeling the Discrete with the Continuous*

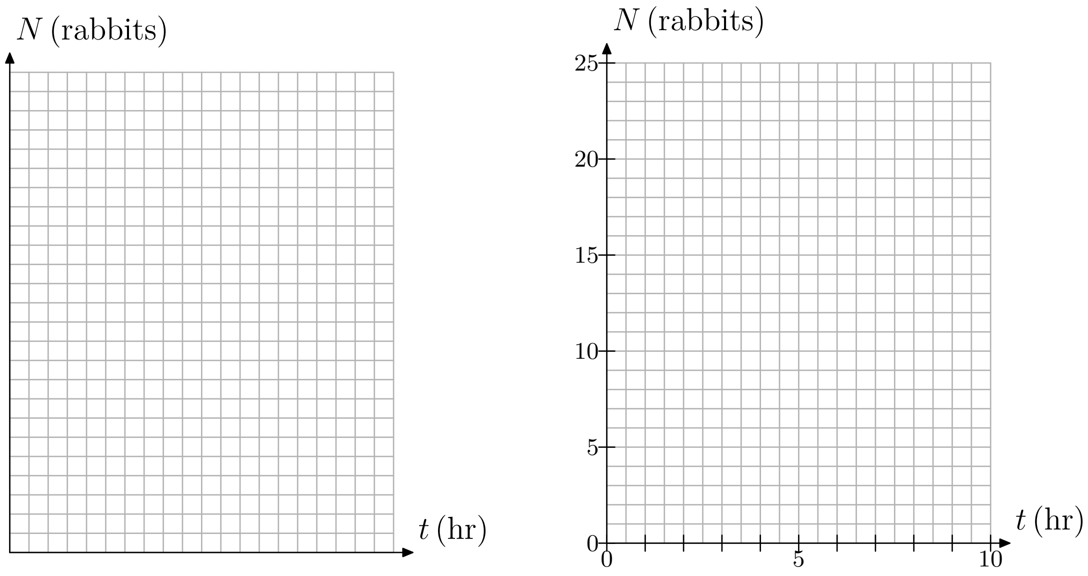
Jenny builds a rabbit hutch behind her barn. She places 25 rabbits in the hutch, then locks the door and leaves. Unfortunately, there is a flaw in the design of the hutch and the rabbits begin to escape at a constant rate of 5 rabbits every 2 hours. Again, we'll model the number $N$ of rabbits remaining in the hutch as a function of time $t$. First, we propose a graphical model.

Note that the number of rabbits remaining in the hutch *depends* on the amount of time that has passed. This makes the number $N$ of rabbits remaining in the hutch the dependent variable, which we will place on the vertical axis in **Figure 7**(a). Time $t$ is the independent variable and is placed on the horizontal axis.

We'll again choose a scale for our axes that accommodates the fact that the rabbit population is decreasing at a constant rate of 5 rabbits every 2 hours. In **Figure 7**(b), we let each box on the vertical axis represent 1 rabbit, while two boxes on the horizontal axis represents 1 hour. We could just as easily let each box on the horizontal axis represent one hour, but our choice makes a graph that is a bit larger. Larger graphs are a bit easier to read and interpret.

At time $t = 0$ hours, the rabbit population is $N = 25$ rabbits. This fact is represented by the point $(t, N) = (0, 25)$ in **Figure 8**(a). Because the rabbit population decreases at a constant rate of 5 rabbits every 2 hours, we start at the point $(0, 25)$, then move 2 hours (4 boxes) to the right, and 5 rabbits (5 boxes) down to the point $(2, 20)$, also shown in **Figure 8**(a).

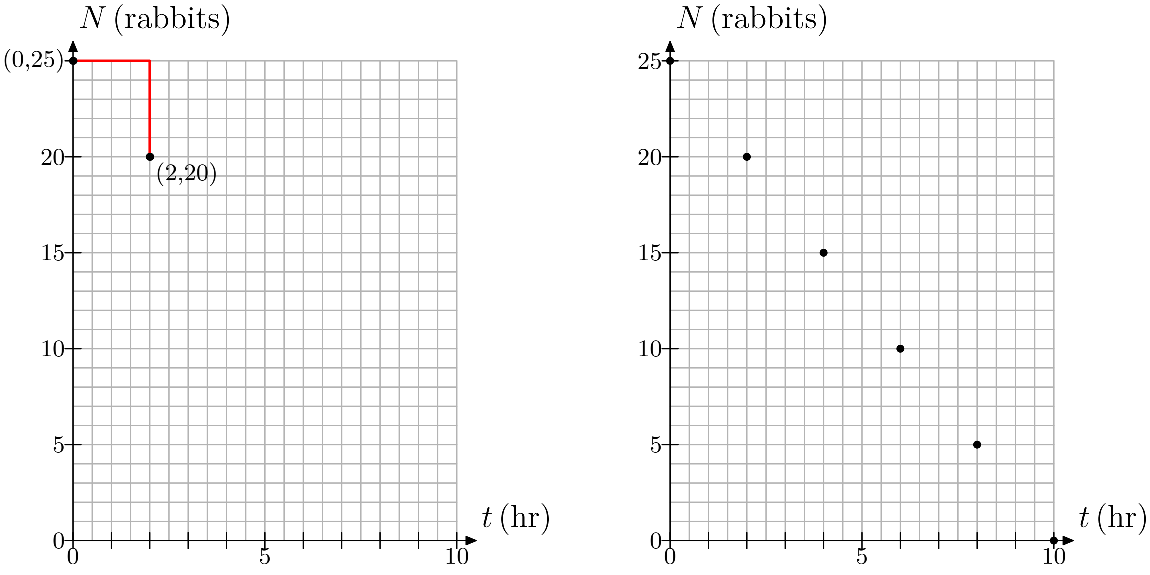
The rate at which the rabbits are decreasing is constant at 5 rabbits every 2 hours, so continue indefinitely in the manner of **Figure 8**(a), moving 2 hours (4 boxes) to the right, then 5 rabbits (5 boxes) downward. This will produce the linear relationship between the number of rabbits $N$ and the time $t$ shown in **Figure 8**(b).



(a) Dependent and independent variables.

(b) Scaling the axes to accommodate the rate.

**Figure 7.**

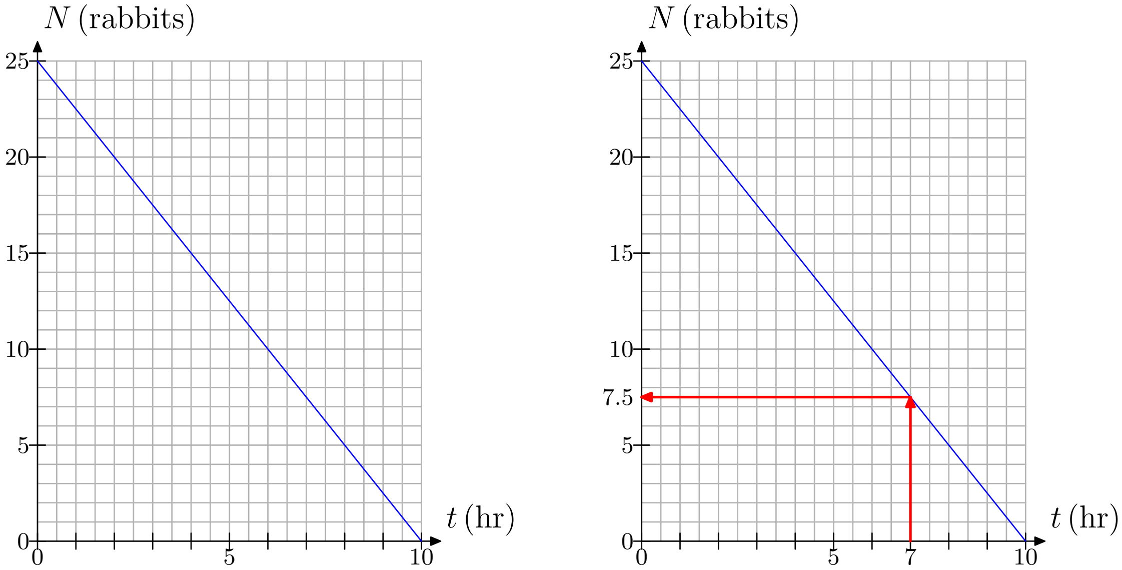


(a) Initial rabbit population is 25. Five rabbits escape every 2 hours.

(b) A discrete model.

**Figure 8.**

We can draw a line through the data points in **Figure 8**(b) to produce the continuous model in **Figure 9**(a). However, we need to be aware of the shortcoming imposed by this continuous approximation. For example, consider the prediction in **Figure 9**(b). Is it reasonable to say that 7.5 rabbits remain in the hutch after 7 hours?



(a) A continuous model.

(b) Predicting the rabbit population after 7 hours.

**Figure 9.**

In our first model, the distance between the brothers can be any real number, so a continuous model was appropriate. However, in the case of Jenny's rabbit hutch, the remaining population must be a whole number of rabbits (unless a fox gets in), so modeling the population with the continuous line in **Figure 9**(b) is at best an approximation of reality. However, mathematicians will frequently model a discrete situation with a continuous model. As long as we are aware of its limitations, we can still use the model to make reasonable predictions. For example, we might say that there are approximately 7 rabbits left in the hutch after 7 hours.

We saw the advantage of using function notation at the end of our previous model, so let's employ function notation a bit earlier in this model. We will let

$$N(t) = \text{the number of rabbits remaining after } t \text{ hours.}$$

Initially, at time $t = 0$, there are 25 rabbits in the hutch. Thus, we write

$$N(0) = 25.$$

It might be easier to think of losing 5 rabbits every 2 hours as being equivalent to losing "on average" 2.5 rabbits every hour. Thus, at the end of 1 hour, the number of rabbits decreases by one increment of 2.5 rabbits, and we write

$$N(1) = 25 - 2.5(1).$$

At the end of 2 hours, the rabbit population decreases by 2 increments of 2.5 rabbits and we can write

$$N(2) = 25 - 2.5(2).$$

At the end of 3 hours, the rabbit population decreases by 3 increments of 2.5 rabbits and we can write

$$N(3) = 25 - 2.5(3).$$

A clear pattern develops, particularly when we summarize these results in **Table 2**.

| $t$ | $N(t)$ |
|---|---|
| 0 | 25 |
| 1 | $25 - 2.5(1)$ |
| 2 | $25 - 2.5(2)$ |
| 3 | $25 - 2.5(3)$ |

**Table 2.** Determining a model equation.

**Table 2** reveals a relationship between the number of rabbits $N$ and time $t$ that can be described by the equation

$$N(t) = 25 - 2.5t. \tag{4}$$

The careful reader will again check that **equation (4)** returns the correct number of rabbits at times $t = 0$, 1, 2, and 3, as recorded in **Table 2**.

There are two important observations we can make about **equation (4)**.

1. The 25 in $N(t) = 25 - 2.5t$ is the initial rabbit population and corresponds to the point $(0, 25)$ in **Figure 8**(a).
2. The $-2.5$ in $N(t) = 25 - 2.5t$ is the rate at which the rabbit population is decreasing "on average" (2.5 rabbits per hour).

The **equation (4)** can be used to predict the number of rabbits remaining in the hutch after $t = 7$ hours. Simply substitute $t = 7$ in **equation (4)**.

$$N(7) = 25 - 2.5(7) = 7.5$$

It is important to note that the prediction made by the model equation is identical to that made by the model graph in **Figure 9**(b).

However, again note that this equation is a continuous model, and its prediction that 7.5 rabbits remain in the hutch is not realistic (unless that fox got loose again). However, if we are aware of the model's shortcomings, the equation can still be used as a good predictive tool. For example, we might again say that approximately 7 rabbits remain in the hutch after 7 hours. This can be written $N(7) \approx 7$, which means that "after 7 hours, there are approximately 7 rabbits remaining in the hutch."

## *Determining the Equation Model from the Graph*

Mrs. Burke sets up a motion detector at the front of her classroom, then positions one of her students a fixed distance from the detector and asks the student to approach the detector at a constant speed. The detector measures the distance $d$ (in meters) of the student from the detector as a function of time $t$ (in seconds). The graph of distance $d$ versus time $t$ is given in **Figure 10**.

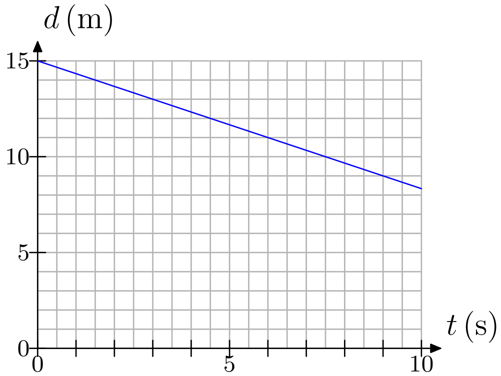
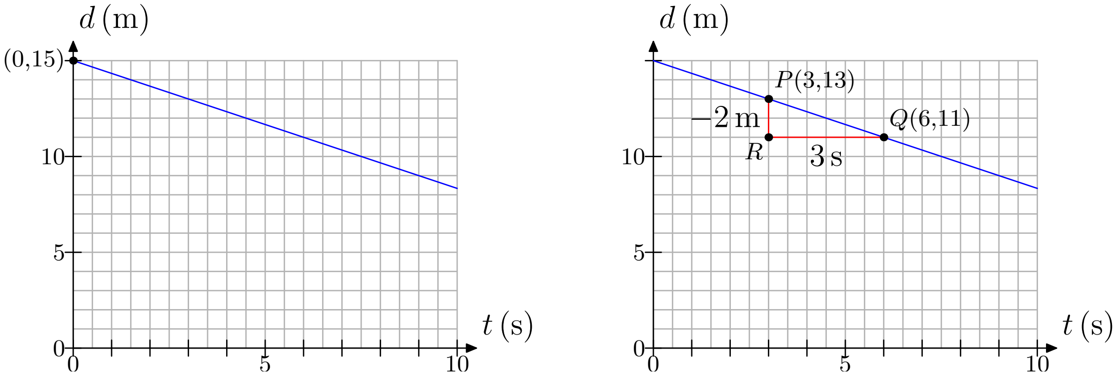


**Figure 10.** The detector measures the distance between student and detector versus time.

It is a simple matter to determine the student's initial distance from the detector. We need only determine the value of $d$ at time $t = 0$ seconds. The result is located at the point $(0, 15)$, as shown in **Figure 11**(a). Thus, the student sets up at an initial distance of 15 meters from the detector.

To determine the rate at which the student approaches the detector, we need to do a bit more work. Examine the graph and pick two points on the line. It makes things a bit easier if you pick points on the line that are situated at the intersection of two grid lines, but as we will show, this is not necessary. With this thought in mind, we've picked the points $P(3, 13)$ and $Q(6, 11)$ on the line, as shown in **Figure 11**(b).



(a) Initial distance from detector.

(b) Determining the rate.

**Figure 11.** Determining initial distance and rate.

Draw a right triangle $\triangle PQR$ with sides parallel to the axes, as shown in **Figure 11**(b). Determine the length of each side of the right triangle.

- Side $PR$ is 2 boxes in length, but each box represents 1 meter, so side $PR$ represents a decrease of 2 meters in distance from the detector. That is why we've used the minus sign in labeling the side $PR$ with $-2\,\mathrm{m}$ in **Figure 11**(b).
- Side $RQ$ is 6 boxes in length, but 2 boxes represents 1 second, so side $RQ$ represents an increase of 3 seconds in time. That is why we've labeled side $RQ$ with $3\,\mathrm{s}$ in **Figure 11**(b).

Thus, the distance between the student and detector is *decreasing* at a rate of 2 meters every 3 seconds.

What would happen if we picked two different points on the line? Consider the case in **Figure 12**, where we've picked the points on the line at $P(3, 13)$ and $Q(9, 9)$. We've also decided to draw right triangle $\triangle PQR$ on the opposite side of the line. However, note again that the sides of the right triangle $\triangle PQR$ are parallel to the horizontal and vertical axes.

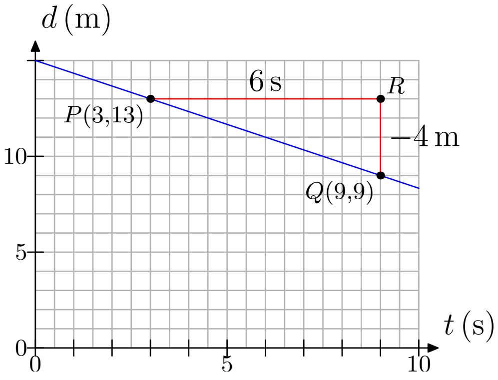


**Figure 12.** Determining the rate.

Determine the length of each side of triangle $\triangle PQR$.

- Side $PR$ is 12 boxes in length, but 2 boxes represent 1 second, so side $PR$ represents an increase of 6 seconds in time. That is why we've labeled side $PR$ with $6\,\mathrm{s}$ in **Figure 12**.
- Side $RQ$ is 4 boxes in length, but each box represents 1 meter, so side $RQ$ represents a decrease of 4 meters in distance from the detector. That is why we've used a minus sign in labeling the side $RQ$ with $-4\,\mathrm{m}$ in **Figure 12**.

Thus, the distance between the student and detector is *decreasing* at a rate of 4 meters every 6 seconds. In symbols, we would write that the rate is

$$\text{Rate} = \frac{-4\,\mathrm{m}}{6\,\mathrm{s}} = -\frac{4}{6}\,\mathrm{m/s}.$$

Note, however, that this reduces to

$$\text{Rate} = -\frac{2}{3}\,\mathrm{m/s},$$

which is identical to the rate found earlier when using the points $P$ and $Q$ in **Figure 11**(b).

The fact that these rates are equivalent is due to the fact that the triangles $\triangle PQR$ in **Figure 11**(b) and **Figure 12** are *similar triangles*, so their sides are proportional. Thus, it doesn't matter which two points you pick on the line, nor does it matter which side of the line you place your right triangle. Thus, the only requirement is that you draw a right triangle with sides parallel to the coordinate axes.

Finally, let's see if we can develop a model equation. We will define

$$d(t) = \text{the distance from the detector at time } t.$$

Initially, the student is 15 meters from the detector. That is, at time $t = 0$, the distance from the detector is 15 meters. In symbols, we write

$$d(0) = 15.$$

The distance decreases at a rate of 2 meters every 3 seconds. This is equivalent to saying that the distance decreases 2/3 meters every second. At the end of 1 second, the distance has decreased by 1 increment of 2/3 meters, so the distance from the detector is given by

$$d(1) = 15 - \frac{2}{3}(1).$$

At the end of 2 seconds, the distance has decreased by 2 increments of 2/3 meters, so the distance from the detector is given by

$$d(2) = 15 - \frac{2}{3}(2).$$

At the end of 3 seconds, the distance has decreased by 3 increments of 2/3 meters, so the distance from the detector is given by

$$d(3) = 15 - \frac{2}{3}(3).$$

A clear pattern emerges, particularly if you summarize the results as we have in **Table 3**.

| $t$ | $d(t)$ |
|---|---|
| 0 | 15 |
| 1 | $15 - (2/3)(1)$ |
| 2 | $15 - (2/3)(2)$ |
| 3 | $15 - (2/3)(3)$ |

**Table 3.** Determining a model equation.

**Table 3** reveals that the linear relationship (see **Figure 10**) between the distance $d$ from the detector at time $t$ can be modeled by the equation

$$d(t) = 15 - \frac{2}{3}t. \tag{5}$$

Again, the careful reader will check that **equation (5)** returns the correct distance $d$ at the times $t = 0$, 1, 2, and 3 recorded in **Table 3**.

There are two important observations to be made about **equation (5)**.

1. The 15 in $d(t) = 15-(2/3)t$ is the initial distance from the detector and corresponds to the point $(0, 15)$ in **Figure 11**(a).
2. The $-2/3$ in $d(t) = 15-(2/3)t$ is the rate at which the distance between the student and detector is changing as determined in **Figure 11**(b). It is negative because the distance is decreasing with time.

Equation (**5**) can be used to make predictions. For example, to determine the distance between the student and detector at the end of 9 seconds, insert $t = 9$ into **equation (5)**.

$$d(9) = 15 - \frac{2}{3}(9) = 15 - 6 = 9.$$

Of course, the notation $d(9) = 9$ is interpreted to mean "the distance between the student and the detector after 9 seconds is 9 meters."

## 3.1 Exercises

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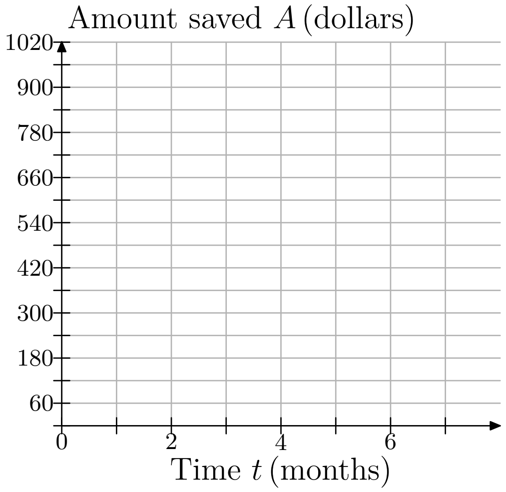
**1.**

Jodiah is saving his money to buy a Playstation 3 gaming system. He estimates that he will need $950 to buy the unit itself, accessories, and a few games. He has $600 saved right now, and he can reasonably put $60 into his savings at the end of each month.

Since the amount of money saved depends on how many months have passed, choose time, in months, as your independent variable and place it on the horizontal axis. Let t represent the number of months passed, and make a mark for every month.

Choose money saved, in dollars, as your dependent variable and place it on the vertical axis. Let $A$ represent the amount saved in dollars. Since Jodiah saves $60 each month, it will be convenient to let each box represent $60.

Copy the following coordinate system onto a sheet of graph paper.



**a)** At month 0, Jodiah has $600 saved. This corresponds to the point $(0, 600)$. Plot this point on your coordinate system.

**b)** For the next month, he saved $60 more. Beginning at point $(0, 600)$, move 1 month to the right and $60 up and plot a new data point. What are the coordinates of this point?

**c)** Each time you go right 1 month, you must go up by $60 and plot a new data point. Repeat this process until you reach the edge of the coordinate system.

**d)** Keeping in mind that we are modeling this discrete situation continuously, draw a line through your data points.

**e)** Use your graph to estimate how much money Jodiah will have saved after 7 months.

**f)** Using your graph, estimate how many months it will take him to have saved

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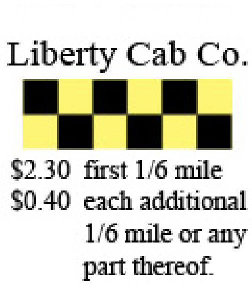
[2] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

up enough money to buy his gaming system, accessories, and games.

**2.**



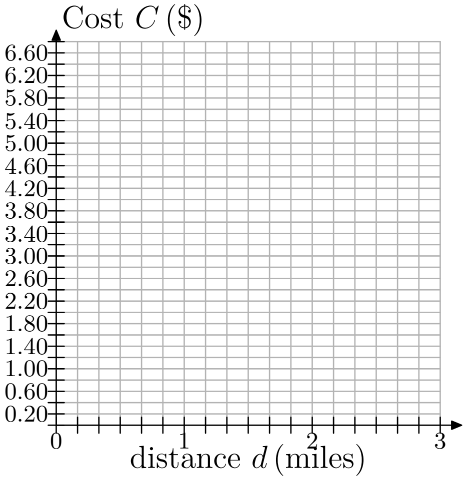
Liberty Cab Co.

$2.30 first 1/6 mile
$0.40 each additional 1/6 mile or any part thereof.

The sign above shows the prices for a taxi ride from Liberty Cab Company. Since the cost depends on the distance traveled, make the distance be the independent variable and place it on the horizontal axis. Let d represent the distance traveled, in miles. Because the cab company charges per 1/6 mile, it is convenient to mark every 1/6 mile.

Make price, in \$, your dependent variable and place it on the vertical axis. Let C represent the cost, in \$. Because the cost occurs in increments of 40¢, mark every 40¢ along the vertical axis.

Copy the following coordinate system onto a sheet of graph paper.



**a)** For the first 1/6 mile of travel, the cost is \$2.30. This corresponds to the point (1/6,\$2.30). Plot this point on your coordinate system.

**b)** For the next 1/6 of a mile, the cost goes up by 40¢. Beginning at point (1/6,\$2.30), move 1/6 of a mile to the right and 40¢ up and plot a new data point. What are the coordinates of this point?

**c)** Each time you go right 1/6 of a mile, you must go up by 40¢ and plot a new data point. Repeat this process until you reach the edge of your coordinate system.

**d)** Keeping in mind that we are modeling this discrete situation continuously, draw a line through your data points.

**e)** Melissa steps into a cab in the city of Niagara Falls, about 2 miles from Niagara Falls State Park. Use your graph to estimate the fare to the park.

**f)** Elsewhere in the area, Georgina takes a cab. She has only \$5 for the fare. Use the graph to estimate how far she can travel, in miles, with only \$5 for the fare.

**3.** A boat is 200 ft from a buoy at sea. It approaches the buoy at an average speed of 15 ft/s.

**a)** Choosing time, in seconds, as your independent variable and distance from the buoy, in feet, as your dependent variable, make a graph of a coordinate system on a sheet of graph paper showing the axes and units. Use tick marks to identify your scales.

**b)** At time t=0, the boat is 200 ft from the buoy. To what point does this correspond? Plot this point on your coordinate system.

**c)** After 1 second, the boat has drawn 15 ft closer to the buoy. Beginning at the previous point, move 1 second to the right and 15 ft down (since the distance is decreasing) and plot a new data point. What are the coordinates of this point?

**d)** Each time you go right 1 second, you must go down by 15 ft and plot a new data point. Repeat this process until you reach 12 seconds.

**e)** Draw a line through your data points.

**f)** When the boat is within 50 feet of the buoy, the driver wants to begin to slow down. Use your graph to estimate how soon the boat will be within 50 feet of the buoy.

**4.** Joe owes $24,000 in student loans. He has finished college and is now working. He can afford to pay $1500 per month toward his loans.

**a)** Choose time in months as your independent variable and amount owed, in $, as the dependent variable. On a sheet of graph paper, make a sketch of the coordinate system, using tick marks and labeling the axes appropriately.

**b)** At time t = 0, Joe has not yet paid anything toward his loans. To what point does this correspond? Plot this point on your coordinate system.

**c)** After one month, he pays $1500. Beginning at the previous point, move 1 month to the right and $1500 down (down because the debt is decreasing). Plot this point. What are its coordinates?

**d)** Each time you go 1 month to the right, you must move $1500 down. Continue doing this until his loans have been paid off.

**e)** Keeping in mind that we are modeling this discrete situation continuously, draw a line through your data points.

**f)** Use the graph to determine how many months it will take him to pay off the full amount of his loans.

**5.**

Earl the squirrel has only ten more days until hibernation. He needs to save 50 more acorns. He is tired of collecting acorns and so he is only able to gather 8 acorns every 2 days.

**a)** Let $t$ represent time in days and make it your independent variable. Let $N$ represent the number of acorns collected and make it your dependent variable. Set up an appropriately scaled coordinate system on a sheet of graph paper.

**b)** At time $t = 0$, Earl has collected zero of the acorns he needs. To what point does this correspond? Plot this point on your coordinate system.

**c)** After two days ($t = 2$), Earl has collected 8 acorns. Beginning at the previous point, move 2 days to the right and 8 acorns up. Plot this point. What are its coordinates?

**d)** Each time you go 2 days to the right, you must move 8 acorns up and plot a point. Continue doing this until you reach 14 days.

**e)** Keeping in mind that we are modeling this discrete situation continuously, draw a line through your data points.

**f)** Use the graph to determine how many acorns he will have collected after 10 days. Will Earl have collected enough acorns for his winter hibernation?

**g)** Notice that the number of acorns collected is increasing at a rate of 8 acorns every 2 days. Reduce this to a rate that tells the average number of acorns that is collected each day.

**h)** The table below lists the number of acorns Earl will have collected at various times. Some of the entries have been completed for you. For example, at $t = 0$, Earl has no acorns, so $N = 0$. After one day, the amount increases by 4, so $N = 0 + 4(1)$. After two days, two increases have occurred, so $N = 0 + 4(2)$. The pattern continues. Fill in the missing entries.

| $t$ | $N$ |
|---|---|
| 0 | 0 |
| 1 | $0 + 4(1)$ |
| 2 | $0 + 4(2)$ |
| 3 | $0 + 4(3)$ |
| 4 | |
| 6 | |
| 8 | |
| 10 | |
| 12 | |
| 14 | |

**i)** Express the number of acorns collected, $N$, as a function of the time $t$, in days.

**j)** Use your function to predict the number of acorns that Earl will have after 10 days. Does this answer agree with your estimate from part (f)?

**6.** On network television, a typical hour of programming contains 15 minutes of commercials and advertisements and 45 minutes of the program itself.

**a)** Choose amount of television watched as your independent variable and place it on the horizontal axis. Let T represent the amount of television watched, in hours. Choose total amount of commercials/ads watched as your dependent variable and place it on the vertical axis. Let C represent the total amount of commercials/ads watched, in minutes. Using a sheet of graph paper, make a sketch of a coordinate system and label appropriately.

**b)** For 0 hours of programming watched, 0 minutes of commercials have been watched. To what point does this correspond? Plot it on your coordinate system.

**c)** After watching 1 hour of programming, 15 minutes of commercials/ads have been watched. Beginning at the previous point, move 1 hour to the right and 15 minutes up. Plot this point. What are its coordinates?

**d)** Each time you go 1 hour to the right, you must move 15 minutes up and plot a point. Continue doing this until you reach 5 hours of programming.

**e)** Draw a line through your data points.

**f)** Billy watches TV for five hours on Monday. Use the graph to determine how many minutes of commercials he has watched during this time.

**g)** Suppose a person has watched one hour of commercials/ads. Use the graph to estimate how many hours of television he watched.

**h)** The following table shows numbers of hours of programming watched as it relates to number of minutes of commercials/ads watched. For 0 hours of TV, 0 minutes of commercials/ads are watched. For each hour of TV watched, we must count 15 minutes of commercials/ads. So, for 1 hour, $0 + 15(1)$ minutes of commercials are watched. For 2 hours, $0 + 15(2)$ minutes; and so on. Fill in the missing entries.

| $T$ (hrs) | $C$ (mins) |
|---|---|
| 0 | 0 |
| 1 | $0 + 15(1)$ |
| 2 | $0 + 15(2)$ |
| 3 | |
| 4 | |
| 5 | |

**i)** Express the amount of commercials/ads watched, C, as a function of the amount of television watched T. Use your equation to predict the amount of commercials/ads watched for 5 hours of television programming. Does this answer agree with your estimate from part (f)?

**7.** According to NATO (the National Association of Theatre Owners), the average price of a movie ticket was 5.65 dollars in the year 2001. Since then, the average price has been rising each year by about 20¢.

**a)** Choose year, beginning with 2000, as the independent variable and make marks every year on the axis. Choose average ticket price, in dollars, as your dependent variable and begin at 5.65 dollars, with marks every 10¢ above. Make a sketch of a coordinate system and label appropriately.

**b)** In 2001, the average ticket price was 5.65 dollars, corresponding to the point $(2001, 5.65)$. Plot it on your coordinate system.

**c)** In 2002, one year later, the average price rose by about 20¢. Beginning at the previous point, move right by 1 year and up by 20¢ and plot the point. What are its coordinates?

**d)** Each time you go 1 year to the right, you must move up by 20¢ and plot a point. Continue doing this until the year 2010.

**e)** Keeping in mind that we are modeling this discrete situation continuously, draw a line through your data points.

**f)** Use the graph to estimate what year the average price of a ticket will pass 7.00 dollars.

**8.** When Jessica drives her car to a work-related conference, her employer reimburses her approximately 45 cents per mile to cover the cost of gas and the wear-and-tear on the vehicle.

**a)** Using distance traveled $d$, in miles, as the independent variable and amount reimbursed $A$, in dollars, as the dependent variable, make a sketch of a coordinate system and label appropriately. Mark distance every 5 miles and amount reimbursed every \$0.45.

**b)** For traveling 0 miles, the reimbursement is 0. This corresponds to the point $(0, 0)$. Plot it on your coordinate system.

**c)** For a trip that requires her to drive a total of 5 miles, she is reimbursed $5(0.45) = \$2.25$. This corresponds to the point $(5, \$2.25)$. Plot it.

**d)** For each 5 miles you go to the right, you must go up \$2.25 and plot the point. Do this until you reach 20 miles.

**e)** Keeping in mind that we are modeling this discrete situation continuously, draw a line through your data points.

**f)** In March, Jessica attends a conference that is only 5 miles away. Counting roundtrip, she travels 10 total miles. Use the graph to determine how much she is reimbursed.

**g)** In December, she attends a conference 10 miles away. How long is her trip in total? Use the graph to determine how much she will be reimbursed.

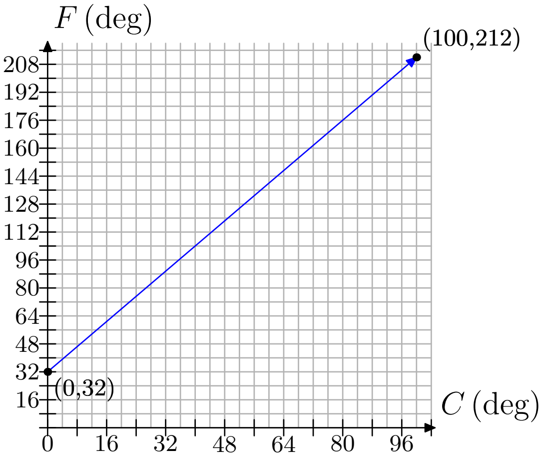
**h)** For longer trips, such as 200 total miles, you will probably need to make a much larger graph. And what if she travels 400 miles? Or further? It is limitations such as these that make it useful to find an equation that describes what the graph shows. To find the equation, we start with a table that helps us to understand the relationship between the dependent and independent variables. Complete the table below.

| $d$ (miles) | $A$ (\$) |
|---|---|
| 0 | 0 |
| 1 | $0 + 0.45(1)$ |
| 2 | $0 + 0.45(2)$ |
| 3 | |
| 4 | |
| 5 | |
| 10 | |
| 20 | |
| 50 | |
| 100 | |

**i)** Use the table from part (h) to come up with an equation that relates $d$ and $A$.

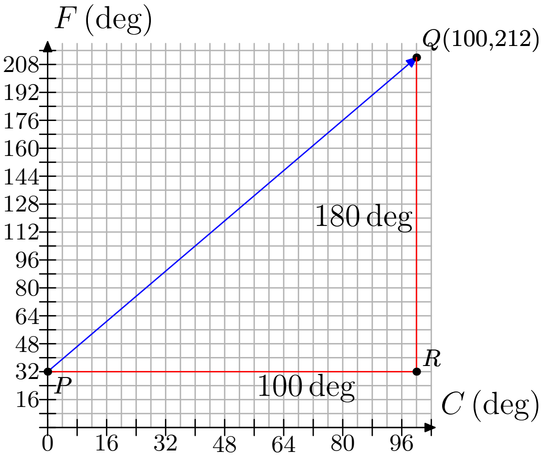
**j)** Now, use the equation to determine the reimbursement amounts for trips of 200 miles and 400 miles.

**9.** Temperature is typically measured in degrees Fahrenheit in the United States; but it is measured in degrees Celsius in many other countries. The relationship between Fahrenheit and Celsius is linear. Let's choose the measurement of degrees in Celsius to be our independent variable and the measurement of degrees in Fahrenheit to be our dependent variable. Water freezes at 0 degrees Celsius, which corresponds to 32 degrees Fahrenheit; and water boils at 100 degrees Celsius, which corresponds to 212 degrees Fahrenheit. We can plot this information as the two points (0,32) and (100,212). The relationship is linear, so have the following graph:



**a)** Use the graph to approximate the equivalent Fahrenheit temperature for 48 degree Celsius.

**b)** To determine the rate of change of Fahrenheit with respect to Celsius, we draw a right triangle with sides parallel to the axes that connects the two points we know...



Side $PR$ is 100 degrees long, representing an increase in 100 degrees Celsius. Side $RQ$ is 180 degrees, representing an increase in 180 degrees Fahrenheit. Find the rate of increase of Fahrenheit per Celsius.

**c)** The following table shows some values of temperatures in Celsius and their corresponding Fahrenheit readings. Zero degrees Celsius corresponds to 32 degrees Fahrenheit. Our rate is 9 degrees Fahrenheit for every 5 degrees Celsius, or 9/5 of a degree Fahrenheit for every 1 degree Celsius. So, for 1 degree Celsius, we increase the Fahrenheit reading by 9/5 degree, getting $32 + 9/5(1)$. For 2 degrees Celsius, we increase by two occurrences of 9/5 degree to get $32 + 9/5(2)$. Fill in the missing entries, following the pattern.

| $C$ (deg) | $F$ (deg) |
|---|---|
| 0 | 32 |
| 1 | $32 + \frac{9}{5}(1)$ |
| 2 | $32 + \frac{9}{5}(2)$ |
| 3 | $32 + \frac{9}{5}(3)$ |
| 4 | |
| 5 | |
| 10 | |
| 20 | |
| 48 | |
| 100 | |

**d)** Use the table to form an equation that gives degrees Fahrenheit in terms of degrees Celsius.

**10.** On June 16, 2006, the conversion rate from Euro to U.S. dollars was approximately 0.8 to 1, meaning that every 0.8 Euros were worth 1 U.S. dollar.

**a)** Choosing dollars to be the independent variable and Euros to be the dependent variable, make a graph of coordinate system. Mark every dollar on the dollar axis and every 0.8 Euros on the Euro axis. Label appropriately.

**b)** Zero dollars are worth 0 Euros. This corresponds to the point $(0, 0)$. Plot it on your coordinate system.

**c)** One dollar is worth 0.8 Euros. Plot this as a point on your coordinate system.

**d)** For every dollar you move to the right, you must go up 0.8 Euros and plot a point. Do this until you reach \$10.

**e)** Draw a line through your data points.

**f)** Use the graph to estimate how many Euros \$8 are worth.

**g)** Use the graph to estimate how many dollars 5 Euros are worth.

**h)** The following table shows some values of dollars and their corresponding value in Euros. Fill in the missing entries.

| *Dollars* | *Euros* |
|---|---|
| 0 | 0 |
| 1 | $0 + 0.8(1)$ |
| 2 | $0 + 0.8(2)$ |
| 3 | |
| 4 | |
| 5 | |
| 10 | |

**i)** Use the table to make an equation that can be used to convert dollars to Euros.

**j)** Use the equation from (i) to convert \$8 to Euros. Does your answer agree with the answer from (f) that you obtained using the graph?

**11.** The Tower of Pisa in Italy has its famous lean to the south because the clay and sand ground on which it is built is softer on the south side than the north. The tilt is often found by measuring the distance that the upper part of the tower overhangs the base, indicated by $h$ in the figure below. In 1980, the tower had a tilt of $h = 4.49$m, and this tilt was increasing by about 1 mm/year.

**Figure 13.** $h$ measures the tilt of the Tower of Pisa.

We will investigate how the tilt of the tower changed from 1980 to 1995.

**a)** First, note that our units do not match: The tilt in 1980 was given as 4.49 m, but the annual increase in the tilt is given as 1 mm/year. Our first goal is to make the units the same. We will use millimeters (mm). Convert 4.49 m to mm.

**b)** Get a sheet of graph paper. Since the tilt of the tower depends on the year, make the year the independent variable and place it on the horizontal axis. Let $t$ represent the year.

Make the tilt the dependent variable and place it on the vertical axis. Let $h$ represent the tilt, measured in millimeters (mm).

Choose 1980 as the first year on the horizontal axis and mark every year thereafter, until 1995. Let the vertical axis begin at 4.49 m, converted to mm from part (a), since that was our first measurement; and then we mark every 1 mm thereafter up to 4510 mm.

**c)** Think of 1980 as the starting year. Together with the tilt measurement from that year, it forms a point. What are the coordinates of this point? Plot the point on your coordinate system.

**d)** Beginning at the first point, from part (c), move one year to the right (to 1981) and 1 mm up (because the tilt increases) and plot a new data point.

**e)** Each time you move one year to the right, you must move 1 mm up and plot a new point. Repeat this process until you reach the year 1995.

**f)** Keeping in mind that we are modeling this discrete situation continuously, draw a line through your data points. We can use this model to make predictions.

**g)** According to computer simulation models, which use sophisticated mathematics, the tower would be in danger of collapsing when $h$ reaches about 4495 mm. Use your graph to estimate what year this would happen.

**h)** In reality, the tilt of the tower passed 4495 mm and the tower did not collapse. In fact, the tilt increased to 4500 mm before the tower was closed on January 7, 1990, to undergo renovations to decrease the tilt. (The tower was reopened in 2001, after engineers used weights and removed dirt from under the base to decrease the tilt by 450 mm.) What might be some reasons why the prediction of the computer model was wrong?

**i)** The following table lists the tilt of the tower, $h$, the year, and the number of years since 1980. In 1980, the tilt was 4490 mm and no occurrences of the 1 mm increase had happened yet, so we fill in $4490 + 0(1) = 4490$. In 1981, one occurrence of the 1 mm increase had occurred because one year had passed since 1980. Therefore, the tilt was $4490 + 1(1)$. In 1982, two occurrences of the 1 mm increase had occurred, because 2 years had passed since 1980. Thus, the tilt was $4490 + 2(1)$. And the pattern continues in this manner. Fill in the remaining entries.

| Year | yrs $x$ after '80 | tilt $h$ |
|---|---|---|
| 1980 | 0 | $4490 + 0(1)$ |
| 1981 | 1 | $4490 + 1(1)$ |
| 1982 | 2 | $4490 + 1(2)$ |
| 1983 | | |
| 1984 | | |
| 1985 | | |
| 1986 | | |
| 1987 | | |
| 1988 | | |
| 1989 | | |
| 1990 | | |
| 1991 | | |
| 1992 | | |
| 1993 | | |
| 1994 | | |
| 1995 | | |

**j)** Let $x$ represent the number of years since 1980 and $h$ represent the tilt. Using the table above, write an equation that relates $h$ and $x$.

**k)** Use your equation to predict the tilt in 1990. Does it agree with the actual value from 1990? Does it agree with the value that is shown on the graph you made?

**l)** In part (g), you used the graph to predict the year in which the tilt would be 4495mm. Use your equation to make the same prediction. Do the answers agree?

**12.** According to the Statistical Abstract of the United States (www.census.gov), there were approximately 31, 000 crimes reported in the United States in 1998, and this was dropping by a rate of about 2900 per year.

**a)** On a sheet of graph paper, make a coordinate system and plot the 1998 data as a point. Note that you will only need to graph the first quadrant of a coordinate system, since there are no data for years before 1998 and there cannot be a negative number of crimes reported. Use the given rate to find points for 1999 through 2006, and then draw a line through your data. We are constructing a continuous model for our discrete situation.

**b)** The following table lists the number of crimes reported, $C$, the year, and the number of years since 1998. In 1998, the number was 31, 000 and no occurrences of the 2900 decrease had happened yet, so we fill in $31000 - 2900(0)$. In 1999, one occurrence of the 2900 decrease had happened because one year had passed since 1998. Therefore, the number of crimes reported was $31000-2900(1)$. And the pattern continues in this manner. Fill in the remaining entries.

| Year | yrs $x$ after 1998 | No. of crimes $C$ |
|---|---|---|
| 1998 | 0 | $31000 - 2900(0)$ |
| 1999 | 1 | $31000 - 2900(1)$ |
| 2000 | | |
| 2001 | | |
| 2002 | | |

**c)** Observing the pattern in the table, we come up with the equation $C = 31000 - 2900x$ to relate the number of crimes $C$ to the number of years $x$ after 1998. Here, $C$ is a function of $x$, and so we can use the notation $C(x) = 31000 - 2900x$ to emphasize this.

i. Compute $C(5)$.
ii. In a complete sentence, explain what $C(5)$ represents.
iii. Compute $C(8)$.
iv. In a complete sentence, explain what $C(8)$ represents.

**13.** According to the Statistical Abstract of the United States (www.census.gov), there were approximately 606, 000 inmates in United States prisons in 1999, and this was increasing by a rate of about 14, 000 per year.

**a)** On a sheet of graph paper, make a coordinate system and plot the 1999 data as a point. Note that you will only need to graph the first quadrant of a coordinate system, since there are no data for years before 1999 and there cannot be a negative number of crimes reported. Use the given rate to find points for 2000 through 2006, and then draw a line through your data. We are constructing a continuous model for our discrete situation.

**b)** The following table lists the number of inmates, $N$, the year, and the number of years since 1999. In 1999, the number was $606,000$ and no occurrences of the $14,000$ increase had happened yet, so we fill in $606000 + 14000(0)$. In 2000, one occurrence of the $14,000$ increase had happened because one year had passed since 1999. Therefore, the number of crimes reported was $606000 + 14000(1)$. And the pattern continues in this manner. Fill in the remaining entries.

| Year | yrs $x$ after '99 | No. of inmates $N$ |
|---|---|---|
| 1999 | 0 | $606000+14000(0)$ |
| 2000 | 1 | $606000+14000(1)$ |
| 2001 | | |
| 2002 | | |

**c)** Observing the pattern in the table, we come up with the equation $N = 606000+14000x$ to relate the number of crimes $C$ to the number of years $x$ after 1999. Here, $N$ is a function of $x$, and so we can use the notation $N(x) = 606000+14000x$ to emphasis this.

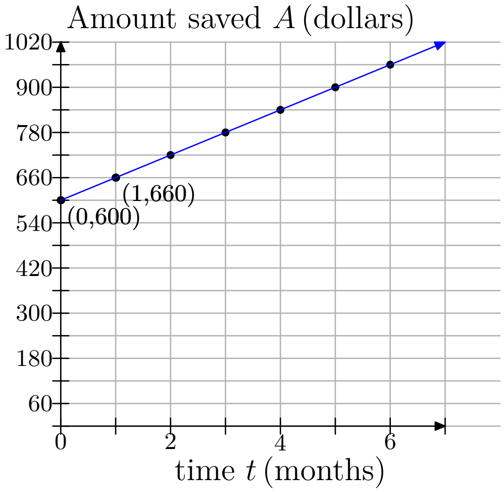
i. Compute $N(5)$.
ii. In a complete sentence, explain what $N(5)$ represents.
iii. Compute $N(7)$.
iv. In a complete sentence, explain what $N(7)$ represents.

## 3.1 Answers

**1.**

**b)** $(1, \$660)$

**d)**



**e)** $1020

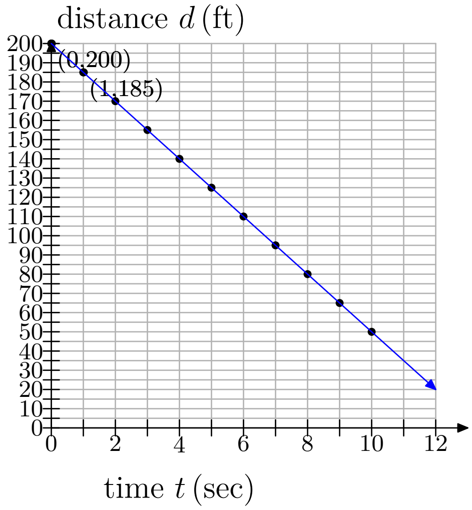
**f)** 6 months

**3.**

**b)** $(0, 200)$

**c)** $(1, 185)$

**e)**



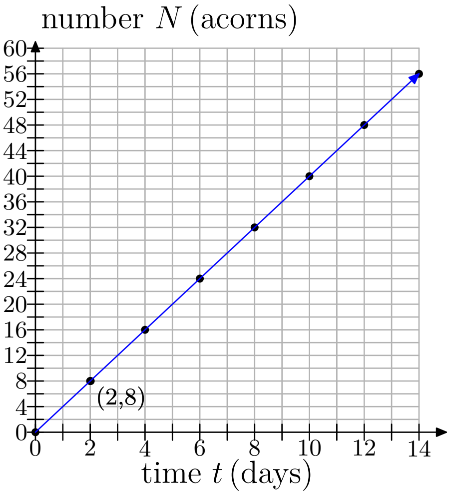
**f)** 10 seconds

**5.**

**b)** $(0, 0)$

**c)** $(2, 8)$

**e)**



**f)** 40 acorns

**g)** 4 acorns/day

**h)**

| $t$ | $N$ |
|---|---|
| 0 | 0 |
| 1 | $0 + 4(1)$ |
| 2 | $0 + 4(2)$ |
| 3 | $0 + 4(3)$ |
| 4 | $0 + 4(4)$ |
| 6 | $0 + 4(6)$ |
| 8 | $0 + 4(8)$ |
| 10 | $0 + 4(10)$ |
| 12 | $0 + 4(12)$ |
| 14 | $0 + 4(14)$ |

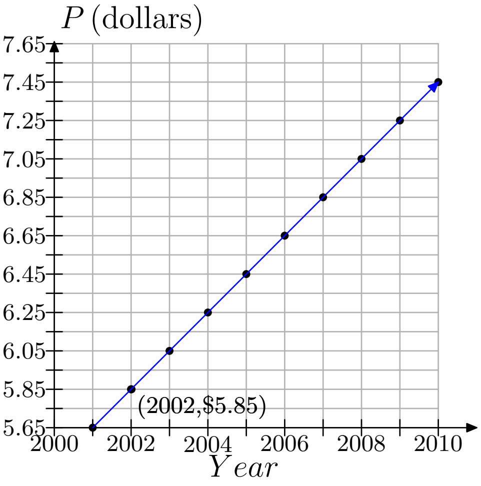
**i)** $N = 0 + 4t$ or $N = 4t$

**j)** $N = 40$; yes.

**7.**

**c)** $(2002, 5.85)$

**e)**



**f)** 2008

**9.**

**a)** The estimate should be approximately 120 degrees Fahrenheit.

**b)** $\frac{9}{5}$

**c)**

| $C$ (deg) | $F$ (deg) |
|---|---|
| 0 | 32 |
| 1 | $32 + \frac{9}{5}(1)$ |
| 2 | $32 + \frac{9}{5}(2)$ |
| 3 | $32 + \frac{9}{5}(3)$ |
| 4 | $32 + \frac{9}{5}(4)$ |
| 5 | $32 + \frac{9}{5}(5)$ |
| 10 | $32 + \frac{9}{5}(10)$ |
| 20 | $32 + \frac{9}{5}(20)$ |
| 48 | $32 + \frac{9}{5}(48)$ |
| 100 | $32 + \frac{9}{5}(100)$ |

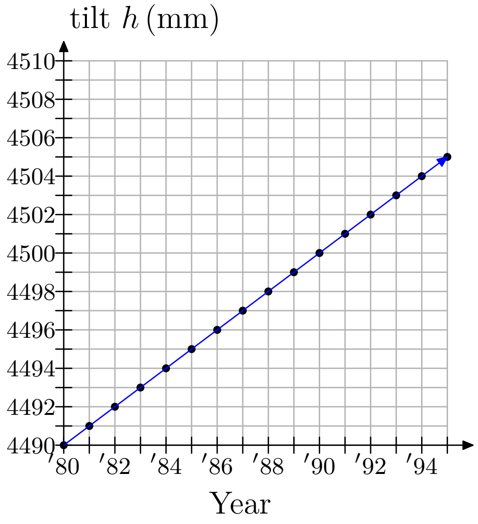
**d)** $F = \frac{9}{5}C + 32$

**11.**

**a)** 4490mm

**c)** $(1980, 4490)$

**f)**



**g)** 1985

**h)** The computer model must not have taken into consideration certain unexpected factors.

**i)**

| Year | yrs after 1980 | tilt $h$ |
|---|---|---|
| 1980 | 0 | 4490 |
| 1981 | 1 | $4490 + 1(1)$ |
| 1982 | 2 | $4490 + 1(2)$ |
| 1983 | 3 | $4490 + 1(3)$ |
| 1984 | 4 | $4490 + 1(4)$ |
| 1985 | 5 | $4490 + 1(5)$ |
| 1986 | 6 | $4490 + 1(6)$ |
| 1987 | 7 | $4490 + 1(7)$ |
| 1988 | 8 | $4490 + 1(8)$ |
| 1989 | 9 | $4490 + 1(9)$ |
| 1990 | 10 | $4490 + 1(10)$ |
| 1991 | 11 | $4490 + 1(11)$ |
| 1992 | 12 | $4490 + 1(12)$ |
| 1993 | 13 | $4490 + 1(13)$ |
| 1994 | 14 | $4490 + 1(14)$ |
| 1995 | 15 | $4490 + 1(15)$ |

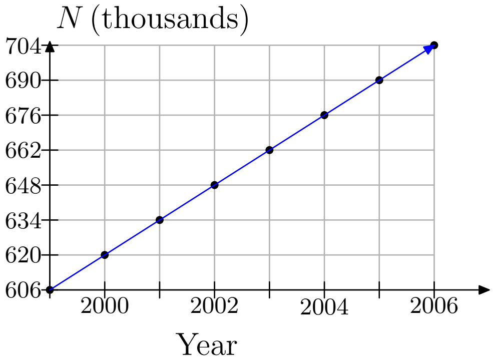
**j)** $h = 4490 + 1x$

**k)** 4500mm. Yes, it agrees with the actual value in 1990.

**l)** 1985. Yes, it agrees with our answer from (g).

**13.**

**a)**



**b)**

| Year | yrs $x$ after ’99 | No. of inmates $N$ |
|---|---|---|
| 1999 | 0 | $606000+14000(0)$ |
| 2000 | 1 | $606000+14000(1)$ |
| 2001 | 2 | $606000+14000(2)$ |
| 2002 | 3 | $606000+14000(3)$ |

**c)**

i. $676,000$.
ii. It means that, according to our model, 5 years after 1999 (that is, in 2004), the number of inmates will be $676,000$.
iii. $704,000$.
iv. It means that, according to our model, in 2006, the number of inmates will be $704,000$

## 3.2 Slope

In the previous section on Linear Models, we saw that if the dependent variable was changing at a *constant* rate with respect to the independent variable, then the graph was a line. If the rate was positive, then as we swept our eyes from left to right, the line rose upward, the dependent variable increasing with increasing changes in the independent variable. If the rate was negative, then the graph fell downward, the dependent variable decreasing with increasing changes in the independent variable. You may have also learned that higher rates led to steeper lines (lines that rose more quickly) and lower rates led to lines that were less steep.

In this section, we will connect the intuitive concept of rate developed in the previous section with a formal definition of the *slope* of a line. To start, let's state up front what is meant by the slope of a line.

**Slope** is a number that tells us how quickly a line rises or falls.

If slope is a number that is directly connected to the "steepness" of a line, then we should have certain expectations.

**Expectations.**

1. Lines with positive slope should slant uphill (as our eyes sweep from left to right).
2. Lines with negative slope should slant downhill (as our eyes sweep from left to right).
3. Because any horizontal line neither slants uphill nor downhill, we expect that it should have slope equal to zero.
4. Lines with a larger positive slope should rise more quickly than lines with a smaller positive slope.
5. If two lines have negative slope, then the line having the slope with larger absolute value should fall more quickly than the other line.

It remains to define how to compute the slope of a particular line. Whatever definition we choose, it should conform with the expectations outlined above. We also would like the definition of slope to conform with the concept of rate developed in the previous section. Thus, we make the following definition.

**Definition 1.** *The* **slope** *of a line is the rate at which the dependent variable is changing with respect to the independent variable.*

Note how the word "change" is used **Definition 1**. It is important to understand that the change in some quantity can be positive, negative, or zero. For example, if

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[3] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

the temperature outside is 40° F when I leave my home at 6 AM, and at noon the temperature is 65° F, then the change in temperature is a positive 25° F. On the other hand, if the temperature outside is 65° F at noon, and the temperature is 50° F when I return home in the evening, then the change in temperature is a negative 15 degrees Fahrenheit.

In calculating the change in a quantity, follow this rule.

**Definition 2.**

$$\textit{Change in Quantity} = \textit{Latter Measurement} - \textit{Former Measurement}.$$

Thus, if $T$ represents the temperature and $\Delta T$ represents the change in the temperature[4], then in our first case (taking the temperature in the morning then later at noon), the change in temperature is

$$\Delta T = \text{Latter} - \text{Former} = 65^\circ\,\text{F} - 40^\circ\,\text{F} = 25^\circ\,\text{F}.$$

This positive result represents an *increase* in the temperature of 25° F.

In the second case (taking the temperature at noon then later in the evening), the change in temperature is

$$\Delta T = \text{Latter} - \text{Former} = 50^\circ\,\text{F} - 65^\circ\,\text{F} = -15^\circ\,\text{F}.$$

This negative result represents a *decrease* in the temperature of 15° F.

**Tip 3.** *Readers should note that the direction of subtraction is extremely important. To detect the change in a quantity, always subtract the former (earlier) measurement* from *the latter (later) measurement.*

▶ **Example 4.** *A ball is perched at rest at the top of a long ramp. It's given a little tap and it begins to roll down the ramp. The speed $v$ of the ball (in meters per second) is plotted versus the time $t$ (in seconds) in* **Figure 1**.

*Determine the slope of the line.*

We've defined the slope as the rate at which the dependent variable is changing with respect to the independent variable. In this case, the speed $v$ of the ball "depends" upon the amount of time $t$ that has elapsed. Consequently, $v$ is the dependent variable and has been placed on the vertical axis.[5] On the other hand, $t$ is the independent variable and has been assigned the horizontal axis.

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[4] The first four letters of the Greek alphabet are $\alpha$ ("alpha"), $\beta$ ("beta"), $\gamma$ ("gamma"), and $\delta$ ("delta"), which have similar meanings to the letters a, b, c, and d of the English alphabet. The symbol $\Delta$ is the uppercase equivalent of the letter $\delta$, so think of $\Delta$ as a "capital D." Note that the word "difference" starts with the letter "d," so our choice of $\Delta T$ (read "delta-T") for the "change in $T$" is no accident. The change in $T$ is found by taking the difference in $T$.

[5] It is traditional to place the dependent variable on the vertical axis and the independent variable on the horizontal axis. Although not required, we will try to follow this tradition when possible.

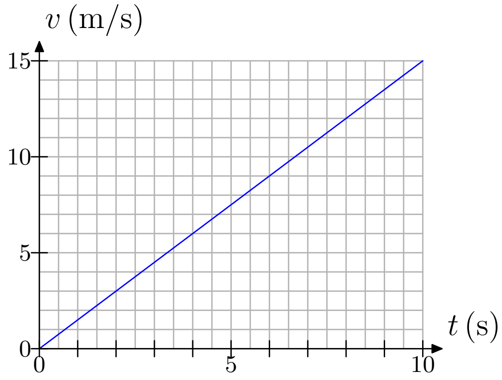


**Figure 1.** Speed versus time.

To determine the rate at which $v$ is changing with respect to $t$ (the slope of the line), we first select two points $P(2, 3)$ and $Q(8, 12)$ on the line, as shown in **Figure 2**. As we sweep our eyes from left to right (a convention we will always follow when dealing with slope), the point $P$ occurs *before* the point $Q$. Hence, we consider $P$ the "former" measurement and point $Q$ the "latter" measurement.

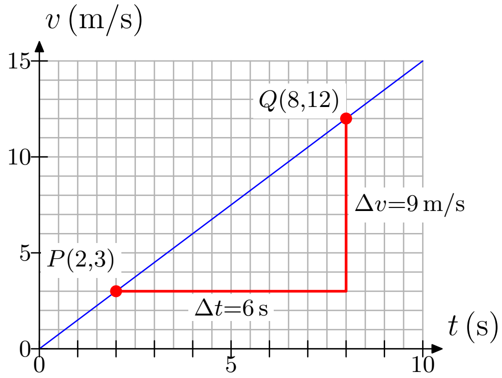


**Figure 2.** Determining the slope of the line.

At point $P$, the time is $t = 2$ seconds, then at point $Q$ the time is $t = 8$ seconds. The change in $t$ is found by subtracting the former measurement from the latter measurement.

$$\Delta t = 8\,\mathrm{s} - 2\,\mathrm{s} = 6\,\mathrm{s}.$$

At point $P$, the speed is $v = 3$ meters per second, then at point $Q$ the speed is $v = 12$ meters per second. Hence, the change in $v$ is

$$\Delta v = 12\,\mathrm{m/s} - 3\,\mathrm{m/s} = 9\,\mathrm{m/s}.$$

Finally, the slope of the line is defined as the rate at which the dependent variable $v$ is changing with respect to the independent variable $t$. That is,

$$\text{Slope} = \frac{\Delta v}{\Delta t} = \frac{9\,\mathrm{m/s}}{6\,\mathrm{s}} = \frac{3}{2}\frac{\mathrm{m/s}}{\mathrm{s}}.$$

Scientists prefer to write this as $1.5\,\mathrm{m/s}^2$, but this might not be as intuitive as writing $1.5\,(\mathrm{m/s})/\mathrm{s}$, which indicates that the speed is increasing at a rate of $1.5\,\mathrm{m/s}$ every second.[6] This makes good sense as a ball rolling down a ramp will pick up speed with the passage of time. The slope provides an exact numerical description of how the speed increases with respect to time.

Note that our definition of the slope of the line satisfies one of our goals: the slope is precisely the same as the notion of rate described in the previous section. Indeed, note the right triangle we've drawn in **Figure 2**. The bottom edge of the triangle is 12 boxes long, but every 2 boxes represents one second, so this displacement in the time $t$ direction is 6 seconds. The vertical side of the right triangle is 9 boxes in height where each box represents 1 meter per second. Consequently, this vertical edge of the right triangle represents a positive displacement of 9 meters per second. Thus, every 6 seconds, there is an increase in speed of 9 meters per second. Hence, the ball is picking up speed at the rate of 9 meters per second every 6 seconds, or equivalently, 1.5 meters per second every second.

**Remark.** In **Figure 2**, the rate at which the speed is increasing with respect to time is equivalent to the slope of the line.

Suppose that we had labeled our points $P(t_{\text{initial}}, v_{\text{initial}})$ and $Q(t_{\text{final}}, v_{\text{final}})$ as shown in **Figure 3**.

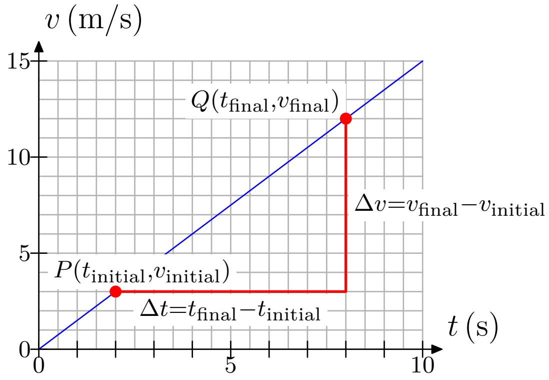


**Figure 3.** Initial and final measurements.

Now the change in speed $v$ would be

$$\Delta v = v_{\text{final}} - v_{\text{initial}},$$

and the change in time $t$ would be

$$\Delta t = t_{\text{final}} - t_{\text{initial}}.$$

---

[6] Scientists call this the *acceleration* of the ball. We'll have more to say about acceleration in upcoming sections.

Therefore, the slope of the line would be computed with the following formula.

$$\text{Slope} = \frac{\Delta v}{\Delta t} = \frac{v_{\text{final}} - v_{\text{initial}}}{t_{\text{final}} - t_{\text{initial}}}.$$

With $P(t_{\text{initial}}, v_{\text{initial}}) = (2\,\text{s}, 3\,\text{m/s})$ and $Q(t_{\text{final}}, v_{\text{final}}) = (8\,\text{s}, 12\,\text{m/s})$, this becomes

$$\text{Slope} = \frac{12\,\text{m/s} - 3\,\text{m/s}}{8\,\text{s} - 2\,\text{s}} = \frac{9\,\text{m/s}}{6\,\text{s}} = 1.5\,\text{m/s}^2.$$

◆

## The Slope Formula

The last calculation in **Example 4** allows us to discuss the slope of a line as a purely mathematical concept, one that is not rooted in a supporting application as in **Example 4**. Take, for example, the line shown in **Figure 4** that passes through the points $P(-3, -3)$ and $Q(2, 1)$.

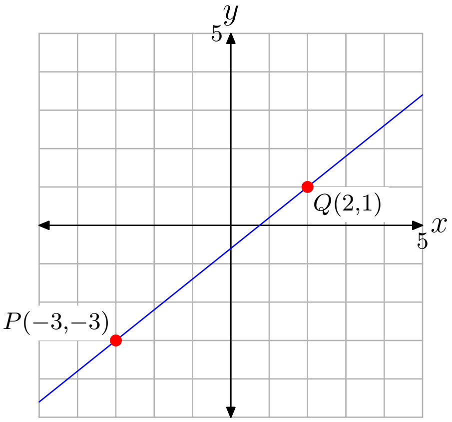


**Figure 4.** Computing the slope of a line in an $xy$-coordinate system.

In this example, the dependent variable is $y$ and the independent variable is $x$, so the slope of the line is $\Delta y$ (the change in $y$) divided by $\Delta x$ (the change in $x$).

$$\text{Slope} = \frac{\Delta y}{\Delta x}.$$

Sweeping our eyes from left to right, the point $P$ comes first, followed by the point $Q$. Keeping "latter minus former" in mind, the change in $y$ is computed by subtracting the $y$-value of point $P$ from the $y$-value of point $Q$. That is,

$$\Delta y = 1 - (-3) = 4.$$

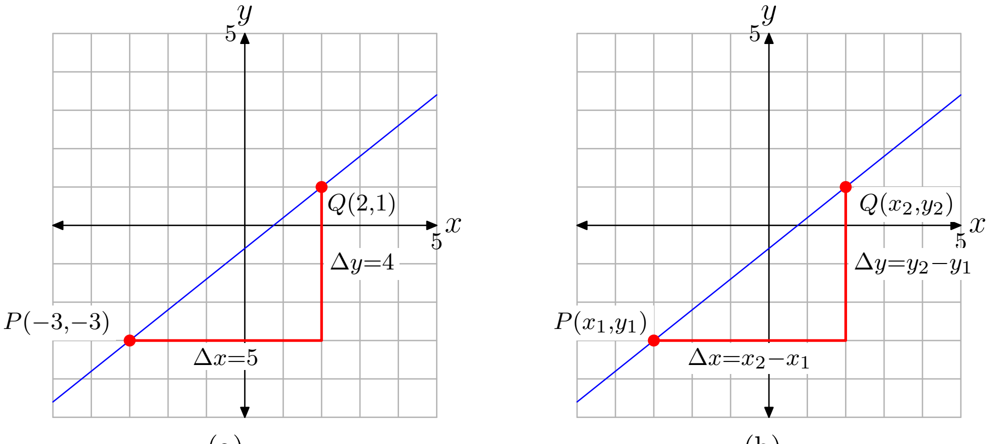
Similarly, the change in $x$ is computed by subtracting the $x$-value of point $P$ from the $x$-value of point $Q$. That is,

$$\Delta x = 2 - (-3) = 5.$$

Thus, the slope of the line is

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{4}{5}.$$

Alternatively, we can use the points $P$ and $Q$ as vertices of a right triangle with sides parallel to the axes (shown in **Figure 5**(a)). The horizontal edge of the right triangle is 5 boxes (each representing 1 unit), so the displacement in $x$ is 5 units. The vertical edge is 4 boxes (each representing 1 unit), so the displacement in $y$ is 4 units. Hence, each time $x$ is increased by 5 units, $y$ experiences an increase of 4 units. Therefore, the slope of the line is again 4/5.



(a) (b)

**Figure 5.** Using a right triangle to determine the slope.

Suppose that we had labeled our points $P(x_1, y_1)$ and $Q(x_2, y_2)$ as shown in **Figure 5**(b). Now the change in $y$ would be[7]

$$\Delta y = y_2 - y_1,$$

and the change in $x$ would be

$$\Delta x = x_2 - x_1.$$

Therefore, the slope of the line would be computed with the following formula.

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

With $P(x_1, y_1) = (-3, -3)$ and $Q(x_2, y_2) = (2, 1)$, this becomes

$$\text{Slope} = \frac{1 - (-3)}{2 - (-3)} = \frac{4}{5}.$$

The slope formula is worth summarizing in a definition.

---

[7] Again, note that the Greek letter Δ is like our uppercase "D." "D" is for "difference." The change in $y$, represented by $\Delta y$ (read "delta-Y") , is calculated by taking the "difference in $y$," or $y_2 - y_1$.

**Definition 5.** *The slope of the line that passes through the points* $P(x_1, y_1)$ *and* $Q(x_2, y_2)$ *is given by the formula*

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Let's look at some more examples.

▶ **Example 6.** *Find the slope of the line passing through the points* $P(-3, -2)$ *and* $Q(3, 1)$.

We can use the slope formula in **Definition 5** to determine the slope. With $(x_1, y_1) = P(-3, -2)$ and $(x_2, y_2) = Q(3, 1)$,

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - (-2)}{3 - (-3)} = \frac{3}{6} = \frac{1}{2}.$$

Readers will sometimes ask, "Which point should be $(x_1, y_1)$ and which should be $(x_2, y_2)$?" The short answer is, "It doesn't matter!" Suppose instead, that we let $(x_1, y_1) = Q(3, 1)$ and $(x_2, y_2) = P(-3, -2)$. Then,

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 1}{-3 - 3} = \frac{-3}{-6} = \frac{1}{2}.$$

Because the change in any quantity is found by subtracting the earlier measurement from the later measurement, we will continue to stress the first order. However, if we reverse the points as we did in our second calculation, both numerator and denominator reverse sign with this interchange, so we get the same answer.

Of course, we can also determine the slope by plotting $P(-3, -2)$ and $Q(3, 1)$ and the line that passes through $P$ and $Q$, as we've done in **Figure 6**.

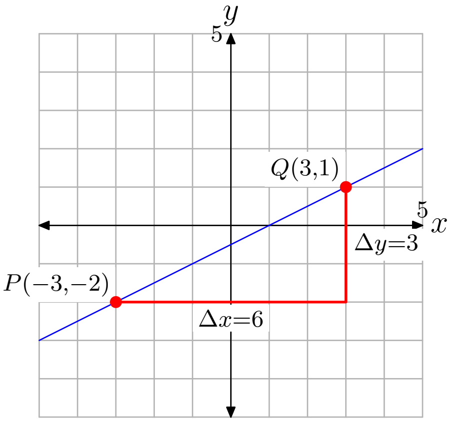


**Figure 6.** Determining the slope from the graph.

Starting at the point $P$, to get to the point $Q$, we move 6 boxes to the right, then 3 boxes up, as shown in **Figure 6**. Hence, the slope of the line is

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{3}{6} = \frac{1}{2}.$$

Note that two of our expectations regarding the slope of a line are met with this example.

1. The line through $P(-3, -2)$ and $Q(3, 1)$ in **Figure 6** has slope 1/2. This is a positive number and the line slants uphill (as expected) as we sweep our eyes from left to right.
2. The slope in this example is 1/2, which is less than the slope of the line in **Figure 5**(a), which was 4/5. Note that the line in **Figure 6** is less steep than the line in **Figure 5**(a), which was another of our earlier expectations regarding the slope of a line.

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► **Example 7.** *Find the slope of the line passing through the points $P(-4, 4)$ and $Q(4, -2)$.*

We can use the slope formula in **Definition 5** to determine the slope. With $(x_1, y_1) = P(-4, 4)$ and $(x_2, y_2) = Q(4, -2)$,

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 4}{4 - (-4)} = \frac{-6}{8} = -\frac{3}{4}.$$

We can also get the slope of the line from the graph in **Figure 7**. Starting at the point $P(-4, 4)$, move 8 units to the right, then 6 units downward, as shown in **Figure 7**.

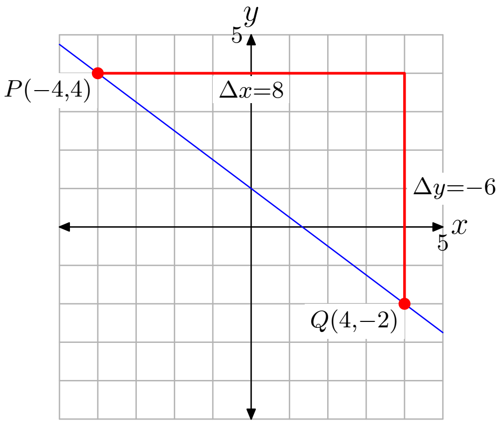

**Figure 7.** Determining the slope from the graph.

Thus, the slope of the line is

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{-6}{8} = -\frac{3}{4}.$$

Again, one of our earlier expectations regarding the slope of a line is met in this example. The slope is $-3/4$, which is a negative number, and the line in **Figure 7** slants downhill (as we sweep our eyes from left to right).

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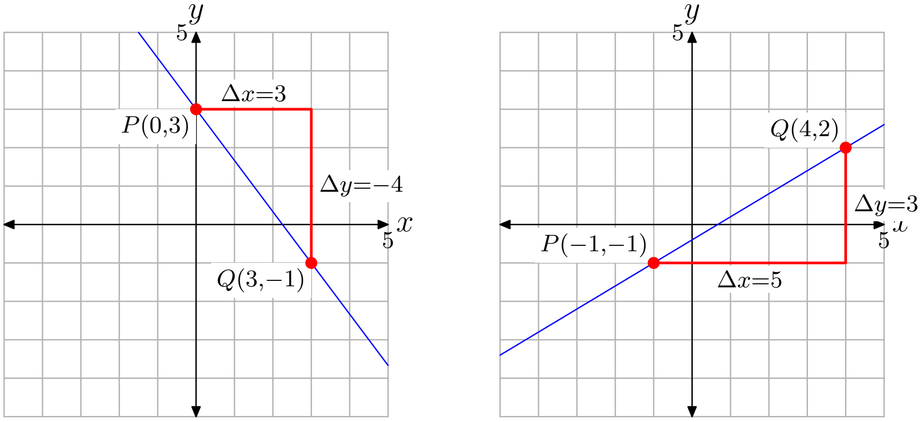
▶ **Example 8.** *Draw a line that intercepts the $y$-axis at $(0, 3)$ so that the line has slope $-4/3$. Draw a second line that passes through the point $P(-1, -1)$ with slope 3/5.*

The slope of the first line is $-4/3$. This means that our line must slant downhill (as we sweep our eyes from left to right). The slope is the change in $y$ over the change in $x$. Therefore, every time $x$ increases by 3 units, $y$ must decrease by 4 units. Plot the point $P(0, 3)$, as shown in **Figure 8**(a). Then, starting at $P$, move 3 units to the right, followed by 4 units downward to the point $Q(3, -1)$, as shown in **Figure 8**(a). Draw the required line, which must pass through the points $P$ and $Q$.

To draw the second line, first plot the point $P(-1, -1)$, as shown in **Figure 8**(b). Starting at the point $P$, move 5 units to the right, then upward 3 units to the point $Q(4, 2)$, as shown in **Figure 8**(b). Draw the required line passing through the points $P$ and $Q$.



(a) A line having $y$-intercept at $(0, 3)$ with slope $-4/3$.

(b) A line having slope 3/5 that passes through the point $P(-1, -1)$.

**Figure 8.**

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## *Parallel Lines*

Because slope controls the "steepness" of a line, it is a simple matter to see that parallel lines must have the same slope.

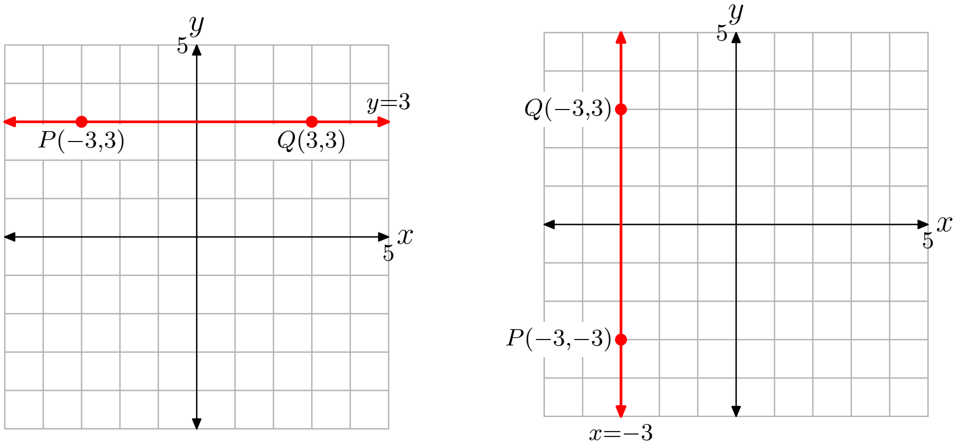
**Property 9.** Let $L_1$ be a line having slope $m_1$. Let $L_2$ be a line having slope $m_2$. If $L_1$ and $L_2$ are **parallel**, then

$$m_1 = m_2 \tag{10}$$

That is, any two parallel lines have the same slope.

▶ **Example 11.** *What is the slope of any horizontal line? What is the slope of any vertical line?*

One would expect that our definition would verify that the slope of any horizontal line is zero. Select, for example, the horizontal line shown in **Figure 9**(a). Select the points $(-3, 3)$ and $(3, 3)$ on this line.



(a) Determine the slope of a horizontal line.

(b) Determine the slope of a vertical line.

**Figure 9.**

With $(x_1, y_1) = (-3, 3)$ and $(x_2, y_2) = (3, 3)$,

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-3}{3-(-3)} = \frac{0}{6} = 0.$$

Thus, the horizontal line in **Figure 9**(a) has slope equal to zero, exactly as expected. Further, all horizontal lines are parallel to this horizontal line and have the same slope. Therefore, all horizontal lines have slope zero.

We would surmise that the vertical line in **Figure 9**(b) has undefined slope (we'll explore this more fully in the exercises). In **Figure 9**(b), we've selected the points

$P(-3,-3)$ and $Q(-3,3)$ on the vertical line. With $(x_1,y_1)=P(-3,-3)$ and $(x_2,y_2)=Q(-3,3)$,

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1} = \frac{3-(-3)}{-3-(-3)} = \frac{6}{0}, \text{ which is undefined.}$$

The slope of the vertical line in **Figure 9**(b) is undefined because division by zero is meaningless. Further, all vertical lines are parallel to this vertical line and have undefined slope.

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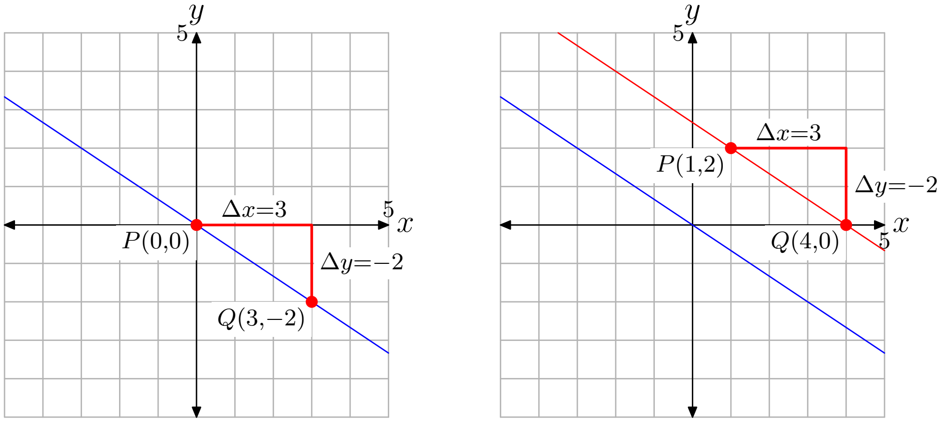
▶ **Example 12.** *Draw a line through the point $P(1,2)$ that is parallel to the line passing through the origin with slope $-2/3$.*

We will first draw a line through the origin with slope $-2/3$. Plot the point $P(0,0)$, then move 3 units to the right and 2 units downward to the point $Q(3,-2)$, as shown in **Figure 10**(a). Draw a line through the points $P$ and $Q$ as shown in **Figure 10**(a).

Next, plot the point $P(1,2)$ as shown in **Figure 10**(b). To draw a line through this point that is parallel to the line through the origin, this second line must have the same slope as the first line. Therefore, start at the point $P(1,2)$, as shown in **Figure 10**(b), then move 3 units to the right and 2 units downward to the point $Q(4,0)$. Draw a line through the points $P$ and $Q$ as shown in **Figure 10**(b). Note that this second line is parallel to the first.



(a) A line through the origin with slope $-2/3$.

(b) A line through $P(1,2)$ that is parallel to the line through the origin.

**Figure 10.**

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## Perpendicular Lines

The relationship between the slopes of two perpendicular lines is not as straightforward as the relation between the slopes of two parallel lines. Let's begin by stating the pertinent property.

**Property 13.** Let $L_1$ be a line having slope $m_1$. Let $L_2$ be a line having slope $m_2$. If $L_1$ and $L_2$ are **perpendicular**, then

$$m_1 m_2 = -1. \tag{14}$$

That is, the product of the slopes of two perpendicular lines is $-1$.

We can solve **equation (14)** for $m_1$ in terms of $m_2$.

$$m_1 = -\frac{1}{m_2} \tag{15}$$

Equation (**15**) tells us that the slope of the first line is the *negative reciprocal* of the slope of the second line.

For example, suppose that $L_1$ and $L_2$ are perpendicular lines with slopes $m_1$ and $m_2$, respectively.

- If $m_2 = 2$, then $m_1 = -\dfrac{1}{2}$.
- If $m_2 = \dfrac{3}{5}$, then $m_1 = -\dfrac{5}{3}$.
- If $m_2 = -\dfrac{2}{3}$, then $m_1 = \dfrac{3}{2}$.

Note that in each bulleted item, the product of the slopes is $-1$.

We won't provide a proof of **equation (15)**, but we will provide some motivating evidence in the form of a graph.

► **Example 16.** *Sketch the graphs of the lines passing through the origin having slopes 2 and* $-1/2$.

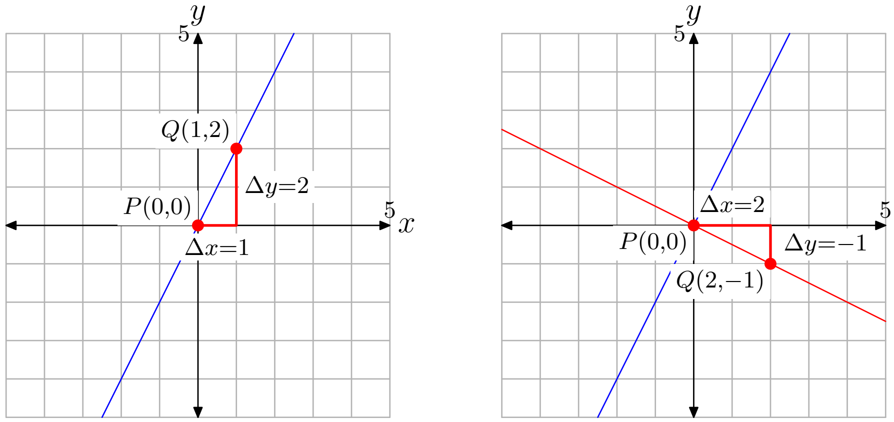
In **Figure 11**(a), we've plotted the point $P(0, 0)$ at the origin, then moved 1 unit to the right and 2 units upward to the point $Q(1, 2)$. The resulting line passes through the origin and has slope $m_1 = 2$ (alternatively, $m_1 = 2/1$).

In **Figure 11**(b), we've again plotted the point $P(0, 0)$ at the origin, then moved 2 units to the right and 1 unit downward to the point $Q(2, -1)$. The resulting line passes through the origin and has slope $m_2 = -1/2$.

There are two important points that need to be made about the lines in **Figure 11**(b).

1. The two lines in **Figure 11**(b) are perpendicular. They meet and form a right angle of 90°. If you have a protractor available, you might want to measure the angle between the two lines and note that the measure of the angle is 90°.
2. The product of the two slopes is

$$m_1m_2 = 2 \cdot \left(-\frac{1}{2}\right) = -1.$$



(a) $m_1 = 2$. (b) $m_2 = -1/2$.

**Figure 11.** Sketching perpendicular lines.

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## 3.2 Exercises

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**1.** Suppose you are riding a bicycle up a hill as shown below.

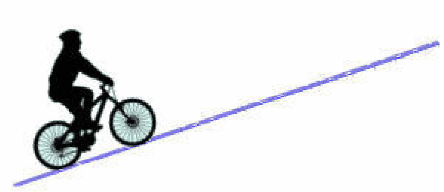
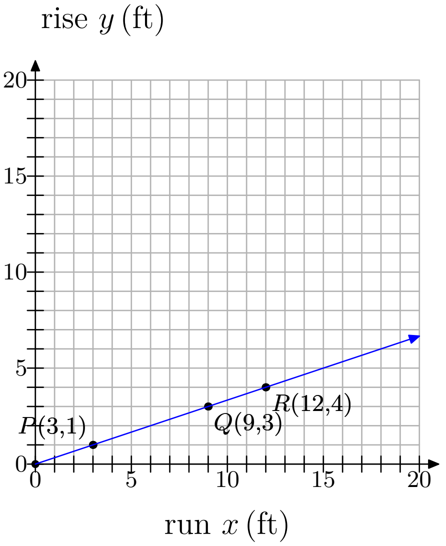


**Figure 12.** Riding a bicycle up a hill.

**a)** If the hill is straight as shown, consider the slant, or steepness, of its incline. As you ride up the hill, what can you say about the slant? Does it change? If so, how?

**b)** The slant is what mathematicians call the slope. To confirm your answer to part (a), you will place the hill on a coordinate system and compute its slope along various segments of the hill. See the figure below. Three points–$P$, $Q$ and $R$–have been labeled along the hill. We call the vertical distance (height) the rise and the horizontal distance the run. As you ride up the hill from point $P$ to point $Q$, what is the rise? What is the run? Use these values to compute the slope from $P$ to $Q$.



**c)** Now consider as you ride from $P$ to $R$. What is the rise? What is the run? Use these values to compute the slope from $P$ to $R$.

**d)** Finally, consider as you ride from $Q$ to $R$. What is the rise? What is the run? Use these values to compute the slope from $Q$ to $R$.

**e)** How do the values for slope from parts (b)-(d) compare? Do these results confirm your answer to part (a)?

**f)** Notice that the slope is positive in this example. In this context of riding a bicycle over a hill, what would negative slope mean?

**2.** Set up a coordinate system on a sheet of graph paper, plotting the points $P(3,4)$ and $Q(-2,-7)$ and drawing the line through them.

**a)** What can you say about the slope of the line? Is it positive, zero, negative or undefined? Is the slope the same everywhere along the line, or does it change in places? If it does change, where are the slopes different?

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[8] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

**b)** Use your graph to determine the change in $y$ (rise) and the change in $x$ (run). Use these results to compute the slope of the line.

**c)** Use the slope formula to compute the slope of the line.

**d)** Does your numerical solution from part (c) agree with your graphical solution from part (b)? If not, check your work for errors.

**3.** Set up a coordinate system on a sheet of graph paper, plotting the points $P(-1, 3)$ and $Q(5, -3)$ and drawing the line through them.

**a)** What can you say about the slope of the line? Is it positive, zero, negative or undefined? Is the slope the same everywhere along the line, or does it change in places? If it does change, where are the slopes different?

**b)** Use your graph to determine the change in $y$ (rise) and the change in $x$ (run). Use these results to compute the slope of the line.

**c)** Use the slope formula to compute the slope of the line.

**d)** Does your numerical solution from part (c) agree with your graphical solution from part (b)? If not, check your work for errors.

---

In **Exercises 4-10**, perform each of the following tasks.

i. Make a sketch of a coordinate system; plot the given points, and draw the line through the points.

ii. Use the slope formula to compute the slope of the line through the given points. Reduce the slope where possible.

**4.** $(0, 0)$ and $(3, 4)$

**5.** $(-5, 2)$ and $(0, 3)$

**6.** $(-3, -3)$ and $(6, -5)$

**7.** $(2, 0)$ and $(2, 2)$

**8.** $(-9, -3)$ and $(6, -3)$

**9.** $(-8, 4)$ and $(3, -8)$

**10.** $(-2, 6)$ and $(5, -2)$

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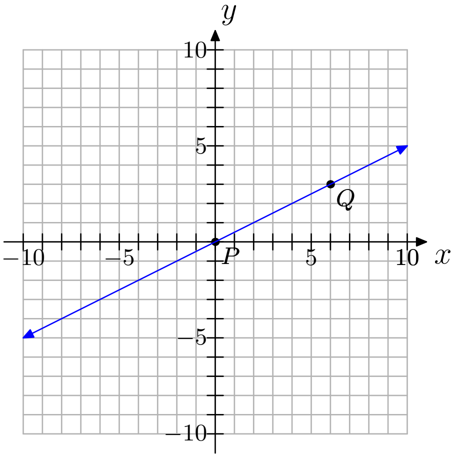
**11.** For the following line, two convenient points $P$ and $Q$ have been chosen. We chose two points that were at the corners of boxes on our grid so their coordinates are easy to read.



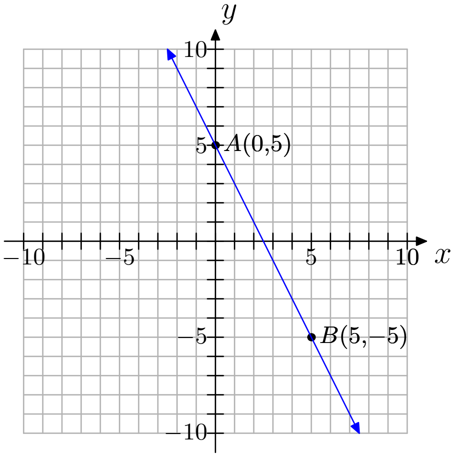
**a)** Label their coordinates.

**b)** Thinking of $P$ as the starting point and $Q$ as the ending point, draw a right triangle joining the points.

**c)** Clearly state the change in $y$ (rise) and the change in $x$ (run) from $P$ to $Q$.

**d)** Compute the slope.

**12.** For the following line, two convenient points $A$ and $B$ have been chosen. We chose two points that were at the corners of boxes on our grid so their coordinates are easy to read.



**a)** Label their coordinates.

**b)** Thinking of $A$ as the starting point and $B$ as the ending point, draw a right triangle joining the points.

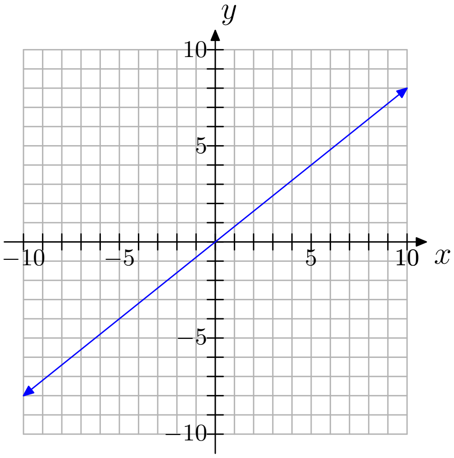
**c)** Clearly state the change in $y$ (rise) and the change in $x$ (run) from $A$ to $B$.

**d)** Compute the slope.

**13.** Copy the coordinate system below onto a sheet of graph paper. Then do the following:

**a)** Select any two convenient points $P$ and $Q$ on the graph of the line. Label each point with its coordinates.

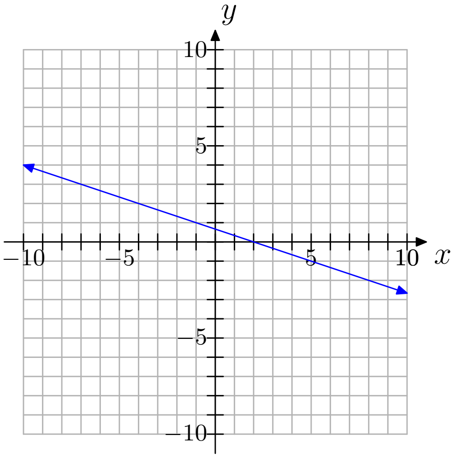
**b)** Clearly state the change in $y$ (rise) and the change in $x$ (run). Compute the slope of the line.



**14.** Copy the coordinate system below onto a sheet of graph paper. Then do the following:

**a)** Select any two convenient points $P$ and $Q$ on the graph of the line. Label each point with its coordinates.

**b)** Clearly state the change in $y$ (rise) and the change in $x$ (run). Compute the slope of the line.

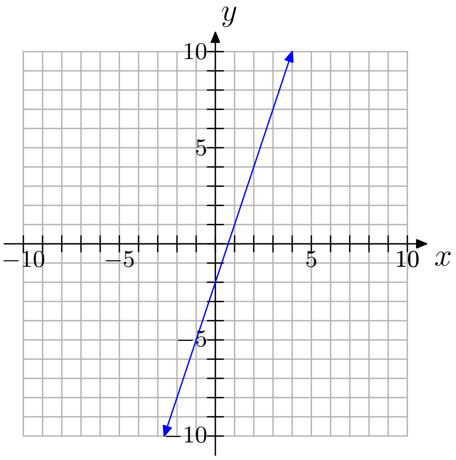


**15.** Copy the coordinate system below onto a sheet of graph paper. Then do the following:

**a)** Select any two convenient points $P$

and $Q$ on the graph of the line. Label each point with its coordinates.

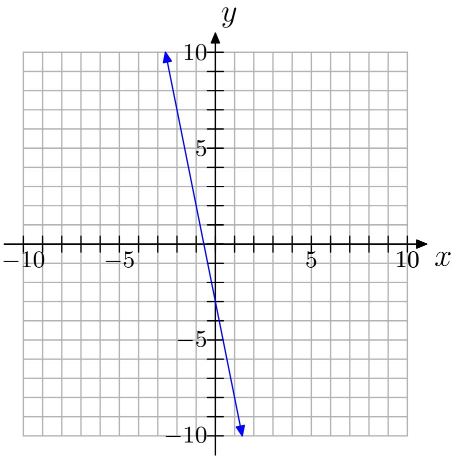
**b)** Clearly state the change in $y$ (rise) and the change in $x$ (run). Compute the slope of the line.



**16.** Copy the coordinate system below onto a sheet of graph paper. Then do the following:

**a)** Select any two convenient points $P$ and $Q$ on the graph of the line. Label each point with its coordinates.

**b)** Clearly state the change in $y$ (rise) and the change in $x$ (run). Compute the slope of the line.

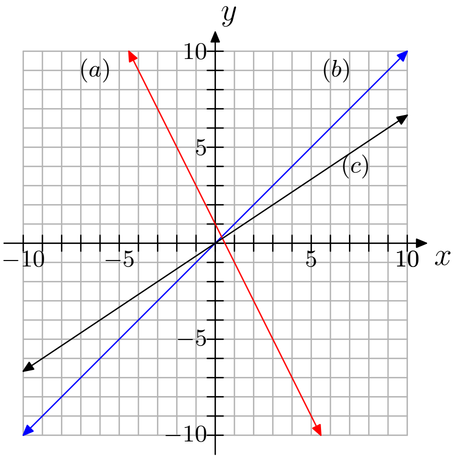


**17.** The following coordinate system shows the graphs of three lines, each with different slope. Match each slope with (a), (b), or (c) appropriately.

slope = 1

slope = 2/3

slope = −2

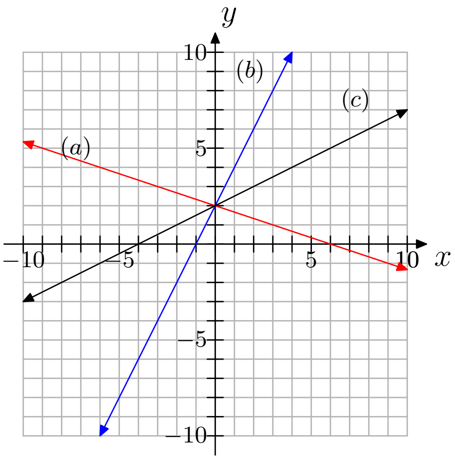


**18.** The following coordinate system shows the graphs of three lines, each with different slope. Match each slope with (a), (b), or (c) appropriately.

slope = 2

slope = −1/3

slope = 1/2



**19.** Draw a coordinate system on a sheet of graph paper for which the $x$- and $y$-axes both range from $-10$ to 10.

**a)** Draw a line that contains the point $(0, 1)$ and has slope 2. Label the line as (a).

**b)** On the same coordinate system, draw a line that contains the point $(0, 1)$ and has slope $-1/2$. Label it as (b).

**c)** Use the slopes of these two lines to show that they are perpendicular.

**20.** Draw a coordinate system on a sheet of graph paper for which the $x$- and $y$-axes both range from $-10$ to 10.

**a)** Draw a line that contains the point $(1, -2)$ and has slope $1/3$. Label the line as (a).

**b)** On the same coordinate system, draw a line that contains the point $(0, 1)$ and has slope $-3$. Label it as (b).

**c)** Use the slopes of these two lines to show that they are perpendicular.

**21.** Draw a line through the point $P(1, 3)$ that is parallel to the line through the origin with slope $-1/4$.

**22.** Draw a line through the point P(1,3) that is parallel to the line through the origin with slope 3/5.

**23.** Draw a coordinate system on a sheet of graph paper for which the $x$- and $y$-axes both range from $-10$ to 10.

**a)** Draw a line that contains the point $(-1, -2)$ and has slope $3/4$. Label the line as (a).

**b)** On the same coordinate system, draw a line that contains the point $(0, 1)$ and has slope $4/3$. Label it as (b).

**c)** Are these lines parallel, perpendicular or neither? Show using their slopes.

**24.** Graph a coordinate system on a sheet of graph paper for which the $x$- and $y$-axes both range from $-10$ to 10.

**a)** Draw a line that contains the point $(-4, 0)$ and has slope 1. Label the line as (a).

**b)** On the same coordinate system, draw a line that contains the point $(0, 2)$ and has slope $-1$. Label it as (b).

**c)** Are these lines parallel, perpendicular or neither? Show using their slopes.

**25.**

**Figure 13.** A grade is a way of expressing slope.

On the road from Fort Bragg to Willits or from Fort Bragg to Santa Rosa, one often passes signs like that shown above. A grade is just slope expressed as a percent instead of a fraction or decimal. In other words, the grade measures the steepness of the road just as slope does.

**a)** An 8% grade means that, for every horizontal distance of 100 ft, the road rises or drops 8 ft (depending on whether you are going uphill or downhill). Write 8% grade as slope in reduced fractional form.

**b)** Suppose a hill drops 16 ft for every 180 ft horizontally. Find the grade of the hill to the nearest tenth of a percent.

**c)** Explain in a complete sentence or sentences what a grade of 0% would represent.

## 3.2 *Answers*

**1.**

**a)** No.

**b)** 1/3

**c)** 1/3

**d)** 1/3

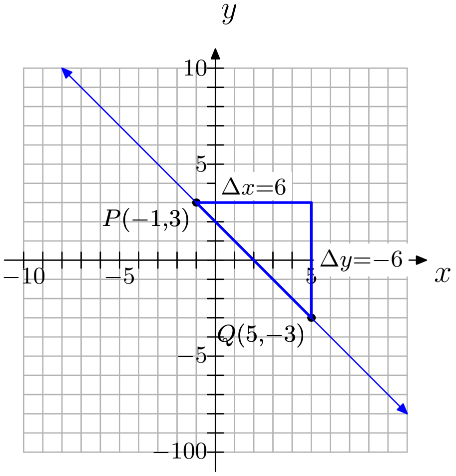
**e)** All are the same because the steepness of the hill is the same everywhere.

**f)** Negative slope would mean that you are riding downhill.

**3.**

**a)** The slope is negative because the line slants downhill. The slope is the same everywhere along the line because the slant of the line does not change.

**b)**



slope $= -1$

**c)** $\Delta y = -6$; $\Delta x = 6$; slope $= -1$

**d)** Yes.

**5.** $\frac{1}{5}$

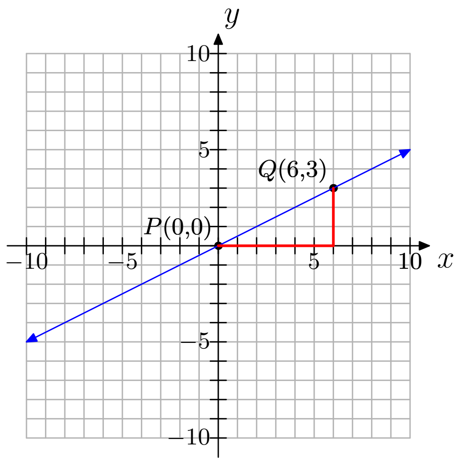
**7.** undefined

**9.** $-\frac{12}{11}$

**11.**

**a)** $(0, 0)$ and $(6, 3)$

**b)**

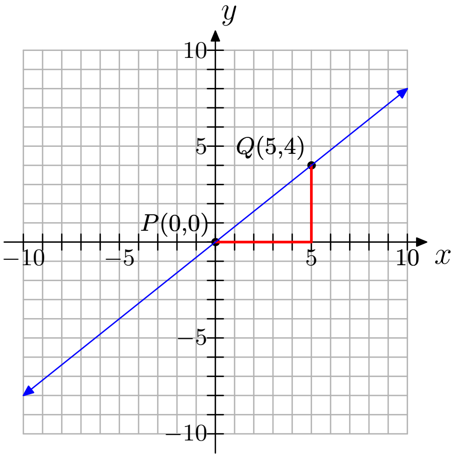


**c)** $\Delta y = 3 - 0 = 3$; $\Delta x = 6 - 0 = 6$

**d)** slope $= \frac{\Delta y}{\Delta x} = \frac{3}{6} = \frac{1}{2}$

**13.**

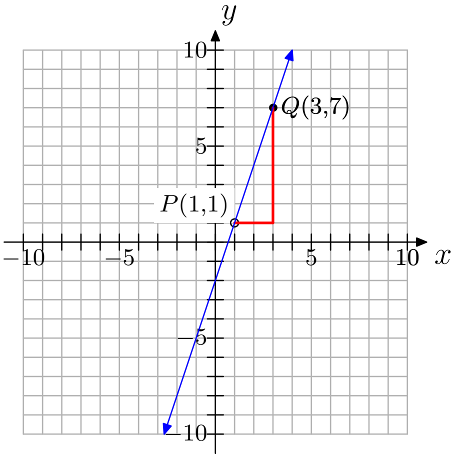
**a)** You can pick any two points on the line; for example, $(0,0)$ and $(5,4)$ as shown below.



**b)** Changes in $y$ and $x$ will vary depending on points chosen, but slope $= \frac{4}{5}$.

**15.**

**a)** You can pick any two points on the line; for example, $(1,1)$ and $(3,7)$ as shown below.



**b)** Changes in $y$ and $x$ will vary depending on points chosen, but slope $= 3$.

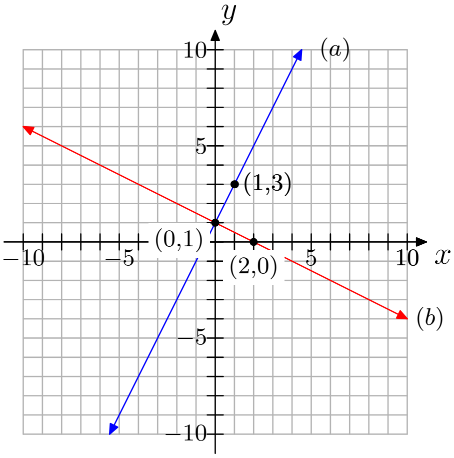
**17.** slope $= 1$: (b)

slope $= 2/3$: (c)

slope $= -2$: (a)

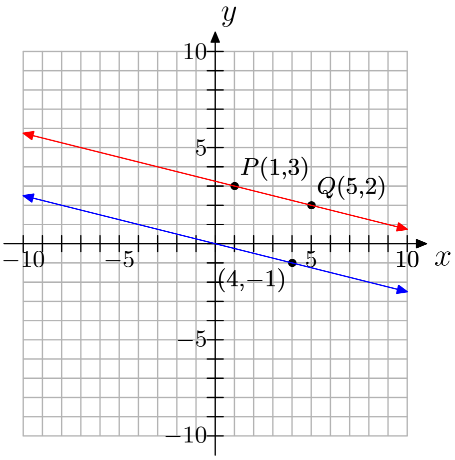
**19.**

**b)**



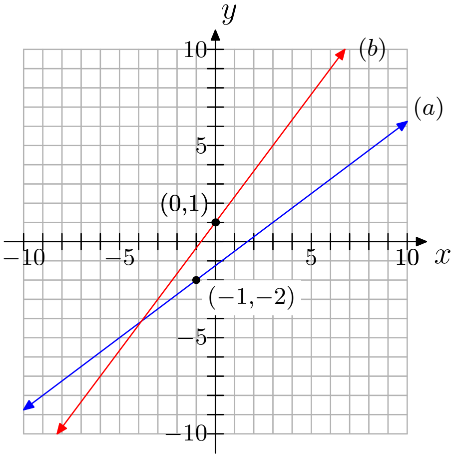
**c)** Yes.

**21.**



**23.**

**b)**



**c)** The lines are neither parallel nor perpendicular.

**25.**

**a)** $\frac{2}{25}$

**b)** 8.9%

**c)** 0% grade represents no grade or slope; that is, a flat road.

# 3.3 Equations of Lines

In this section we will develop the *slope-intercept form* of a line. When you have completed the work in this section, you should be able to look at the graph of a line and determine its equation in slope-intercept form.

## *The Slope-Intercept Form*

In the previous section, we developed the formula for the slope of a line. Let's assume that the dependent variable is $y$ and the independent variable is $x$ and we have a line passing through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$, as shown in **Figure 1**.

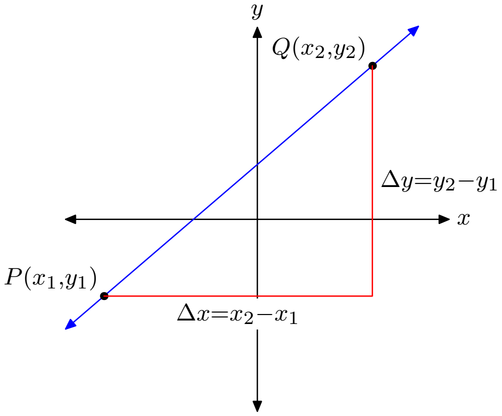


**Figure 1.** Determining the slope of a line through two points.

As we sweep our eyes from left to right, note that the change in $x$ is $\Delta x = x_2 - x_1$ and the change in $y$ is $\Delta y = y_2 - y_1$. Thus, the slope of the line is determined by the formula

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{1}$$

Now consider the line in **Figure 2**. Suppose that we are given two facts about this line:

1. The point where the line crosses the $y$-axis (the $y$-intercept) is $(0, b)$.
2. The "slope" of the line is some number $m$.

To find the equation of the line pictured in **Figure 2**, select an arbitrary point $Q(x, y)$ on the line, then compute the slope of the line using $(x_1, y_1) = P(0, b)$ and $(x_2, y_2) = Q(x, y)$ in the slope formula (**1**).

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - b}{x - 0}$$

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[9] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

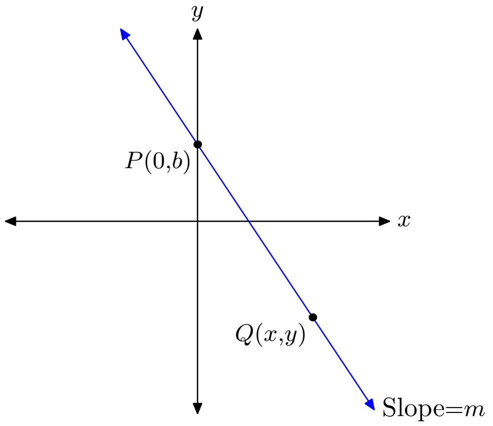


**Figure 2.** Find the equation of the line in slope-intercept form.

Simplify.

$$\text{Slope} = \frac{y-b}{x}$$

We're given that the slope is the number $m$, so substitute this number for the word "Slope" in the last result.

$$m = \frac{y-b}{x}$$

Multiply both sides of the last equation by $x$.

$$mx = y - b$$

Add $b$ to both sides of the last equation to obtain

$$mx + b = y,$$

or upon exchanging sides of the equation,

$$y = mx + b.$$

The above discussion leads to the following result.

**The Slope-Intercept Form of a Line**. If the line $L$ intercepts the $y$-axis at the point $(0, b)$ and has slope $m$, then the equation of the line is

$$y = mx + b. \tag{2}$$

This form of the equation of a line is called the **slope-intercept form**. The function defined by the equation

$$f(x) = mx + b$$

is called a **linear function**.

It is important to note two key facts about the slope-intercept form $y = mx + b$.

- The coefficient of $x$ (the $m$ in $y = mx + b$) is the slope of the line.
- The constant term (the $b$ in $y = mx+b$) is the $y$-coordinate of the $y$-intercept $(0, b)$.

**Procedure for Using the Slope-Intercept Form of a Line**. When given the slope of a line and the $y$-intercept of the line, use the slope-intercept form as follows:

1. Substitute the given slope for $m$ in the formula $y = mx + b$.
2. Substitute the $y$-coordinate of the $y$-intercept for $b$ in the formula $y = mx + b$.

For example, if the line has slope $-2$ and the $y$-intercept (the point where the line crosses the $y$-axis) is $(0, 3)$, then substitute $m = -2$ and $b = 3$ in the formula $y = mx + b$ to obtain

$$y = -2x + 3.$$

Let's look at some examples of use of this all-important formula.

▶ **Example 3.** *What is the equation of the line having slope $-2/3$ and $y$-intercept at $(0, 3)$? Sketch the line on graph paper.*

The equation of the line is

$$y = mx + b. \tag{4}$$

We're given that the slope is $-2/3$. Hence, $m = -2/3$. Secondly, we're given that the line intercepts the $y$-axis at the point $(0, 3)$. In the slope-intercept form $y = mx + b$, recall that $b$ represents the $y$-coordinate of the $y$-intercept. Hence, $b = 3$. Substitute $m = -2/3$ and $b = 3$ in **equation (4)**, obtaining

$$y = -\frac{2}{3}x + 3.$$

To sketch the graph of the line, first locate the $y$-intercept at $P(0, 3)$, as shown in **Figure 3**. Starting from the $y$-intercept at $P(0, 3)$, move 3 units to the right and 2 units downward to the point $Q(3, 1)$. The required line passes through the points $P$ and $Q$.

Note that the line "intercepts" the $y$-axis at 3 and slants downhill, in accordance with the fact that the slope is negative in this example.

◆

▶ **Example 5.** *Given the graph of the line in* **Figure 4***(a), determine the equation of the line.*

First, locate the $y$-intercept of the line, which we've labeled $P(0, -1)$ in **Figure 4**(b). In the formula $y = mx + b$, recall that $b$ represents the $y$-coordinate of the $y$-intercept. Thus, $b = -1$.

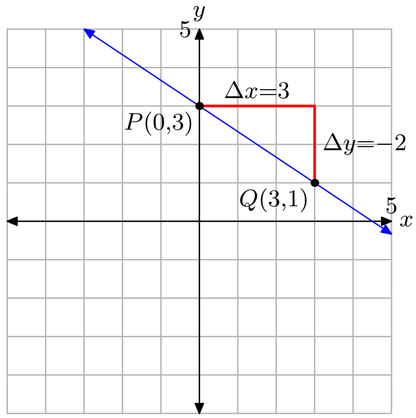


**Figure 3.** The line has $y$-intercept at $(0, 3)$ and slope $-2/3$.

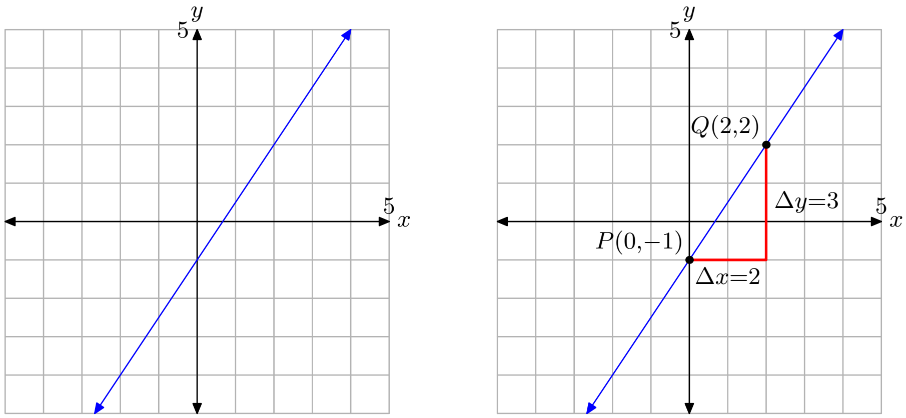
Secondly, we need to determine the slope of the line. In **Figure 4**(b), start at the point $P$, move 2 units to the right, then 3 units upward to the point $Q(2, 2)$. This makes the slope

$$m = \frac{\Delta y}{\Delta x} = \frac{3}{2}.$$

Substitute $m = 3/2$ and $b = -1$ into the slope-intercept form $y = mx + b$ to obtain

$$y = \frac{3}{2}x - 1,$$

which is the desired equation of the line.



(a) (b)

**Figure 4.** Determining the equation of a line from its graph.

◇

## Making Connections

If the connection between rate and slope is still not clear, let's recall an example we did earlier in the chapter.

> *Sebastian waves good-bye to his brother, who is talking to a group of his friends approximately 20 feet away. Sebastian then begins to walk away from his brother at a* constant rate *of 4 feet per second.*

The distance between the brothers depends upon the amount of time that has passed, so we set distance $d$ on the vertical axis and time $t$ on the horizontal axis, as shown in **Figure 5**. Note that $d$ and $t$ are taking the "usual" place of $y$ and $x$, respectively. The distance separating the brothers at time $t = 0$ is $d = 20$ feet. This is indicated with the "$d$-intercept" at $P(0, 20)$ in **Figure 5**.

Next, the distance between the brothers is increasing at a rate of 4 feet per second. Starting at the point $P(0, 20)$, move 1 second to the right (2 boxes) and 4 units up (1 box) to the point $Q(1, 24)$, as shown in **Figure 5**. The line through the points $P$ and $Q$ then models the distance between the brothers as a function of time.

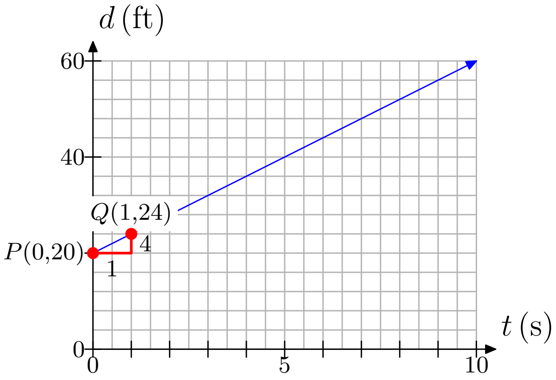


**Figure 5.** Distance between brothers as a function of time.

If you recall, we then determined a relation between distance $d$ and time $t$ by examining the distance between the brothers at times $t = 0$, 1, 2, and 3, and summarizing the results in **Table 1**.

| $t$ | $d$ |
|---|---|
| 0 | 20 |
| 1 | $20 + 4(1)$ |
| 2 | $20 + 4(2)$ |
| 3 | $20 + 4(3)$ |

**Table 1.** Determining a model equation.

Intuition then led to the following model, which provides the distance $d$ between the brothers as a function of time $t$.

$$d = 20 + 4t \tag{6}$$

Again, readers should check that **equation (6)** produces the results in **Table 1** at $t = 0$, 1, 2, and 3.

Alternatively, with the theory developed in this section, we would develop the equation of the line by using the slope-intercept form of the line; that is,

$$y = mx + b. \tag{7}$$

However, in this case, the dependent variable is $d$, not $y$, and the independent variable is $t$, not $x$. So, replace $y$ and $x$ in **equation (7)** with $d$ and $t$, respectively, obtaining

$$d = mt + b. \tag{8}$$

Next, the line intercepts the $d$-axis at $P(0, 20)$, so $b = 20$. Furthermore, the slope of the line is 4 feet per second, so $m = 4$. Substitute $m = 4$ and $b = 20$ in **equation (8)** to obtain

$$d = 4t + 20, \tag{9}$$

or using function notation, $d(t) = 4t + 20$. Note that **equation (9)** is identical to the intuitively generated model of **equation (6)**.

Hopefully, this development should cement for all time the idea that slope of the line is the rate at which the dependent variable is changing with respect to the independent variable.

## *The Standard Form of a Line*

We now know that if our equation has the form $y = mx + b$ (or can be manipulated into this form), the graph will be a line. Let's take a moment to demonstrate that the graph of the equation $Ax + By = C$, where $A$, $B$, and $C$ are constants, is a line.

If we can place the form $Ax + By = C$ into slope-intercept form $y = mx + b$, then that will demonstrate that the graph of $Ax + By = C$ is a line. So, start with $Ax + By = C$ and subtract $Ax$ from both sides of the equation.

$$By = -Ax + C$$

Divide both sides of this last equation by $B$. Note that there is an assumption here that $B \neq 0$. We will handle the case when $B = 0$ separately, at the end of this section.

$$\begin{aligned} \frac{By}{B} &= \frac{-Ax + C}{B} \\ y &= -\frac{A}{B}x + \frac{C}{B} \end{aligned}$$

When we compare $y = -(A/B)x + (C/B)$ with $y = mx + b$, we note that the slope is $m = -A/B$ and the $y$-coordinate of the $y$-intercept is $b = C/B$. Because we were successful in placing the equation $Ax + By = C$ into slope-intercept form, we now know that the graph of $Ax + By = C$ is a line (we'll need this result in later work).[10]

**The Standard Form of a Line**. The graph of the equation $Ax + By = C$ is a line. The form

$$Ax + By = C \tag{10}$$

is called the **standard form** of a line.

Let's look at an example.

▶ **Example 11.** *The equation $3x+4y = 12$ is in standard form. Place this equation in slope-intercept form, determine the slope and y-intercept, then use these results to draw the graph of the line.*

First, solve the equation $3x + 4y = 12$ for $y$.

$$\begin{aligned} 3x + 4y &= 12 \\ 4y &= -3x + 12 \\ y &= -\frac{3}{4}x + 3 \end{aligned}.$$

Note in the last step how the distributive property came into play. When we divided $-3x + 12$ by 4, we divided each term by 4, getting $(-3/4)x + 3$.

When we compare $y = (-3/4)x+3$ with the general slope-intercept form $y = mx+b$, we determine that the slope is $m = -3/4$ and the $y$-coordinate of the $y$-intercept is $b = 3$. To sketch the graph of the line, as we've done in **Figure 6**, plot the $y$-intercept at $P(0, 3)$, then move 4 units to the right and 3 units down to the point $Q(4, 0)$. The line passing through the points $P$ and $Q$ is the required line.

Note again that the slope is $m = -3/4$ and the line slants "downhill." Also, $b = 3$ and the line "intercepts" the $y$-axis at $P(0, 3)$.

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Let's look at another example.

▶ **Example 12.** *In* **Example 5**, *we determined that the given line had the equation*

$$y = \frac{3}{2}x - 1.$$

*Place this equation in standard from $Ax + By = C$, where $A$, $B$, and $C$ are integers and $A > 0$.*

---

[10] Some would argue that it is useful to memorize that the line $Ax + By = C$ has slope $m = -A/B$ and the $y$-coordinate of the $y$-intercept is $b = C/B$. If you memorize these facts, then you can quickly determine that the slope of the line $3x + 4y = 12$ is $m = -A/B = -3/4$ and the $y$-coordinate of the $y$-intercept is $b = C/B = 12/4 = 3$.

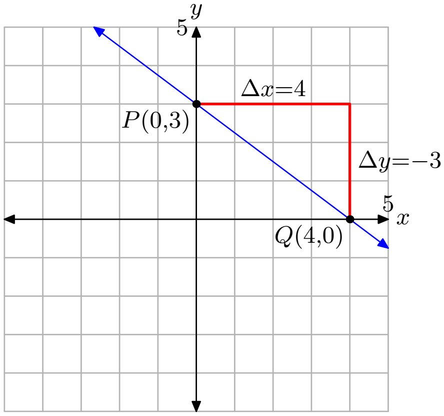


**Figure 6.** The graph of $3x + 4y = 12$.

We're requested to place the equation $y = (3/2)x - 1$ in the form $Ax + By = C$, where $A$, $B$, and $C$ are integers, so let's begin by clearing fractions from the equation. Multiply both sides of the equation by the common denominator 2.

$$\begin{aligned} y &= \frac{3}{2}x - 1 \\ 2y &= 2\left(\frac{3}{2}x - 1\right) \\ 2y &= 3x - 2 \end{aligned}$$

Now, subtract $2y$ from both sides of the equation, then add 2 to both sides of the equation to obtain

$$2 = 3x - 2y,$$

or equivalently,

$$3x - 2y = 2.$$

Note that this last result is in standard form $Ax + By = C$, where $A$, $B$, and $C$ are integers and $A > 0$.[11]

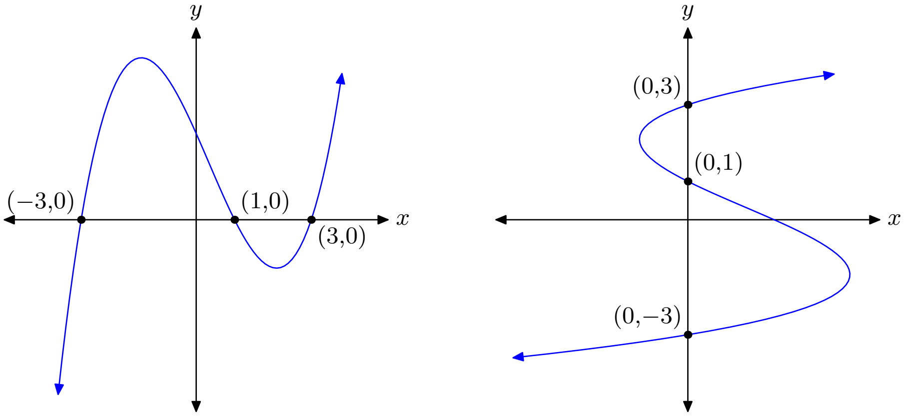
## Intercepts

We now know that the graph of the equation $Ax + By = C$, where $A$, $B$, and $C$ are constants, is a line. Because the graph of $Ax + By = C$ is a line, to draw the graph of the line, we need only find two points that satisfy the equation, plot them, then draw a line through them. Our two favorite points to work with are the $x$- and $y$-intercepts, because each involves the number zero, an easy number to work with.

---

[11] This is a traditional request. When placing a linear function in the form $Ax + By = C$, we will always require that $A$, $B$, and $C$ be integers and that $A > 0$.

Consider the graph in **Figure 7**(a). Note that the graph passes through the $x$-axis three times. The points where the graph intercepts the $x$-axis are called *$x$-intercepts*. Note that each of these points has a defining value in common: the $y$-value of each of these $x$-intercepts is equal to zero.



(a) $x$-intercepts. (b) $y$-intercepts.

**Figure 7.**

**How to Find an $x$-intercept.** To find an $x$ intercept, let $y = 0$ in the equation and solve for $x$.

On the other hand, consider the graph in **Figure 7**(b). Note that this is not a function (fails the vertical line test) but the graph intercepts the $y$-axis three separate times. The points where the graph intercepts the $y$-axis are called *$y$-intercepts*. Note that each of the $y$-intercepts in **Figure 7**(b) has a defining value in common: the $x$-value of each of the $y$-intercepts is equal to zero.

**How to Find a $y$-intercept.** To find a $y$-intercept, let $x = 0$ in the equation and solve for $y$.

Let's put what we've learned into practice.

▶ **Example 13.** *Sketch the graph of* $3x + 4y = 12$.

We drew the graph of the equation $3x + 4y = 12$ in **Figure 6**. There we solved the equation for $y$ to determine the slope and the $y$-intercept. These, in turn, were used to draw the graph of $3x + 4y = 12$ in **Figure 6**.

Here, our approach will be different. We will first determine the $x$- and $y$-intercepts, plot them, then draw a line through these intercepts. Hopefully, we will get a result that matches that in **Figure 6**.

To find the $x$-intercept, let $y = 0$ in $3x + 4y = 12$ and solve for $x$.

$$\begin{aligned} 3x + 4y &= 12 \\ 3x + 4(0) &= 12 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

Hence, the $x$-intercept is the point $Q(4, 0)$. To find the $y$-intercept, let $x = 0$ in $3x + 4y = 12$ and solve for $y$.

$$\begin{aligned} 3x + 4y &= 12 \\ 3(0) + 4y &= 12 \\ 4y &= 12 \\ y &= 3 \end{aligned}$$

Hence, the $y$-intercept is the point $P(0, 3)$. In **Figure 8**, we've plotted both $x$- and $y$-intercepts and drawn a line through them. Note that the resulting line in **Figure 8** matches the same line drawn in **Figure 6** (where we used a different method).

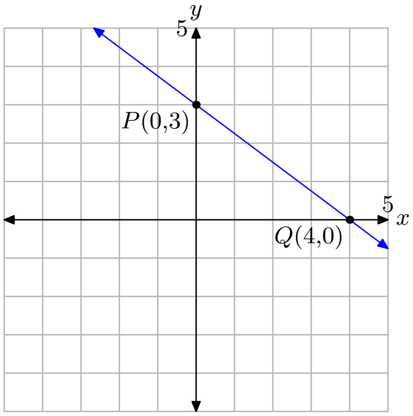


**Figure 8.** Plotting the $x$- and $y$-intercepts.

We recommend that whenever the line is given in Standard Form $Ax + By = C$, find the $x$- and $y$-intercepts, plot them, then draw a line through them. This technique is quite efficient because working with the number zero greatly simplifies the calculations.

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## Horizontal and Vertical Lines

We've introduced the standard form of the line $Ax + By = C$. The case where $A$ and $B$ are simultaneously equal to zero is not very interesting.[12] However, the following two cases are of interest.

---

[12] In the case of $0x + 0y = C$, where $C$ is non zero, there are no points satisfying this equation. Hence, no graph. In the case $0x + 0y = 0$, all points in the plane satisfy this equation, so the graph would consist of every point in the Cartesian plane.

1. If we let $A = 0$ and $B \neq 0$ in the standard form $Ax + By = C$, then $By = C$, or equivalently $y = C/B$. Note that this has the form $y = b$, where $b$ is some constant.
2. Similarly, if we let $B = 0$ and $A \neq 0$ in the standard form $Ax + By = C$, then $Ax = C$, or equivalently, $x = C/A$. Note that this has the form $x = a$, where $a$ is some constant.

The lines having the form $x = a$ and $y = b$ are two of the easiest lines to plot. Let's look at an example of each.

► **Example 14.** *Sketch the graph of the equation* $x = 3$.

The direction "sketch the graph of the equation $x = 3$" can be quite vexing unless one remembers that a graph of an equation is the set of all points that satisfy the equation. Thus, the direction is better posed if we say "sketch the set of all points $(x, y)$ that satisfy $x = 3$," or equivalently, "sketch the set of all points $(x, y)$ that have an $x$-value of 3." Then it is an easy matter to sketch the vertical line shown in **Figure 9**.

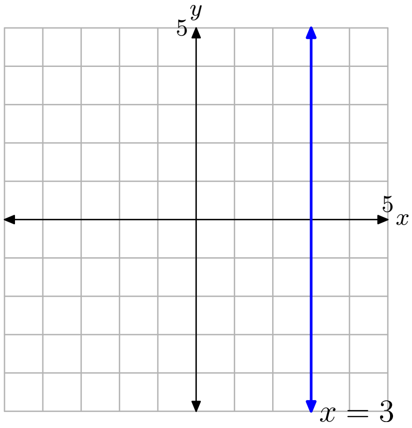

**Figure 9.** The graph of the equation $x = 3$.

Note that each point on the line has an $x$-value equal to 3. Also, note that the slope of this vertical line is undefined.

► **Example 15.** *Sketch the graph of the equation* $y = 3$.

This direction is better posed if we say "sketch the set of all points $(x, y)$ that satisfy $y = 3$," or equivalently, "sketch the set of all points $(x, y)$ that have a $y$-value of 3." Then it is an easy matter to sketch the horizontal line shown in **Figure 10**.

Note that each point on the line has a $y$-value equal to 3. Also, note that this horizontal line has slope zero.

Two final comments are in order. Because the line in **Figure 10** has slope zero and $y$-intercept $(0, 3)$, we can insert $m = 0$ and $b = 3$ into the slope intercept form $y = mx + b$ and obtain

$$y = 0x + 3,$$

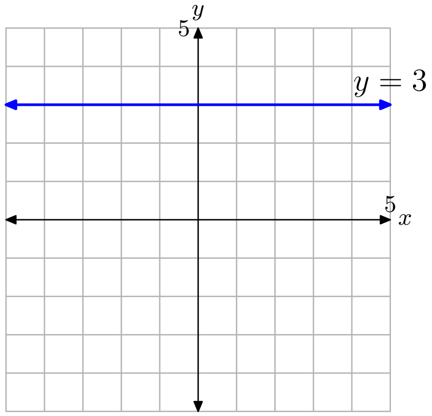

**Figure 10.** The graph of the equation $y = 3$.

which of course, is equivalent to $y = 3$. However, the vertical line shown in **Figure 9** has "undefined" slope, so this approach is unavailable. We must simply recognize that the vertical line in **Figure 9** consists of all points having an $x$-value equal to 3, and then intuit that its equation is $x = 3$.

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## 3.3 Exercises

In **Exercises 1-6**, perform each of the following tasks for the given linear function.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Identify the slope and $y$-intercept of the graph of the given linear function.
iii. Use the slope and $y$-intercept to draw the graph of the given linear function on your coordinate system. Label the $y$-intercept with its coordinate and the graph with its equation.

**1.** $f(x) = 2x + 1$

**2.** $f(x) = -2x + 3$

**3.** $f(x) = 3 - x$

**4.** $f(x) = 2 - 3x$

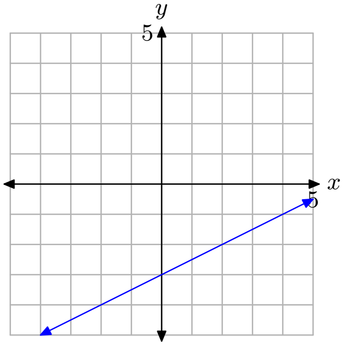
**5.** $f(x) = -\dfrac{3}{4}x + 3$

**6.** $f(x) = \dfrac{2}{3}x - 2$

In **Exercises 7-12**, perform each of the following tasks.

i. Make a copy of the given graph on a sheet of graph paper.
ii. Label the $y$-intercept with its coordinates, then draw a right triangle and label the sides to help identify the slope.
iii. Label the line with its equation.

**7.**



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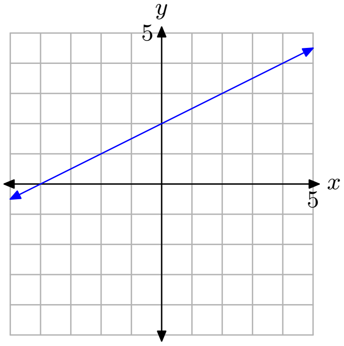

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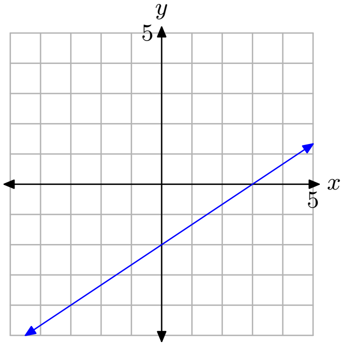

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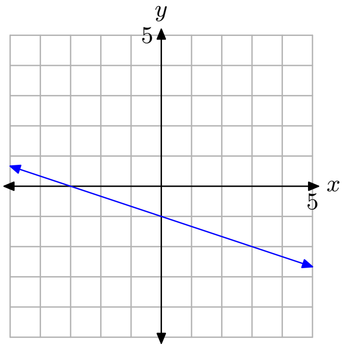
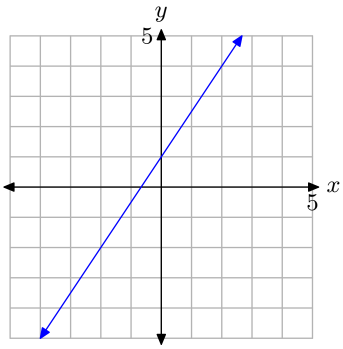
**10.**



**11.**



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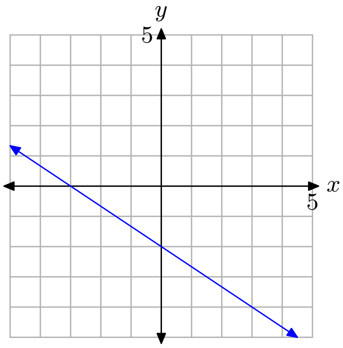

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**13.** Kate makes $39,000 per year and gets a raise of $1000 each year. Since her salary depends on the year, let time $t$ represent the year, with $t = 0$ being the present year, and place it along the horizontal axis. Let salary $S$, in thousands of dollars, be the dependent variable and place it along the vertical axis.

We will assume that the rate of increase of $1000 per year is constant, so we can model this situation with a linear function.

**a)** On a sheet of graph paper, make a graph to model this situation, going as far as $t = 10$ years.

**b)** What is the $S$-intercept?

**c)** What is the slope?

**d)** Suppose we want to predict Kate's salary in 20 years or 30 years. We cannot use the graphical model because it only shows up to $t = 10$ years. We could draw a larger graph, but what if we then wanted to predict 50 years into the future? The point is that a graphical model is limited to what it shows. A model algebraic function, however, can be used to predict for any year!

Find the slope-intercept form of the linear function that models Kate's salary.

**e)** Write the function using function notation, which emphasizes that $S$ is a function of $t$.

**f)** Now use the algebraic model from (e) to predict Kate's salary 10 years, 20 years, 30 years, and 50 years into the future.

**g)** Compute S(40).

**h)** In a complete sentence, explain what the value of $S(40)$ from part (g) means in the context of the problem.

**14.** For each DVD that Blue Charles Co. sells, they make 5¢ profit. Profit depends on the number of DVD's sold, so let number sold $n$ be the independent variable and profit $P$, in \$, be the dependent variable.

**a)** On a sheet of graph paper, make a graph to model this situation, going as far as $n = 15$.

**b)** Use the graph to predict the profit if $n = 10$ DVD's are sold.

**c)** The graphical model is limited to predicting for values of $n$ on your graph. Any larger value of $n$ necessitates a larger graph, or a different kind of model. To begin finding an algebraic model, identify the $P$-intercept of the graph.

**d)** What is the slope of the line in you graphical model?

**e)** Find a slope-intercept form of a linear function that models Blue Charles Co.'s sales.

**f)** Write the function using function notation.

**g)** Explain why this model does not have the same limitation as the graphical model.

**h)** Find $P(100)$, $P(1000)$, and $P(10000)$.

**i)** In complete sentences, explain what the values of $P(100)$, $P(1000)$, and $P(10000)$ mean in the context of the problem.

**15.** Enrique had \$1,000 saved when he began to put away an additional \$25 each month.

**a)** Let $t$ represent time, in months, and $S$ represent Enrique's savings, in \$. Identify which should be the independent and dependent variables.

**b)** To begin finding a linear function to model this situation, identify the $S$-intercept and slope.

**c)** Find a slope-intercept form of a linear function to model Enrique's savings over time.

**d)** Write the linear function in function notation.

**e)** Use the function model to predict how much will be in his savings in one year.

**f)** Use the function model to predict when will he have \$2000 saved.

**g)** Graph the function on a coordinate system.

**h)** At the same time, Anne-Marie also begins to save \$25 per month, but she begins with \$1200 already in her savings. Make a graphical model of her situation and place it on the same coordinate system as the graphical model for Enrique's savings. Label it appropriately.

**i)** How do the lines compare to each other? Say something about their slopes.

**j)** Find a slope-intercept form of a linear function that models Anne-Marie's savings. Use the same variables as you did for Enrique's model.

**k)** Write the function using function notation.

**l)** Prove that the graphs of the two functions are parallel lines.

**m)** For Anne-Marie, looking at the graphs, do you think it will take her more time or less time than Enrique to save up \$2000?.

**n)** Use the linear function model for Anne-Marie to predict how long it will take her to save \$2000. Does this agree with your expectation from (m)?

**16.** Jose is initially 400 meters away from the bus stop. He starts running toward the stop at a rate of 5 meters per second.

**a)** Express Jose's distance $d$ from the bus stop as a function of time $t$.

**b)** Use your model to determine Jose's distance from the bus stop after one minute.

**c)** Use your model to determine the time it will take Jose to reach the bus stop.

**17.** A ball is dropped from rest above the surface of the earth. As it falls, its speed increases at a constant rate of 32 feet per second per second.

**a)** Express the speed $v$ of the ball as a function of time $t$.

**b)** Use your model to determine the speed of the ball after 5 seconds.

**c)** Use your model to determine the time it will take for the ball to achieve a speed of 256 feet per second.

**18.** A ball is thrown into the air with an initial speed of 200 meters per second. It immediately begins to lose speed at a rate of 9.5 meters per second per second.

**a)** Express the speed $v$ of the ball as a function of time $t$.

**b)** Use your model to determine the speed of the ball after 5 seconds.

**c)** Use your model to determine the time it will take for the ball to achieve its maximum height.

---

In **Exercises 19-24**, a linear function is given in standard form $Ax + By = C$. In each case, solve the given equation for $y$, placing the equation in slope-intercept form. Use the slope and intercept to draw the graph of the equation on a sheet of graph paper.

**19.** $3x - 2y = 6$

**20.** $3x + 5y = 15$

**21.** $3x + 2y = 6$

**22.** $4x - y = 4$

**23.** $x - 3y = -3$

**24.** $x + 4y = -4$

---

In **Exercises 25-30**, you are given a linear function in slope-intercept form. Place the linear function in standard form $Ax + By = C$, where $A$, $B$, and $C$ are integers and $A > 0$.

**25.** $y = \frac{2}{3}x - 5$

**26.** $y = \frac{5}{6}x + 1$

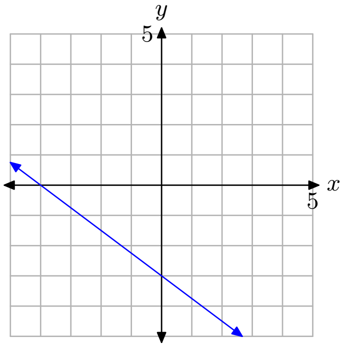
**27.** $y = -\frac{4}{5}x + 3$

**28.** $y = -\frac{3}{7}x + 2$

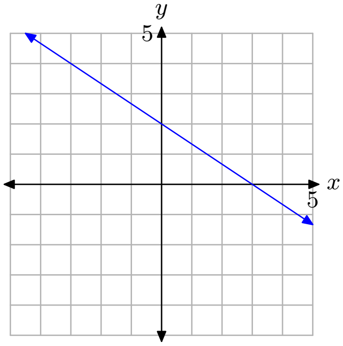
**29.** $y = -\frac{2}{5}x - 3$

**30.** $y = -\frac{1}{4}x + 2$

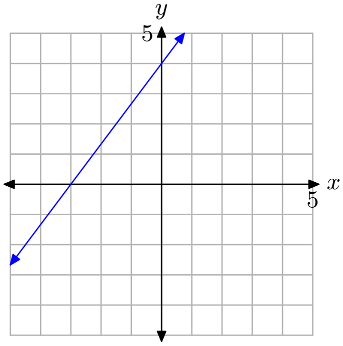
**31.** What is the $x$-intercept of the line?



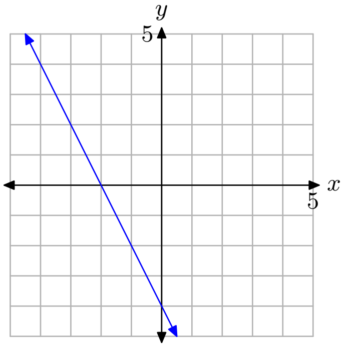
**32.** What is the $y$-intercept of the line?



**33.** What is the $y$-intercept of the line?



**34.** What is the $x$-intercept of the line?



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In **Exercises 35-40**, find the $x$- and $y$-intercepts of the linear function that is given in standard form. Use the intercepts to plot the graph of the line on a sheet of graph paper.

**35.** $3x - 2y = 6$

**36.** $4x + 5y = 20$

**37.** $x - 2y = -2$

**38.** $6x + 5y = 30$

**39.** $2x - y = 4$

**40.** $8x - 3y = 24$

---

**41.** Sketch the graph of the horizontal line that passes through the point $(3, -3)$. Label the line with its equation.

**42.** Sketch the graph of the horizontal line that passes through the point $(-9, 9)$. Label the line with its equation.

**43.** Sketch the graph of the vertical line that passes through the point $(2, -1)$. Label the line with its equation.

**44.** Sketch the graph of the vertical line that passes through the point $(15, -16)$. Label the line with its equation.

---

In **Exercises 45-48**, find the domain and range of the given linear function.

**45.** $f(x) = -37x - 86$

**46.** $f(x) = 98$

**47.** $f(x) = -12$

**48.** $f(x) = -2x + 8$

## 3.3 *Answers*

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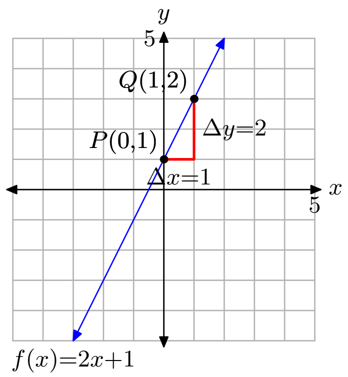
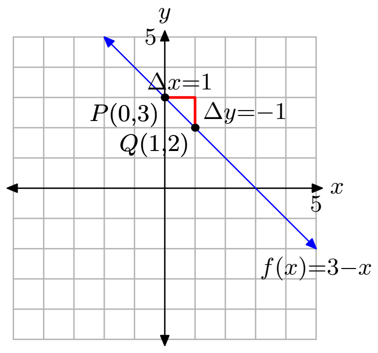
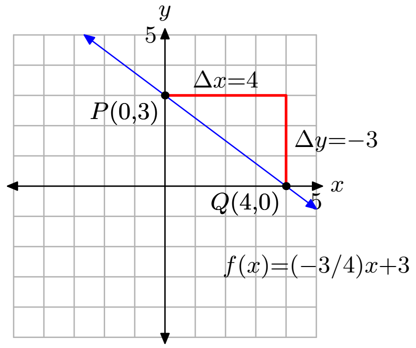
**1.** Slope = 2, $y$-intercept = $(0, 1)$



**3.** Slope = −1, $y$-intercept = $(0, 3)$



**5.** Slope = −3/4, $y$-intercept = $(0, 3)$



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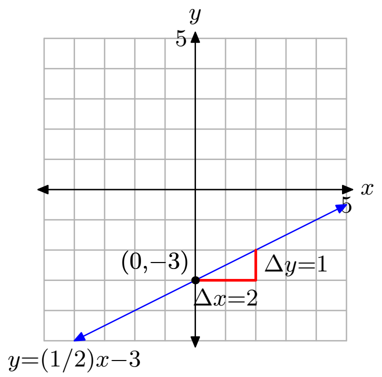

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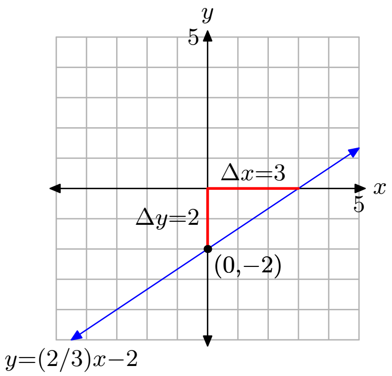

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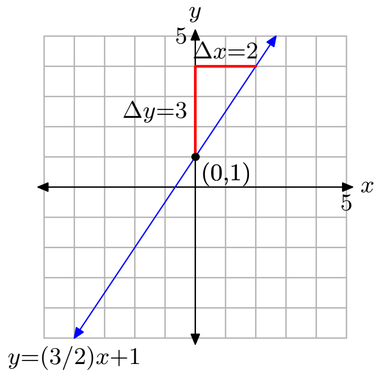
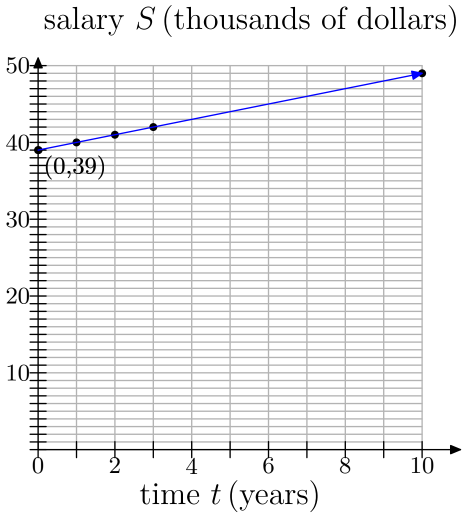


**13.**

**a)**



**b)** $(0, 39)$

**c)** 1

**d)** $S = t + 39$

**e)** $S(t) = t + 39$

**f)** \$49000, \$59000, \$69000, and \$89000

**g)** 79

**h)** If the current rate of increase continues, in 40 years Kate's salary will be \$79,000.

**15.**

**a)** $t$ should be the independent variable and $S$ should be the dependent variable.

**b)** $S$-intercept $= (0, 1000)$; slope $= 25$

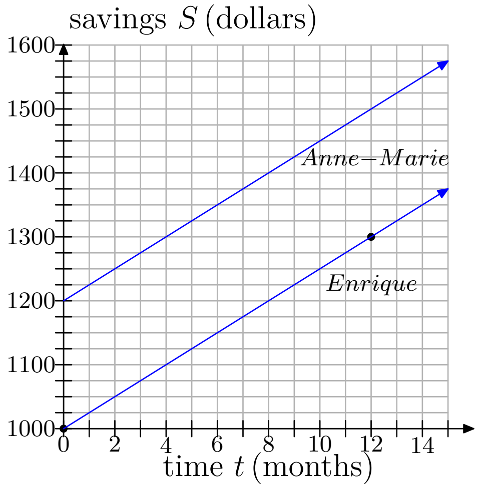
**c)** $S = 25t + 1000$

**d)** $S(t) = 25t + 1000$

**e)** 1300

**f)** It will take 40 months for him to reach \$2000.

**h)**



**i)** The lines have the same slope; they are parallel.

**j)** $S = 25t + 1200$

**k)** $S(t) = 25t + 1200$

**l)** They are lines because they are in the $y = mx + b$ form. They are parallel because their slopes are equal (both are 25).

**m)** It should take her less time.

**n)** It will take 32 months for her to reach \$2000. This agrees with our expectation from (m).

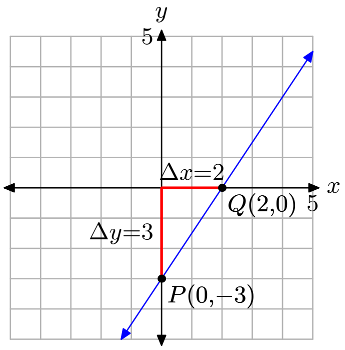
**17.**

**a)** $v = 32t$

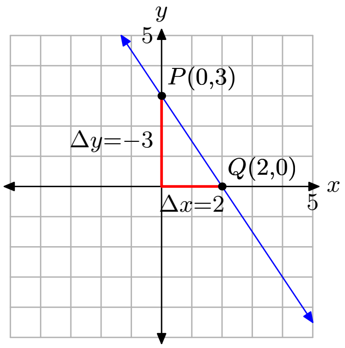
**b)** $v = 160$ feet per second

**c)** $t = 8$ seconds

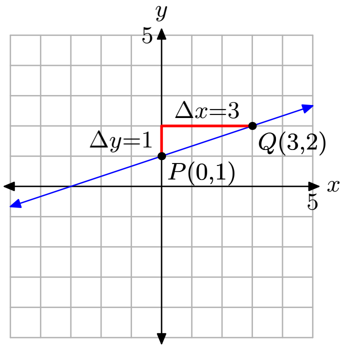
**19.** $y = (3/2)x - 3$



**21.** $y = (-3/2)x + 3$



**23.** $y = (1/3)x + 1$



**25.** $2x - 3y = 15$

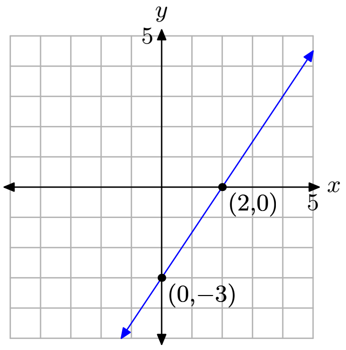
**27.** $4x + 5y = 15$

**29.** $2x + 5y = -15$

**31.** $(-4, 0)$

**33.** $(0, 4)$

**35.**



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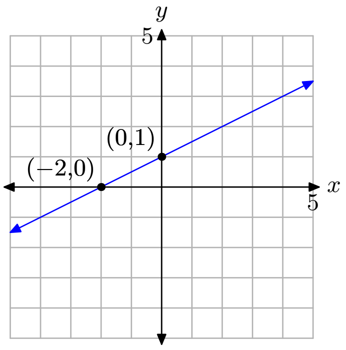
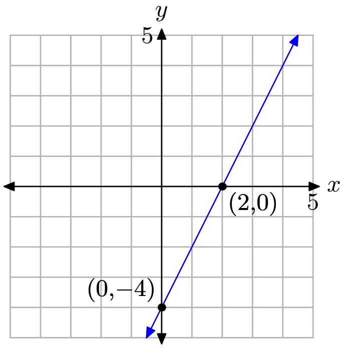
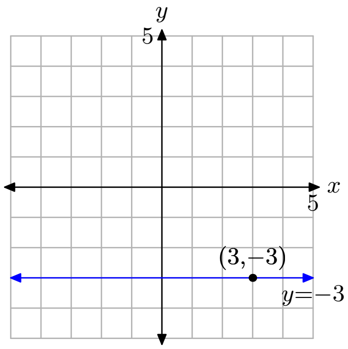


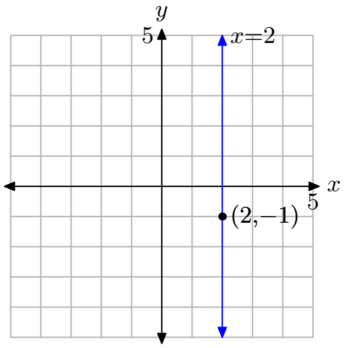
**39.**



**41.**



**43.**



**45.** Domain=$(-\infty, \infty)$ and Range=$(-\infty, \infty)$

**47.** Domain=$(-\infty, \infty)$ and Range=$\{-12\}$

# 3.4 The Point-Slope Form of a Line

In the last section, we developed the slope-intercept form of a line ($y = mx + b$). The slope-intercept form of a line is applicable when you're given the slope and $y$-intercept of the line. However, there will be times when the $y$-intercept is unknown.

Suppose for example, that you are asked to find the equation of a line that passes through a particular point $P(x_0, y_0)$ with slope $= m$. This situation is pictured in **Figure 1**.

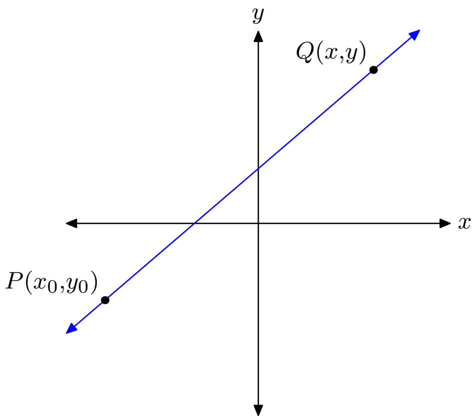


**Figure 1.** A line through $(x_0, y_0)$ with slope $m$.

Let the point $Q(x, y)$ be an arbitrary point on the line. We can determine the equation of the line by using the slope formula with points $P$ and $Q$. Hence,

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y - y_0}{x - x_0}.$$

Because the slope equals $m$, we can set Slope $= m$ in this last result to obtain

$$m = \frac{y - y_0}{x - x_0}.$$

If we multiply both sides of this last equation by $x - x_0$, we get

$$m(x - x_0) = y - y_0,$$

or exchanging sides of this last equation,

$$y - y_0 = m(x - x_0).$$

This last result is the equation of the line.

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**The Point-Slope Form of a Line.** If line $L$ passes through the point $(x_0, y_0)$ and has slope $m$, then the equation of the line is

$$y - y_0 = m(x - x_0). \tag{1}$$

This form of the equation of a line is called the **point-slope form**.

To use the point-slope form of a line, follow these steps.

**Procedure for Using the Point-Slope Form of a Line.** When given the slope of a line and a point on the line, use the point-slope form as follows:

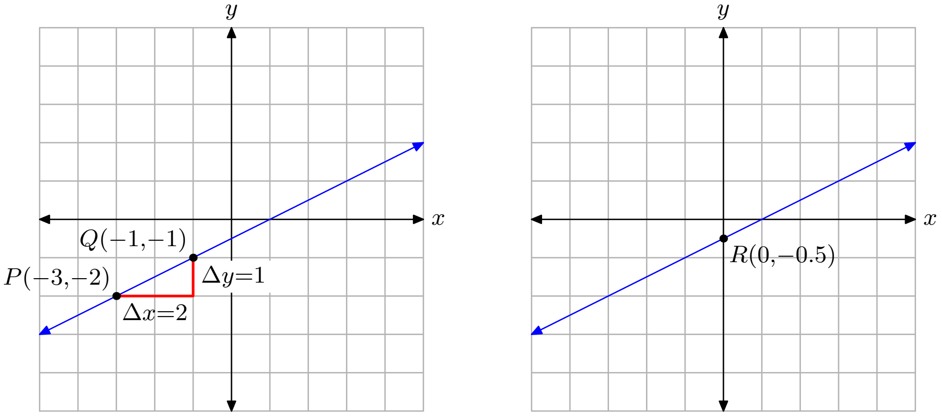
1. Substitute the given slope for $m$ in the formula $y - y_0 = m(x - x_0)$.
2. Substitute the coordinates of the given point for $x_0$ and $y_0$ in the formula $y - y_0 = m(x - x_0)$.

For example, if the line has slope $-2$ and passes through the point $(3, 4)$, then substitute $m = -2$, $x_0 = 3$, and $y_0 = 4$ in the formula $y - y_0 = m(x - x_0)$ to obtain

$$y - 4 = -2(x - 3).$$

► **Example 2.** *Draw the line that passes through the point $P(-3, -2)$ and has slope $m = 1/2$. Use the point-slope form to determine the equation of the line.*

First, plot the point $P(-3, -2)$, as shown in **Figure 2**(a). Starting from the point $P(-3, -2)$, move 2 units to the right and 1 unit up to the point $Q(-1, -1)$. The line through the points $P$ and $Q$ in **Figure 2**(a) now has slope $m = 1/2$.



(a) The line through $P(-3, -2)$ with slope $m = 1/2$.

(b) Checking the $y$-intercept.

**Figure 2.**

To determine the equation of the line in **Figure 2**(a), we will use the point-slope form of the line

$$y - y_0 = m(x - x_0). \tag{3}$$

The slope of the line is $m = 1/2$ and the given point is $P(-3, -2)$, so $(x_0, y_0) = (-3, -2)$. In **equation (3)**, set $m = 1/2$, $x_0 = -3$, and $y_0 = -2$, obtaining

$$y - (-2) = \frac{1}{2}(x - (-3)),$$

or equivalently,

$$y + 2 = \frac{1}{2}(x + 3). \tag{4}$$

This is the equation of the line in **Figure 2**(a).

As a check, we've estimated the $y$-intercept of the line in **Figure 2**(b) as $R(0, -0.5)$. Let's place **equation (4)** in slope-intercept form to determine the exact value of the $y$-intercept. First, distribute $1/2$ to get

$$y + 2 = \frac{1}{2}x + \frac{3}{2}.$$

Subtract 2 from both sides of this last equation.

$$y = \frac{1}{2}x + \frac{3}{2} - 2$$

Make equivalent fractions with a common denominator and simplify.

$$\begin{aligned} y &= \frac{1}{2}x + \frac{3}{2} - \frac{4}{2} \\ y &= \frac{1}{2}x - \frac{1}{2} \end{aligned} \tag{5}$$

Comparing **equation (5)** with $y = mx + b$ gives us $b = -1/2$. This is the exact $y$-value of the $y$-intercept. Note that this result compares exactly with the $y$-value of point $R$ in **Figure 2**(b). This is a bit lucky. Don't expect to get an exact comparison every time. However, if the comparison is not close, look for an error in your work, either in your computations or in your graph.

◆

Let's look at another example.

▶ **Example 6.** *Find the equation of the line passing through the points $P(-3, 2)$ and $Q(2, -1)$. Place your final answer in standard form.*

Again, to help keep our focus, we draw the line passing through the points $P(-3, 2)$ and $Q(2, -1)$ in **Figure 3**.

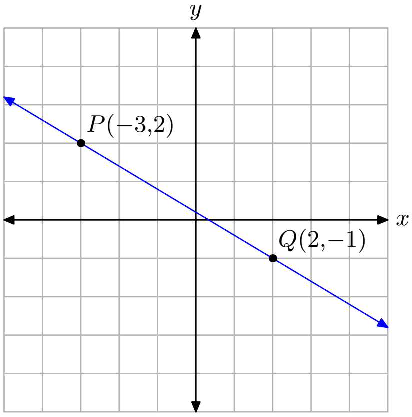


**Figure 3.** The line through points $P(-3, 2)$ and $Q(2, -1)$.

Use the slope formula to determine the slope of the line through the points $P(-3, 2)$ and $Q(2, -1)$.

$$m = \frac{\Delta y}{\Delta x} = \frac{-1 - 2}{2 - (-3)} = -\frac{3}{5}$$

We'll use the point-slope form of the line

$$y - y_0 = m(x - x_0). \tag{7}$$

Let's use point $P(-3, 2)$ as the given point $(x_0, y_0)$. That is, $(x_0, y_0) = (-3, 2)$. Substitute $m = -3/5$, $x_0 = -3$, and $y_0 = 2$ in **equation (7)**, obtaining

$$y - 2 = -\frac{3}{5}(x - (-3)). \tag{8}$$

This is the equation of the line passing through the points $P$ and $Q$.

Alternatively, we could also use the point $Q(2, -1)$ as the given point $(x_0, y_0)$. That is, $(x_0, y_0) = (2, -1)$. Substitute $m = -3/5$, $x_0 = 2$, and $y_0 = -1$ in the point-slope form (7), obtaining

$$y - (-1) = -\frac{3}{5}(x - 2). \tag{9}$$

This too, is the equation of the line passing through the points $P$ and $Q$.

How can the equations (8) and (9) both be the equation of the line through $P$ and $Q$, yet look so distinctly different? Let's place each equation in standard form $Ax + By = C$ and compare the results.

If we start with **equation (8)** and distribute the slope,

$$\begin{aligned} y - 2 &= -\frac{3}{5}(x - (-3)) \\ y - 2 &= -\frac{3}{5}x - \frac{9}{5}. \end{aligned}$$

Multiply both sides by the common denominator 5 to clear the fractions.

$$\begin{aligned}5(y-2) &= 5\left(-\frac{3}{5}x-\frac{9}{5}\right)\\ 5y-10 &= -3x-9\end{aligned}$$

Add $3x$ to both sides of the equation, then add 10 to both sides of the equation to obtain

$$3x+5y=1. \tag{10}$$

Place **equation (9)** in standard form in a similar manner. First, start with **equation (9)** and distribute the slope,

$$\begin{aligned}y-(-1) &= -\frac{3}{5}(x-2)\\ y+1 &= -\frac{3}{5}x+\frac{6}{5}.\end{aligned}$$

Next, multiply both sides of this last result by 5 to clear the fractions from the equation.

$$\begin{aligned}5\,(y+1) &= 5\left(-\frac{3}{5}x+\frac{6}{5}\right)\\ 5y+5 &= -3x+6\end{aligned}$$

Finally, add $3x$ to both sides of the equation, then subtract 5 from both sides of the equation to obtain

$$3x+5y=1. \tag{11}$$

Note that **equation (11)** is identical to **equation (10)**. Thus, it doesn't matter which point you use in the point-slope form. Both lead to the same result.

◇

## *Parallel Lines*

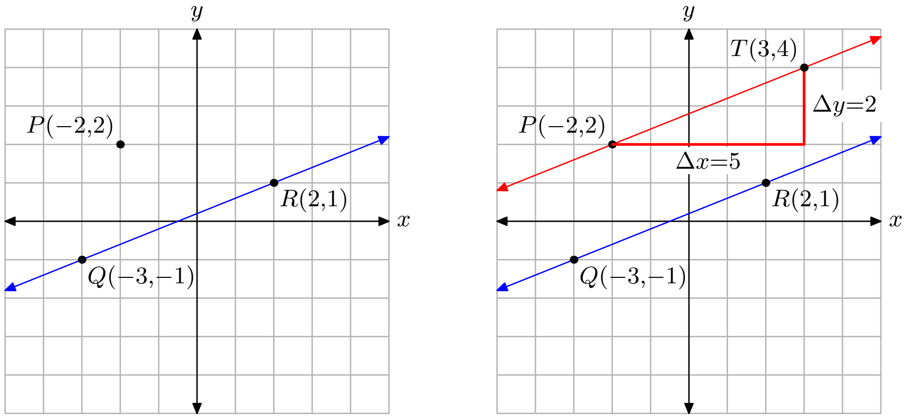
Recall that slope controls the "steepness" of a line. Consequently, if two lines are parallel, they must have the same "steepness" or slope. Let's look at an example of parallel lines.

▶ **Example 12.** *Find the equation of the line that passes through the point $P(-2,2)$ that is parallel to the line passing through the points $Q(-3,-1)$ and $R(2,1)$.*

First, to help us stay focused, we draw the line through the points $Q(-3,-1)$ and $R(2,1)$, then plot the point $P(-2,2)$, as shown in **Figure 4**(a).

We can use the slope formula to calculate the slope of the line passing through the points $Q(-3,-1)$ and $R(2,1)$.

$$m=\frac{\Delta y}{\Delta x}=\frac{1-(-1)}{2-(-3)}=\frac{2}{5}$$



(a) The line through $Q(-3,-1)$ and $R(2,1)$.

(b) The line through $P(-2,2)$ that is parallel to the line through $Q$ and $R$.

**Figure 4.**

We now draw a line through the point $P(-2,2)$ that is parallel to the line through the points $Q$ and $R$. Parallel lines must have the same slope, so we start at the point $P(-2,2)$, "run" 5 units to the right, then "rise" 2 units up to the point $T(3,4)$, as shown in **Figure 4**(b).

We seek the equation of the line through the points $P$ and $T$. We'll use the point-slope form of the line

$$y - y_0 = m(x - x_0). \tag{13}$$

We'll use the point $P(-2,2)$ as the given point $(x_0, y_0)$. That is, $(x_0, y_0) = (-2,2)$. The line through $P$ has slope $2/5$. Substitute $m = 2/5$, $x_0 = -2$, and $y_0 = 2$ in **equation (13)** to obtain

$$y - 2 = \frac{2}{5}(x - (-2)). \tag{14}$$

Let's place the **equation (14)** in standard form. Distribute the slope, then clear fractions by multiplying both sides of the resulting equation by 5.

$$\begin{aligned} y - 2 &= \frac{2}{5}x + \frac{4}{5} \\ 5(y-2) &= 5\left(\frac{2}{5}x + \frac{4}{5}\right) \\ 5y - 10 &= 2x + 4 \end{aligned}$$

Finally, subtract $5y$ from both sides of the last equation, then subtract 4 from both sides of the equation, obtaining

$$-14 = 2x - 5y,$$

or equivalently,

$$2x - 5y = -14.$$

This is the standard form of the equation of the line passing through the point $P$ and parallel to the line passing through the points $Q$ and $R$.

◆

## Perpendicular Lines

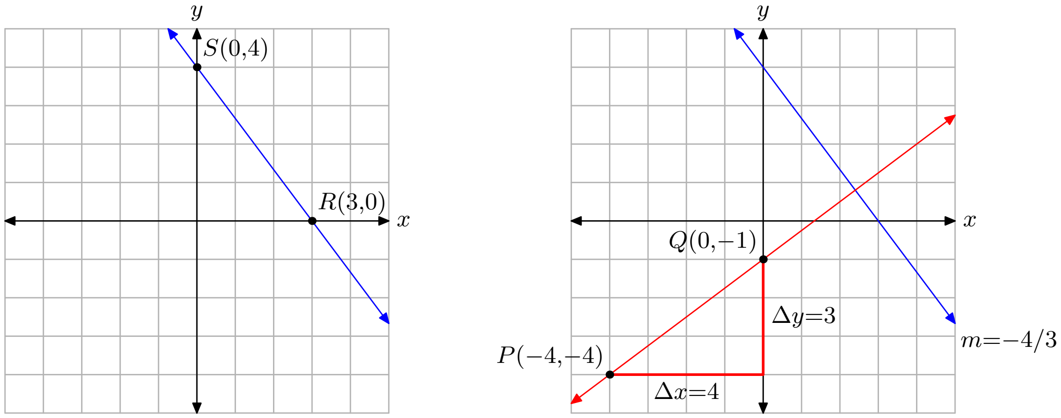
Suppose that two lines $L_1$ and $L_2$ have slopes $m_1$ and $m_2$, respectively. Recall (see the section on Slope) that if $L_1$ and $L_2$ are perpendicular, then the product of their slopes is $m_1 m_2 = -1$. Alternatively, the slope of the first line is the negative reciprocal of the second line, and vice-versa; i.e., $m_1 = -1/m_2$ and $m_2 = -1/m_1$. Let's look at an example of perpendicular lines.

▶ **Example 15.** *Find the equation of the line passing through the point $P(-4, -4)$ that is perpendicular to the line $4x + 3y = 12$.*

It will help our focus if we draw the given line $4x + 3y = 12$. The easiest way to plot a line in standard form $Ax + By = C$ is to find the $x$- and $y$-intercepts.

$$\begin{aligned} 4x + 3y &= 12 \\ 4x + 3(0) &= 12 \\ 4x &= 12 \\ x &= 3 \end{aligned} \qquad\qquad \begin{aligned} 4x + 3y &= 12 \\ 4(0) + 3y &= 12 \\ 3y &= 12 \\ y &= 4 \end{aligned}$$

Plot the $x$- and $y$-intercepts $R(3, 0)$ and $S(0, 4)$ as shown in **Figure 5**(a). The line through points $R$ and $S$ is the graph of the equation $4x + 3y = 12$.



(a) Plot the $x$ and $y$-intercepts of $4x + 3y = 12$.

(b) A line through $P(-4, -4)$ that is perpendicular to $4x + 3y = 12$.

**Figure 5.**

Next, determine the slope of the line $4x + 3y = 12$ by placing this equation in slope-intercept form (i.e., solve the equation $4x + 3y = 12$ for $y$).[15]

$$\begin{aligned} 4x + 3y &= 12 \\ 3y &= -4x + 12 \\ y &= -\frac{4}{3}x + 4 \end{aligned}$$

If two lines are perpendicular, then their slopes are negative reciprocals of one another. Therefore, the slope of the line that is perpendicular to the line $4x + 3y = 12$ (which has slope $-4/3$) is $m = 3/4$. Our second line must pass through the point $P(-4, -4)$. To draw this second line, first plot the point $P(-4, -4)$, then move 4 units to the right and 3 units upward to the point $Q(0, -1)$, as shown in **Figure 5**(b). The line through the points $P$ and $Q$ is perpendicular to the line $4x + 3y = 12$.[16]

To determine the equation of the line through the points $P$ and $Q$, we will use the point-slope form of the line, namely

$$y - y_0 = m(x - x_0). \tag{16}$$

The slope of the line through points $P$ and $Q$ is $m = 3/4$. If we use the point $P(-4, -4)$, then $(x_0, y_0) = (-4, -4)$. Set $m = 3/4$, $x_0 = -4$, and $y_0 = -4$ in **equation (16)**, obtaining

$$y - (-4) = \frac{3}{4}(x - (-4)),$$

or equivalently,

$$y + 4 = \frac{3}{4}(x + 4). \tag{17}$$

Alternatively, we could use the slope-intercept form of the line. We know that the line through points $P$ and $Q$ in **Figure 5**(b) crosses the $y$-axis at $Q(0, -1)$. So, with slope $m = 3/4$ and $y$-coordinate of the $y$-intercept $b = -1$, the slope-intercept form $y = mx + b$ becomes

$$y = \frac{3}{4}x - 1.$$

On the other hand, if we solve **equation (17)** for $y$,

$$\begin{aligned} y + 4 &= \frac{3}{4}(x + 4) \\ y + 4 &= \frac{3}{4}x + 3 \\ y &= \frac{3}{4}x - 1. \end{aligned} \tag{18}$$

Note that this is identical to the result found using the slope-intercept form above.

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[15] If you also remember that the slope of $Ax + By = C$ is $m = -A/B$, then the slope of $4x + 3y = 12$ is $m = -A/B = -4/3$.

[16] It's a good exercise to measure the angle between the two lines with a protractor. If the angle measures 90 degrees, then you know the lines are truly perpendicular.

It is comforting to note that the two forms (point-slope and slope-intercept) give the same result, but how do we determine the most efficient form to use for a particular problem? Here's a good hint.

**Determining the Form of the Line to Use**. Here is some sound advice when you are trying to determine whether to use the slope-intercept form or the point-slope form of a line.

- If you are given the slope and the $y$-intercept, use the slope-intercept form $y = mx + b$.
- If you are given a point (other than the $y$-intercept) and the slope, use the point-slope form $y - y_0 = m(x - x_0)$.

## Applications of Linear Functions

In this section we will look at some applications of linear functions. We begin by developing a function relating Fahrenheit and Celsius temperature.

▶ **Example 19.** *Water freezes at 32° F and 0° C. Water boils at 212° F and 100° C. F and C are abbreviations for Fahrenheit and Celsius temperature scales, respectively. Assuming a linear relationship, develop a model relating Fahrenheit and Celsius temperature.*

First, to help keep our focus, we set up a coordinate system on a sheet of graph paper. In **Figure 6**, we've decided to make the Celsius temperature the dependent variable and have assigned the Celsius temperature to the vertical axis. Similarly, we've declared the Fahrenheit temperature the independent variable and assigned it to the horizontal axis.[17]

Interpret the given data:

- Water freezes at 32° F and 0° C. This gives us the point $(F, C) = (32, 0)$, which we plot in **Figure 6**.
- Water boils at 212° F and 100° C. This gives us the point $(F, C) = (212, 100)$, which we plot in **Figure 6**.

Now we are on familiar ground. We want to find the equation of the line through these two points, which is the same type of problem we tackled in **Example 6**. First, use the points $(32, 0)$ and $(212, 100)$ to determine the slope of the line.

$$m = \frac{\Delta C}{\Delta F} = \frac{100 - 0}{212 - 32} = \frac{100}{180} = \frac{5}{9}.$$

We will now use the point-slope form of the line, $y - y_0 = m(x - x_0)$ with $m = 5/9$ and $(x_0, y_0) = (32, 0)$. Substitute $m = 5/9$, $x_0 = 32$, and $y_0 = 0$ in $y - y_0 = m(x - x_0)$ to obtain

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[17] We could easily reverse these assignments, placing the Fahrenheit temperature on the vertical axis and the Celsius temperature on the horizontal axis.

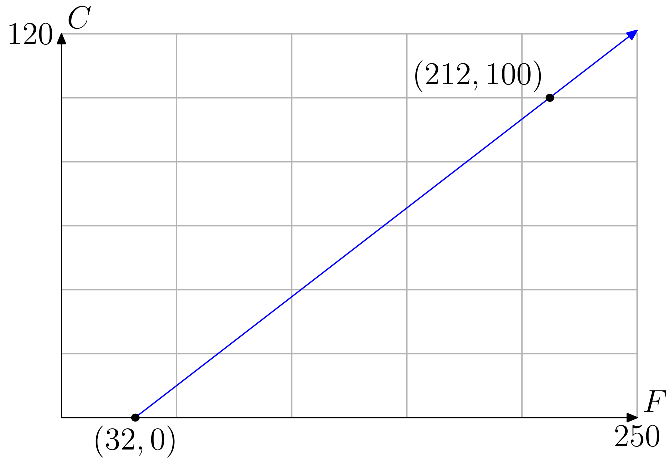

**Figure 6.** Plotting Celsius temperature versus Fahrenheit temperature.

$$y - 0 = \frac{5}{9}(x - 32). \tag{20}$$

However, our dependent axis is labeled $C$, not $y$, and our independent axis is labeled $F$, not $x$. So, we must replace $y$ and $x$ in **equation (20)** with $C$ and $F$, respectively, obtaining

$$C = \frac{5}{9}(F - 32). \tag{21}$$

This result in **equation (21)** expresses the Celsius temperature as a function of the Fahrenheit temperature. Alternatively, we could also use function notation and write

$$C(F) = \frac{5}{9}(F - 32).$$

Suppose that we know that the Fahrenheit temperature outside is 80° F and we wish to express this using the Celsius scale. To do so, we simply evaluate $C(80)$, as in

$$C(80) = \frac{5}{9}(80 - 32) \approx 26.6.$$

Hence, the Celsius temperature is approximately 26.6° C.

On the other hand, suppose that we know the Celsius temperature on a metal roof is 80° C and we wish to find the Fahrenheit temperature. To do so, we need to solve

$$C(F) = 80$$

for $F$, or equivalently,

$$\frac{5}{9}(F - 32) = 80.$$

Multiply both sides by 9 to obtain

$$5(F - 32) = 720,$$

then divide both sides of the result by 5 to obtain

$$F - 32 = 144.$$

Adding 32 to both sides of this last result produces the Fahrenheit temperature $F =$ 176° F. Wow, that's hot!

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## 3.4 Exercises

In **Exercises 1-4**, perform each of the following tasks.

i. Draw the line on a sheet of graph paper with the given slope $m$ that passes through the given point $(x_0, y_0)$.
ii. Estimate the $y$-intercept of the line.
iii. Use the point-slope form to determine the equation of the line. Place your answer in slope-intercept form by solving for $y$. Compare the exact value of the $y$-intercept with the approximation found in part (ii).

**1.** $m = 2/3$ and $(x_0, y_0) = (-1, -1)$

**2.** $m = -2/3$ and $(x_0, y_0) = (1, -1)$

**3.** $m = -3/4$ and $(x_0, y_0) = (-2, 3)$

**4.** $m = 2/5$ and $(x_0, y_0) = (-3, -2)$

**5.** Find the equation of the line in slope-intercept form that passes through the point $(1, 3)$ and has a slope of 1.

**6.** Find the equation of the line in slope-intercept form that passes through the point $(0, 2)$ and has a slope of $1/4$.

**7.** Find the equation of the line in slope-intercept form that passes through the point $(1, 9)$ and has a slope of $-2/3$.

**8.** Find the equation of the line in slope-intercept form that passes through the point $(1, 9)$ and has a slope of $-3/4$.

In **Exercises 9-12**, perform each of the following tasks.

i. Set up a coordinate system on a sheet of graph paper and draw the line through the two given points.
ii. Use the point-slope form to determine the equation of the line.
iii. Place the equation of the line in standard form $Ax + By = C$, where $A$, $B$, and $C$ are integers and $A > 0$. Label the line in your plot with this result.

**9.** $(-2, -1)$ and $(3, 2)$

**10.** $(-1, 4)$ and $(2, -3)$

**11.** $(-2, 3)$ and $(4, -3)$

**12.** $(-4, 4)$ and $(2, -4)$

**13.** Find the equation of the line in slope-intercept form that passes through the points $(-5, 5)$ and $(6, 8)$.

**14.** Find the equation of the line in slope-intercept form that passes through the points $(6, -6)$ and $(9, -7)$.

**15.** Find the equation of the line in slope-intercept form that passes through the points $(-4, 6)$ and $(2, -4)$.

**16.** Find the equation of the line in slope-intercept form that passes through the points $(-1, 5)$ and $(4, 4)$.

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[18] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

In **Exercises 17-20**, perform each of the following tasks.

i. Draw the graph of the given linear equation on graph paper and label it with its equation.
ii. Determine the slope of the given equation, then use this slope to draw a second line through the given point $P$ that is *parallel* to the first line.
iii. Estimate the $y$-intercept of the second line from your graph.
iv. Use the point-slope form to determine the equation of the second line. Place this result in slope-intercept form $y = mx + b$, then state the *exact* value of the $y$-intercept. Label the second line with the slope-intercept form of its equation.

**17.** $2x + 3y = 6$, $P = (-2, -3)$

**18.** $3x - 4y = 12$, $P = (-3, 4)$

**19.** $x + 2y = -4$, $P = (3, 3)$

**20.** $5x + 2y = 10$, $P = (-3, -5)$

In **Exercises 21-24**, perform each of the following tasks.

i. Draw the graph of the given linear equation on graph paper and label it with its equation.
ii. Determine the slope of the given equation, then use this slope to draw a second line through the given point $P$ that is *prependicular* to the first line.
iii. Use the point-slope form to determine the equation of the second line. Place this result in standard form $Ax+By = C$, where $A$, $B$, $C$ are integers and $A > 0$. Label the second line with this standard form of its equation.

**21.** $x - 2y = -2$, $P = (3, -4)$

**22.** $3x + y = 3$, $P = (-3, -4)$

**23.** $x - 2y = 4$, $P = (-3, 3)$

**24.** $x - 4y = 4$, $P = (-3, 4)$

**25.** Find the equation of the line in slope-intercept form that passes through the point $(7, 8)$ and is parallel to the line $x - 5y = 4$.

**26.** Find the equation of the line in slope-intercept form that passes through the point $(3, -7)$ and is perpendicular to the line $7x - 2y = -8$.

**27.** Find the equation of the line in slope-intercept form that passes through the point $(1, -2)$ and is perpendicular to the line $-7x + 5y = 4$.

**28.** Find the equation of the line in slope-intercept form that passes through the point $(4, -9)$ and is parallel to the line $9x + 3y = 5$.

**29.** Find the equation of the line in slope-intercept form that passes through the point $(2, -9)$ and is perpendicular to the line $-8x + 3y = 1$.

**30.** Find the equation of the line in slope-intercept form that passes through the point $(-7, -7)$ and is parallel to the line $8x + y = 2$.

**31.** A ball is thrown vertically upward on a distant planet. After 1 second, its velocity is 100 meters per second. After 5 seconds, the velocity is 50 meters per second. Assume that the velocity $v$ of the ball is a *linear function* of the time $t$.

**a)** On graph paper, sketch the graph of the velocity $v$ versus the time $t$. Assume that the velocity is the dependent variable and place it on the vertical axis.

**b)** Determine the slope of the line, including its units, then give a real world explanation of the meaning of this slope.

**c)** Determine an equation that models the velocity $v$ of the ball as a function of time $t$.

**d)** Determine the time it takes the ball to reach its maximum height.

**32.** A ball is thrown vertically upward on a distant planet. After 2 seconds, its velocity is 320 feet per second. After 8 seconds, the velocity is 200 feet per second. Assume that the velocity $v$ of the ball is a *linear function* of the time $t$.

**a)** On graph paper, sketch the graph of the velocity $v$ versus the time $t$. Assume that the velocity is the dependent variable and place it on the vertical axis.

**b)** Determine the slope of the line, including its units, then give a real world explanation of the meaning of this slope.

**c)** Determine an equation that models the velocity $v$ of the ball as a function of time $t$.

**d)** Determine the time it takes the ball to reach its maximum height.

**33.** An automobile is traveling down the autobahn and the driver applies its brakes. After 2 seconds, the car's speed is 60 km/h. After 4 seconds, the car's speed is 50 km/h.

**a)** On graph paper, sketch the graph of the velocity $v$ versus the time $t$. Assume that the velocity is the dependent variable and place it on the vertical axis.

**b)** Determine the slope of the line, including its units, then give a real world explanation of the meaning of this slope.

**c)** Determine an equation that models the velocity $v$ of the automobile as a function of time $t$.

**d)** Determine the time it takes the automobile to stop.

**34.** An automobile is traveling down the autobahn and its driver steps on the accelerator. After 2 seconds, the car's velocity is 30 km/h. After 4 seconds, the car's velocity is 40 km/h.

**a)** On graph paper, sketch the graph of the velocity $v$ versus the time $t$. Assume that the velocity is the dependent variable and place it on the vertical axis.

**b)** Determine the slope of the line, including its units, then give a real world explanation of the meaning of this slope.

**c)** Determine an equation that models the velocity $v$ of the automobile as a function of time $t$.

**d)** Determine the speed of the vehicle after 8 seconds.

**35.** Suppose that the demand $d$ for a particular brand of teakettle is a linear function of its unit price $p$. When the unit price is fixed at \$30, the demand for teakettles is 100. This means the public buys 100 teakettles. If the unit price is fixed at \$50, then the demand for teakettles is 60.

**a)** On graph paper, sketch the graph of the demand $d$ versus the unit price $p$. Assume that the demand is the dependent variable and place it on the vertical axis.

**b)** Determine the slope of the line, including its units, then give a real world explanation of the meaning of this slope.

**c)** Determine an equation that models the demand $d$ for teakettles as a function of unit price $p$.

**d)** Compute the demand if the unit price is set at \$40.

**36.** It's perfect kite-flying weather on the coast of Oregon. Annie grabs her kite, climbs up on the roof of her two story home, and begins playing out kite string. In 10 seconds, Annie's kite is 120 feet above the ground. After 20 seconds, it is 220 feet above the ground. Assume that the height $h$ of the kite above the ground is a linear function of the amount of time $t$ that has passed since Annie began playing out kite string.

**a)** On graph paper, sketch the graph of the height $h$ of the kite above ground versus the time $t$ . Assume that the height is the dependent variable and place it on the vertical axis.

**b)** Determine the slope of the line, including its units, then give a real world explanation of the meaning of this slope.

**c)** Determine an equation that models the height $h$ of the kite as a function of time $t$.

**d)** Determine the height of the kite after 20 seconds.

**e)** Determine the height of Annie's second story roof above ground.

## 3.4 *Answers*

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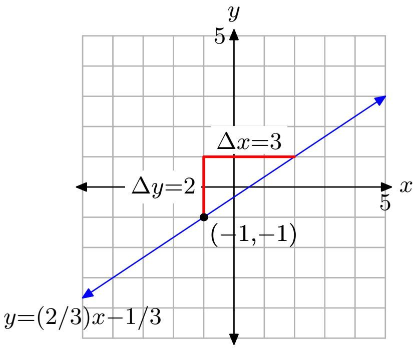
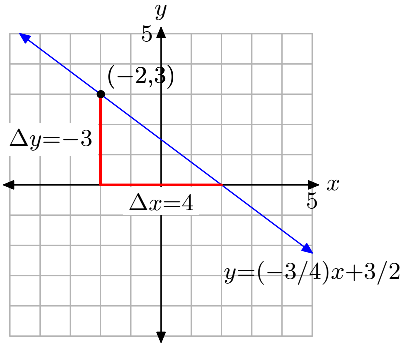
**1.** Approximate $y$-intercept is $(0, -0.3)$. Exact is $(0, -1/3)$.



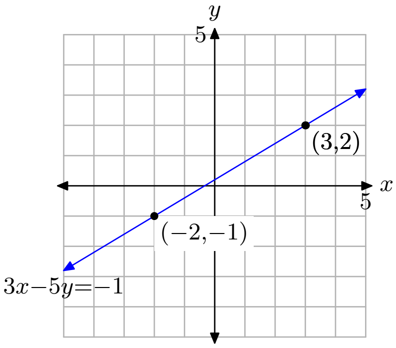
**3.** Approximate $y$-intercept is $(0, 1.5)$. Exact is $(0, 3/2)$.



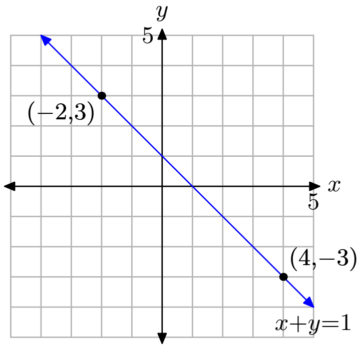
**5.** $y = x + 2$

**7.** $y = (-2/3)x + 29/3$

**9.**



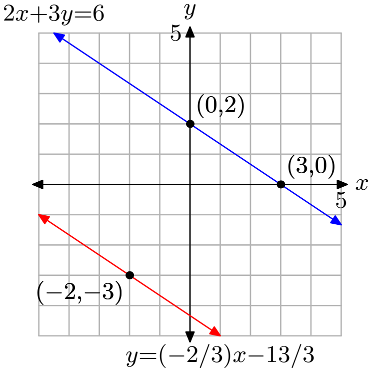
**11.**



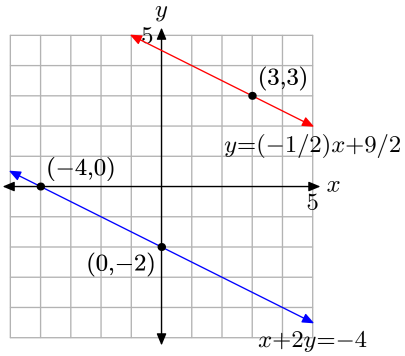
**13.** $y = \frac{3}{11}x + \frac{70}{11}$

**15.** $y = -\frac{5}{3}x - \frac{2}{3}$

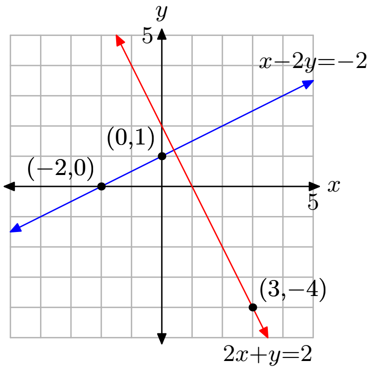
**17.** Approximate $y$-intercept: $(0, -4.3)$. Exact $y$-intercept: $(0, -13/3)$.



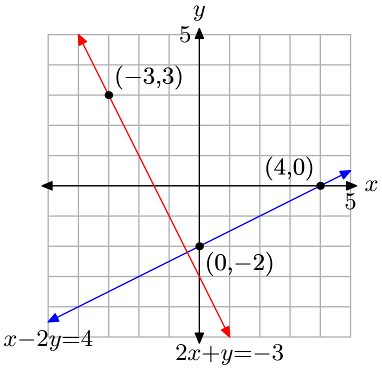
**19.** Approximate $y$-intercept: $(0, 4.5)$. Exact $y$-intercept: $(0, 9/2)$.



**21.**



**23.**



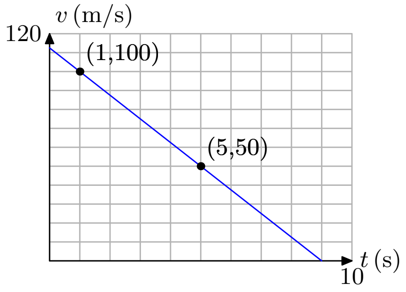
**25.** $y = \frac{1}{5}x + \frac{33}{5}$

**27.** $y = -\frac{5}{7}x - \frac{9}{7}$

**29.** $y = -\frac{3}{8}x - \frac{33}{4}$

**31.**

**a)**



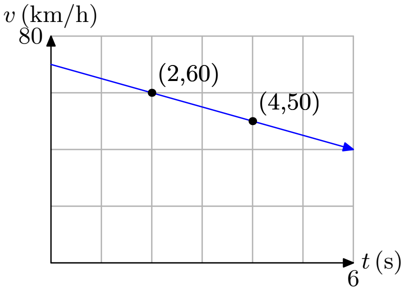
**b)** $-12.5$ (m/s)/s

**c)** $v = -12.5t + 112.5$

**d)** 9 seconds

**33.**

**a)**



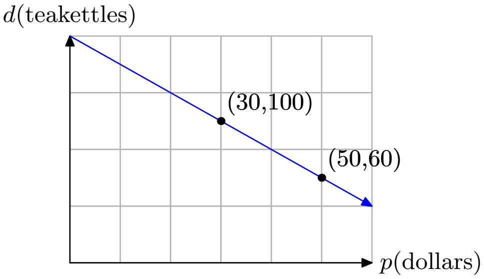
**b)** $-5$ (km/h)/s

**c)** $v = -5t + 70$

**d)** $t = 14$ seconds

**35.**

**a)**



**b)** $-2$ teakettles/dollar

**c)** $d = -2p + 160$

**d)** 80 teakettles

# 3.5 The Line of Best Fit

When gathering data in the real world, a plot of the data often reveals a “linear trend,” but the data don’t fall precisely on a single line. In this case, we seek to find a linear model that approximates the data. Let’s begin by looking at an extended example.

Aditya and Tami are lab partners in Dr. Mills’ physics class. They are hanging masses from a spring and measuring the resulting stretch in the spring. See **Table 1** for their data.

| $m$ (mass in grams) | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|
| $x$ (stretch in cm) | 6.8 | 10.2 | 13.9 | 21.2 | 24.2 |

**Table 1.** Aditya and Tami’s data set.

The goal is to find a model that describes the data, in both the form of a graph and of an equation. The first step is to plot the data. Recall some of the guidelines provided in the first section of the current chapter.

**Guidelines.** When plotting real data, we follow these guidelines.

1. You don’t want small graphs. It’s best to scale your graph so that it fills a full sheet of graph paper. This will make it much easier to read and interpret the graph.
2. You may have different scales on each axis, but once chosen, you must remain consistent.
3. You want to choose a scale which facilitates our first objective, but which also makes the data easy to plot.

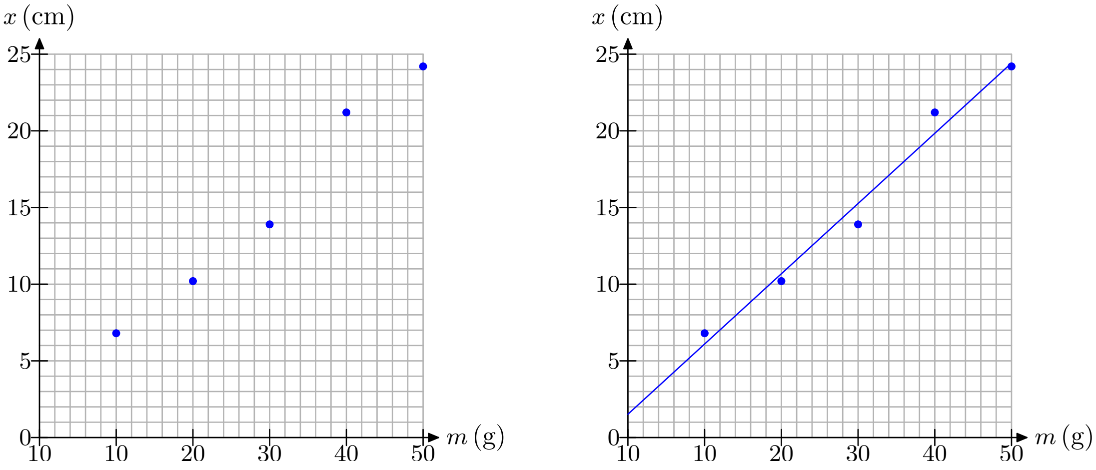
Aditya and Tami are free to choose the masses which they hang on the spring. Hence, the mass $m$ is the independent variable. Consequently, we will scale the horizontal axis to accommodate the mass. The distance the spring stretches depends upon the amount of mass that is hanging from the spring, so the distance stretched $x$ is the dependent variable. We will scale the vertical axis to accommodate the distance stretched.

On the horizontal axis, we need to fit the masses 10, 20, 30, 40, and 50 grams. To avoid a smallish graph, we will let every 5 boxes represent 10 grams. On the vertical axis, we need to fit distances ranging from 6.8 centimeters up to and including 24.2 centimeters. Making each box represent 1 cm gives a nice sized graph and will allow for easy plotting of our data points, which we’ve done in **Figure 1**(a).

Note the linear trend displayed by the data in **Figure 1**(a). It’s not possible to draw a single line that will pass through every one of the data points, so a linear model will not exactly “fit” the data. However, the data are “approximately linear,” so let’s try to draw a line that “nearly fits” the data.

It is not our goal here to try to draw a line that passes through as many data points as possible. If we do, then we are essentially saying that the points through which the line does not pass have no influence on the overall model, nor do they have any influence on any predictions we might make with our model. This is not a reasonable assumption.

Indeed, the goal is to draw a line that comes as close to as many points as possible. Some points will lie above the line, some will lie below, and what we'll try to do is "balance" the overestimates and the underestimates in an attempt to minimize the overall error. The best way to do this is to take a clear plastic ruler, something you can see through, and rotate and shift the ruler until you think you have a line that balances the overestimates and underestimates. We've done this for you in **Figure 1**(b). The resulting line is called the "line of best fit."



(a) Scaling the axes and plotting the data.

(b) Drawing the "line of best fit."

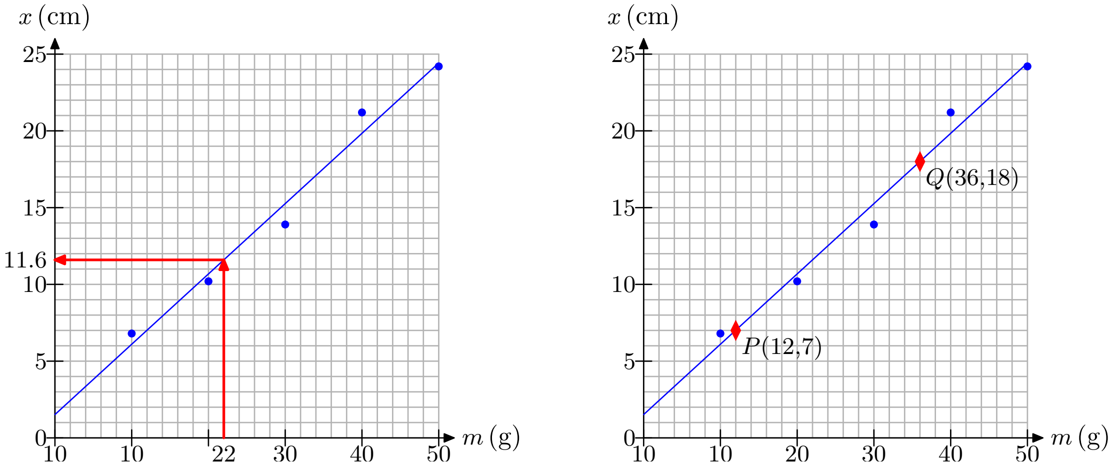
**Figure 1.**

We can use the "line of best fit" in **Figure 1**(b) to make predictions. For example, if we wanted to predict how much the spring will stretch when Aditya and Tami attach a 22 gram mass, then we would locate 22 grams on the horizontal axis, draw a vertical line upward to the "line of best fit," followed by a horizontal line to the vertical axis, as shown in **Figure 2**(a). Note that the $x$-value on the vertical axis appears to be approximately 11.6 centimeters.

Alternatively, we will develop an equation model. First, select two points on the "line of best fit" using the following guidelines.

**Guidelines.**

1. Pick two points on the "line of best fit" that are **not** data points.
2. Try to pick points passing through a lattice point of the grid. It makes interpreting the coordinates of the point a lot easier.
3. The further apart the two selected points, the better the accuracy. Don't pick points that are too close together.



(a) Predicting the stretch when the mass is 22 grams.

(b) Pick two points on the line that are not data points.

**Figure 2.**

In **Figure 2**(b), we've selected points $P(12, 7)$ and $Q(36, 18)$. The first point indicates that a mass of 12 grams stretches the spring 7 centimeters. The interpretation for the second point is similar. We can find the slope of the line through the points $P$ and $Q$ with the slope formula.

$$m = \frac{\Delta x}{\Delta m} = \frac{18\,\mathrm{cm} - 7\,\mathrm{cm}}{36\,\mathrm{g} - 12\,\mathrm{g}} = \frac{11}{24}\,\frac{\mathrm{cm}}{\mathrm{g}}.$$

The slope of the line is the rate at which the distance stretched is changing with respect to how the mass is changing. In this case, for every additional 24 grams of mass that is hung, the spring stretches an additional 11 centimeters.

The next step is to use the point-slope formula to determine the equation of the line.

$$y - y_0 = m(x - x_0) \tag{1}$$

Let's use point $P(12, 7)$. That is, set $(x_0, y_0) = (12, 7)$. Substitute $m = 11/24$, $x_0 = 12$, and $y_0 = 7$ into **equation (1)** to obtain

$$y - 7 = \frac{11}{24}(x - 12). \tag{2}$$

In the spring-mass application, the dependent variable is $x$, not $y$, and the independent variable is $m$, not $x$. Replace the $y$ on the left-hand side of **equation (2)** with $x$, then replace $x$ on the right-hand side of **equation (2)** with $m$ to obtain

$$x - 7 = \frac{11}{24}(m - 12). \tag{3}$$

Solve **equation (3)** for $x$.

$$\begin{aligned} x - 7 &= \frac{11}{24}m - \frac{132}{24} \\ x &= \frac{11}{24}m - \frac{132}{24} + 7 \\ x &= \frac{11}{24}m - \frac{132}{24} + \frac{168}{24} \\ x &= \frac{11}{24}m + \frac{36}{24} \end{aligned}$$

Reduce 36/24 to 3/2 to obtain

$$x = \frac{11}{24}m + \frac{3}{2}.$$

Recall that $x$ represents the distance stretched and $m$ represents the amount of mass hung from the spring. That is, $x$ is a function of $m$. We can use function notation to write the last equation as follows.

$$x(m) = \frac{11}{24}m + \frac{3}{2} \tag{4}$$

We can use the model in **equation (4)** to determine the amount of stretch when a mass of 22 grams is attached to the spring. Substitute $m = 22$ in **equation (4)**, then use a calculator to approximate the stretch in the spring.

$$x(22) = \frac{11}{24}(22) + \frac{3}{2} \approx 11.6\,\text{cm}$$

Note the agreement with the graphical solution found in **Figure 2**(a). Readers should understand that this kind of accuracy is not the usual norm. There are a number of factors that can introduce error.

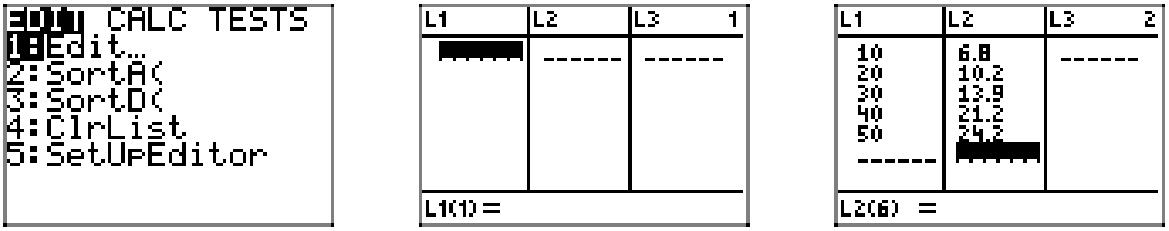
- Aditya and Tami might not have taken accurate measurements in the lab, so the data could be flawed to begin with.
- There could be errors made when we scale the axes and plot the data.
- The "eyeball" line of best fit that we drew was very subjective. A slight rotation or translation of the ruler during the drawing of the supposed "line of best fit" can produce different results.
- Our calculations could contain mistakes and round-off error.

## *Using the Graphing Calculator to Find the Line of Best Fit*

Statisticians have developed a particular method, called the "method of least squares," which is used to find a "line of best fit" for a set of data that shows a linear trend. The algorithm seeks to find the line that minimizes the total error. These algorithms are programmed into the graphing calculator and are available for student use.

To use the graphing calculator to determine the line of best fit, the first thing you have to learn how to do is load the data from **Table 1** into your calculator.

- Locate and push the `STAT` button on your keyboard, which will open the menu shown in **Figure 3**(a).
- Select `1:Edit` from this menu, which will open the edit window shown in **Figure 3**(b).[19]
- Enter the data from **Table 1** into lists $L_1$ and $L_2$, as shown in **Figure 3**(c)



(a)

```
L1     L2     L3    1
------ ------ ------
L1(1)=
```

(b)

```
L1     L2     L3    2
 10     6.8   ------
 20     10.2
 30     13.9
 40     21.2
 50     24.2
------
L2(6) =
```

(c)

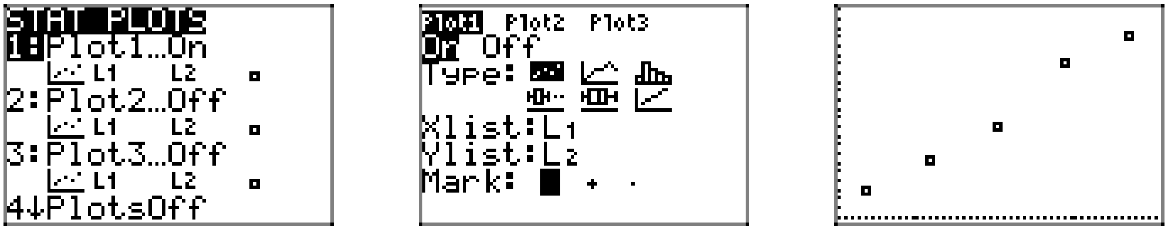
**Figure 3.** Enter the data from **Table 1** into lists $L_1$ and $L_2$ in your graphing calculator.

The next step is to plot the data you've entered into lists $L_1$ and $L_2$.

- Press the `2ND` key, followed by `STAT PLOT` (located above the `Y=` menu). This opens the window shown in **Figure 4**(a).
- Select `1:Plot1` to open the plot selection window shown in **Figure 4**(b).
- In the plot selection window of **Figure 4**(b), there are several things you need to check.
  1. Use the arrow keys to place the cursor over the word "On" and press the `ENTER` key to highlight this selection.
  2. There are six "Types" of plots: scatterplot, lineplot, histogram, modified box plot, box plot, and normal probability plot. These choices are arranged in two rows of three plots. Move your cursor to the first plot of the first row, the scatterplot, then press the `ENTER` key to highlight your selection.
  3. The next selection is the `XList`. This is the list that goes on the horizontal axis. In the case of **Table 1**, we want to place the mass data on the horizontal axis. We entered the mass data in list $L_1$, so enter `2ND` $L_1$ ($L_1$ is located above the 1 on the keyboard).
  4. The next selection is the `Ylist`. Enter `2ND` $L_2$ ($L_2$ is located above the 2 on the keyboard). This lists the distance stretched and will be placed on the vertical axis.
  5. The last item is the marker. Choose the first one with the arrow keys (it's the easiest to see) and press the `ENTER` key to highlight this choice.
- Push the `ZOOM` button on the first row of keys on your keyboard. Use the arrow keys to scroll the menu downward until you can select `9:ZoomStat`. This will produce the image shown in **Figure 4**(c).

---

[19] You may have to clear out existing data sets. The easiest way to do this is to use the arrow keys on your calculator to move the cursor into the header of the column, press the `CLEAR` button on your keyboard, followed by the `ENTER` key. This should clear the data out of the corresponding column.



(a) (b) (c)

**Figure 4.** Plotting the data points from **Table 1**

The final step is to calculate and plot the line of best fit.

- Press the `STAT` button again, but then use the right-arrow to select the `CALC` submenu highlighted in **Figure 5**(a).
- Select `4:LinReg(ax+b)` from the `CALC` submenu.[20] This places the command `LinReg(ax+b)` on your home screen, as shown in **Figure 5**(b). You must then type `2ND` $L_1$, a comma (located on its own key just above the 7 key), then `2ND` $L_2$, as shown in **Figure 5**(b).
- Press the `ENTER` key to execute the command `LinReg` $L_1$, $L_2$, which produces the equation of the line of best fit shown in **Figure 5**(c).

```
LinReg(ax+b) L1,
L2
```

```
LinReg
 y=ax+b
 a=.458
 b=1.52
```

(a) (b) (c)

**Figure 5.** Finding the equation of the line of best fit.

The screen in **Figure 5**(c) is quite informative. It tells us two things.

1. The equation of the line of best fit is `y = ax + b`.
2. The slope is `a = .458` and the $y$-intercept is `b = 1.52`.

Substituting $a = 0.458$ and $b = 1.52$ into the equation $y = ax + b$ gives us the equation of the line of best fit.

$$y = 0.458x + 1.52 \tag{5}$$

We can superimpose the plot of the line of best fit on our data set in two easy steps.

- Press the `Y=` key and enter the equation `0.458*X+1.52` in $Y_1$, as shown in **Figure 6**(a).
- Press the `GRAPH` button on the top row of keys on your keyboard to produce the line of best fit in **Figure 6**(b).

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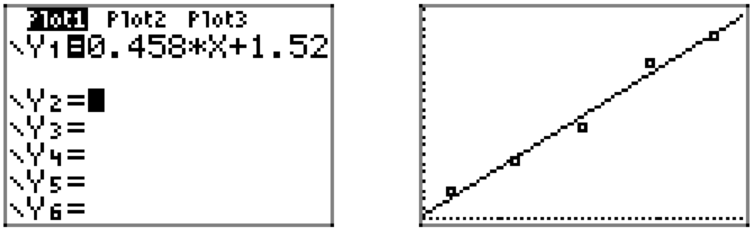
[20] The technical name of the process for finding the line of best fit is *linear regression*. Hence, the abbreviation `LinReg`.



```
Plot1 Plot2 Plot3
\Y1=0.458*X+1.52
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

(a) (b)

**Figure 6.** Superimpose the line of best fit on the scatterplot of the data from **Table 1**.

On the left-hand side of **equation (5)**, replace $y$ with $x$ (the distance stretched); on the right-hand side, replace $x$ with $m$ (amount of mass). This leads to the result

$$x = 0.458m + 1.52 \tag{6}$$

You might recall that our hand calculation produced **equation (4)**, which we repeat here for convenience.

$$x = \frac{11}{24}x + \frac{3}{2}.$$

Note that $11/24 \approx 0.4583$ and $3/2 = 1.5$, so **equation (6)** agrees closely with our hand-calculated equation of the line of best fit.

It is rather unusual to have a hand-calculated line of best fit agree so closely with the sophisticated and very accurate result produced by the graphing calculator. So, don't be disappointed when your homework results don't match as nicely as they have in this example. If you are in the ballpark with your hand-calculated equation for the line of best fit, that will usually be good enough. However, if your hand-calculated equation is not even close to what your calculator produces, it's "back to the drawing board." Recheck your plot and your calculations. Be stubborn! Don't be satisfied with your results until you have reasonable agreement.

## 3.5 *Exercises*

---

**1.** The following set of data about revolving consumer credit (debt) in the United States is from Google.com. This is primarily made up of credit card debt, but also includes other consumer non-mortgage credit, like those offered by commercial banks, credit unions, Sallie Mae, and the federal government.

| Year | yrs $x$ after 2001 | all revolving credit $C$ in billions of $ |
|---|---|---|
| 2001 | 0 | 721.0 |
| 2002 | 1 | 741.2 |
| 2003 | 2 | 759.3 |
| 2004 | 3 | 786.1 |
| 2005 | 4 | 805.4 |

**a)** Set up a coordinate system on graph paper, placing the credit $C$ on the vertical axis, and the years $x$ after 2001 on the horizontal axis. Label and scale each axis appropriately. Draw what you feel is the line of best fit. *Remember to draw all lines with a ruler.*

**b)** Select two points on your line of best fit that are not from the data table above. Use these two points to determine the slope of the line. Include units with your answer. Write a sentence or two explaining the real world significance of the slope of the line of best fit.

**c)** Use one of the two points on the line and the slope to determine the equation of the line of best fit in point-slope form. Use $C$ and $x$ for the dependent and independent variables, respectively. Solve the resulting equation for $C$ and write your result using function notation.

**d)** Use the equation developed in part (c) to predict the revolving credit debt in the year 2008.

**e)** If the linear trend predicted by the line of best fit continues, in what year will the revolving credit debt reach 1.0 trillion dollars?

**2.** The following set of data about non-revolving credit (debt) in the United States is from Google.com. The largest component of non-revolving credit is automobile loans, but it is also includes student loans and other defined-term consumer loans.

| Year | yrs $x$ after 2001 | Non-revolving debt $D$ in billions of $ |
|---|---|---|
| 2001 | 0 | 1121.3 |
| 2002 | 1 | 1184.1 |
| 2003 | 2 | 1247.3 |
| 2004 | 3 | 1305.0 |
| 2005 | 4 | 1342.3 |

**a)** Set up a coordinate system on graph paper, placing the non-revolving credit debt $D$ on the vertical axis, and the years $x$ after 2001 on the horizontal axis. Label and scale each axis appropriately. Draw what you feel is the line of best fit. *Remember to draw all lines with a ruler.*

**b)** Select two points on your line of best

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[21] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

fit that are not from the data table above. Use these two points to determine the slope of the line. Include units with your answer. Write a sentence or two explaining the real world significance of the slope of the line of best fit.

**c)** Use one of the two points on the line and the slope to determine the equation of the line of best fit in point-slope form. Use $D$ and $x$ for the dependent and independent variables, respectively. Solve the resulting equation for $D$ and write your result using function notation.

**d)** Use the equation developed in part (c) to predict the non-revolving credit debt in the year 2008.

**e)** If the linear trend predicted by the line of best fit continues, in what year will the non-revolving credit debt reach 2.0 trillion dollars?

**3.** According to the U.S. Bureau of Transportation (www.bts.gov), retail sales of new cars declined every year from 2000-2004, as shown in the following table.

| Year | yrs $x$ after 2000 | Sales $S$ in thousands |
|---|---|---|
| 2000 | 0 | 8847 |
| 2001 | 1 | 8423 |
| 2002 | 2 | 8103 |
| 2003 | 3 | 7610 |
| 2004 | 4 | 7506 |

**a)** Set up a coordinate system on graph paper, placing the sales $S$ on the vertical axis, and the years $x$ after 2000 on the horizontal axis. Label and scale each axis appropriately. Draw what you feel is the line of best fit. *Remember to draw all lines with a ruler.*

**b)** Select two points on your line of best fit that are not from the data table above. Use these two points to determine the slope of the line. Include units with your answer. Write a sentence or two explaining the real world significance of the slope of the line of best fit.

**c)** Use one of the two points on the line and the slope to determine the equation of the line of best fit in point-slope form. Use $S$ and $x$ for the dependent and independent variables, respectively. Solve the resulting equation for $S$ and write your result using function notation.

**d)** Use the equation developed in part (c) to predict sales in the year 2006.

**e)** If the linear trend predicted by the line of best fit continues, when will sales drop to 7 million cars per year?

**4.** The following table shows total midyear population of the world according to the U.S. Census Bureau, (www.census.gov) for recent years.

| Year | yrs $x$ after 2000 | Population $P$ in billions |
|---|---|---|
| 2000 | 0 | 6.08 |
| 2001 | 1 | 6.16 |
| 2002 | 2 | 6.23 |
| 2003 | 3 | 6.30 |
| 2004 | 4 | 6.38 |
| 2005 | 5 | 6.45 |
| 2006 | 6 | 6.53 |

**a)** Set up a coordinate system on graph paper, placing the population $P$ on the vertical axis, and the years $x$ after 2000 on the horizontal axis. Label and scale each axis appropriately.

Draw what you feel is the line of best fit. *Remember to draw all lines with a ruler.*

**b)** Select two points on your line of best fit that are not from the data table above. Use these two points to determine the slope of the line. Include units with your answer. Write a sentence or two explaining the real world significance of the slope of the line of best fit.

**c)** Use one of the two points on the line and the slope to determine the equation of the line of best fit in point-slope form. Use $P$ and $x$ for the dependent and independent variables, respectively. Solve the resulting equation for $P$ and write your result using function notation.

**d)** Use the equation developed in part (c) to predict the population in 2010.

**e)** If the linear trend predicted by the line of best fit continues, when will world population reach 7 billion?

**5.** The following table shows an excerpt from the U.S. Census Bureau's 2005 data (www.census.gov) on annual sales of new homes in the United States.

| Price Range (thousands of \$) | Number sold (thousands) |
|---|---|
| 150 – 199 | 246 |
| 200 – 249 | 200 |
| 250 – 299 | 152 |

We cannot use price ranges as coordinate values (we must have single values), so we replace each price range in the table with a single price in the middle of the range–the average value of a home in that range. This gives us the following modified table:

| Avg Price $P$ (thousands of \$) | Number sold $N$ (thousands) |
|---|---|
| 175 | 246 |
| 225 | 200 |
| 275 | 152 |

We can now plot the data on a coordinate system.

**a)** Enter the data into your calculator and make a scatter plot. Copy it down onto your paper, labeling appropriately.

**b)** Use your calculator to determine a line of best fit. This is called a linear demand function, because it allows you to predict the demand for houses with a certain price. Write it using function notation and round to the nearest thousandth. Graph it on your calculator and copy it onto your coordinate system.

**c)** Use the linear demand function to predict annual sales of homes priced at $\$200,000$. Try to use the TABLE feature on your calculator to make this prediction.

**6.** The following table shows data from the National Association of Homebuilders (www.nahb.org), indicating the median price of new homes in the United States.

| Year | Median Price (thousands of $) |
|---|---|
| 2000 | 169 |
| 2001 | 175 |
| 2002 | 188 |
| 2003 | 195 |
| 2004 | 221 |
| 2005 | 238 |

**a)** Enter the data into your calculator and make a scatter plot. Copy it down onto your paper, labeling appropriately.

**b)** Use your calculator to determine a line of best fit that can be used to predict the median price of new homes in future years. Write it using function notation. Graph it on your calculator and copy it onto your coordinate system.

**c)** Use the linear demand function to predict the median price of a new home in 2010. Try to use the TABLE feature on your calculator to make this prediction.

**d)** Looking at the graph, do you think the linear demand function models the actual data points well? If not, why not? What does this mean about the prediction you made in part (c)?

**7.** Jim is hanging blocks of various mass on a spring in the physics lab. He notices that the spring will stretch further if he adds more mass to the end of the spring. He is soon convinced that the distance the spring will stretch depends on the amount of mass attached to it. He decides to take some measurements. He records the amount of mass attached to the end of the spring and then measures the distance that the spring stretched. Here is Jim's data.

| Mass (grams) | Distance Stretched (cm) |
|---|---|
| 50 | 1.2 |
| 100 | 1.9 |
| 150 | 3.1 |
| 200 | 4.0 |
| 250 | 4.8 |
| 300 | 6.2 |

**a)** Enter the data into your calculator and make a scatter plot. Copy it down onto your paper, labeling appropriately.

**b)** Use your calculator to determine a line of best fit that can be used to predict the distance the spring stretches. Write it using function notation. Graph it on your calculator and copy it onto your coordinate system.

**c)** Use the function from part (c) to predict the distance the spring will stretch if 175 grams is attached to the spring. Try to use the TABLE feature on your calculator to make this prediction.

**8.** Dave and Melody are lab partners in Tony Sartori's afternoon chemistry lab. Professor Sartori has prepared an experiment to help them discover the relationship between the Celsius and Fahrenheit temperature scales. The experiment consists of a beaker full of ice and two thermometers, one calibrated in the Fahrenheit scale, the other in the Celsius scale. Dave and Melody use a Bunsen burner to heat the beaker, eventually bringing the water in the beaker to the boiling point. Every few minutees they make two temperature readings, one in Fahrenheit, one in Celsius. The data that they record during the laboratory session follows.

| Celsius | Fahrenheit |
|---|---|
| 4.0 | 39 |
| 18 | 65 |
| 30 | 85 |
| 51 | 122 |
| 70 | 159 |
| 85 | 186 |
| 100 | 210 |

**a)** Enter the data into your calculator and make a scatter plot. Copy it down onto your paper, labeling appropriately.

**b)** Use your calculator to determine a line of best fit that can be used to predict the Fahrenheit temperature as a function of the Celsius temperature. Write it using function notation. Graph it on your calculator and copy it onto your coordinate system.

**c)** Use the function from part (c) to predict the Fahrenheit temperature if the Celsius temperature is 40. Try to use the TABLE feature on your calculator to make this prediction.

**d)** Use the function from part (c) to predict the Celsius temperature if the Fahrenheit temperature is 100.

**9.** The following table shows data on home sales at the Mendocino Coast in 2005.

| Price Range (thousands of \$) | Number sold (thousands) |
|---|---|
| 200 – 299 | 14 |
| 300 – 399 | 55 |
| 400 – 499 | 62 |

We cannot use price ranges as coordinate values (we must have single values), so we replace each price range in the table with a single price in the middle of the range–the average value of a home in that range. This gives us the following modified table:

| Avg Price $P$ (thousands of \$) | Number sold $N$ (thousands) |
|---|---|
| 250 | 14 |
| 350 | 55 |
| 450 | 62 |

We can now plot the data on a coordinate system.

**a)** Enter the data into your calculator and make a scatter plot. Copy it down onto your paper, labeling appropriately.

**b)** Use your calculator to determine a line of best fit. Write it using function notation and round to the nearest thousandth. Graph it on your calculator and copy it onto your coordinate system.

**c)** Use the linear function to predict the sales for houses in the price range $\$500,000$–$\$599,000$. Use the average price of $\$550,000$ for this estimate.

**d)** The actual number of houses sold in the price range $\$500,000 - \$599,000$ was 41. Plot this as a point on your coordinate system and compare it to your linear function model's prediction. Notice that this actual value is pretty different from the prediction.

**e)** What this means is that a linear model is not very good for the data for home sales! Draw a simple curve that goes through each of the data points. Notice that it does not very closely resemble the shape of a line! More sophisticated functions are required to model this example–such as quadratic functions, which we study in a later chapter. The moral of the story here is that not every data set can be modeled linearly!

**10.** The following from the July 14, 2006 edition of the Beijing Today newspaper shows how high-heels affect the ball of the foot. The table shows the increase in percent of pressure on the ball of the foot for given heights of heels.

| Heel height $h$ (inches) | %increase in pressure |
|---|---|
| 1 | 22 |
| 2 | 57 |

**a)** Enter the data into your calculator and make a scatter plot. Copy it down onto your paper, labeling appropriately.

**b)** Notice that, because we have exactly two data points, the line of best fit is the line that goes through both points. To begin finding the equation, use the slope formula to compute the slope.

**c)** Use the point-slope form to find an equation for the line. Write it in slope-intercept form.

**d)** Use the linear function to predict the percent of stress increase for a 3-inch heel.

**e)** The actual percent of pressure increase for a 3-inch heel is 76 %. Plot this as a point on your coordinate system and compare it to your linear function model's prediction. Notice that this actual value is pretty different from the prediction.

**f)** What this means is that a linear model is not very good for the data! Draw a simple curve that goes through each of the data points. Notice that it does not very closely resemble the shape of a line! More sophisticated functions are required to model this example. Not every data set should be modeled linearly!

## 3.5 *Answers*

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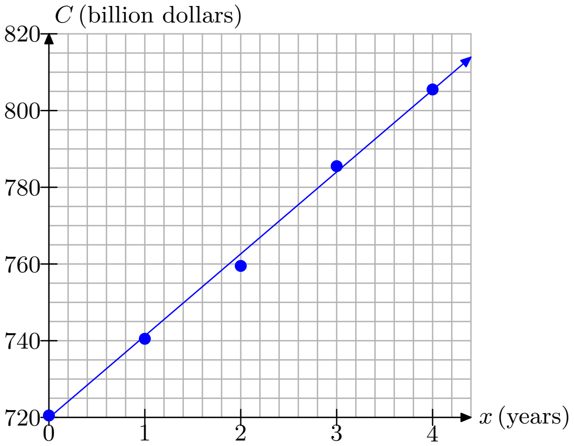
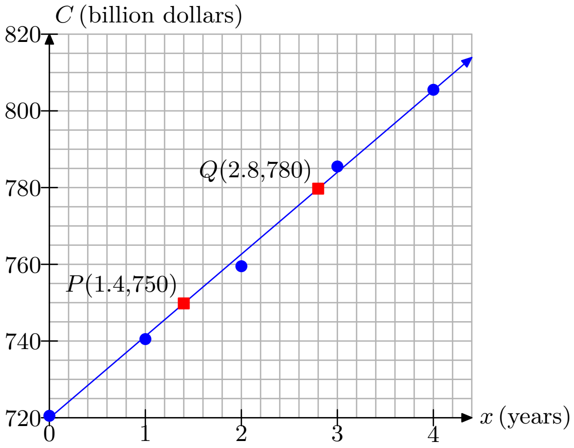
**1.**



**a)**

**b)**



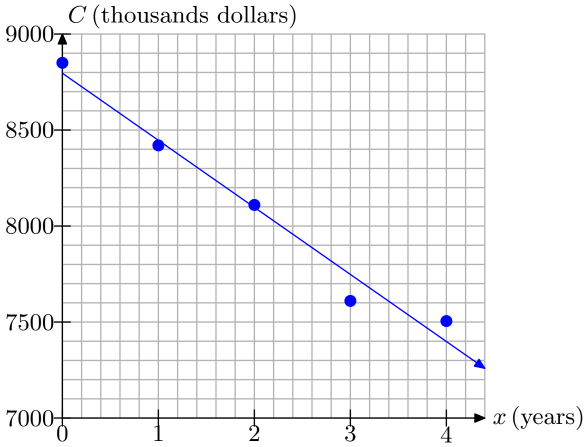
**c)** $C(x) = 21.42x + 720.012$

**d)** Approximately 869 billion dollars.

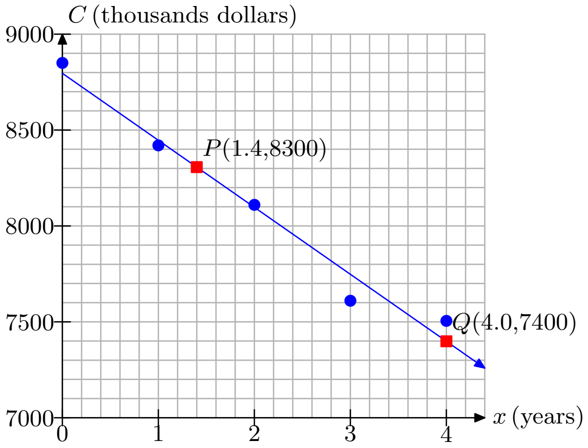
**e)** 2014

**3.**

**a)**



**b)**



$m = -346.15$ thousand cars per year

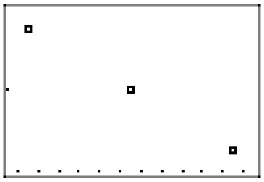
**c)** $S(x) = -346.15x + 8784.61$

**d)** $S(6) \approx 6707$ thousand cars.

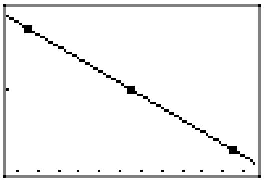
**e)** In the year 2005-2006.

**5.**

**a)**

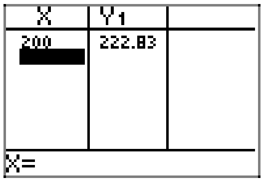


**b)**



$N(P) = -0.94P + 410.833$

**c)**



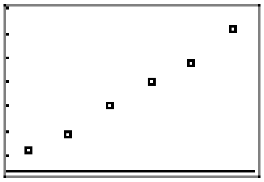
| X | $Y_1$ |  |
|---|---|---|
| 200 | 222.83 |  |

X=

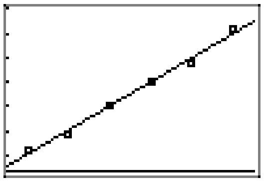
Approximately $222,830$ homes.

**7.**

**a)**

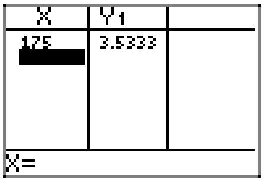


**b)**



$d(m) = 0.01977m + 0.07333$

**c)**



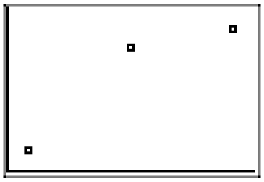
| X | $Y_1$ |  |
|---|---|---|
| 175 | 3.5333 |  |

X=

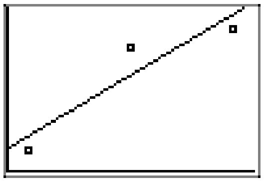
Approximately 3.53 centimeters.

**9.**

**a)**

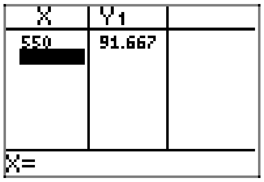


**b)**



$N(P) = 0.24P - 40.33.$

**c)**

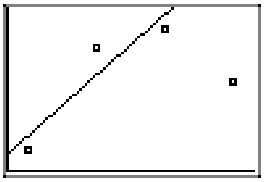


| X | $Y_1$ |  |
|---|---|---|
| 550 | 91.667 |  |

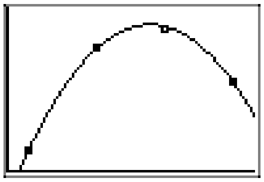
X=

Approximately $91,667$ homes.

**d)**



**e)**



## 3.6 Index

# 4 Absolute Value Functions

In this chapter we will introduce the *absolute value function*, one of the more useful functions that we will study in this course. Because it is closely related to the concept of distance, it is a favorite among statisticians, mathematicians, and other practitioners of science.

Most readers probably already have an intuitive understanding of absolute value. You've probably seen that the absolute value of seven is seven, i.e., $|7| = 7$, and the absolute value of negative seven is also seven, i.e., $|-7| = 7$. That is, the absolute value function takes a number as input, and then makes that number positive (if it isn't already). Technically, because $|0| = 0$, which is not a positive number, we are forced to say that the absolute value function takes a number as input, and then makes it *nonnegative* (positive or zero).

However, as you advance in your coursework, you will quickly discover that this intuitive notion of absolute value is insufficient when tackling more sophisticated problems. In this chapter, as we try to raise our understanding of absolute value to a higher plane, we will encounter piecewise-defined functions and use them to create piecewise definitions for absolute value functions. These piecewise definitions will help us draw the graphs of a variety of absolute value functions.

Finally we'll conclude our work in this chapter by developing techniques for solving equations and inequalities containing expressions that implement the absolute value function.

## Table of Contents

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## 4.1 Piecewise-Defined Functions

In preparation for the definition of the absolute value function, it is extremely important to have a good grasp of the concept of a piecewise-defined function. However, before we jump into the fray, let's take a look at a special type of function called a *constant function.*

One way of understanding a constant function is to have a look at its graph.

▶ **Example 1.** *Sketch the graph of the constant function* $f(x) = 3$.

Because the notation $f(x) = 3$ is equivalent to the notation $y = 3$, we can sketch a graph of $f$ by drawing the graph of the horizontal line having equation $y = 3$, as shown in **Figure 1**.

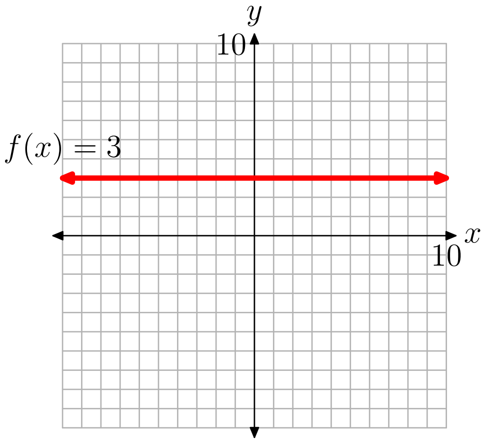


**Figure 1.** The graph of a constant function is a horizontal line.

When you look at the graph in **Figure 1**, note that every point on the horizontal line having equation $f(x) = 3$ has a $y$-value equal to 3. We say that the $y$-values on this horizontal line are *constant*, for the simple reason that they are constantly equal to 3.

The function form works in precisely the same manner. Look again at the notation

$$f(x) = 3.$$

Note that no matter what number you substitute for $x$ in the left-hand side of $f(x) = 3$, the right-hand side is constantly equal to 3. Thus,

$$f(-5) = 3, \quad f(-1/2) = 3, \quad f(\sqrt{2}) = 3, \quad \text{or} \quad f(\pi) = 3.$$

◆

The above discussion leads to the following definition.

---

[1] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

**Definition 2.** *The function defined by $f(x) = c$, where $c$ is a constant (fixed real number), is called a* constant function*.*

Two comments are in order:

1. $f(x) = c$ for all real numbers $x$.
2. The graph of $f(x) = c$ is a horizontal line. It consists of all the points $(x, y)$ having $y$-value equal to $c$.

## *Piecewise Constant Functions*

Piecewise functions are a favorite of engineers. Let's look at an example.

▶ **Example 3.** *Suppose that a battery provides no voltage to a circuit when a switch is open. Then, starting at time $t = 0$, the switch is closed and the battery provides a constant 5 volts from that time forward. Create a piecewise function modeling the problem constraints and sketch its graph.*

This is a fairly simple exercise, but we will have to introduce some new notation. First of all, if the time $t$ is less than zero ($t < 0$), then the voltage is 0 volts. If the time $t$ is greater than or equal to zero ($t \geq 0$), then the voltage is a constant 5 volts. Here is the notation we will use to summarize this description of the voltage.

$$V(t) = \begin{cases} 0, & \text{if } t < 0, \\ 5, & \text{if } t \geq 0 \end{cases} \tag{4}$$

Some comments are in order:

- The voltage difference provide by the battery in the circuit is a function of time. Thus, $V(t)$ represents the voltage at time $t$.
- The notation used in (4) is universally adopted by mathematicians in situations where the function changes definition depending on the value of the independent variable. This definition of the function $V$ is called a "piecewise definition." Because each of the pieces in this definition is constant, the function $V$ is called a *piecewise constant* function.
- This particular function has two pieces. The function is the constant function $V(t) = 0$, when $t < 0$, but a different constant function, $V(t) = 5$, when $t \geq 0$.

If $t < 0$, $V(t) = 0$. For example, for $t = -1$, $t = -10$, and $t = -100$,

$$V(-1) = 0, \quad V(-10) = 0, \quad \text{and} \quad V(-100) = 0.$$

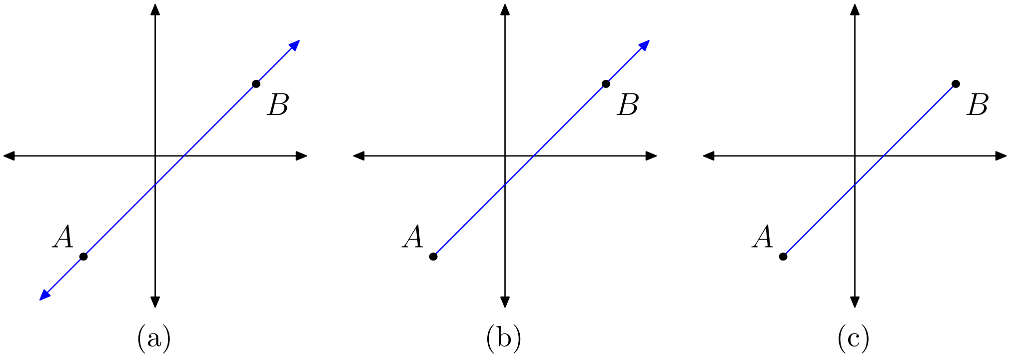
On the other hand, if $t \geq 0$, then $V(t) = 5$. For example, for $t = 0$, $t = 10$, and $t = 100$,

$$V(0) = 5, \quad V(10) = 5, \quad \text{and} \quad V(100) = 5.$$

Before we present the graph of the piecewise constant function $V$, let's pause for a moment to make sure we understand some standard geometrical terms.

**Geometrical Terms.**

- A *line* stretches indefinitely in two directions, as shown in **Figure 2**(a).
- If a line has a fixed endpoint and stretches indefinitely in only one direction, as shown in **Figure 2**(b), then it is called a *ray*.
- If a portion of the line is fixed at each end, as shown in **Figure 2**(c), then it is called a *line segment*.



(a) (b) (c)

**Figure 2.** Lines, rays, and segments.

With these terms in hand, let's turn our attention to the graph of the voltage defined by **equation (4)**. When $t < 0$, then $V(t) = 0$. Normally, the graph of $V(t) = 0$ would be a horizontal line where each point on the line has $V$-value equal to zero. However, $V(t) = 0$ only if $t < 0$, so the graph is the *horizontal ray* that starts at the origin, then moves indefinitely to the left, as shown in **Figure 3**. That is, the horizontal line $V = 0$ has been restricted to the domain $\{t : t < 0\}$ and exists only to the left of the origin.

Similarly, when $t \geq 0$, then $V(t) = 5$ is the horizontal ray shown in **Figure 3**. Each point on the ray has a $V$-value equal to 5.

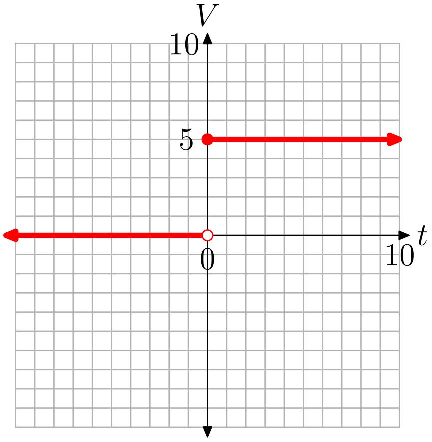


**Figure 3.** The voltage as a function of time $t$.

Two comments are in order:

- Because $V(t) = 0$ only when $t < 0$, the point $(0, 0)$ is unfilled (it is an open circle). The open circle at $(0, 0)$ is a mathematician's way of saying that this particular point is **not** plotted or shaded.
- Because $V(t) = 5$ when $t \geq 0$, the point $(0, 5)$ is filled (it is a filled circle). The filled circle at $(0, 5)$ is a mathematician's way of saying that this particular point **is** plotted or shaded.

◆

Let's look at another example.

▶ **Example 5.** *Consider the piecewise-defined function*

$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } 0 \leq x < 2, \\ 2, & \text{if } x \geq 2. \end{cases} \tag{6}$$

*Evaluate $f(x)$ at $x = -1$, 0, 1, 2, and 3. Sketch the graph of the piecewise function $f$.*

Because each piece of the function in (6) is constant, evaluation of the function is pretty easy. You just have to select the correct piece.

- Note that $x = -1$ is less than 0, so we use the first piece and write $f(-1) = 0$.
- Note that $x = 0$ satisfies $0 \leq x < 2$, so we use the second piece and write $f(0) = 1$.
- Note that $x = 1$ satisfies $0 \leq x < 2$, so we use the second piece and write $f(1) = 1$.
- Note that $x = 2$ satisfies $x \geq 2$, so we use the third piece and write $f(2) = 2$.
- Finally, note that $x = 3$ satisfies $x \geq 2$, so we use the third piece and write $f(3) = 2$.

The graph is just as simple to sketch.

- Because $f(x) = 0$ for $x < 0$, the graph of this piece is a horizontal ray with endpoint at $x = 0$. Each point on this ray will have a $y$-value equal to zero and the ray will lie entirely to the left of $x = 0$, as shown in **Figure 4**.
- Because $f(x) = 1$ for $0 \leq x < 2$, the graph of this piece is a horizontal segment with one endpoint at $x = 0$ and the other at $x = 2$. Each point on this segment will have a $y$-value equal to 1, as shown in **Figure 4**.
- Because $f(x) = 2$ for $x \geq 2$, the graph of this piece is a horizontal ray with endpoint at $x = 2$. Each point on this ray has a $y$-value equal to 2 and the ray lies entirely to the right of $x = 2$, as shown in **Figure 4**.

Several remarks are in order:

- The function is zero to the left of the origin (for $x < 0$), but not at the origin. This is indicated by an empty circle at the origin, an indication that we are not plotting that particular point.
- For $0 \leq x < 2$, the function equals 1. That is, the function is constantly equal to 1 for all values of $x$ between 0 and 2, including zero but not including 2. This is why you see a filled circle at $(0, 1)$ and an empty circle at $(2, 1)$.
- Finally, for $x \geq 2$, the function equals 2. That is, the function is constantly equal to 2 whenever $x$ is greater than or equal to 2. That is why you see a filled circle at $(2, 2)$.

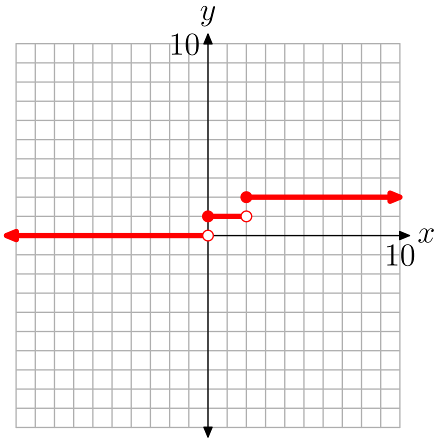


**Figure 4.** Sketching the graph of the piecewise function (6).

◆

## Piecewise-Defined Functions

Now, let's look at a more generic situation involving piecewise-defined functions—one where the pieces are not necessarily constant. The best way to learn is by doing, so let's start with an example.

▶ **Example 7.** *Consider the piecewise-defined function*

$$f(x)=\begin{cases} -x+2, & \text{if } x<2,\\ x-2, & \text{if } x\geq 2.\end{cases} \tag{8}$$

*Evaluate $f(x)$ for $x=0$, 1, 2, 3 and 4, then sketch the graph of the piecewise-defined function.*

The function changes definition at $x=2$. If $x<2$, then $f(x)=-x+2$. Because both 0 and 1 are strictly less than 2, we evaluate the function with this first piece of the definition.

$$\begin{aligned} f(x)&=-x+2 \qquad \text{and} \qquad & f(x)&=-x+2\\ f(0)&=-0+2 & f(1)&=-1+2\\ f(0)&=2 & f(1)&=1. \end{aligned}$$

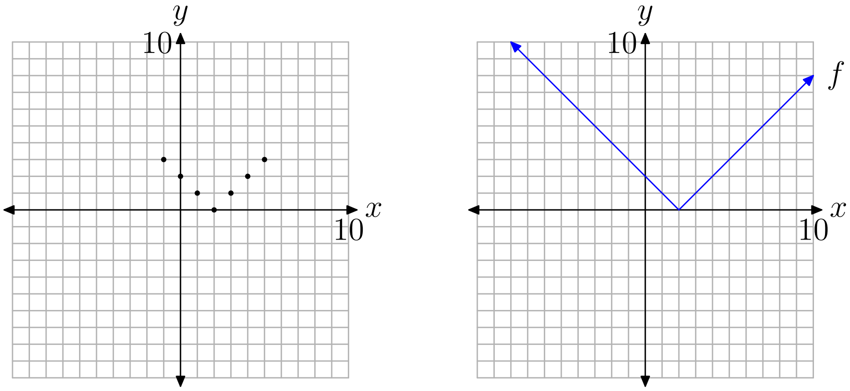
On the other hand, if $x\geq 2$, then $f(x)=x-2$. Because 2, 3, and 4 are all greater than or equal to 2, we evaluate the function with this second piece of the definition.

$$\begin{aligned} f(x)&=x-2 \qquad \text{and} \qquad & f(x)&=x-2 \qquad \text{and} \qquad & f(x)&=x-2\\ f(2)&=2-2 & f(3)&=3-2 & f(4)&=4-2\\ f(2)&=0 & f(3)&=1 & f(4)&=2. \end{aligned}$$

One possible approach to the graph of $f$ is to place the points we've already calculated, plus a couple extra, in a table (see **Figure 5**(a)), plot them (see **Figure 5**(b)), then intuit the shape of the graph from the evidence provided by the plotted points. This is done in **Figure 5**(c).

| $x$ | $f(x)$ |
|---|---|
| $-1$ | 3 |
| 0 | 2 |
| 1 | 1 |
| 2 | 0 |
| 3 | 1 |
| 4 | 2 |
| 5 | 3 |

(a)



(b)

(c)

**Figure 5.** Plotting the graph of the piecewise function defined in (**8**).

However pragmatic, this point-plotting approach is a bit tedious; but, more importantly, it does not provide the background necessary for the discussion of the absolute value function in the next section. We need to stretch our understanding to a higher level. Fortunately, all the groundwork is in place. We need only apply what we already know about the equations of lines to fit this piecewise situation.

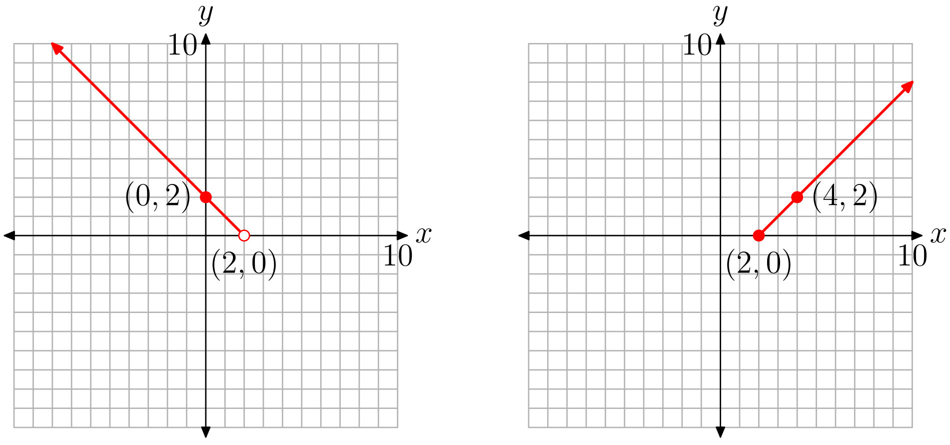
**Alternative approach**. Let's use our knowledge of the equation of a line (i.e. $y = mx + b$) to help sketch the graph of the piecewise function defined in (**8**).

Let's sketch the first piece of the function $f$ defined in (**8**). We have $f(x) = -x+2$, provided $x < 2$. Normally, this would be a line (with slope $-1$ and intercept 2), but we are to sketch only a part of that line, the part where $x < 2$ ($x$ is to the left of 2). Thus, this piece of the graph will be a *ray*, starting at the point where $x = 2$, then moving indefinitely to the left.

The easiest way to sketch a ray is to first calculate and plot its fixed endpoint (in this case at $x = 2$), then plot a second point on the ray having $x$-value less than 2, then use a ruler to draw the ray.

With this thought in mind, to find the coordinates of the endpoint of the ray, substitute $x = 2$ in $f(x) = -x + 2$ to get $f(2) = 0$. Now, technically, we're not supposed to use this piece of the function unless $x$ is strictly less than 2, but we could use it with $x = 1.9$, or $x = 1.99$, or $x = 1.999$, etc. So let's go ahead and use $x = 2$ in this piece of the function, but indicate that we're not actually supposed to use this point by drawing an "empty circle" at $(2, 0)$, as shown in **Figure 6**(a).

To complete the plot of the ray, we need a second point that lies to the left of its endpoint at $(2, 0)$. Note that $x = 0$ is to the left of $x = 2$. Evaluate $f(x) = -x + 2$ at $x = 0$ to obtain $f(0) = -0 + 2 = 2$. This gives us the second point $(0, 2)$, which we plot as shown in **Figure 6**(a). Finally, draw the ray with endpoint at $(2, 0)$ and second point at $(0, 2)$, as shown in **Figure 6**(a).



(a) (b)

**Figure 6.** Sketch each piece separately.

We now repeat this process for the second piece of the function defined in (**8**). The equation of the second piece is $f(x) = x - 2$, provided $x \geq 2$. Normally, $f(x) = x - 2$ would be a line (with slope 1 and intercept $-2$), but we're only supposed to sketch that part of the line that lies to the right of or at $x = 2$. Thus, the graph of this second piece is a ray, starting at the point with $x = 2$ and continuing to the right. If we evaluate $f(x) = x - 2$ at $x = 2$, then $f(2) = 2 - 2 = 0$. Thus, the fixed endpoint of the ray is at the point $(2, 0)$. Since we're actually supposed to use this piece with $x = 2$, we indicate this fact with a filled circle at $(2, 0)$, as shown in **Figure 6**(b).

We need a second point to the right of this fixed endpoint, so we evaluate $f(x) = x - 2$ at $x = 4$ to get $f(4) = 4 - 2 = 2$. Thus, a second point on the ray is the point $(4, 2)$. Finally, we simply draw the ray, starting at the endpoint $(2, 0)$ and passing through the second point at $(4, 2)$, as shown in **Figure 6**(b).

To complete the graph of the piecewise function $f$ defined in **equation (8)**, simply combine the two pieces in **Figure 6**(a) and **Figure 6**(b) to get the finished graph in **Figure 7**. Note that the graph in **Figure 7** is identical to the earlier result in **Figure 5**(c).

◆

Let's try this alternative procedure in another example.

▶ **Example 9.** *A source provides voltage to a circuit according to the piecewise definition*

$$V(t) = \begin{cases} 0, & \text{if } t < 0, \\ t, & \text{if } t \geq 0. \end{cases} \tag{10}$$

*Sketch the graph of the voltage $V$ versus time $t$.*

For all time $t$ that is less than zero, the voltage $V$ is zero. The graph of $V(t) = 0$ is a constant function, so its graph is normally a horizontal line. However, we must restrict

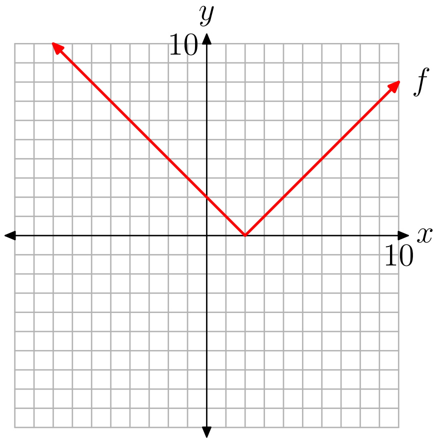
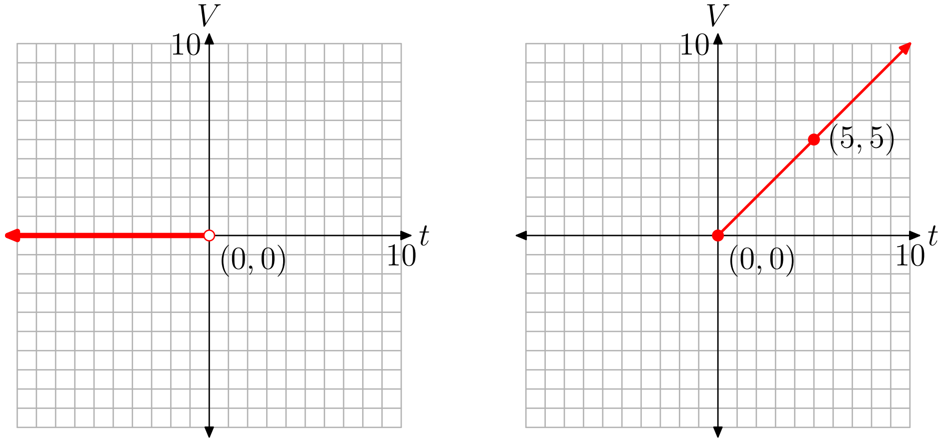


**Figure 7.** Combining both pieces.

the graph to the domain $(-\infty, 0)$, so this piece of **equation (10)** will be a horizontal ray, starting at the origin and moving indefinitely to the left, as shown in **Figure 8**(a).

On the other hand, $V(t) = t$ for all values of $t$ that are greater than or equal to zero. Normally, this would be a line with slope 1 and intercept zero. However, we must restrict the domain to $[0, \infty)$, so this piece of **equation (10)** will be a ray, starting at the origin and moving indefinitely to the right.

- The endpoint of this ray starts at $t = 0$. Because $V(t) = t$, $V(0) = 0$. Hence, the endpoint of this ray is at the point $(0, 0)$.
- Choose any value of $t$ that is greater than zero. We'll choose $t = 5$. Because $V(t) = t$, $V(5) = 5$. This gives us a second point on the ray at $(5, 5)$, as shown in **Figure 8**(b).



(a) $V(t) = 0$ for $t < 0$.

(b) $V(t) = t$ for $t \geq 0$.

**Figure 8.**

Finally, to provide a complete graph of the voltage function defined by **equation (10)**, we combine the graphs of each piece of the definition shown in **Figures 8**(a) and (b).

The result is shown in **Figure 9**. Engineers refer to this type of input function as a "ramp function."

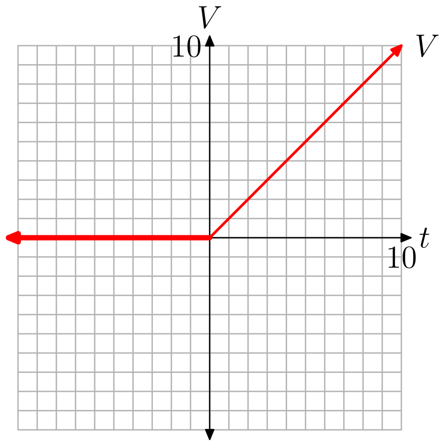


**Figure 9.** The graph of the ramp function defined by **equation (10)**.

Let's look at a very practical application of piecewise functions.

▶ **Example 11.** *The federal income tax rates for a single filer in the year 2005 are given in* **Table 1**.

| *Income* | *Tax Rate* |
|---|---|
| *Up to $7,150* | *10%* |
| *$7,151-$29,050* | *15%* |
| *$29,051-$70,350* | *25%* |
| *$70,351-$146,750* | *28%* |
| *$146,751-$319,100* | *33%* |
| *$319,101 or more* | *35%* |

**Table 1.** 2005 Federal Income Tax rates for single filer.

*Create a piecewise definition that provides the tax rate as a function of personal income.*

In reporting taxable income, amounts are rounded to the nearest dollar on the federal income tax form. Technically, the domain is discrete. You can report a taxable income of $35,000 or $35,001, but numbers between these two incomes are not used on the federal income tax form. However, we will think of the income as a continuum, allowing the income to be any real number greater than or equal to zero. If we did not do this, then our graph would be a series of dots–one for each dollar amount. We would have to plot *lots* of dots!

We will let $R$ represent the tax rate and $I$ represent the income. The goal is to define $R$ as a function of $I$.

- If income $I$ is any amount greater than or equal to zero, and less than or equal to \$7,150, the tax rate $R$ is 10% (i.e., $R = 0.10$). Thus, if $\$0 \le I \le \$7,150$, $R(I) = 0.10$.
- If income $I$ is any amount that is strictly greater than \$7,150 but less than or equal to \$29,050, then the tax rate $R$ is 15% (i.e., $R = 0.15$). Thus, if $\$7,150 < I \le \$29,050$, then $R(I) = 0.15$.

Continuing in this manner, we can construct a piecewise definition of rate $R$ as a function of taxable income $I$.

$$R(I) = \begin{cases} 0.10, & \text{if } \$0 \le I \le \$7,150, \\ 0.15, & \text{if } \$7,150 < I \le \$29,050, \\ 0.25, & \text{if } \$29,050 < I \le \$70,350, \\ 0.28, & \text{if } \$70,350 < I \le \$146,750, \\ 0.33, & \text{if } \$146,750 < I \le \$319,100, \\ 0.35, & \text{if } I > \$319,100. \end{cases} \tag{12}$$

Let's turn our attention to the graph of this piecewise-defined function. All of the pieces are constant functions, so each piece will be a horizontal line at a level indicating the tax rate. However, each of the first five pieces of the function defined in **equation (12)** are segments, because the rate is defined on an interval with a starting and ending income. The sixth and last piece is a ray, as it has a starting endpoint, but the rate remains constant for *all* incomes above \$319,100. We use this knowledge to construct the graph shown in **Figure 10**.

The first rate is 10% and this is assigned to taxable income starting at \$0 and ending at \$7,150, inclusive. Thus, note the first horizontal line segment in **Figure 10** that runs from \$0 to \$7,150 at a height of $R = 0.10$. Note that each of the endpoints are filled circles.

The second rate is 15% and this is assigned to taxable incomes greater than \$7,150, but less than or equal to \$29,050. The second horizontal line segment in **Figure 10** runs from \$7,150 to \$29,050 at a height of $R = 0.15$. Note that the endpoint at the left end of this horizontal segment is an open circle while the endpoint on the right end is a filled circle because the taxable incomes range on $\$7,150 < I \le \$29,050$. Thus, we exclude the left endpoint and include the right endpoint.

The remaining segments are drawn in a similar manner.

The last piece assigns a rate of $R = 0.35$ to all taxable incomes strictly above \$319,100. Hence, the last piece is a horizontal ray, starting at $(\$319\,100, 0.35)$ and extending indefinitely to the right. Note that the left endpoint of this ray is an open circle because the rate $R = 0.35$ applies to taxable incomes $I > \$319,100$.

Let's talk a moment about the domain and range of the function $R$ defined by **equation (12)**. The graph of $R$ is depicted in **Figure 10**. If we project all points on the graph onto the horizontal axis, the entire axis will "lie in shadow." Thus, at first

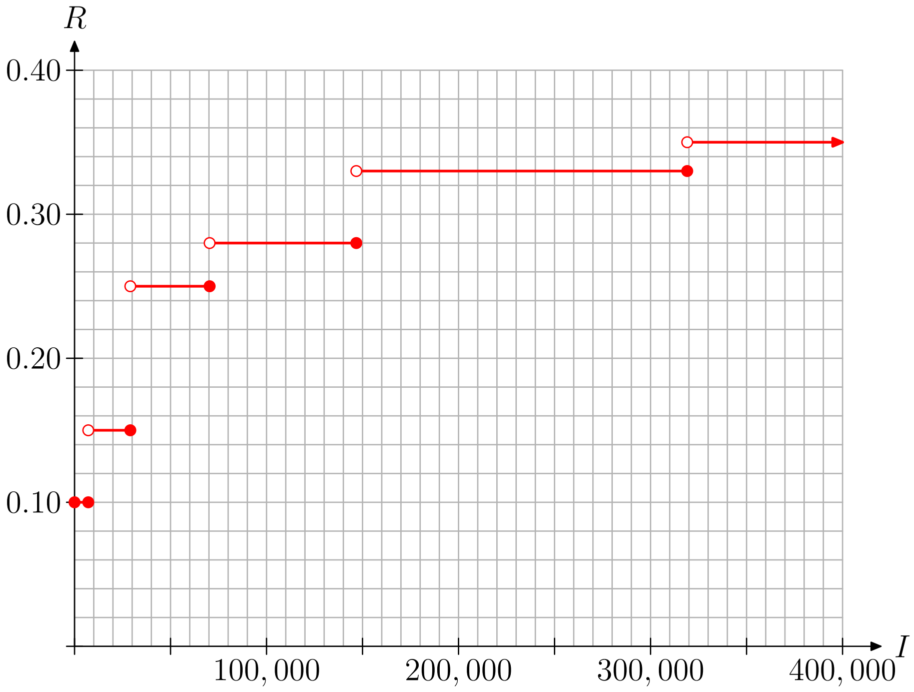


**Figure 10.** The graph of the tax rate $R$ versus taxable income $I$.

glance, one would state that the domain of $R$ is the set of all real numbers that are greater than or equal to zero.

However, remember that we chose to model a discrete situation with a continuum. Taxable income is always rounded to the nearest dollar on federal income tax forms. Therefore, the domain is actually all whole numbers greater than or equal to zero. In symbols,

$$\text{Domain} = \{I \in \mathbb{W} : \; I \geq 0\}.$$

To find the range of $R$, we would project all points on the graph of $R$ in **Figure 10** onto the vertical axis. The result would be that six points would be shaded on the vertical axis, one each at 0.10, 0.15, 0.25, 0.28, 0.33, and 0.35. Thus, the range is a finite discrete set, so it's best described by simply listing its members.

$$\text{Range} = \{0.10, 0.15, 0.25, 0.28, 0.33, 0.35\}$$

◆

## 4.1 Exercises

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**1.** Given the function defined by the rule $f(x) = 3$, evaluate $f(-3)$, $f(0)$ and $f(4)$, then sketch the graph of $f$.

**2.** Given the function defined by the rule $g(x) = 2$, evaluate $g(-2)$, $g(0)$ and $g(4)$, then draw the draw the graph of $g$.

**3.** Given the function defined by the rule $h(x) = -4$, evaluate $h(-2)$, $h(a)$, and $h(2x+3)$, then draw the graph of $h$.

**4.** Given the function defined by the rule $f(x) = -2$, evaluate $f(0)$, $f(b)$, and $f(5-4x)$, then draw the graph of $f$.

**5.** The speed of an automobile traveling on the highway is a function of time and is described by the constant function $v(t) = 30$, where $t$ is measured in hours and $v$ is measured in miles per hour. Draw the graph of $v$ versus $t$. Be sure to label each axis with the appropriate units. Shade the area under the graph of $v$ over the time interval $[0, 5]$ hours. What is the area under the graph of $v$ over this time interval and what does it represent?

**6.** The speed of a skateboarder as she travels down a slope is a function of time and is described by the constant function $v(t) = 8$, where $t$ is measured in seconds and $v$ is measured in feet per second. Draw the graph of $v$ versus $t$. Be sure to label each axis with the appropriate units. Shade the area under the graph of $v$ over the time interval $[0, 60]$ seconds. What is the area under the graph of $v$ over this time interval and what does it represent?

**7.** An unlicensed plumber charges 15 dollars for each hour of labor. Let's define this rate as a function of time by $r(t) = 15$, where $t$ is measured in hours and $r$ is measured in dollars per hour. Draw the graph of $r$ versus $t$. Be sure to label each axis with appropriate units. Shade the area under the graph of $r$ over the time interval $[0, 4]$ hours. What is area under the graph of $r$ over this time interval and what does it represent?

**8.** A carpenter charges a fixed rate for each hour of labor. Let's describe this rate as a function of time by $r(t) = 25$, where $t$ is measured in hours and $r$ is measured in dollars per hour. Draw the graph of $r$ versus $t$. Be sure to label each axis with appropriate units. Shade the area under the graph of $r$ over the time interval $[0, 5]$ hours. What is the area under the graph of $r$ over this time interval and what does it represent?

**9.** Given the function defined by the rule

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 2, & \text{if } x \geq 0, \end{cases}$$

evaluate $f(-2)$, $f(0)$, and $f(3)$, then draw the graph of $f$ on a sheet of graph paper. State the domain and range of $f$.

**10.** Given the function defined by the rule

$$f(x) = \begin{cases} 2, & \text{if } x < 0 \\ 0, & \text{if } x \geq 0, \end{cases}$$

evaluate $f(-2)$, $f(0)$, and $f(3)$, then draw the graph of $f$ on sheet of graph paper. State the domain and range of $f$.

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[2] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

**11.** Given the function defined by the rule

$$g(x) = \begin{cases} -3, & \text{if } x < -2, \\ 1, & \text{if } -2 \leq x < 2, \\ 3, & \text{if } x \geq 2, \end{cases}$$

evaluate $g(-3)$, $g(-2)$, and $g(5)$, then draw the graph of $g$ on a sheet of graph paper. State the domain and range of $g$.

**12.** Given the function defined by the rule

$$g(x) = \begin{cases} 4, & \text{if } x \leq -1, \\ 2, & \text{if } -1 < x \leq 2, \\ -3, & \text{if } x > 2, \end{cases}$$

evaluate $g(-1)$, $g(2)$, and $g(3)$, then draw the graph of $g$ on a sheet of graph paper. State the domain and range of $g$.

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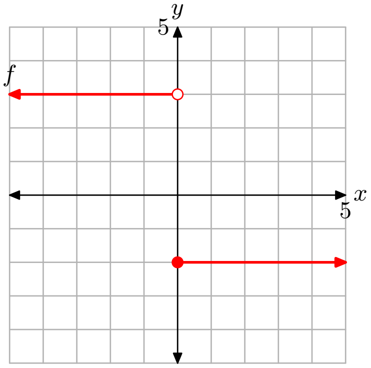
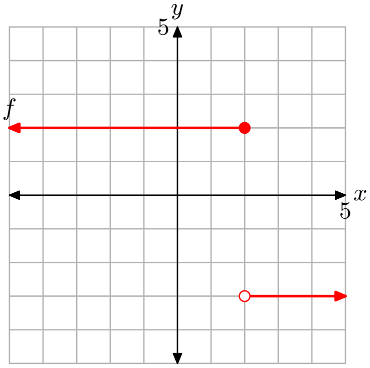
In **Exercises 13-16**, determine a piecewise definition of the function described by the graphs, then state the domain and range of the function.

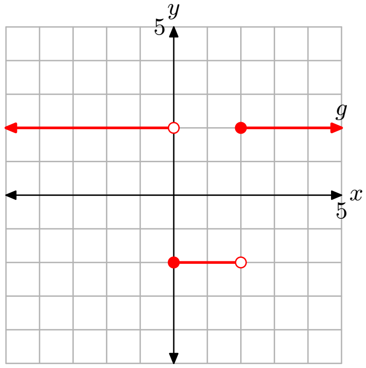
**13.**



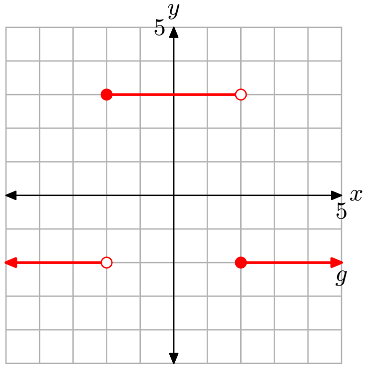
**14.**



**15.**



**16.**



**17.** Given the piecewise definition

$$f(x) = \begin{cases} -x-3, & \text{if } x < -3, \\ x+3, & \text{if } x \geq -3, \end{cases}$$

evaluate $f(-4)$ and $f(0)$, then draw the graph of $f$ on a sheet of graph paper. State the domain and range of the function.

**18.** Given the piecewise definition

$$f(x) = \begin{cases} -x+1, & \text{if } x < 1, \\ x-1, & \text{if } x \geq 1, \end{cases}$$

evaluate $f(-2)$ and $f(3)$, then draw the graph of $f$ on a sheet of graph paper. State the domain and range of the function.

**19.** Given the piecewise definition

$$g(x) = \begin{cases} -2x+3, & \text{if } x < 3/2, \\ 2x-3, & \text{if } x \geq 3/2, \end{cases}$$

evaluate $g(0)$ and $g(3)$, then draw the graph of $g$ on a sheet of graph paper. State the domain and range of the function.

**20.** Given the piecewise definition

$$g(x) = \begin{cases} -3x-4, & \text{if } x < -4/3, \\ 3x+4, & \text{if } x \geq -4/3, \end{cases}$$

evaluate $g(-2)$ and $g(3)$, then draw the graph of $g$ on a sheet of graph paper. State the domain and range of the function.

**21.** A battery supplies voltage to an electric circuit in the following manner. Before time $t = 0$ seconds, a switch is open, so the voltage supplied by the battery is zero volts. At time $t = 0$ seconds, the switch is closed and the battery begins to supply a constant 3 volts to the circuit. At time $t = 2$ seconds, the switch is opened again, and the voltage supplied by the battery drops immediately to zero volts. Sketch a graph of the voltage $v$ versus time $t$, label each axis with the appropriate units, then provide a piecewise definition of the voltage $v$ supplied by the battery as a function of time $t$.

**22.** Prior to time $t = 0$ minutes, a drum is empty. At time $t = 0$ minutes a hose is turned on and the water level in the drum begins to rise at a constant rate of 2 inches every minute. Let $h$ represent water level (in inches) at time $t$ (in minutes). Sketch the graph of $h$ versus $t$, label the axes with appropriate units, then provide a piecewise definition of $h$ as a function of $t$.

## 4.1 *Answers*

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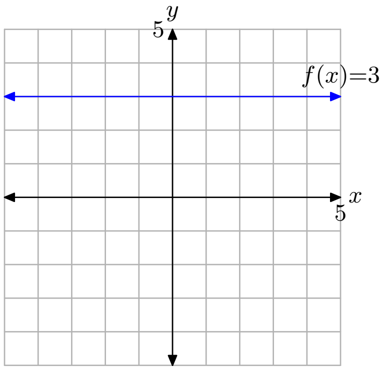
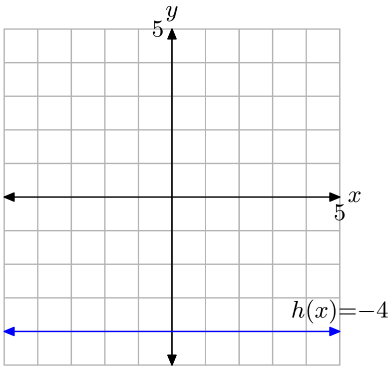
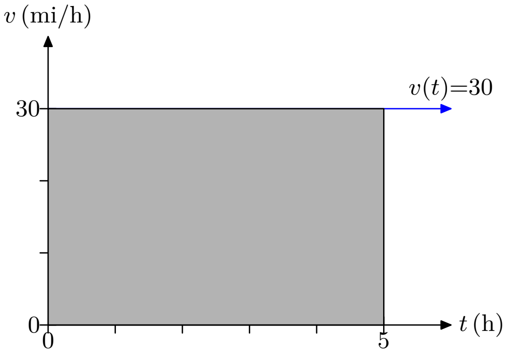
**1.** $f(-3) = 3$, $f(0) = 3$, and $f(4) = 3$.



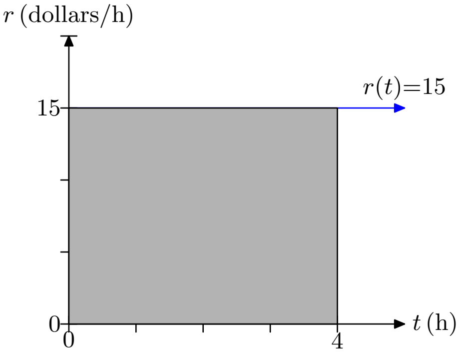
**3.** $h(-2) = -4$, $h(a) = -4$, and $h(2x + 3) = -4$.



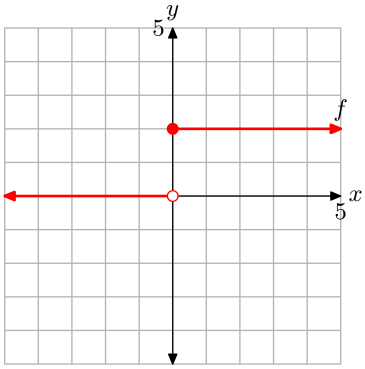
**5.** The area under the curve is 150 miles. This is the distance traveled by the car.



**7.** The area under the curve is 150 miles. This is the distance traveled by the car.

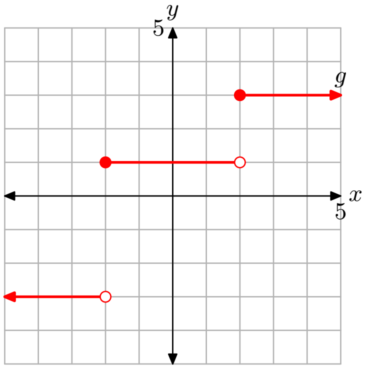


**9.** $f(-2) = 0$, $f(0) = 2$, and $f(3) = 2$.



The domain of $f$ is the set of all real numbers. The range of $f$ is $\{0, 2\}$.

**11.** $g(-3) = -3$, $g(-2) = 1$, and $g(5) = 3$.



The domain of $g$ is all real numbers. The range of $g$ is $\{-3, 1, 3\}$.

**13.**

$$f(x) = \begin{cases} 3, & \text{if } x < 0, \\ -2, & \text{if } x \geq 0. \end{cases}$$

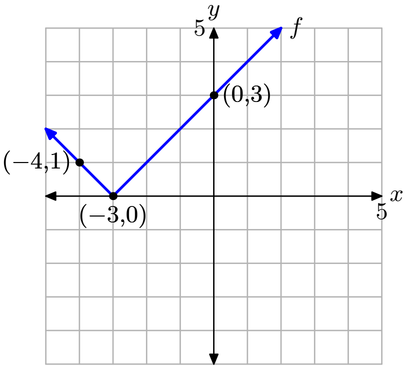
Domain of $f$ is the set of all real numbers. The range of $f$ is $\{-2, 3\}$.

**15.**

$$g(x) = \begin{cases} 2, & \text{if } x < 0, \\ -2, & \text{if } 0 \leq x < 2, \\ 2, & \text{if } x \geq 2. \end{cases}$$

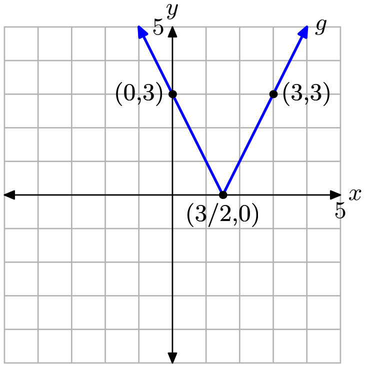
The domain of $f$ is the set of all real numbers. The range of $f$ is $\{-2, 2\}$.

**17.** $f(-4) = 1$ and $f(0) = 3$.



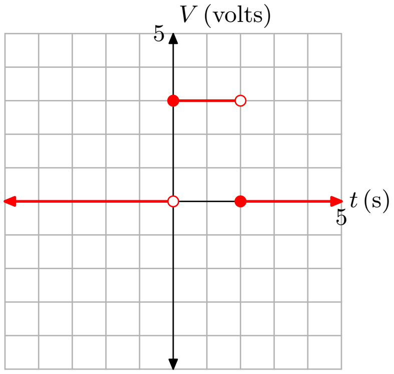
The domain of $f$ is the set of all real numbers. The range of $f$ is $\{y : y \geq 0\}$.

**19.** $g(-2) = 7$ and $g(2) = 1$.



The domain of $g$ is the set of all real numbers. The range of $g$ is $\{y : y \geq 0\}$.

**21.** The graph follows.



The piecewise definition is

$$v(t) = \begin{cases} 0, & \text{if } t < 0, \\ 3, & \text{if } 0 \leq t < 2, \\ 0, & \text{if } t \geq 2. \end{cases}$$

# 4.2 Absolute Value

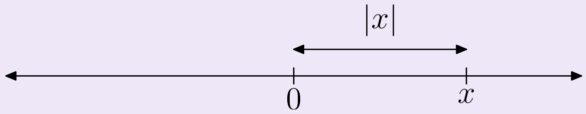
Now that we have the fundamentals of piecewise-defined functions in place, we are ready to introduce the absolute value function. First, let's state a trivial reminder of what it means to take the absolute value of a real number.

In a sense, the absolute value of a number is a measure of its magnitude, sans (without) its sign. Thus,

$$|7| = 7 \qquad \text{and} \qquad |-7] = 7. \tag{1}$$

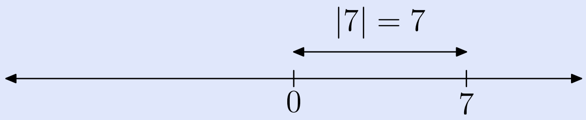
Here is the formal definition of the absolute value of a real number.

**Definition 2.** *To find the absolute value of any real number, first locate the number on the real line.*

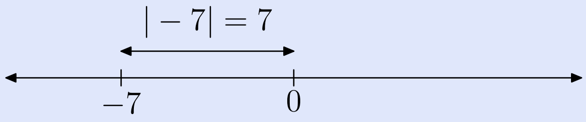


*The absolute value of the number is defined as its* **distance** *from the origin.*

For example, to find the absolute value of 7, locate 7 on the real line and then find its distance from the origin.



To find the absolute value of $-7$, locate $-7$ on the real line and then find its distance from the origin.



Some like to say that taking the absolute value "produces a number that is always positive." However, this ignores an important exception, that is,

$$|0| = 0. \tag{3}$$

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[3] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

Thus, the correct statement is "the absolute value of any real number is either positive or it is zero," i.e., the absolute value of a real number is "not negative."[4] Instead of using the phrase "not negative," mathematicians prefer the word "nonnegative." When we take the absolute value of a number, the result is always *nonnegative*; that is, the result is either positive or zero. In symbols,

$$|x| \geq 0 \text{ for all real numbers } x.$$

This makes perfect sense in light of **Definition 2**. Distance is always nonnegative.

However, the discussion above is not of sufficient depth to handle more sophisticated problems involving absolute value.

## *A Piecewise Definition of Absolute Value*

Because absolute value is intimately connected with distance, mathematicians and scientists find it an invaluable tool for measurement and error analysis. However, we will need a formulaic definition of the absolute value if we want to use this tool in a meaningful way. We need to develop a piecewise definition of the absolute value function, one that will define the absolute value for any arbitrary real number $x$.

We begin with a few observations. Remember, the absolute value of a number is always nonnegative (positive or zero).

1. If a number is negative, negating that number will make it positive.

   $$|-5| = -(-5) = 5, \text{ and similarly, } |-12| = -(-12) = 12.$$

   Thus, if $x < 0$ (if $x$ is negative), then $|x| = -x$.
2. If $x = 0$, then $|x| = 0$.
3. If a number is positive, taking the absolute value of that number will not change a thing.

   $$|5| = 5, \text{ and similarly, } |12| = 12.$$

   Thus, if $x > 0$ (if $x$ is positive), then $|x| = x$.

We can summarize these three cases with a piecewise definition .

$$|x| = \begin{cases} -x, & \text{if } x < 0, \\ 0, & \text{if } x = 0., \\ x, & \text{if } x > 0. \end{cases} \tag{4}$$

It is the first line in our piecewise definition (**4**) that usually leaves students scratching their heads. They might say "I thought absolute value makes a number positive (or zero), yet you have $|x| = -x$; that is, you have the absolute value of $x$ equal to a *negative* $x$." Try as they might, this seems contradictory. Does it seem so to you?

---

[4] A real number is either positive, negative, or zero. If we say that the real number is "not negative," then that implies that it is either "positive" or "zero."

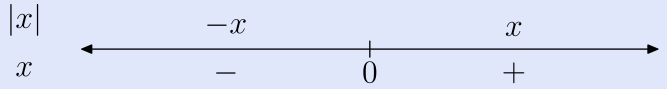
However, there is no contradiction. If $x < 0$, that is, if $x$ is a negative number, then $-x$ is a **positive** number, and our intuitive notion of absolute value is not dissimilar to that of our piecewise definition (4). For example, if $x = -8$, then $-x = 8$, and even though we say "negative $x$," in this case $-x$ is a positive number.

If this still has you running confused, consider the simple fact that $x$ and $-x$ must have "opposite signs." If one is positive, the other is negative, and vice versa. Consequently,

- if $x$ is positive, then $-x$ is negative, but
- if $x$ is negative, then $-x$ is positive.

Let's summarize what we've learned thus far.

**Summarizing the Definition on a Number Line.** We like to use a number line to help summarize the definition of the absolute value of $x$.



Some remarks are in order for this summary on the number line.

- We first draw the real line then mark the "critical value" for the expression inside the absolute value bars on the number line. The number zero is a critical value for the expression $x$, because $x$ changes sign as you move from one side of zero to the other.
- To the left of zero, $x$ is a negative number. We indicate this with the minus sign below the number line. To the right of zero, $x$ is a positive number, indicated with a plus sign below the number line.
- Above the number line, we simplify the expression $|x|$. To the left of zero, $x$ is a negative number (look below the line), so $|x| = -x$. Note how the result $-x$ is placed above the line to the left of zero. Similarly, to the right of zero, $x$ is a positive number (look below the line), so $|x| = x$. Note how the result $x$ is placed above the line to the right of zero.

In the piecewise definition of $|x|$ in (4), note that we have three distinct pieces, one for each case discussed above. However, because $|0| = 0$, we can include this case with the piece $|x| = x$, if we adjust the condition to include zero.

**Definition 5.**

$$|x| = \begin{cases} -x, & \text{if } x < 0, \\ x, & \text{if } x \geq 0. \end{cases} \tag{6}$$

Note that this piecewise definition agrees with our discussion to date.

1. In the first line of **equation (6)**, if $x$ is a negative number (i.e., if $x < 0$), then the absolute value must change $x$ to a positive number by negating. That is, $|x| = -x$.
2. In the second line of **equation (6)**, if $x$ is positive or zero (i.e., if $x \geq 0$), then there's nothing to do except remove the absolute value bars. That is, $|x| = x$.

Because $|0| = -0$, we could just as well include the case for zero on the left, defining the absolute value with

$$|x| = \begin{cases} -x, & \text{if } x \leq 0, \\ x, & \text{if } x > 0. \end{cases}$$

However, in this text we will always include the critical value on the right, as shown in **Definition 5**.

## *Constructing Piecewise Definitions*

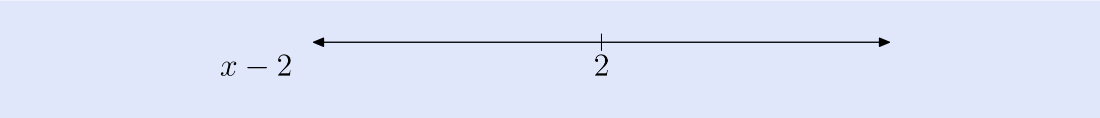
Let's see if we can determine piecewise definitions for other expressions involving absolute value.

▶ **Example 7.** *Determine a piecewise definition for* $|x - 2|$.

First, set the expression inside the absolute value bars equal to zero and solve for $x$.

$$\begin{aligned} x - 2 &= 0 \\ x &= 2 \end{aligned}$$

Note that $x - 2 = 0$ at $x = 2$. This is the "critical value" for this expression. Draw a real line and mark this critical value of $x$ on the line. Place the expression $x - 2$ below the line at its left end.

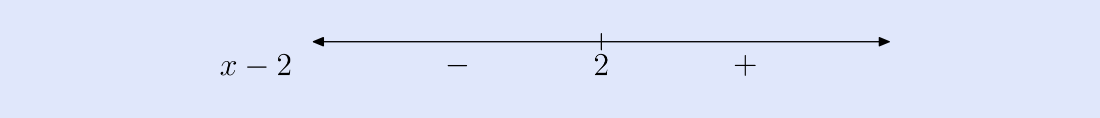


Next, determine the sign of $x - 2$ for values of $x$ on each side of 2. This is easily done by "testing" a point on each side of 2 in the expression $x - 2$.

- Take $x = 1$, which lies to the left of the critical value 2 on our number line. Substitute this value of $x$ in the expression $x - 2$, obtaining

  $$x - 2 = 1 - 2 = -1,$$

  which is negative. Indeed, regardless of which $x$-value you pick to the left of 2, when inserted into the expression $x - 2$, you will get a negative result (you should check this for other values of $x$ to the left of 2). We indicate that the expression $x - 2$ is negative for values of $x$ to the left of 2 by placing a minus ($-$) sign below the number line to the left of 2.



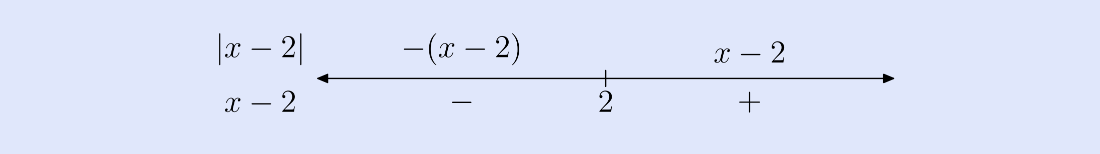
- Next, pick $x = 3$, which lies to the right of the critical value 2 on the number line. Substitute this value of $x$ into the expression $x - 2$, obtaining

  $$x - 2 = 3 - 2 = 1,$$

  which is positive. Indeed, regardless of which $x$-value you pick to the right of 2, when inserted into the expression $x - 2$, you will get a positive result (you should check this for other values of $x$ to the right of 2). We indicate that the expression $x - 2$ is positive for values of $x$ to the right of 2 by placing a plus (+) sign below the number line to the right of 2 (see the number line above).

The next step is to remove the absolute value bars from the expression $|x-2|$, depending on the sign of $x - 2$.

- To the left of 2, the expression $x - 2$ is negative (note the minus sign (−) below the number line), so $|x - 2| = -(x - 2)$. That is, we have to negate $x - 2$ to make it positive. This is indicated by placing $-(x - 2)$ above the line to the left of 2.



- To the right of 2, the expression $x - 2$ is positive (note the plus sign (+) below the line), so $|x - 2| = x - 2$. That is, we simply remove the absolute value bars because the quantity inside is already positive. This is indicated by placing $x - 2$ above the line to the right of 2 (see the number line above).

We can use this last number line summary to construct a piecewise definition of the expression $|x - 2|$.

$$|x-2| = \begin{cases} -(x-2), & \text{if } x < 2, \\ x-2, & \text{if } x \geq 2 \end{cases} = \begin{cases} -x+2, & \text{if } x < 2, \\ x-2, & \text{if } x \geq 2. \end{cases}$$

Our number line and piecewise definition agree: $|x - 2| = -(x - 2)$ to the left of 2 and $|x - 2| = x - 2$ to the right of 2. Further, note how we've included the critical value of 2 "on the right" in our piecewise definition.

◆

Let's summarize the method we followed to construct the piecewise function above.

**Constructing a Piecewise Definition for Absolute Value**. When presented with the absolute value of an algebraic expression, perform the following steps to remove the absolute value bars and construct an equivalent piecewise definition.

1. Take the expression that is inside the absolute value bars, and set that expression equal to zero. Then solve for $x$. This value of $x$ is called a "critical value." (*Note: The expression inside the absolute value bars could have more than one critical value. We will not encounter such problems in this text.*)
2. Place your critical value on a number line.
3. Place the expression inside the absolute value bars below the number line at the left end.
4. Test the sign of the expression inside the absolute value bars by inserting a value of $x$ from each side of the critical value and marking the result with a plus (+) or minus (−) sign below the number line.
5. Place the original expression, the one including the absolute value bars, above the number line at the left end.
6. Use the sign of the expression inside the absolute value bars (indicated by the plus and minus signs below the number line) to remove the absolute value bars, placing the results above the number line on each side of the critical value.
7. Construct a piecewise definition that mimics the results on the number line.

Let's apply this technique to another example.

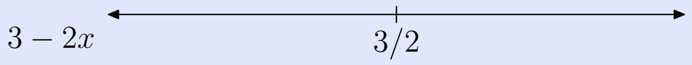
► **Example 8.** *Determine a piecewise definition for* $|3 - 2x|$.

**Step 1**: First set the expression inside the absolute value bars equal to zero and solve for $x$.

$$\begin{aligned} 3 - 2x &= 0 \\ x &= 3/2 \end{aligned}$$

Note that $3 - 2x = 0$ at $x = 3/2$. This is the "critical value" for this expression.

**Steps 2 and 3**: Draw a number line and mark this critical value on the line. The next step requires that we place the expression inside the absolute value bars, namely $3 - 2x$, underneath the line at its left end.

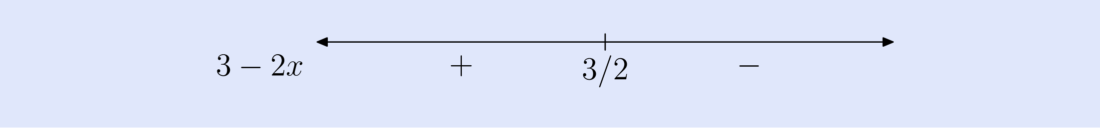


**Step 4**: Next, determine the sign of $3 - 2x$ for values of $x$ on each side of $3/2$. This is easily done by "testing" a point on each side of $3/2$ in the expression $3 - 2x$.

- Take $x = 1$, which lies to the left of $3/2$. Substitute this value of $x$ into the expression $3 - 2x$, obtaining

$$3 - 2x = 3 - 2(1) = 1,$$

which is positive. Indicate this result by placing a plus sign (+) below the number line to the left of 3/2.



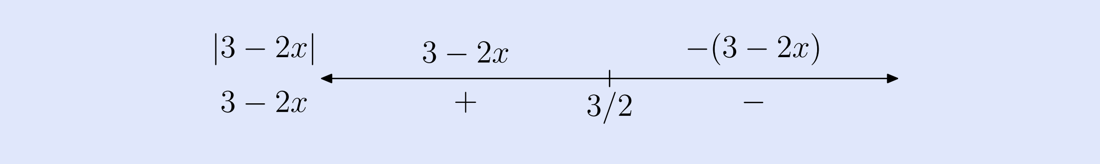
- Next, pick $x = 2$, which lies to the right of 3/2. Substitute this value of $x$ into the expression $3 - 2x$, obtaining

$$3 - 2x = 3 - 2(2) = -1,$$

which is negative. Indicate this result by placing a negative sign (−) below the line to the right of 3/2 (see the number line above).

**Steps 5 and 6**: Place the original expression, namely $|3 - 2x|$, above the number line at the left end. The next step is to remove the absolute value bars from the expression $|3 - 2x|$.

- To the left of 3/2, the expression $3 - 2x$ is positive (note the plus sign (+) below the number line), so $|3 - 2x| = 3 - 2x$. Indicate this result by placing the expression $3 - 2x$ above the number line to the left of 3/2.



- To the right of 3/2, the expression $3 - 2x$ is negative (note the minus sign (−) below the numberline), so $|3 - 2x| = -(3 - 2x)$. That is, we have to negate $3 - 2x$ to make it positive. This is indicated by placing the expression $-(3 - 2x)$ above the line to the right of 3/2 (see the number line above).

**Step 7**: We can use this last number line summary to write a piecewise definition for the expression $|3 - 2x|$.

$$|3 - 2x| = \begin{cases} 3 - 2x, & \text{if } x < 3/2. \\ -(3 - 2x), & \text{if } x \geq 3/2 \end{cases} = \begin{cases} 3 - 2x, & \text{if } x < 3/2, \\ -3 + 2x, & \text{if } x \geq 3/2. \end{cases}$$

Again, note how we've included the critical value of 3/2 "on the right."

◆

## *Drawing the Graph of an Absolute Value Function*

Now that we know how to construct a piecewise definition for an expression containing absolute value bars, we can use what we learned in the previous section to draw the graph.

▶ **Example 9.** *Sketch the graph of the function* $f(x) = |3 - 2x|$.

In **Example 8**, we constructed the following piecewise definition.

$$f(x) = |3 - 2x| = \begin{cases} 3 - 2x, & \text{if } x < 3/2 \\ -3 + 2x, & \text{if } x \geq 3/2 \end{cases} \tag{10}$$

We now sketch each piece of this function.

- If $x < 3/2$, then $f(x) = 3 - 2x$ (see **equation (10)**). This is a ray, starting at $x = 3/2$ and extending to the left. At $x = 3/2$,

  $$f(3/2) = 3 - 2(3/2) = 3 - 3 = 0.$$

  Thus, the endpoint of the ray is located at $(3/2, 0)$.

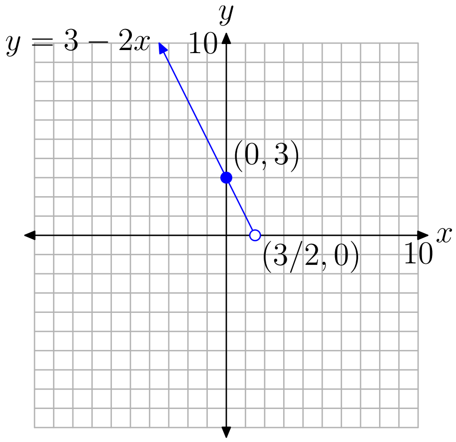
  Next, pick a value of $x$ that lies to the left of $3/2$. At $x = 0$,

  $$f(0) = 3 - 2(0) = 3 - 0 = 3.$$

  Thus, a second point on the ray is $(0, 3)$.

  A table containing the two evaluated points and a sketch of the accompanying ray are shown in **Figure 1**. Because $f(x) = 3 - 2x$ only if $x$ is strictly less than $3/2$, the point at $(3/2, 0)$ is unfilled.

| $x$ | $f(x) = 3 - 2x$ | $(x, f(x))$ |
|---|---|---|
| $3/2$ | $0$ | $(3/2, 0)$ |
| $0$ | $3$ | $(0, 3)$ |



(a) (b)

**Figure 1.** $f(x) = 3 - 2x$ when $x < 3/2$.

- If $x \geq 3/2$, then $f(x) = -3 + 2x$ (see **equation (10)**). This is a ray, starting at $x = 3/2$ and extending to the right. At $x = 3/2$,

  $$f(3/2) = -3 + 2(3/2) = -3 + 3 = 0.$$

  Thus, the endpoint of the ray is located at $(3/2, 0)$.

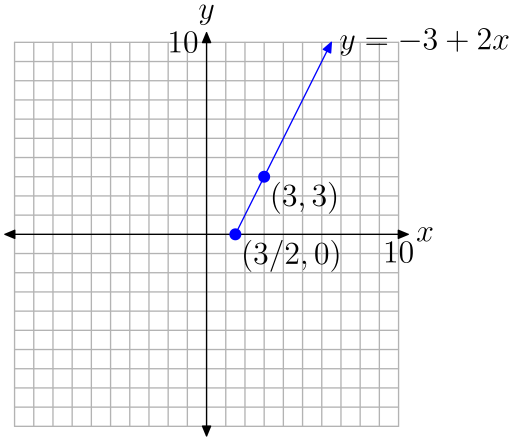
  Next, pick a value of $x$ that lies to the right of $3/2$. At $x = 3$,

  $$f(3) = -3 + 2(3) = -3 + 6 = 3.$$

Thus, a second point on the ray is $(3, 3)$.

A table containing the two evaluated points and a sketch of the accompanying ray are shown in **Figure 2**. Because $f(x) = -3 + 2x$ for all values of $x$ that are greater than **or equal** to $3/2$, the point at $(3/2, 0)$ is filled in this plot.

| $x$ | $f(x) = -3 + 2x$ | $(x, f(x))$ |
|---|---|---|
| $3/2$ | $0$ | $(3/2, 0)$ |
| $3$ | $3$ | $(-3, 3)$ |



(a) (b)

**Figure 2.** $f(x) = -3 + 2x$ when $x \geq 3/2$.

- To sketch the graph of $f(x) = |3 - 2x|$, we need only combine the two pieces from **Figures 1** and **2**. The result is shown in **Figure 3**.

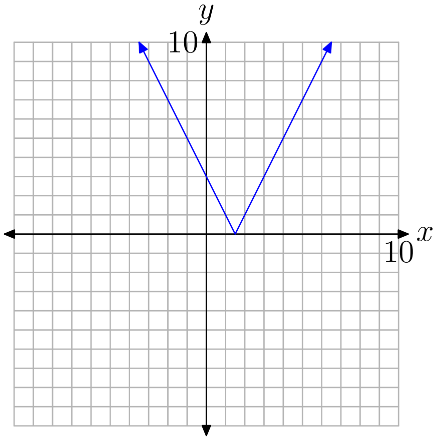


**Figure 3.** The graph of $f(x) = |3 - 2x|$.

Note the "V-shape" of the graph. We will refer to the point at the tip of the "V" as the **vertex** of the absolute value function.

◆

In **Figure 3**, the equation of the left-hand branch of the "V" is $y = 3 - 2x$. An alternate approach to drawing this branch is to note that its graph is contained in the graph of the full line $y = 3 - 2x$, which has slope $-2$ and $y$-intercept at $(0, 3)$. Thus, one could draw the full line using the slope and $y$-intercept, then erase that part of the line that lies to the right of $x = 3/2$. A similar strategy would work for the right-hand branch of $y = |3 - 2x|$.

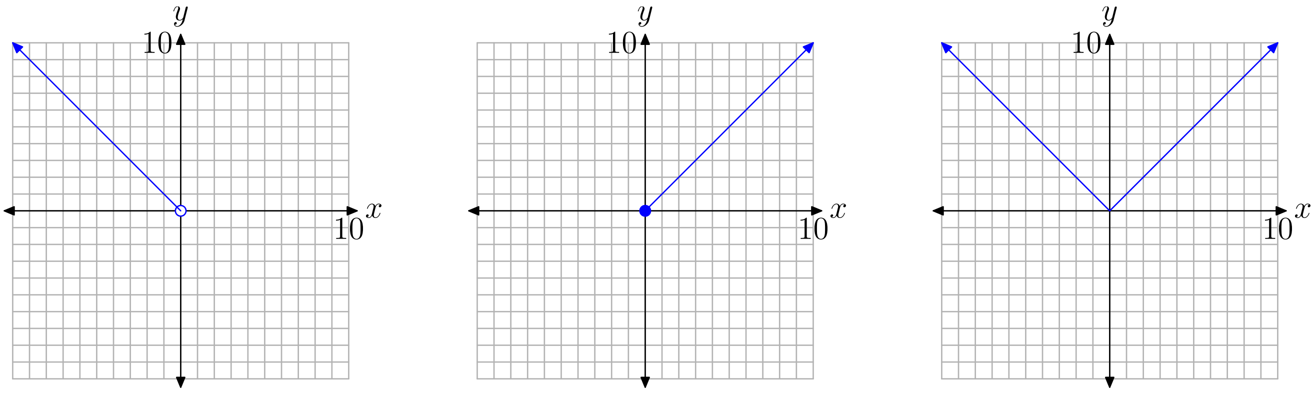
## *Using Transformations*

Consider again the basic definition of the absolute value of $x$.

$$f(x) = |x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases} \tag{11}$$

Some basic observations are:

- If $x < 0$, then $f(x) = -x$. This ray starts at the origin and extends to the left with slope $-1$. Its graph is pictured in **Figure 4**(a).
- If $x \geq 0$, then $f(x) = x$. This ray starts at the origin and extends to the right with slope 1. Its graph is pictured in **Figure 4**(b).
- We combine the graphs in **Figures 4**(a) and **4**(b) to produce the graph of $f(x) = |x|$ in **Figure 4**(c).



(a) $f(x) = -x$, if $x < 0$. (b) $f(x) = x$, if $x \geq 0$. (c) $f(x) = |x|$.

**Figure 4.** Combine left and right branches to produce the basic graph of $f(x) = |x|$.

You should commit the graph of $f(x) = |x|$ to memory. Things to note:

- The graph of $f(x) = |x|$ is "V-shaped."
- The vertex of the graph is at the point $(0, 0)$.
- The left-hand branch has equation $y = -x$ and slope $-1$.
- The right-hand branch has equation $y = x$ and slope 1.
- Each branch of the graph of $f(x) = |x|$ forms a perfect 45° angle with the $x$-axis.

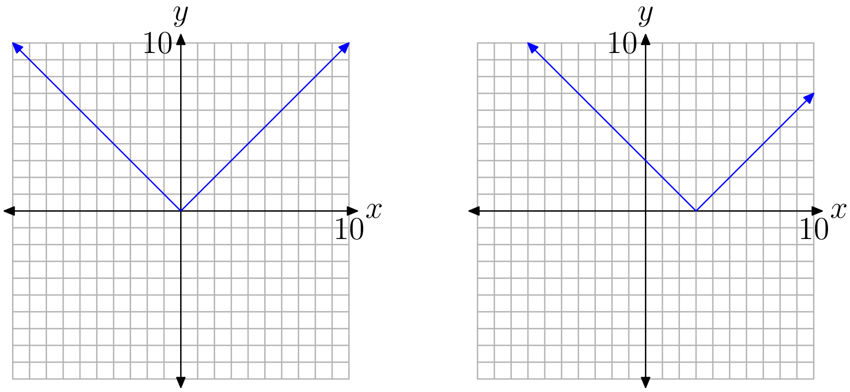
Now that we know how to draw the graph of $f(x) = |x|$, we can use the transformations we learned in Chapter 2 (sections 5 and 6) to sketch a number of simple graphs involving absolute value.

▶ **Example 12.** *Sketch the graph of $f(x) = |x - 3|$.*

First, sketch the graph of $y = f(x) = |x|$, as shown in **Figure 5**(a). Note that if $f(x) = |x|$, then

$$y = f(x - 3) = |x - 3|.$$

To sketch the graph of $y = f(x - 3) = |x - 3|$, shift the graph of $y = f(x) = |x|$ three units to the right, producing the result shown in **Figure 5**(b).



(a) $y = f(x) = |x|$. (b) $y = f(x - 3) = |x - 3|$.

**Figure 5.** To draw the graph of $y = |x - 3|$, shift the graph of $y = |x|$ three units to the right.

We can check this result using the graphing calculator. Load the function $f(x) = |x - 3|$ into `Y1` in the `Y=` menu on your graphing calculator as shown in **Figure 6**(a). Push the `MATH` button, right-arrow to the `NUM` menu, then select `1:abs(` (see **Figure 6**(b)) to enter the absolute value in `Y1`. Push the `ZOOM` button, then select `6:ZStandard` to produce the image shown in **Figure 6**(c).

(a) (b) (c)

**Figure 6.** Using the graphing calculator to draw the graph of $f(x) = |x - 3|$.

◆

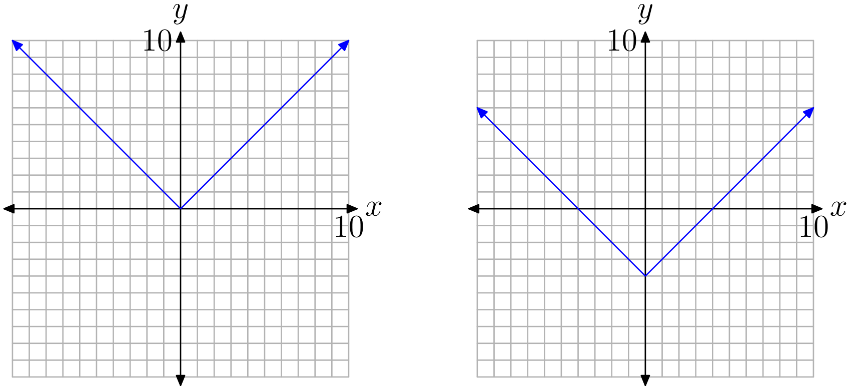
Let's look at another simple example.

▶ **Example 13.** *Sketch the graph of* $f(x) = |x| - 4$.

First, sketch the graph of $y = f(x) = |x|$, as shown in **Figure 7**(a). Note that if $f(x) = |x|$, then

$$y = f(x) - 4 = |x| - 4.$$

To sketch the graph of $y = f(x) - 4 = |x| - 4$, shift the graph of $y = f(x) = |x|$ downward 4 units, producing the result shown in **Figure 5**(b).



(a) $y = f(x) = |x|$. (b) $y = f(x) - 4 = |x| - 4$.

**Figure 7.** To draw the graph of $y = |x| - 4$, shift the graph of $y = |x|$ downward 4 units.

◇

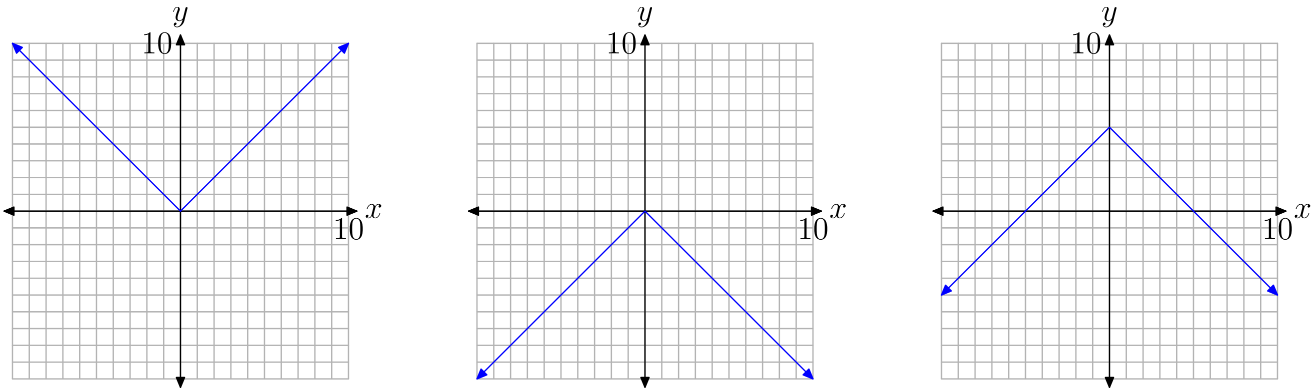
Let's look at one final example.

▶ **Example 14.** *Sketch the graph of* $f(x) = -|x| + 5$. *State the domain and range of this function.*

- First, sketch the graph of $y = f(x) = |x|$, as shown in **Figure 8**(a).
- Next, sketch the graph of $y = -f(x) = -|x|$, which is a reflection of the graph of $y = f(x) = |x|$ across the $x$-axis and is pictured in **Figure 8**(b).
- Finally, we will want to sketch the graph of $y = -f(x) + 5 = -|x| + 5$. To do this, we shift the graph of $y = -f(x) = -|x|$ in **Figure 8**(b) upward 5 units to produce the result in **Figure 8**(c).

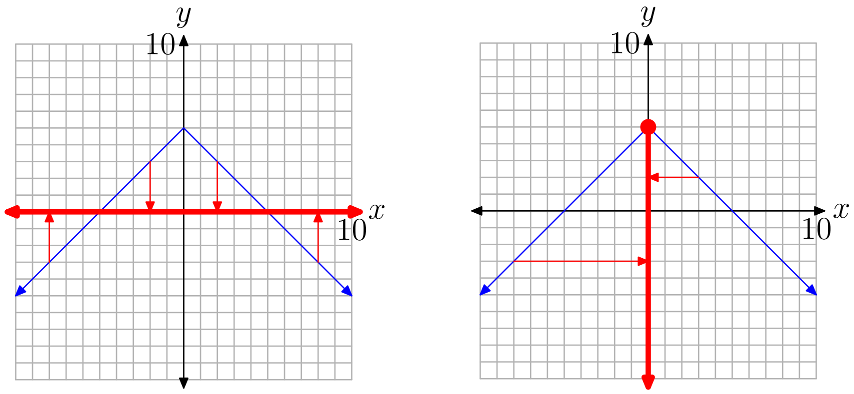
To find the domain of $f(x) = -|x| + 5$, project all points on the graph onto the $x$-axis, as shown in **Figure 9**(a). Thus, the domain of $f$ is $(-\infty, \infty)$. To find the range, project all points on the graph onto the $y$-axis, as shown in **Figure 9**(b). Thus, the range is $(-\infty, 5]$.

◇



(a) $y = f(x) = |x|$. (b) $y = -f(x) = -|x|$. (b) $y = -f(x) + 5 = -|x| + 5$.

**Figure 8.** To draw the graph of $y = -|x| + 5$, first reflect the graph of $y = |x|$ across the $x$-axis to produce the graph of $y = -|x|$, then shift this result up 5 units to produce the graph of $y = -|x| + 5$.



(a) Domain $= (-\infty, \infty)$. (b) Range $= (-\infty, 5]$.

**Figure 9.** Projecting onto the axes to find the domain and range.

## 4.2 Exercises

For each of the functions in **Exercises 1-8**, as in Examples 7 and 8 in the narrative, mark the “critical value” on a number line, then mark the sign of the expression inside of the absolute value bars below the number line. Above the number line, remove the absolute value bars according to the sign of the expression you marked below the number line. Once your number line summary is finished, create a piecewise definition for the given absolute value function.

**1.** $f(x) = |x + 1|$

**2.** $f(x) = |x - 4|$

**3.** $g(x) = |4 - 5x|$

**4.** $g(x) = |3 - 2x|$

**5.** $h(x) = | - x - 5|$

**6.** $h(x) = | - x - 3|$

**7.** $f(x) = x + |x|$

**8.** $f(x) = \dfrac{|x|}{x}$

For each of the functions in **Exercises 9-16**, perform each of the following tasks.

i. Create a piecewise definition for the given function, using the technique in **Exercises 1-8** and Examples 7 and 8 in the narrative.
ii. Following the lead in Example 9 in the narrative, use your piecewise definition to sketch the graph of the given function on a sheet of graph paper. Please place each exercise on its own coordinate system.

**9.** $f(x) = |x - 1|$

**10.** $f(x) = |x + 2|$

**11.** $g(x) = |2x - 1|$

**12.** $g(x) = |5 - 2x|$

**13.** $h(x) = |1 - 3x|$

**14.** $h(x) = |2x + 1|$

**15.** $f(x) = x - |x|$

**16.** $f(x) = x + |x - 1|$

**17.** Use a graphing calculator to draw the graphs of $y = |x|$, $y = 2|x|$, $y = 3|x|$, and $y = 4|x|$ on the same viewing window. In your own words, explain what you learned in this exercise.

**18.** Use a graphing calculator to draw the graphs of $y = |x|$, $y = (1/2)|x|$, $y = (1/3)|x|$, and $y = (1/4)|x|$ on the same viewing window. In your own words, explain what you learned in this exercise.

**19.** Use a graphing calculator to draw the graphs of $y = |x|$, $y = |x - 2|$, $y = |x-4|$, and $y = |x-6|$ on the same viewing window. In your own words, explain what you learned in this exercise.

**20.** Use a graphing calculator to draw the graphs of $y = |x|$, $y = |x + 2|$, $y = |x+4|$, and $y = |x+6|$ on the same viewing window. In your own words, explain what you learned in this exercise.

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[5] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

In **Exercises 21-36**, perform each of the following tasks. Feel free to check your work with your graphing calculator, but you should be able to do all of the work by hand.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Create an accurate plot of the function $y = |x|$ on your coordinate system and label this graph with its equation.
ii. Use the technique of Examples 12, 13, and 14 in the narrative to help select the appropriate geometric transformations to transform the equation $y = |x|$ into the form of the function given in the exercise. On the same coordinate system, use a different colored pencil or pen to draw the graph of the function resulting from your applied transformation. Label the resulting graph with its equation.
iii. Use interval notation to describe the domain and range of the given function.

**21.** $f(x) = |-x|$

**22.** $f(x) = -|x|$

**23.** $f(x) = (1/2)|x|$

**24.** $f(x) = -2|x|$

**25.** $f(x) = |x + 4|$

**26.** $f(x) = |x - 2|$

**27.** $f(x) = |x| + 2$

**28.** $f(x) = |x| - 3$

**29.** $f(x) = |x + 3| + 2$

**30.** $f(x) = |x - 3| - 4$

**31.** $f(x) = -|x - 2|$

**32.** $f(x) = -|x| - 2$

**33.** $f(x) = -|x| + 4$

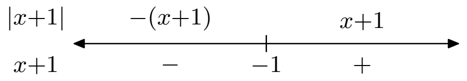
**34.** $f(x) = -|x + 4|$

**35.** $f(x) = -|x - 1| + 5$

**36.** $f(x) = -|x + 5| + 2$

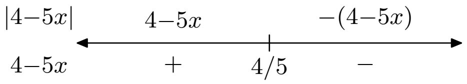
## 4.2 Answers

**1.**



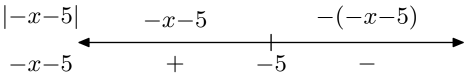
$$f(x) = \begin{cases} -x-1, & \text{if } x < -1, \\ x+1, & \text{if } x \geq -1. \end{cases}$$

**3.**



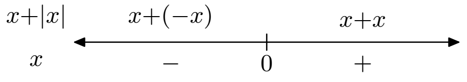
$$g(x) = \begin{cases} 4-5x, & \text{if } x < 4/5, \\ -4+5x, & \text{if } x \geq 4/5. \end{cases}$$

**5.**



$$h(x) = \begin{cases} -x-5, & \text{if } x < -5, \\ x+5, & \text{if } x \geq -5. \end{cases}$$

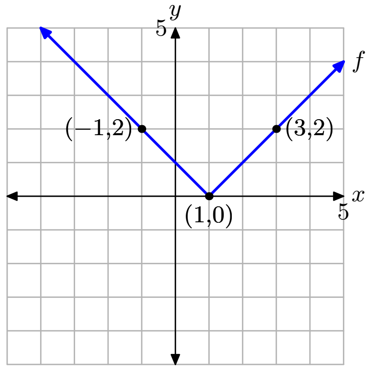
**7.**



$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ 2x, & \text{if } x \geq 0. \end{cases}$$

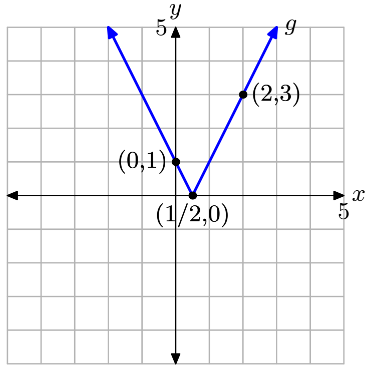
**9.**

$$f(x) = \begin{cases} -x+1, & \text{if } x < 1, \\ x-1, & \text{if } x \geq 1. \end{cases}$$



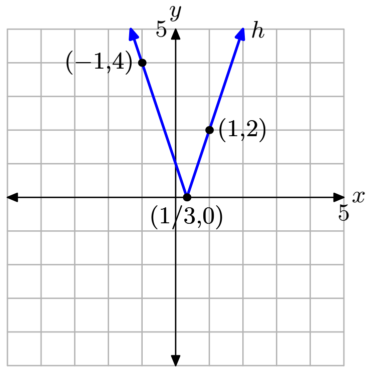
**11.**

$$g(x) = \begin{cases} -2x+1, & \text{if } x < 1/2, \\ 2x-1, & \text{if } x \geq 1/2. \end{cases}$$



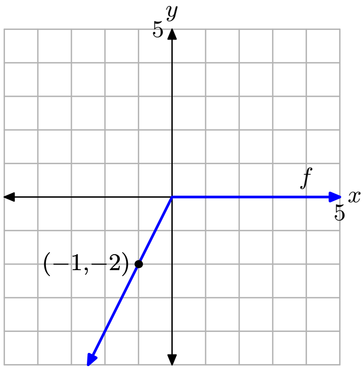
**13.**

$$h(x) = \begin{cases} 1-3x, & \text{if } x < 1/3, \\ -1+3x, & \text{if } x \geq 1/3. \end{cases}$$



**15.**

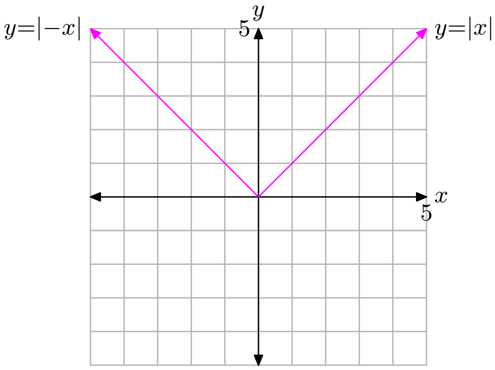
$$f(x)=\begin{cases}2x, & \text{if } x<0,\\ 0, & \text{if } x\ge 0.\end{cases}$$



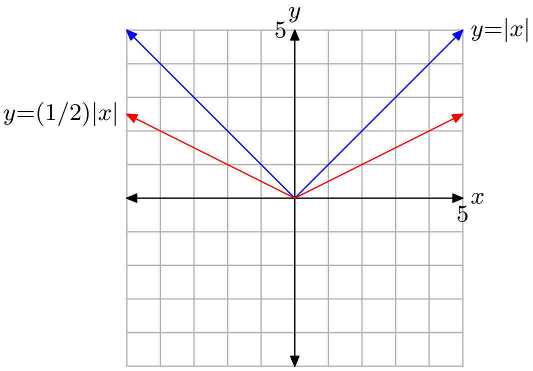
**17.** Multiplying by a factor of $a>1$, as in $y=a|x|$, stretches the graph of $y=|x|$ vertically by a factor of $a$. The higher the value of $a$, the more it stretches vertically.

**19.** Subtracting a positive value of $a$, as in $y=|x-a|$, shifts the graph $a$ units to the right.

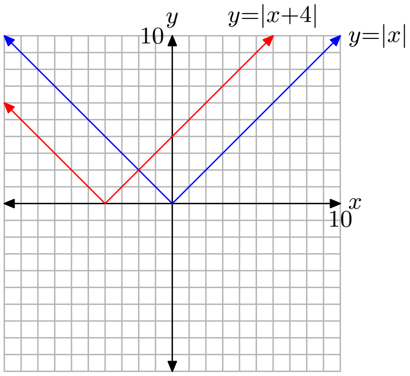
**21.** The graphs of $y=|x|$ and $y=|-x|$ coincide. The domain is $(-\infty,\infty)$ and the range is $[0,\infty)$.



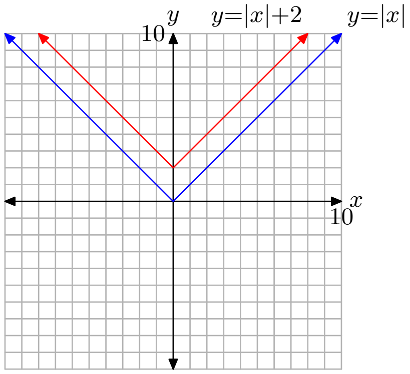
**23.** The domain is $(-\infty,\infty)$ and the range is $[0,\infty)$.



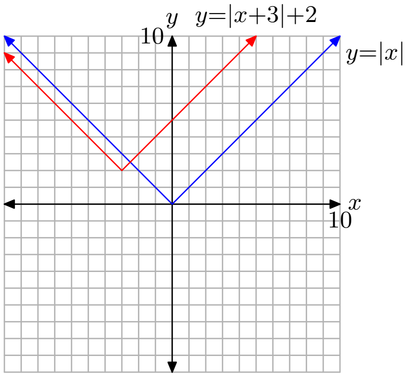
**25.** The domain is $(-\infty,\infty)$ and the range is $[0,\infty)$.



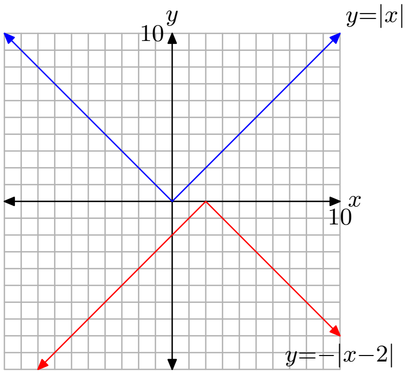
**27.** The domain is $(-\infty,\infty)$ and the range is $[2,\infty)$.



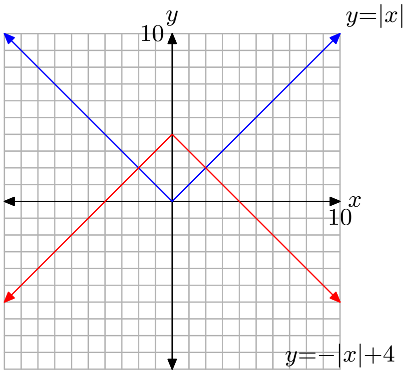
**29.** The domain is $(-\infty, \infty)$ and the range is $[2, \infty)$.



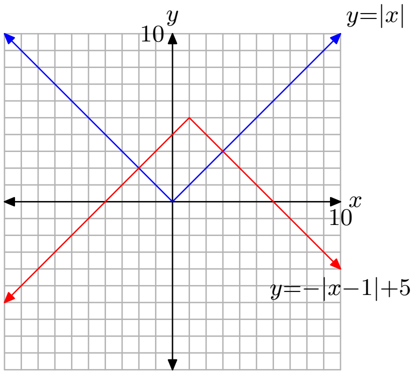
**31.** The domain is $(-\infty, \infty)$ and the range is $(-\infty, 0]$.



**33.** The domain is $(-\infty, \infty)$ and the range is $(-\infty, 4]$.



**35.** The domain is $(-\infty, \infty)$ and the range is $(-\infty, 5]$.



# 4.3 Absolute Value Equations

In the previous section, we defined

$$|x| = \begin{cases} -x, & \text{if } x < 0. \\ x, & \text{if } x \geq 0, \end{cases} \tag{1}$$

and we saw that the graph of the absolute value function defined by $f(x) = |x|$ has the "V-shape" shown in **Figure 1**.

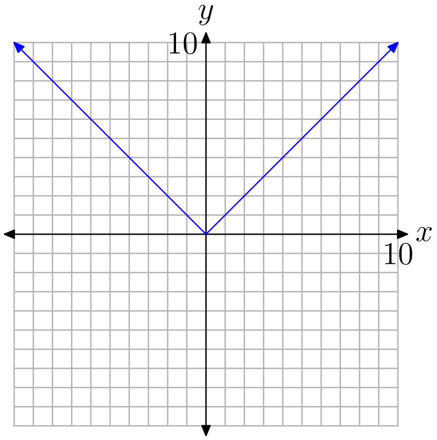


**Figure 1.** The graph of the absolute value function $f(x) = |x|$.

It is important to note that the equation of the left-hand branch of the "V" is $y = -x$. Typical points on this branch are $(-1, 1)$, $(-2, 2)$, $(-3, 3)$, etc. It is equally important to note that the right-hand branch of the "V" has equation $y = x$. Typical points on this branch are $(1, 1)$, $(2, 2)$, $(3, 3)$, etc.

## *Solving* $|x| = a$

We will now discuss the solutions of the equation

$$|x| = a.$$

There are three distinct cases to discuss, each of which depends upon the value and sign of the number $a$.

- Case I: $a < 0$

  If $a < 0$, then the graph of $y = a$ is a horizontal line that lies strictly below the $x$-axis, as shown in **Figure 2**(a). In this case, the equation $|x| = a$ has no solutions because the graphs of $y = a$ and $y = |x|$ do not intersect.

---

[6] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

- Case II: $a = 0$

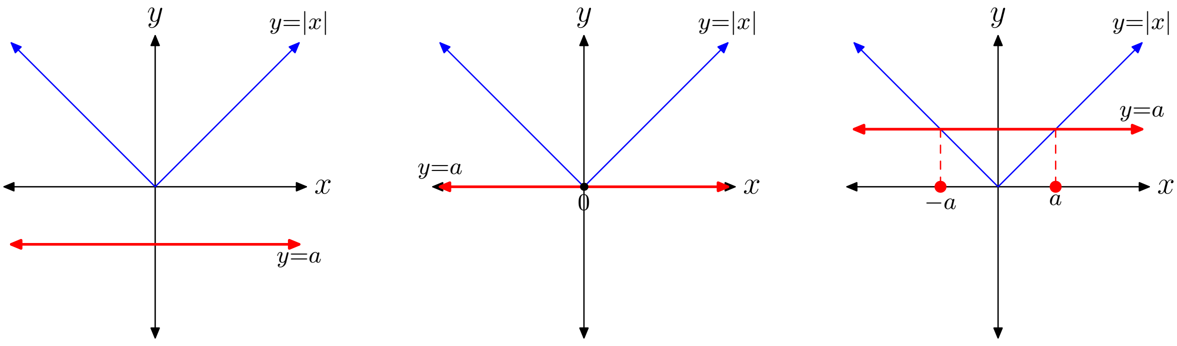
  If $a = 0$, then the graph of $y = 0$ is a horizontal line that coincides with the $x$-axis, as shown in **Figure 2**(b). In this case, the equation $|x| = 0$ has the single solution $x = 0$, because the horizontal line $y = 0$ intersects the graph of $y = |x|$ at exactly one point, at $x = 0$.

- Case III: $a > 0$

  If $a > 0$, then the graph of $y = a$ is a horizontal line that lies strictly above the $x$-axis, as shown in **Figure 2**(c). In this case, the equation $|x| = a$ has two solutions, because the graphs of $y = a$ and $y = |x|$ have two points of intersection.

  Recall that the left-hand branch of $y = |x|$ has equation $y = -x$, and points on this branch have the form $(-1, 1)$, $(-2, 2)$, etc. Because the point where the graph of $y = a$ intersects the left-hand branch of $y = |x|$ has $y$-coordinate $y = a$, the $x$-coordinate of this point of intersection is $x = -a$. This is one solution of $|x| = a$.

  Recall that the right-hand branch of $y = |x|$ has equation $y = x$, and points on this branch have the form $(1, 1)$, $(2, 2)$, etc. Because the point where the graph of $y = a$ intersects the right-hand branch of $y = |x|$ has $y$-coordinate $y = a$, the $x$-coordinate of this point of intersection is $x = a$. This is the second solution of $|x| = a$.



(a) $a < 0$. (b) $a = 0$. (c) $a > 0$.

**Figure 2.** The solution of $|x| = a$ has three cases.

This discussion leads to the following key result.

**Property 2.** The solution of $|x| = a$ depends upon the value and sign of $a$.

- Case I: $a < 0$

  The equation $|x| = a$ has no solutions.

- Case II: $a = 0$

  The equation $|x| = 0$ has one solution, $x = 0$.

- Case III: $a > 0$

  The equation $|x| = a$ has two solutions, $x = -a$ or $x = a$.

Let's look at some examples.

▶ **Example 3.** *Solve* $|x| = -3$ *for* $x$.

The graph of the left-hand side of $|x| = -3$ is the "V" of **Figure 2**(a). The graph of the right-hand side of $|x| = -3$ is a horizontal line three units below the $x$-axis. This has the form of the sketch in **Figure 2**(a). The graphs do not intersect. Therefore, the equation $|x| = -3$ has no solutions.

An alternate approach is to consider the fact that the absolute value of $x$ can never equal $-3$. The absolute value of a number is always nonnegative (either zero or positive). Hence, the equation $|x| = -3$ has no solutions.

◇

▶ **Example 4.** *Solve* $|x| = 0$ *for* $x$.

This is the case shown in **Figure 2**(b). The graph of the left-hand side of $|x| = 0$ intersects the graph of the right-hand side of $|x| = 0$ at $x = 0$. Thus, the only solution of $|x| = 0$ is $x = 0$.

Thinking about this algebraically instead of graphically, we know that $0 = 0$, but there is no other number with an absolute value of zero. So, intuitively, the only solution of $|x| = 0$ is $x = 0$.

◇

▶ **Example 5.** *Solve* $|x| = 4$ *for* $x$.

The graph of the left-hand side of $|x| = 4$ is the "V" of **Figure 2**(c). The graph of the right-hand side is a horizontal line 4 units above the $x$-axis. This has the form of the sketch in **Figure 2**(c). The graphs intersect at $(-4, 4)$ and $(4, 4)$. Therefore, the solutions of $|x| = 4$ are $x = -4$ or $x = 4$.

Alternatively, $|-4| = 4$ and $|4| = 4$, but no other real numbers have absolute value equal to 4. Hence, the only solutions of $|x| = 4$ are $x = -4$ or $x = 4$.

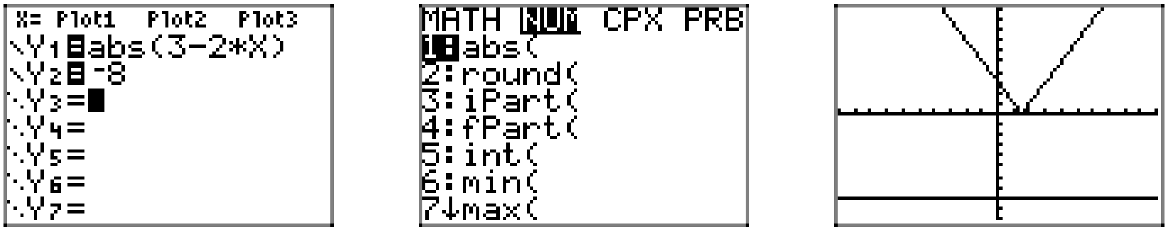
◇

▶ **Example 6.** *Solve the equation* $|3-2x|=-8$ *for* $x$.

If the equation were $|x|=-8$, we would not hesitate. The equation $|x|=-8$ has no solutions. However, the reasoning applied to the simple case $|x|=-8$ works equally well with the equation $|3-2x|=-8$. The left-hand side of this equation must be nonnegative, so its graph must lie above or on the $x$-axis. The right-hand side of $|3-2x|=-8$ is a horizontal line 8 units below the $x$-axis. The graphs cannot intersect, so there is no solution.

We can verify this argument with the graphing calculator. Load the left and right-hand sides of $|3-2x|=-8$ into `Y1` and `Y2`, respectively, as shown in **Figure 3**(a). Push the `MATH` button on your calculator, then right-arrow to the `NUM` menu, as shown in **Figure 3**(b). Use `1:abs(` to enter the absolute value shown in `Y1` in **Figure 3**(a). From the `ZOOM` menu, select `6:ZStandard` to produce the image shown in **Figure 3**(c).

Note, that as predicted above, the graph of $y=|3-2x|$ lies on or above the $x$-axis and the graph of $y=-8$ lies strictly below the $x$-axis. Hence, the graphs cannot intersect and the equation $|3-2x|=-8$ has no solutions.



```
Plot1  Plot2  Plot3
\Y1=abs(3-2*X)
\Y2=-8
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

(a) (b) (c)

**Figure 3.** Using the graphing calculator to examine the solution of $|3-2x|=-8$.

Alternatively, we can provide a completely intuitive solution of $|3-2x|=-8$ by arguing that the left-hand side of this equation is nonnegative, but the right-hand side is negative. This is an impossible situation. Hence, the equation has no solutions.

◆

▶ **Example 7.** *Solve the equation* $|3-2x|=0$ *for* $x$.

We have argued that the only solution of $|x|=0$ is $x=0$. Similar reasoning points out that $|3-2x|=0$ only when $3-2x=0$. We solve this equation independently.

$$\begin{aligned} 3-2x&=0\\ -2x&=-3\\ x&=\frac{3}{2} \end{aligned}$$

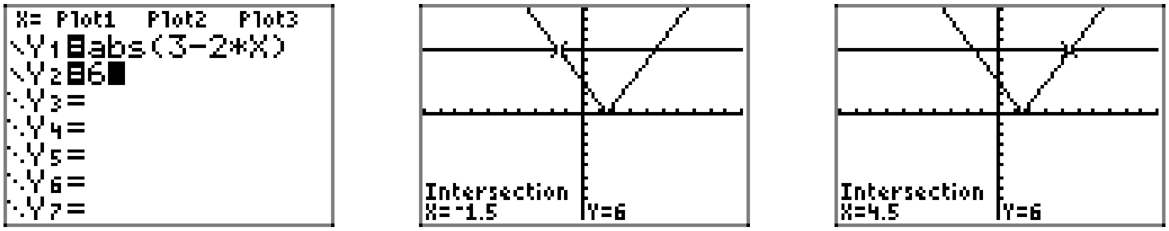
Thus, the only solution of $|3-2x|=0$ is $x=3/2$.

It is worth pointing out that the "tip" or "vertex" of the "V" in **Figure 3**(c) is located at $x=3/2$. This is the only location where the graphs of $y=|3-2x|$ and $y=0$ intersect.

◆

▶ **Example 8.** *Solve the equation* $|3 - 2x| = 6$ *for* $x$.

In this example, the graph of $y = 6$ is a horizontal line that lies 6 units above the $x$-axis, and the graph of $y = |3 - 2x|$ intersects the graph of $y = 6$ in two locations. You can use the intersect utility to find the points of intersection of the graphs, as we have in **Figure 4**(b) and (c).



```
Plot1  Plot2  Plot3
\Y1=abs(3-2*X)
\Y2=6
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
Intersection
X=-1.5        Y=6
```

```
Intersection
X=4.5         Y=6
```

(a) (b) (c)

**Figure 4.** Using the graphing calculator to find two solutions of $|3 - 2x| = 6$.

**Expectations.** We need a way of summarizing this graphing calculator approach on our homework paper. First, draw a reasonable facsimile of your calculator's viewing window on your homework paper. Use a ruler to draw all lines. Complete the following checklist.

- Label each axis, in this case with $x$ and $y$.
- Scale each axis. To do this, press the `WINDOW` button on your calculator, then report the values of `xmin`, `xmax`, `ymin`, and `ymax` on the appropriate axis.
- Label each graph with its equation.
- Drop dashed vertical lines from the points of intersection to the $x$-axis. Shade and label these solutions of the equation on the $x$-axis.

Following the guidelines in the above checklist, we obtain the image in **Figure 5**.

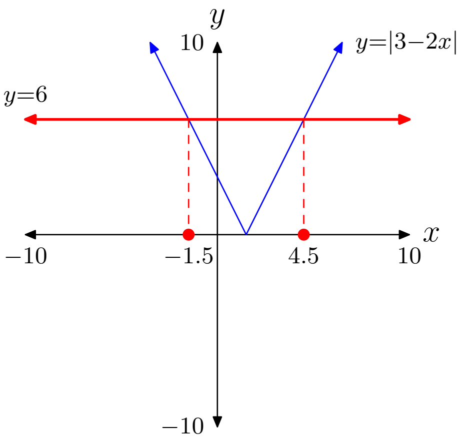


**Figure 5.** Reporting a graphical solution of $|3 - 2x| = 6$.

**Algebraic Approach**. One can also use an algebraic technique to find the two solutions of $|3-2x|=6$. Much as $|x|=6$ has solutions $x=-6$ or $x=6$, the equation

$$|3-2x|=6$$

is possible only if the expression inside the absolute values is either equal to $-6$ or 6. Therefore, write

$$3-2x=-6 \qquad \text{or} \qquad 3-2x=6,$$

and solve these equations independently.

$$\begin{aligned} 3-2x &= -6 &\qquad \text{or} \qquad\qquad 3-2x &= 6\\ -2x &= -9 & -2x &= 3\\ x &= \frac{9}{2} & x &= -\frac{3}{2}. \end{aligned}$$

Because $-3/2=-1.5$ and $9/2=4.5$, these exact solutions agree exactly with the graphical solutions in **Figure 4**(b) and (c).

◆

Let's summarize the technique involved in solving this important case.

**Solving** $|\mathit{expression}|=a$**, when** $a>0$. To solve the equation

$$|\text{expression}|=a, \quad \text{when } a>0,$$

set

$$\text{expression}=-a \qquad \text{or} \qquad \text{expression}=a,$$

then solve each of these equations independently.

For example:

- To solve $|2x+7|=5$, set
$$2x+7=-5 \qquad \text{or} \qquad 2x+7=5,$$
then solve each of these equations independently.
- To solve $|3-5x|=9$, set
$$3-5x=-9 \qquad \text{or} \qquad 3-5x=9,$$
then solve each of these equations independently.
- Note that this technique should **not** be applied to the equation $|2x+11|=-10$, because the right-hand side of the equation is not a positive number. Indeed, in this case, no values of $x$ will make the left-hand side of this equation equal to $-10$, so the equation has no solutions.

Sometimes we have to do a little algebra before removing the absolute value bars.

▶ **Example 9.** *Solve the equation*

$$|x+2|+3=8$$

*for $x$.*

First, subtract 3 from both sides of the equation.

$$\begin{aligned} |x+2|+3&=8 \\ |x+2|+3-3&=8-3 \end{aligned}$$

This simplifies to

$$|x+2|=5$$

Now, either

$$x+2=-5 \qquad \text{or} \qquad x+2=5,$$

each of which can be solved separately.

$$\begin{aligned} x+2&=-5 &\qquad \text{or} \qquad x+2&=5 \\ x+2-2&=-5-2 & x+2-2&=5-2 \\ x&=-7 & x&=3 \end{aligned}$$

◆

▶ **Example 10.** *Solve the equation*

$$3|x-5|=6$$

*for $x$.*

First, divide both sides of the equation by 3.

$$\begin{aligned} 3|x-5|&=6 \\ \frac{3|x-5|}{3}&=\frac{6}{3} \end{aligned}$$

This simplifies to

$$|x-5|=2.$$

Now, either

$$x-5=-2 \qquad \text{or} \qquad x-5=2,$$

each of which can be solved separately.

$$\begin{aligned} x - 5 &= -2 & &\text{or} & x - 5 &= 2 \\ x - 5 + 5 &= -2 + 5 & & & x - 5 + 5 &= 2 + 5 \\ x &= 3 & & & x &= 7 \end{aligned}$$

◇

## *Properties of Absolute Value*

An example will motivate the need for some discussion of the properties of absolute value.

▶ **Example 11.** *Solve the equation*

$$\left|\frac{x}{2} - \frac{1}{3}\right| = \frac{1}{4} \tag{12}$$

*for $x$.*

It is tempting to multiply both sides of this equation by a common denominator as follows.

$$\begin{aligned} \left|\frac{x}{2} - \frac{1}{3}\right| &= \frac{1}{4} \\ 12\left|\frac{x}{2} - \frac{1}{3}\right| &= 12\left(\frac{1}{4}\right) \end{aligned}$$

If it is permissible to move the 12 inside the absolute values, then we could proceed as follows.

$$\begin{aligned} \left|12\left(\frac{x}{2} - \frac{1}{3}\right)\right| &= 3 \\ |6x - 4| &= 3 \end{aligned}$$

Assuming for the moment that this last move is allowable, either

$$6x - 4 = -3 \qquad \text{or} \qquad 6x - 4 = 3.$$

Each of these can be solved separately, first by adding 4 to both sides of the equations, then dividing by 6.

$$\begin{aligned} 6x - 4 &= -3 & &\text{or} & 6x - 4 &= 3 \\ 6x &= 1 & & & 6x &= 7 \\ x &= 1/6 & & & x &= 7/6 \end{aligned}$$

As we've used a somewhat questionable move in obtaining these solutions, it would be wise to check our results. First, substitute $x = 1/6$ into the original equation.

$$\begin{aligned}\left|\frac{x}{2}-\frac{1}{3}\right| &= \frac{1}{4}\\ \left|\frac{1/6}{2}-\frac{1}{3}\right| &= \frac{1}{4}\\ \left|\frac{1}{12}-\frac{1}{3}\right| &= \frac{1}{4}\end{aligned}$$

Write equivalent fractions with a common denominator and subtract.

$$\begin{aligned}\left|\frac{1}{12}-\frac{4}{12}\right| &= \frac{1}{4}\\ \left|-\frac{3}{12}\right| &= \frac{1}{4}\\ \left|-\frac{1}{4}\right| &= \frac{1}{4}\end{aligned}$$

Clearly, $x = 1/6$ checks.[7] We'll leave the check of the second solution to our readers.

◆

Well, we've checked our solutions and they are correct, so it must be the case that

$$12\left|\frac{x}{2}-\frac{1}{3}\right| = \left|12\left(\frac{x}{2}-\frac{1}{3}\right)\right|.$$

But why? After all, absolute value bars, though they do act as grouping symbols, have a bit more restrictive meaning than ordinary grouping symbols such as parentheses, brackets, and braces.

We state the first property of absolute values.

**Property 13.** If $a$ and $b$ are any real numbers, then

$$|ab| = |a||b|.$$

We can demonstrate the validity of this property by simply checking cases.

- If $a$ and $b$ are both positive real numbers, then so is $ab$ and $|ab| = ab$. On the other hand, $|a||b| = ab$. Thus, $|ab| = |a||b|$.
- If $a$ and $b$ are both negative real numbers, then $ab$ is positive and $|ab| = ab$. On the other hand, $|a||b| = (-a)(-b) = ab$. Thus, $|ab| = |a||b|$.

We will leave the proof of the remaining two cases as exercises. We can use $|a||b| = |ab|$ to demonstrate that

---

[7] Note that the check is almost as difficult as the solution. Perhaps that's why we get a bit lazy, not checking our solutions as often as we should.

$$12\left|\frac{x}{2}-\frac{1}{3}\right| = |12|\left|\frac{x}{2}-\frac{1}{3}\right| = \left|12\left(\frac{x}{2}-\frac{1}{3}\right)\right|.$$

This validates the method of attack we used to solve **equation (12)** in **Example 11**.

**Warning 14.** *On the other hand, it is not permissible to multiply by a negative number and simply slide the negative number inside the absolute value bars. For example,*

$$-2|x-3| = |-2(x-3)|$$

*is clearly an error (well, it does work for $x = 3$). For any $x$ except 3, the left-hand side of this result is a negative number, but the right-hand side is a positive number. They are clearly not equal.*

In similar fashion, one can demonstrate a second useful property involving absolute value.

**Property 15.** If $a$ and $b$ are any real numbers, then

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|},$$

provided, of course, that $b \neq 0$.

Again, this can be proved by checking four cases. For example, if $a$ is a positive real number and $b$ is a negative real number, then $a/b$ is negative and $|a/b| = -a/b$. On the other hand, $|a|/|b| = a/(-b) = -a/b$.

We leave the proof of the remaining three cases as exercises.

This property is useful in certain situations. For example, should you desire to divide $|2x-4|$ by 2, you would proceed as follows.

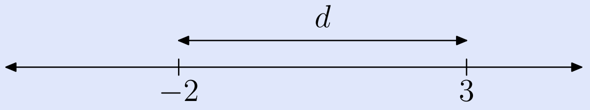
$$\frac{|2x-4|}{2} = \frac{|2x-4|}{|2|} = \left|\frac{2x-4}{2}\right| = |x-2|$$

This technique is useful in several situations. For example, should you want to solve the equation $|2x-4| = 6$, you could divide both sides by 2 and apply the quotient property of absolute values.

## *Distance Revisited*

Recall that for any real number $x$, the absolute value of $x$ is defined as the distance between the real number $x$ and the origin on the real line. In this section, we will push this distance concept a bit further.

Suppose that you have two real numbers on the real line. For example, in the figure that follows, we've located 3 and $-2$ on the real line.



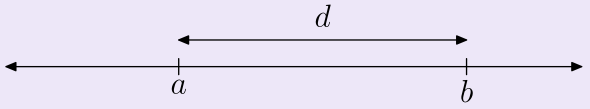
You can determine the distance between the two points by subtracting the number on the left from the number on the right. That is, the distance between the two points is $d = 3 - (-2) = 5$ units. If you subtract in the other direction, you get the negative of the distance, as in $-2 - 3 = -5$ units. Of course, distance is a nonnegative quantity, so this negative result cannot represent the distance between the two points. Consequently, to find the distance between two points on the real line, you must always subtract the number on the left from the number on the right.[8]

However, if you take the absolute value of the difference, you'll get the correct result regardless of the direction of subtraction.

$$d = |3 - (-2)| = |5| = 5 \qquad \text{and} \qquad d = |-2 - 3| = |-5| = 5.$$

This discussion leads to the following key idea.

**Property 16.** Suppose that $a$ and $b$ are two numbers on the real line.



You can determine the distance $d$ between $a$ and $b$ on the real line by taking the absolute value of their difference. That is,

$$d = |a - b|.$$

Of course, you could subtract in the other direction, obtaining $d = |b - a|$. This is also correct.

Now that this geometry of distance has been introduced, it is useful to pronounce the symbolism $|a - b|$ as "the distance between $a$ and $b$" instead of saying "the absolute value of $a$ minus $b$."

► **Example 17.** *Solve the equation*

$$|x - 3| = 8$$

*for $x$.*

Here's the ideal situation to apply our new concept of distance. Instead of saying "the absolute value of $x$ minus 3 is 8," we pronounce the equation $|x - 3| = 8$ as "the distance between $x$ and 3 is 8."

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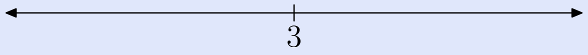
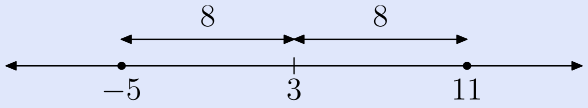
[8] On a vertical line, you would subtract the lower number from the upper number.

Draw a number line and locate the number 3 on the line.



Recall that the "distance between $x$ and 3 is 8." Having said this, mark two points on the real line that are 8 units away from 3.



Thus, the solutions of $|x - 3| = 8$ are $x = -5$ or $x = 11$.

◆

▶ **Example 18.** *Solve the equation*

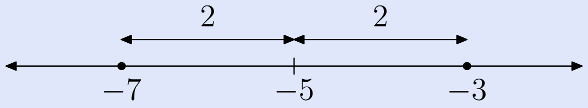
$$|x + 5| = 2$$

*for $x$.*

Rewrite the equation as a difference.

$$|x - (-5)| = 2$$

This is pronounced "the distance between $x$ and $-5$ is 2." Locate two points on the number line that are 2 units away from $-5$.



Hence, the solutions of $|x + 5| = 2$ are $x = -7$ or $x = -3$.

◆

## 4.3 Exercises

For each of the equations in **Exercises 1-4**, perform each of the following tasks.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.
ii. Sketch the graph of each side of the equation without the aid of a calculator. Label each graph with its equation.
iii. Shade the solution of the equation on the $x$-axis (if any) as shown in Figure 5 (read "Expectations") in the narrative. That is, drop dashed lines from the points of intersection to the axis, then shade and label the solution set on the $x$-axis.

**1.** $|x| = -2$

**2.** $|x| = 0$

**3.** $|x| = 3$

**4.** $|x| = 2$

---

For each of the equations in **Exercises 5-8**, perform each of the following tasks.

i. Load each side of the equation into the `Y=` menu of your calculator. Adjust the viewing window so that all points of intersection of the two graphs are visible in the viewing window.
ii. Copy the image in your viewing screen onto your homework paper. Label each axis and scale each axis with xmin, xmax, ymin, and ymax. Label each graph with its equation.
iii. Use the `intersect` utility in the `CALC` menu to determine the points of intersection. Shade and label each solution as shown in Figure 5 (read "Expectations") in the narrative. That is, drop dashed lines from the points of intersection to the axis, then shade and label the solution set on the $x$-axis.

**5.** $|3 - 2x| = 5$

**6.** $|2x + 7| = 4$

**7.** $|4x + 5| = 7$

**8.** $|5x - 7| = 8$

---

For each of the equations in **Exercises 9-14**, provide a purely algebraic solution without the use of a calculator. Arrange your work as shown in Examples 6, 7, and 8 in the narrative, but do not use a calculator.

**9.** $|4x + 3| = 0$

**10.** $|3x - 11| = -5$

**11.** $|2x + 7| = 14$

**12.** $|7 - 4x| = 8$

**13.** $|3 - 2x| = -1$

**14.** $|4x + 9| = 0$

---

For each of the equations in **Exercises 15-20**, perform each of the following tasks.

i. Arrange each of the following parts on your homework paper in the same location. Do not do place the alge-

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braic work on one page and the graphical work on another.

ii. Follow each of the directions given for **Exercises 5-8** to find and record a solution with your graphing calculator.

iii. Provide a purely algebraic solution, showing all the steps of your work. Do these solutions compare favorably with those found using your graphing calculator in part (ii)? If not, look for a mistake in your work.

**15.** $|x-8|=7$

**16.** $|2x-15|=5$

**17.** $|2x+11|=6$

**18.** $|5x-21|=7$

**19.** $|x-12|=6$

**20.** $|x+11|=5$

---

Use a strictly algebraic technique to solve each of the equations in **Exercises 21-28**. Do not use a calculator.

**21.** $|x+2|-3=4$

**22.** $3|x+5|=6$

**23.** $-2|3-2x|=-6$

**24.** $|4-x|+5=12$

**25.** $3|x+2|-5=|x+2|+7$

**26.** $4-3|4-x|=2|4-x|-1$

**27.** $\left|\dfrac{x}{3}-\dfrac{1}{4}\right|=\dfrac{1}{12}$

**28.** $\left|\dfrac{x}{4}-\dfrac{1}{2}\right|=\dfrac{2}{3}$

---

Use the technique of distance on the number line demonstrated in Examples 16 and 17 to solve each of the equations in **Exercises 29-32**. Provide number line sketches on your homework paper as shown in Examples 16 and 17 in the narrative.

**29.** $|x-5|=8$

**30.** $|x-2|=4$

**31.** $|x+4|=3$

**32.** $|x+2|=11$

---

Use the instructions provided in **Exercises 5-8** to solve the equations in **Exercises 33-34**.

**33.** $|x+2|=\dfrac{1}{3}x+5$

**34.** $|x-3|=5-\dfrac{1}{2}x$

---

In **Exercises 35-36**, perform each of the following tasks.

i. Set up a coordinate system on graph paper. Label and scale each axis.

ii. Without the use of a calculator, sketch the graphs of the left- and right-hand sides of the given equation. Label each graph with its equation.

iii. Drop dashed vertical lines from each point of intersection to the $x$-axis. Shade and label each solution on the $x$-axis (you will have to approximate).

**35.** $|x-2|=\dfrac{1}{3}x+2$

**36.** $|x+4|=\dfrac{1}{3}x+4$

---

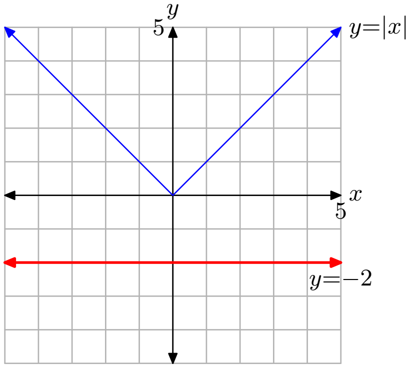
**37.** Given that $a<0$ and $b>0$, prove that $|ab|=|a||b|$.

**38.** Given that $a > 0$ and $b < 0$, prove that $|ab| = |a||b|$.

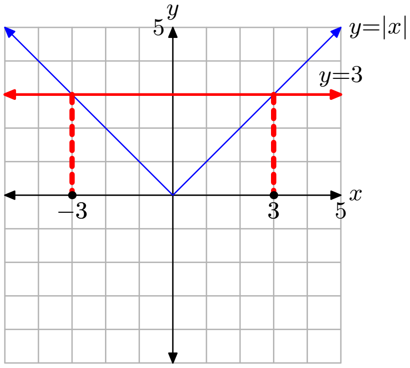
**39.** In the narrative, we proved that if $a > 0$ and $b < 0$, then $|a/b| = |a|/|b|$. Prove the remaining three cases.

## 4.3 Answers

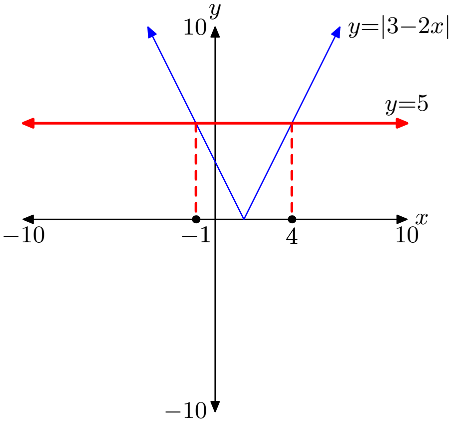
**1.** No solutions.



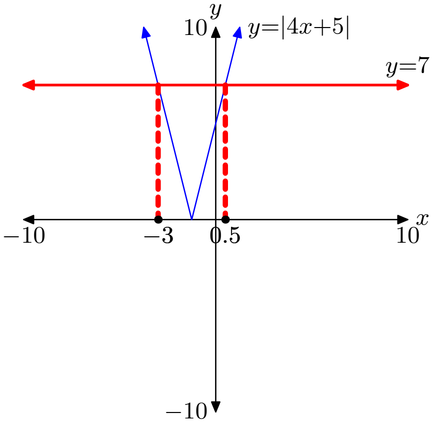
**3.** Solution: $x = -3$ or $x = 3$.



**5.** Solutions: $x = -1$ or $x = 4$



**7.** Solutions: $x = -3$ or $x = 0.5$

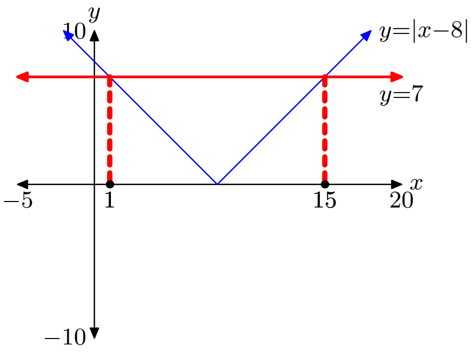


**9.** $x = -3/4$

**11.** $x = -21/2$ or $x = 7/2$

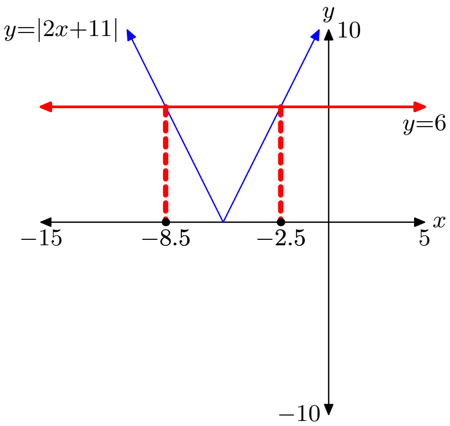
**13.** No solutions.

**15.**



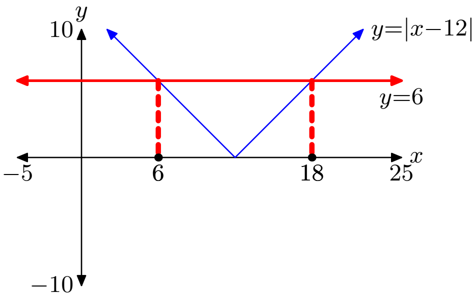
$x = 1$ or $x = 15$

**17.**



$x = -8.5 \quad \text{or} \quad x = -2.5$

**19.**



$x = 6 \quad \text{or} \quad x = 18$

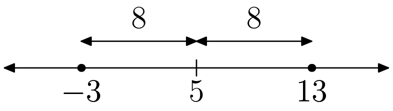
**21.** $x = -9 \quad \text{or} \quad x = 5$

**23.** $x = 0 \quad \text{or} \quad x = 3$

**25.** $x = -8 \quad \text{or} \quad x = 4$

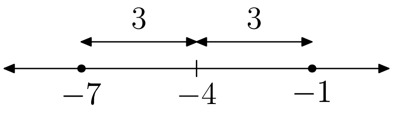
**27.** $x = 1/2 \quad \text{or} \quad x = 1$

**29.**



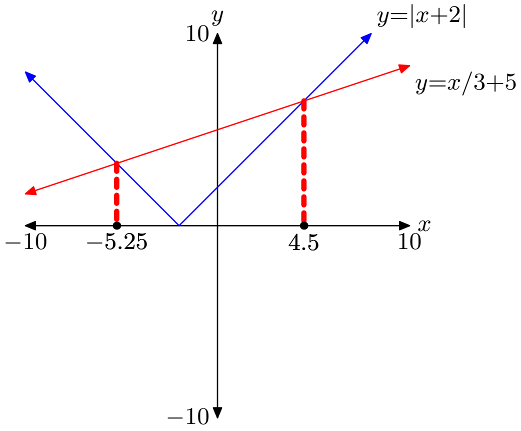
$x = -3 \quad \text{or} \quad x = 13$

**31.**

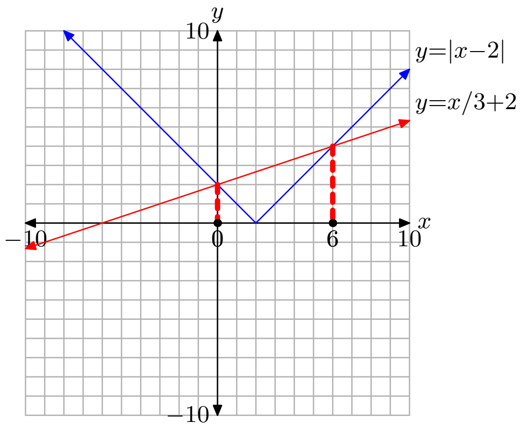


$x = -7 \quad \text{or} \quad x = -1$

**33.**



**35.**



**37.** If $a$ is a negative real number and $b$ is a positive real number, then $ab$ is negative, so $|ab| = -ab$. On the other hand, $a$ negative also means that $|a| = -a$, and $b$ positive means that $|b| = b$, so that $|a||b| = -a(b) = -ab$. Comparing these results, we see that $|ab|$ and $|a||b|$ equal the same thing, and so they must be equal to one another.

**39.** Case I. $(a, b > 0)$ If $a$ and $b$ are both positive real numbers, then $a/b$ is positive and so $|a/b| = a/b$. On the other hand, $a$ positive also means that $|a| = a$, and $b$ positive means that $|b| = b$, so that $|a|/|b| = a/b$. Comparing these two results, we see that $|a/b|$ and $|a|/|b|$ equal the same thing, and so they must be equal to one another.

Case II. $(a, b < 0)$ If $a$ and $b$ are both negative real numbers, then $a/b$ is positive and so $|a/b| = a/b$. On the other hand, $a$ negative also means that $|a| = -a$, and $b$ negative means that $|b| = -b$, so that $|a|/|b| = -a/(-b) = a/b$. Comparing these two results, we see that $|a/b|$ and $|a|/|b|$ equal the same thing, and so they must be equal to one another.

Case III. $(a < 0, b > 0)$ If $a$ is a negative real number and $b$ is a positive real number, then $a/b$ is negative and so $|a/b| = -(a/b)$. On the other hand, $a$ negative also means that $|a| = -a$, and $b$ positive means that $|b| = b$, so that $|a|/|b| = -a/b = -(a/b)$. Comparing these two results, we see that $|a/b|$ and $|a|/|b|$ equal the same thing, and so they must be equal to one another.

# 4.4 Absolute Value Inequalities

In the last section, we solved absolute value equations. In this section, we turn our attention to inequalities involving absolute value.

## *Solving* $|x| < a$

The solutions of

$$|x| < a$$

again depend upon the value and sign of the number $a$. To solve $|x| < a$ graphically, we must determine where the graph of the left-hand side *lies below* the graph of the right-hand side of the inequality $|x| < a$. There are three cases to consider.

- Case I: $a < 0$

  In this case, the graph of $y = a$ lies strictly below the $x$-axis. As you can see in **Figure 1**(a), the graph of $y = |x|$ **never** lies below the graph of $y = a$. Hence, the inequality $|x| < a$ has no solutions.

- Case II: $a = 0$

  In this case, the graph of $y = 0$ coincides with the $x$-axis. As you can see in **Figure 1**(b), the graph of $y = |x|$ **never** lies strictly below the $x$-axis. Hence, the inequality $|x| < 0$ has no solutions.

- Case III: $a > 0$

  In this case, the graph of $y = a$ lies strictly above the $x$-axis. In **Figure 1**(c), the graph of $y = |x|$ and $y = a$ intersect at $x = -a$ and $x = a$. In **Figure 1**(c), we also see that the graph of $y = |x|$ lies strictly below the graph of $y = a$ when $x$ is in-between $-a$ and $a$; that is, when $-a < x < a$.

  In **Figure 1**(c), we've dropped dashed vertical lines from the points of intersection of the two graphs to the $x$-axis. On the $x$-axis, we've shaded the solution of $|x| < a$, that is, $-a < x < a$.

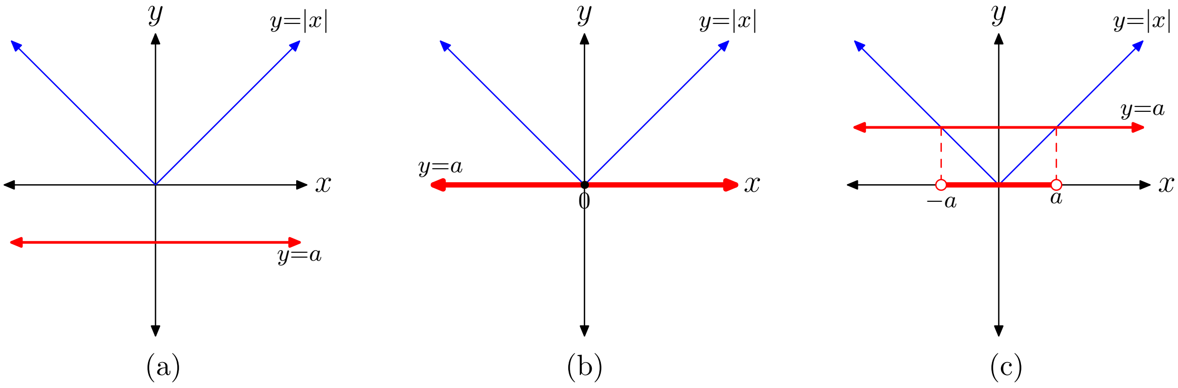


**Figure 1.** The solution of $|x| < a$ has three cases.

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This discussion leads to the following key property.

**Property 1.** The solution of $|x| < a$ depends upon the value and sign of $a$.

- Case I: $a < 0$

  The inequality $|x| < a$ has no solution.

- Case II: $a = 0$

  The inequality $|x| < 0$ has no solution.

- Case III: $a > 0$

  The inequality $|x| < a$ has solution set $\{x : -a < x < a\}$.

Let's look at some examples.

▶ **Example 2.** *Solve the inequality* $|x| < -5$ *for* $x$.

The graph of the left-hand side of $|x| < -5$ is the "V" of **Figure 1**(a). The graph of the right-hand side of $|x| < -5$ is a horizontal line located 5 units below the $x$-axis. This is the situation shown in **Figure 1**(a). The graph of $y = |x|$ is therefore **never** below the graph of $y = -5$. Thus, the inequality $|x| < -5$ has no solution.

An alternate approach is to consider the fact that the absolute value of $x$ is always nonnegative and can never be less than $-5$. Thus, the inequality $|x| < -5$ has no solution.

◇

▶ **Example 3.** *Solve the inequality* $|x| < 0$ *for* $x$.

This is the case shown in **Figure 1**(b). The graph of $y = |x|$ is never strictly below the $x$-axis. Thus, the inequality $|x| < 0$ has no solution.

◇

▶ **Example 4.** *Solve the inequality* $|x| < 8$ *for* $x$.

The graph of the left-hand side of $|x| < 8$ is the "V" of **Figure 1**(c). The graph of the right-hand side of $|x| < 8$ is a horizontal line located 8 units above the $x$-axis. This is the situation depicted in **Figure 1**(c). The graphs intersect at $(-8, 8)$ and $(8, 8)$ and the graph of $y = |x|$ lies strictly below the graph of $y = 8$ for values of $x$ in-between $-8$ and 8. Thus, the solution of $|x| < 8$ is $-8 < x < 8$.

It helps the intuition if you check the results of the last example. Note that numbers between $-8$ and 8, such as $-7.75$, $-3$ and 6.8 satisfy the inequality,

$$|-7.75| < 8 \qquad \text{and} \qquad |-3| < 8 \qquad \text{and} \qquad |6.8| < 8,$$

while values that do not lie between −8 and 8 do not satisfy the inequality. For example, none of the numbers −9.3, 8.2, and 11.7 lie between −8 and 8, and each of the following is a false statement.

$$|-9.3| < 8 \qquad \text{and} \qquad |8.2| < 8 \qquad \text{and} \qquad |11.7| < 8 \qquad \text{(all are false)}$$

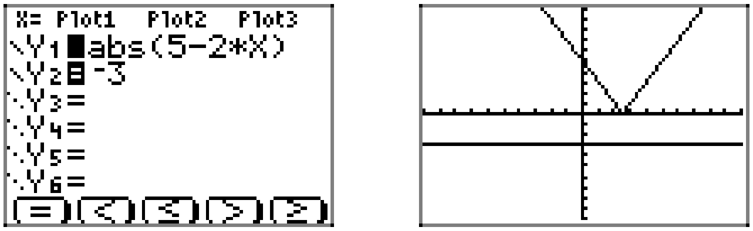
If you reflect upon these results, they will help cement the notion that the solution of $|x| < 8$ is all values of $x$ satisfying $-8 < x < 8$.

▶ **Example 5.** *Solve the inequality* $|5 - 2x| < -3$ *for* $x$.

If the inequality were $|x| < -3$, we would not hesitate. This is the situation depicted in **Figure 1**(a) and the inequality $|x| < -3$ has no solutions. The reasoning applied to $|x| < -3$ works equally well for the inequality $|5 - 2x| < -3$. The left-hand side of this inequality must be nonnegative, so its graph must lie on or above the $x$-axis. The right-hand side of $|5 - 2x| < -3$ is a horizontal line located 3 units below the $x$-axis. Therefore, the graph of $y = |5 - 2x|$ can **never** lie below the graph of $y = -3$ and the inequality $|5 - 2x| < -3$ has no solution.

We can verify this result with the graphing calculator. Load the left- and right-hand sides of $|5-2x| < -3$ into `Y1` and `Y2`, respectively, as shown in **Figure 2**(a). From the `ZOOM` menu, select `6:ZStandard` to produce the image shown in **Figure 2**(b).

As predicted, the graph of $y = |5-2x|$ never lies below the graph of $y = -3$, so the inequality $|5 - 2x| < -3$ has no solution.



```
X= Plot1  Plot2  Plot3
\Y1 ■abs(5-2*X)
\Y2 ■ -3
\Y3=
\Y4=
\Y5=
\Y6=
```

(a) (b)

**Figure 2.** Using the graphing calculator to solve the inequality $|5 - 2x| < -3$.

▶ **Example 6.** *Solve the inequality* $|5 - 2x| < 0$ *for* $x$.

We know that the left-hand side of the inequality $|5 - 2x| < 0$ has the “V” shape indicated in **Figure 1**(b). The graph “touches” the $x$-axis when $|5 - 2x| = 0$, or when

$$\begin{aligned} 5 - 2x &= 0 \\ -2x &= -5 \\ x &= \frac{5}{2}. \end{aligned}$$

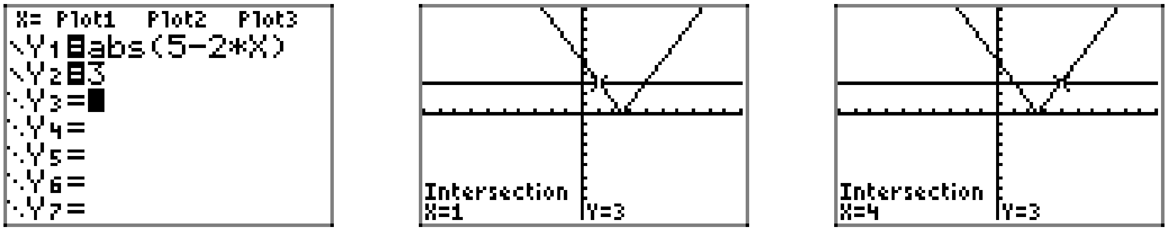
However, the graph of $y = |5 - 2x|$ **never** falls below the $x$-axis, so the inequality $|5 - 2x| < 0$ has no solution.

Intuitively, it should be clear that the inequality $|5-2x| < 0$ has no solution. Indeed, the left-hand side of this inequality is always nonnegative, and can never be strictly less than zero.

◆

▶ **Example 7.** *Solve the inequality* $|5 - 2x| < 3$ *for* $x$.

In this example, the graph of the right-hand side of the inequality $|5 - 2x| < 3$ is a horizontal line located 3 units above the $x$-axis. The graph of the left-hand side of the inequality has the "V" shape shown in **Figure 3**(b) and (c). You can use the intersect utility on the graphing calculator to find the points of intersection of the graphs of $y = |5 - 2x|$ and $y = 3$, as we have done in **Figures 3**(b) and (c). Note that the calculator indicates two points of intersection, one at $x = 1$ and a second at $x = 4$.



```
Plot1  Plot2  Plot3
\Y1=abs(5-2*X)
\Y2=3
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
Intersection
X=1          Y=3
```

```
Intersection
X=4          Y=3
```

(a) (b) (c)

**Figure 3.** Using the graphing calculator to solve the inequality $|5 - 2x| < 3$.

The graph of $y = |5 - 2x|$ falls **below** the graph of $y = 3$ for all values of $x$ between 1 and 4. Hence, the solution of the inequality $|5 - 2x| < 3$ is the set of all $x$ satisfying $1 < x < 4$; i.e. $\{x : 1 < x < 4\}$.

**Expectations.** We need a way of summarizing this graphing calculator approach on our homework paper. First, draw a reasonable facsimile of your calculator's viewing window on your homework paper. Use a ruler to draw all lines. Complete the following checklist.

- Label each axis, in this case with $x$ and $y$.
- Scale each axis. To do this, press the `WINDOW` button on your calculator, then report the values of `xmin`, `xmax`, `ymin`, and `ymax` on the appropriate axis.
- Label each graph with its equation.
- Drop dashed vertical lines from the points of intersection to the $x$-axis. Shade and label the solution set of the inequality on the $x$-axis.

Following the guidelines in the above checklist, we obtain the image in **Figure 4**.

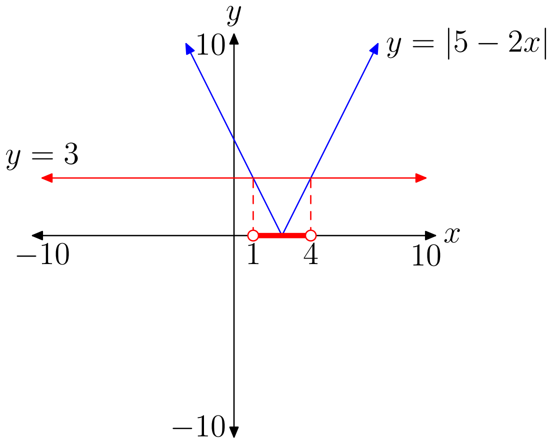


**Figure 4.** Reporting a graphical solution of $|5-2x|<3$.

**Algebraic Approach**. Let's now explore an algebraic solution of the inequality $|5-2x|<3$. Much as $|x|<3$ implies that $-3<x<3$, the inequality

$$|5-2x|<3$$

requires that

$$-3<5-2x<3.$$

We can subtract 5 from all three members of this last inequality, then simplify.

$$\begin{aligned}-3-5<5-2x-5<3-5\\ -8<-2x<-2\end{aligned}$$

Divide all three members of this last inequality by −2, reversing the inequality symbols as you go.

$$4>x>1$$

We prefer that our inequalities read from "small-to-large," so we write

$$1<x<4.$$

This form matches the order of the shaded solution on the number line in **Figure 4**, which we found using the graphing calculator.

◆

The algebraic technique of this last example leads us to the following property.

**Property 8.** If $a>0$, then the inequality $|x|<a$ is equivalent to the inequality $-a<x<a$.

This property provides a simple method for solving inequalities of the form $|x|<a$. Let's apply this algebraic technique in the next example.

▶ **Example 9.** *Solve the inequality* $|4x+5| < 7$ *for* $x$.

The first step is to use **Property 8** to write that

$$|4x+5| < 7$$

is equivalent to the inequality

$$-7 < 4x+5 < 7.$$

From here, we can solve for $x$ by first subtracting 5 from all three members, then dividing through by 4.

$$\begin{aligned} -12 &< 4x < 2 \\ -3 &< x < \frac{1}{2} \end{aligned}$$

We can sketch the solution on a number line.

And we can describe the solution in both interval and set-builder notation as follows.

$$\left(-3, \frac{1}{2}\right) = \left\{x : -3 < x < \frac{1}{2}\right\}$$

◆

Assuming that $a > 0$, the inequality $|x| \leq a$ requires that we find where the absolute value of $x$ is either "less than" $a$ or "equal to" $a$. We know that $|x| < a$ when $-a < x < a$ and we know that $|x| = a$ when $x = -a$ or $x = a$. Thus, the solution of $|x| \leq a$ is the "union" of these two solutions.

This argument leads to the following property.

**Property 10.** If $a > 0$, then the inequality $|x| \leq a$ is equivalent to the inequality $-a \leq x \leq a$.

▶ **Example 11.** *Solve the inequality* $5 - 3|x-4| \geq -4$ *for* $x$.

At first glance, the inequality

$$5 - 3|x-4| \geq -4$$

has a form quite dissimilar from what we've done thus far. However, let's subtract 5 from both sides of the inequality.

$$-3|x-4| \geq -9$$

Now, let's divide both sides of this last inequality by −3, reversing the inequality sign.

$$|x-4| \leq 3$$

Aha! Familiar ground. Using **Property 10**, this last inequality is equivalent to

$$-3 \leq x-4 \leq 3,$$

and when we add 4 to all three members, we have the solution.

$$1 \leq x \leq 7$$

We can sketch the solution on a number line.

And we can describe the solution with interval and set-builder notation.

$$[1,7]=\{x:\ 1 \leq x \leq 7\}$$

◆

## *Solving* $|x| > a$

The solutions of $|x|>a$ again depend upon the value and sign of $a$. To solve $|x|>a$ graphically, we must determine where the graph of $y=|x|$ lies **above** the graph of $y=a$. Again, we consider three cases.

- Case I: $a<0$

  In this case, the graph of $y=a$ lies strictly below the $x$-axis. Therefore, the graph of $y=|x|$ in **Figure 5**(a) **always** lies above the graph of $y=a$. Hence, all real numbers are solutions of the inequality $|x|>a$.

- Case II: $a=0$

  In this case, the graph of $y=0$ coincides with the $x$-axis. As shown in **Figure 5**(b), the graph of $y=|x|$ will lie strictly above the graph of $y=0$ for all values of $x$ with one exception, namely, $x$ cannot equal zero. Hence, every real number except $x=0$ is a solution of $|x|>0$. In **Figure 5**(b), we've shaded the solution of $|x|>0$, namely the set of all real numbers except $x=0$.

- Case III: $a>0$

  In this case, the graph of $y=a$ lies strictly above the $x$-axis. In **Figure 5**(c), the graph of $y=|x|$ intersects the graph of $y=a$ at $x=-a$ and $x=a$. In **Figure 5**(c), we see that the graph of $y=|x|$ lies strictly above the graph of $y=a$ if $x$ is less than $-a$ or greater than $a$.

  In **Figure 5**(c), we've dropped dashed vertical lines from the points of intersection to the $x$-axis. On the $x$-axis, we've shaded the solution of $|x|>a$, namely the set of all real numbers $x$ such that $x<-a$ or $x>a$.

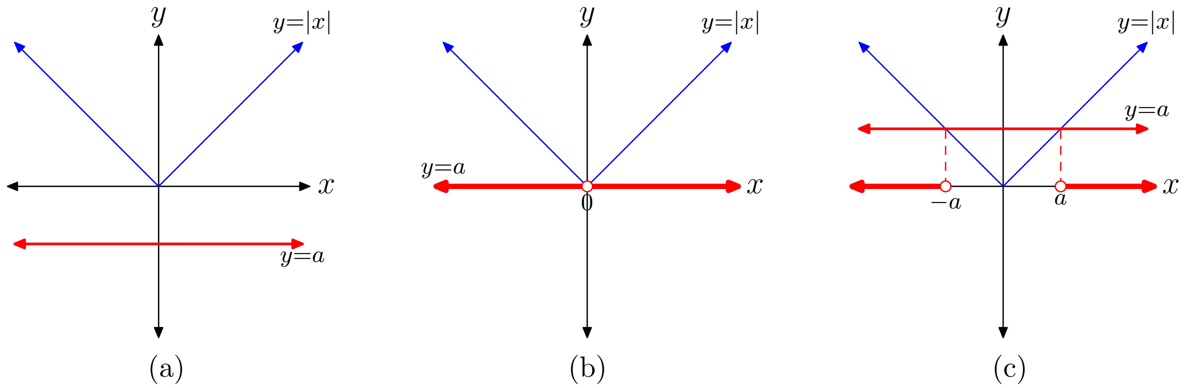


**Figure 5.** The solution of $|x| > a$ has three cases.

This discussion leads to the following property.

**Property 12.** The solution of $|x| > a$ depends upon the value and sign of $a$.

- Case I: $a < 0$

  All real numbers are solutions of the inequality $|x| > a$.

- Case II: $a = 0$

  All real numbers, with the exception of $x = 0$, are solutions of $|x| > 0$.

- Case III: $a > 0$

  The inequality $|x| > a$ has solution set $\{x : x < -a \text{ or } x > a\}$.

▶ **Example 13.** *State the solution of each of the following inequalities.*

a. $|x| > -5$   b. $|x| > 0$   c. $|x| > 4$

**Solution**:

a. The solution of $|x| > -5$ is all real numbers.
b. The solution of $|x| > 0$ is all real numbers except zero.
c. The solution of $|x| > 4$ is the set of all real numbers less than $-4$ or greater than 4.

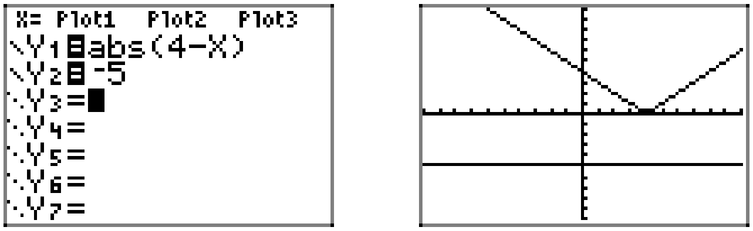
◆

▶ **Example 14.** *Solve the inequality* $|4 - x| > -5$ *for* $x$.

The left-hand side of the inequality $|4 - x| > -5$ is nonnegative, so the graph of $y = |4 - x|$ must lie above or on the $x$-axis. The graph of the right-hand side of $|4 - x| > -5$ is a horizontal line located 5 units below the $x$-axis. Therefore, the graph of $y = |4 - x|$ **always** lies above the graph of $y = -5$. Thus, all real numbers are solutions of the inequality $|4 - x| > -5$.

We can verify our thinking with the graphing calculator. Load the left- and right-hand sides of the inequality $|4 - x| > -5$ into `Y1` and `Y2`, respectively, as shown in **Figure 6**(a). From the `ZOOM` menu, select `6:ZStandard` to produce the image shown in **Figure 6**(b).

As predicted, the graph of $y = |4 - x|$ lies above the graph of $y = -5$ for all real numbers.



```
Plot1  Plot2  Plot3
\Y1=abs(4-X)
\Y2=-5
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

(a) (b)

**Figure 6.** Using the graphing calculator to solve $|4 - x| > -5$.

Intuitively, the absolute value of any number is always nonnegative, so $|4-x| > -5$ for all real values of $x$.

◆

▶ **Example 15.** *Solve the inequality* $|4 - x| > 0$ *for* $x$.

As we saw in **Figure 6**(b), the graph of $y = |4 - x|$ lies on or above the $x$-axis for all real numbers. It "touches" the $x$-axis at the "vertex" of the "V," where

$$|4 - x| = 0.$$

This can occur only if

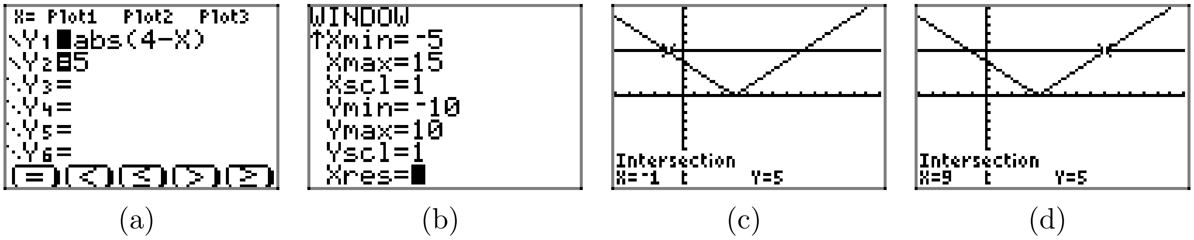
$$\begin{aligned} 4 - x &= 0 \\ -x &= -4 \\ x &= 4. \end{aligned}$$

Thus, the graph of $y = |4 - x|$ is strictly above the $x$-axis for all real numbers except $x = 4$. That is, the solution of $|4 - x| > 0$ is $\{x : x \neq 4\}$.

◆

▶ **Example 16.** *Solve the inequality* $|4 - x| > 5$ *for* $x$.

In this example, the graph of the right-hand side of $|4 - x| > 5$ is a horizontal line located 5 units above the $x$-axis. The graph of $y = |4 - x|$ has the "V" shape shown in **Figure 6**(c). You can use the intersect utility on the graphing calculator to approximate the points of intersection of the graphs of $y = |4 - x|$ and $y = 5$, as we have done in **Figure 7**(c) and (d). The calculator indicates two points of intersection, one at $x = -1$ and a second at $x = 9$.



```
Plot1 Plot2 Plot3
\Y1■abs(4-X)
\Y2■5
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
Xmin=-5
Xmax=15
Xscl=1
Ymin=-10
Ymax=10
Yscl=1
Xres=■
```

```
Intersection
X=-1          Y=5
```

```
Intersection
X=9           Y=5
```

(a) (b) (c) (d)

**Figure 7.** Using the graphing calculator to solve the inequality $|4 - x| > 5$.

The graph of $y = |4 - x|$ lies **above** the graph of $y = 5$ for all values of $x$ that lie either to the left of $-1$ or to the right of 9. Hence, the solution of $|4 - x| > 5$ is the set $\{x : x < -1 \text{ or } x > 9\}$.

Following the guidelines established in **Example 7**, we create the image shown in **Figure 8** on our homework paper. Note that we've labeled each axis, scaled each axis with `xmin`, `xmax`, `ymin`, and `ymax`, labeled each graph with its equation, and shaded and labeled the solution on $x$-axis.

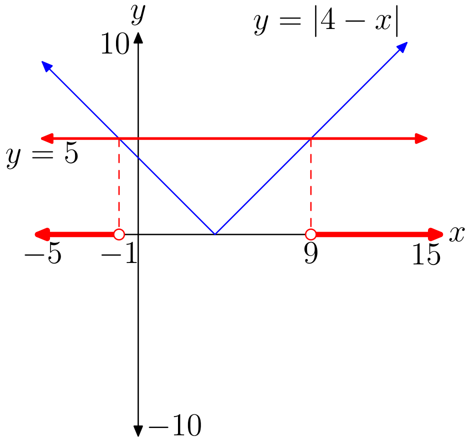


**Figure 8.** Reporting a graphical solution of $|4 - x| > 5$.

**Algebraic Approach**. Let's explore an algebraic solution of $|4 - x| > 5$. In much the same manner that $|x| > 5$ leads to the conditions $x < -5$ or $x > 5$, the inequality

$$|4 - x| > 5$$

requires that

$$4 - x < -5 \quad \text{or} \quad 4 - x > 5.$$

We can solve each of these independently by first subtracting 4 from each side of the inequality, then multiplying both sides of each inequality by $-1$, reversing each inequality as we do so.

$$\begin{aligned} 4 - x &< -5 &\quad\text{or}\quad&& 4 - x &> 5 \\ -x &< -9 &&& -x &> 1 \\ x &> 9 &&& x &< -1 \end{aligned}$$

We prefer to write this solution in the order

$$x < -1 \qquad \text{or} \qquad x > 9,$$

as it then matches the order of the graphical solution shaded in **Figure 8**. That is, the solution set is $\{x : x < -1 \quad \text{or} \quad x > 9\}$.

◆

The algebraic technique of this last example leads to the following property.

**Property 17.** If $a > 0$, then the inequality $|x| > a$ is equivalent to the compound inequality $x < -a$ or $x > a$.

This property provides a simple algebraic technique for solving inequalities of the form $|x| > a$, when $a > 0$. Let's concentrate on this technique in the examples that follow.

▶ **Example 18.** *Solve the inequality* $|4x - 3| > 1$ *for* $x$.

The first step is to use **Property 17** to write that

$$|4x - 3| > 1$$

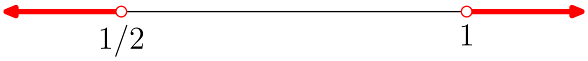
is equivalent to

$$4x - 3 < -1 \qquad \text{or} \qquad 4x - 3 > 1.$$

We can now solve each inequality independently. We begin by adding 3 to both sides of each inequality, then we divide both sides of the resulting inequalities by 4.

$$\begin{aligned} 4x - 3 &< -1 &\quad\text{or}\quad&& 4x - 3 &> 1 \\ 4x &< 2 &&& 4x &> 4 \\ x &< \frac{1}{2} &&& x &> 1 \end{aligned}$$

We can sketch the solutions on a number line.



And we can describe the solution using interval and set-builder notation.

$$(-\infty, 1/2) \cup (1, \infty) = \{x : \; x < 1/2 \text{ or } x > 1\}$$

◆

Again, let $a > 0$. As we did with $|x| \le a$, we can take the union of the solutions of $|x| = a$ and $|x| > a$ to find the solution of $|x| \ge a$. This leads to the following property.

**Property 19.** If $a > 0$, then the inequality $|x| \ge a$ is equivalent to the inequality $x \le -a$ or $x \ge a$.

▶ **Example 20.** *Solve the inequality* $3|1-x| - 4 \ge |1-x|$ *for* $x$.

Again, at first glance, the inequality

$$3|1-x| - 4 \ge |1-x|$$

looks unlike any inequality we've attempted to this point. However, if we subtract $|1-x|$ from both sides of the inequality, then add 4 to both sides of the inequality, we get

$$3|1-x| - |1-x| \ge 4.$$

On the left, we have like terms. Note that $3|1-x|-|1-x| = 3|1-x|-1|1-x| = 2|1-x|$. Thus,

$$2|1-x| \ge 4.$$

Divide both sides of the last inequality by 2.

$$|1-x| \ge 2$$

We can now use **Property 19** to write

$$1 - x \le -2 \qquad \text{or} \qquad 1 - x \ge 2.$$

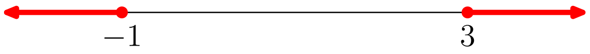
We can solve each of these inequalities independently. First, subtract 1 from both sides of each inequality, then multiply both sides of each resulting inequality by $-1$, reversing each inequality as you go.

$$\begin{aligned} 1 - x &\le -2 \\ -x &\le -3 \\ x &\ge 3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 1 - x &\ge 2 \\ -x &\ge 1 \\ x &\le -1 \end{aligned}$$

We prefer to write this in the order

$$x \le -1 \qquad \text{or} \qquad x \ge 3.$$

We can sketch the solutions on a number line.



And we can describe the solutions using interval and set-builder notation.

$$(-\infty, -1] \cup [3, \infty) = \{x : x \leq -1 \text{ or } x \geq 3\}$$

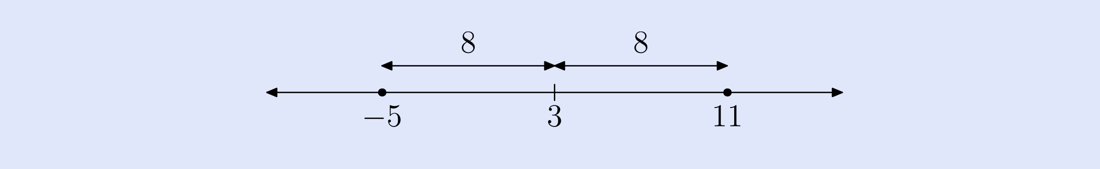
◆

## *Revisiting Distance*

If $a$ and $b$ are any numbers on the real line, then the distance between $a$ and $b$ is found by taking the absolute value of their difference. That is, the distance $d$ between $a$ and $b$ is calculated with $d = |a - b|$. More importantly, we've learned to pronounce the symbolism $|a - b|$ as "the distance between $a$ and $b$." This pronunciation is far more useful than saying "the absolute value of $a$ minus $b$."

► **Example 21.** *Solve the inequality* $|x - 3| < 8$ *for* $x$.

This inequality is pronounced "the distance between $x$ and 3 is less than 8." Draw a number line, locate 3 on the line, then note two points that are 8 units away from 3.



Now, we need to shade the points that are *less than* 8 units from 3.

Hence, the solution of the inequality $|x - 3| < 8$ is

$$(-5, 11) = \{x : -5 < x < 11\}.$$

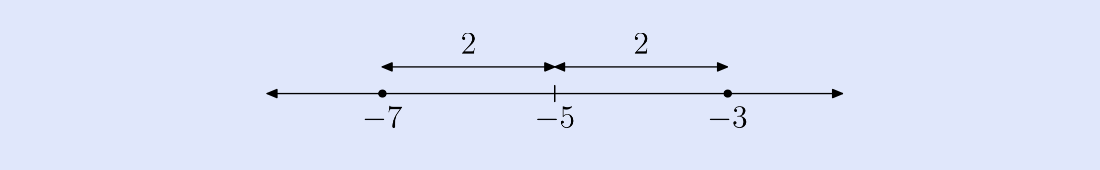
◆

► **Example 22.** *Solve the inequality* $|x + 5| > 2$ *for* $x$.

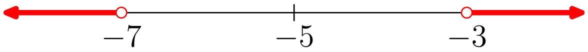
First, write the inequality as a difference.

$$|x - (-5)| > 2$$

This last inequality is pronounced "the distance between $x$ and $-5$ is greater than 2." Draw a number line, locate $-5$ on the number line, then note two points that are 2 units from $-5$.



Now, we need to shade the points that are *greater than* 2 units from −5.



Hence, the solution of the inequality $|x + 5| > 2$ is

$$(-\infty, -7) \cup (-3, \infty) = \{x : \ x < -7 \quad \text{or} \quad x > -3\}.$$

◆

## 4.4 Exercises

For each of the inequalities in **Exercises 1-10**, perform each of the following tasks.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.
ii. Sketch the graph of each side of the inequality without the aid of a calculator. Label each graph with its equation.
iii. Shade the solution of the inequality on the $x$-axis (if any) in the manner shown in Figures 4 and 8 in the narrative. That is, drop dashed lines from the points of intersection to the axis, then shade and label the solution set on the $x$-axis. Use set-builder and interval notation (when possible) to describe your solution set.

**1.** $|x| > -2$

**2.** $|x| > 0$

**3.** $|x| < 3$

**4.** $|x| > 2$

**5.** $|x| > 1$

**6.** $|x| < 4$

**7.** $|x| \leq 0$

**8.** $|x| \leq -2$

**9.** $|x| \leq 2$

**10.** $|x| \geq 1$

For each of the inequalities in **Exercises 11-22**, perform each of the following tasks.

i. Load each side of the inequality into the `Y=` menu of your calculator. Adjust the viewing window so that all points of intersection of the two graphs are visible in the viewing window.
ii. Copy the image in your viewing screen onto your homework paper. Label each axis and scale each axis with xmin, xmax, ymin, and ymax. Label each graph with its equation.
iii. Use the `intersect` utility in the `CALC` menu to determine the points of intersection. Shade the solution of the inequality on the $x$-axis (if any) in the manner shown in Figures 4 and 8 in the narrative. That is, drop dashed lines from the points of intersection to the axis, then shade and label the solution set on the $x$-axis. Use set-builder and interval notation (when appropriate) to describe your solution set.

**11.** $|3 - 2x| > 5$

**12.** $|2x + 7| < 4$

**13.** $|4x + 5| < 7$

**14.** $|5x - 7| > 8$

**15.** $|4x + 5| > -2$

**16.** $|3x - 5| < -3$

**17.** $|2x - 9| \geq 6$

**18.** $|3x + 25| \geq 8$

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[11] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

**19.** $|13 - 2x| \leq 7$

**20.** $|2x + 15| \leq 7$

**21.** $|3x - 11| > 0$

**22.** $|4x + 19| \leq 0$

---

For each of the inequalities in **Exercises 23-32**, provide a purely algebraic solution without the use of a calculator. Show all of your work that leads to the solution, shade your solution set on a number line, then use set-builder and interval notation (if possible) to describe your solution set.

**23.** $|4x + 3| < 8$

**24.** $|3x - 5| > 11$

**25.** $|2x - 3| \leq 10$

**26.** $|3 - 5x| \geq 15$

**27.** $|3x - 4| < 7$

**28.** $|5 - 2x| > 10$

**29.** $|3 - 7x| \geq 5$

**30.** $|2 - 11x| \leq 6$

**31.** $|x + 2| \geq -3$

**32.** $|x + 5| < -4$

---

For each of the inequalities in **Exercises 33-38**, perform each of the following tasks.

i. Arrange each of the following parts on your homework paper in the same location. Do not do place the algebraic work on one page and the graphical work on another.
ii. Follow each of the directions given for **Exercises 11-22** to find and record a solution with your graphing calculator.
iii. Provide a purely algebraic solution, showing all the steps of your work. Sketch your solution on a number line, then use set-builder and interval notation to describe your solution set. Do these solutions compare favorably with those found using your graphing calculator in part (ii)? If not, look for a mistake in your work.

**33.** $|x - 8| < 7$

**34.** $|2x - 15| > 5$

**35.** $|2x + 11| \geq 6$

**36.** $|5x - 21| \leq 7$

**37.** $|x - 12| > 6$

**38.** $|x + 11| < 5$

---

Use a strictly algebraic technique to solve each of the equations in **Exercises 39-46**. Do not use a calculator. Shade the solution set on a number line and describe the solution set using both set-builder and interval notation.

**39.** $|x + 2| - 3 > 4$

**40.** $3|x + 5| < 6$

**41.** $-2|3 - 2x| \leq -6$

**42.** $|4 - x| + 5 \geq 12$

**43.** $3|x + 2| - 5 > |x + 2| + 7$

**44.** $4 - 3|4 - x| > 2|4 - x| - 1$

**45.** $\left|\dfrac{x}{3} - \dfrac{1}{4}\right| \leq \dfrac{1}{12}$

**46.** $\left|\dfrac{x}{4} - \dfrac{1}{2}\right| \geq \dfrac{2}{3}$

Use the technique of distance on the number line demonstrated in Examples 21 and 22 to solve each of the inequalities in **Exercises 47-50**. Provide number line sketches as in Example 17 in the narrative. Describe the solution set using both set-builder and interval notation.

**47.** $|x-5|<8$

**48.** $|x-2|>4$

**49.** $|x+4|\geq 3$

**50.** $|x+2|\leq 11$

Use the instructions provided in **Exercises 11-22** to solve the inequalities in **Exercises 51-52**. Describe the solution set using both set-builder and interval notation.

**51.** $|x+2|<\dfrac{1}{3}x+5$

**52.** $|x-3|>5-\dfrac{1}{2}x$

In **Exercises 53-54**, perform each of the following tasks.

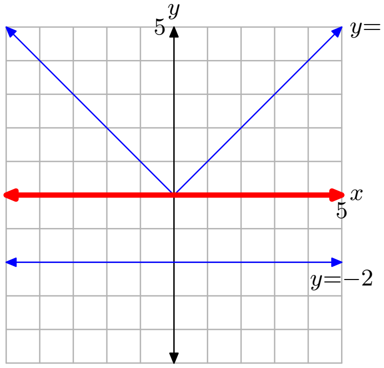
i. Set up a coordinate system on graph paper. Label and scale each axis.
ii. Without the use of a calculator, sketch the graphs of the left- and right-hand sides of the given inequality. Label each graph with its equation.
iii. Shade the solution of the inequality on the $x$-axis (if any) in the manner shown in Figures 4 and 8 in the narrative. That is, drop dashed lines from the points of intersection to the axis, then shade and label the solution set on the $x$-axis (you will have to approximate). Describe the solution set using both set-builder and interval notation.

**53.** $|x-2|>\dfrac{1}{3}x+2$

**54.** $|x+4|<\dfrac{1}{3}x+4$

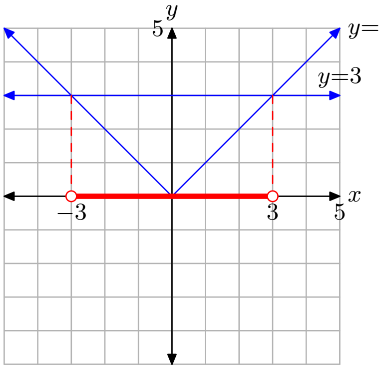
## 4.4 Answers

**1.**



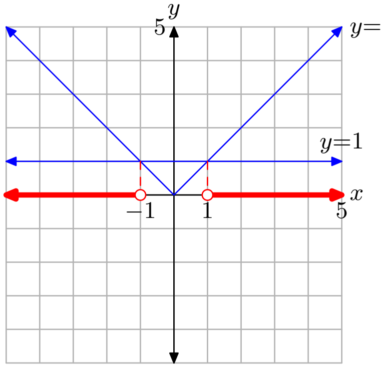
Solution: $\mathbb{R} = (-\infty, \infty)$

**3.**



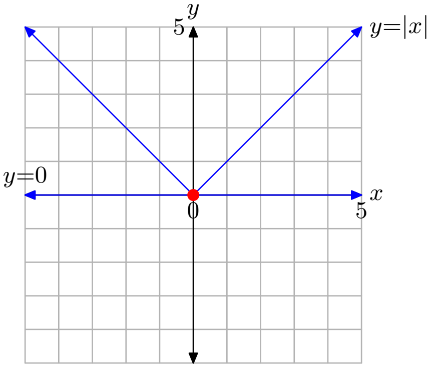
Solution: $(-3, 3) = \{x : -3 < x < 3\}$.

**5.**



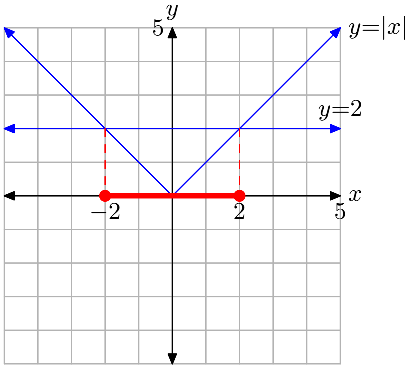
Solution: $(-\infty, -1) \cup (1, \infty) = \{x : x < -1 \text{ or } x > 1\}$.

**7.**



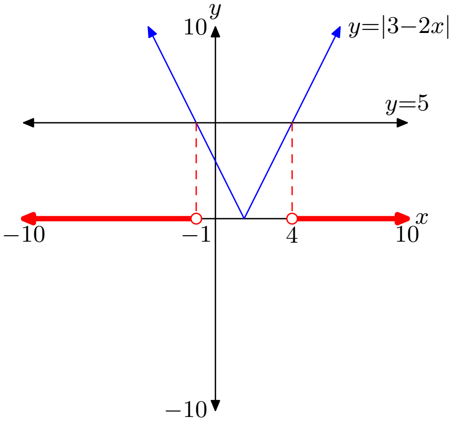
Solution: $\{x : x = 0\}$.

**9.**



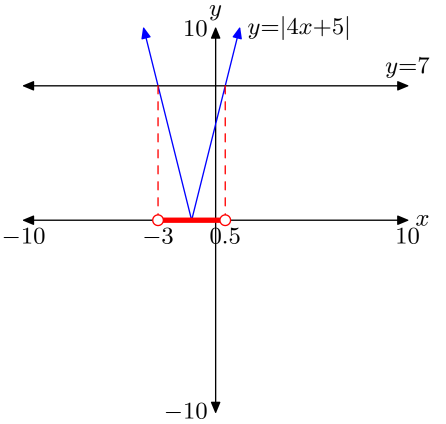
Solution: $[-2, 2] = \{x : -2 \leq x \leq 2\}$.

**11.**



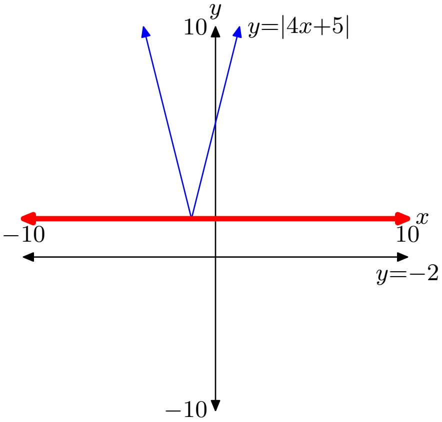
Solution: $(-\infty,-1)\cup(4,\infty)=\{x : x < -1 \text{ or } x > 4\}$.

**13.**



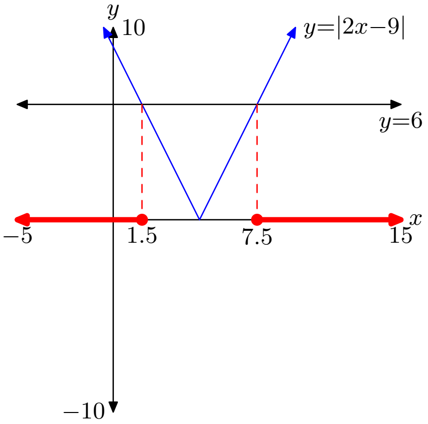
Solution: $(-3,0.5)=\{x : -3 < x < 0.5\}$.

**15.**



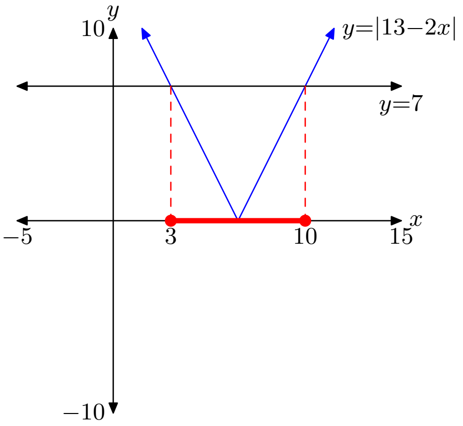
Solution: $\mathbb{R}=(-\infty,\infty)$.

**17.**



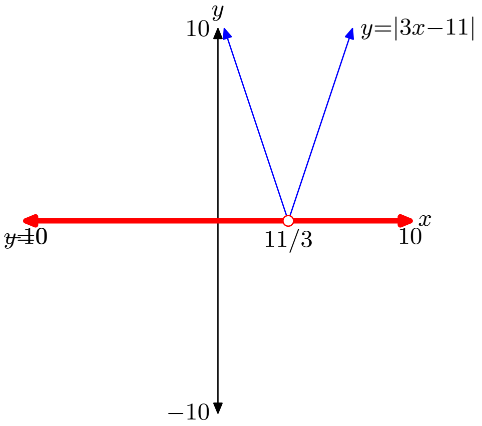
Solution: $(-\infty,1.5]\cup[7.5,\infty)=\{x : x \leq 1.5 \text{ or } x \geq 7.5\}$.

**19.**



Solution: $[3, 10] = \{x : 3 \leq x \leq 10\}$.

**21.**



Solution: $\{x : x \neq 11/3\}$.

**23.**

$(-11/4, 5/4) = \{x : -11/4 < x < 5/4\}$

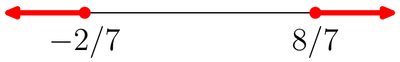
**25.**

$[-7/2, 13/2] = \{x : -7/2 \leq x \leq 13/2\}$

**27.**

$(-1, 11/3) = \{x : -1 < x < 11/3\}$

**29.**

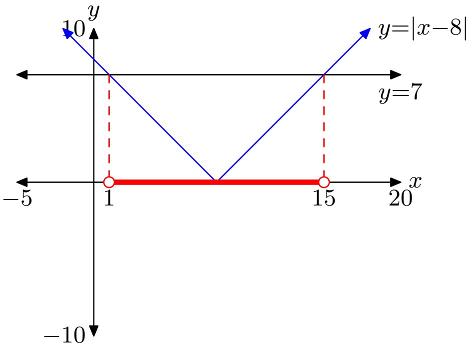


$(-\infty, -2/7] \cup [8/7, \infty) = \{x : x \leq -2/7 \text{ or } x \geq 8/7\}$

**31.**

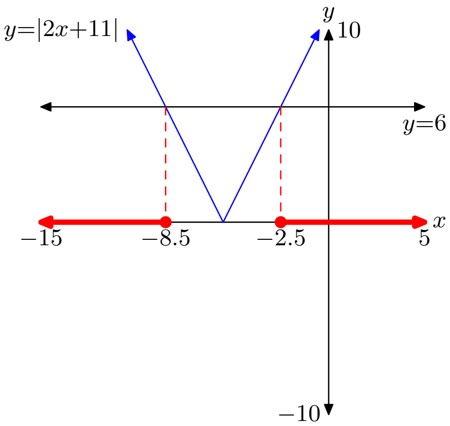
$\mathbb{R} = (-\infty, \infty)$

**33.**



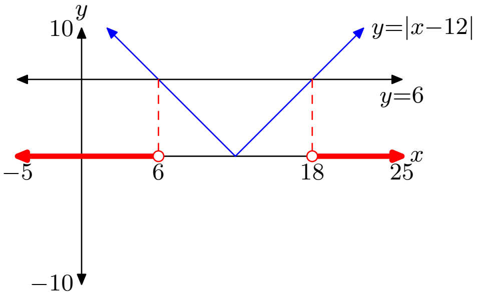
$(1, 15) = \{x : 1 < x < 15\}$

**35.**



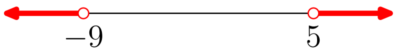
$(-\infty,-8.5]\cup[-2.5,\infty)=\{x:\ x\le -8.5 \text{ or } x\ge -2.5\}$

**37.**



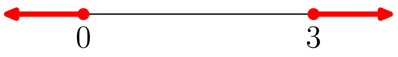
$(-\infty,6)\cup(18,\infty)=\{x:\ x<6 \text{ or } x>18\}$

**39.**



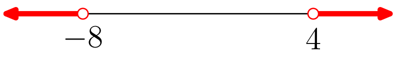
$(-\infty,-9)\cup(5,\infty)=\{x:\ x<-9 \text{ or } x>5\}$

**41.**



$(-\infty,0]\cup[3,\infty)=\{x:\ x\le 0 \text{ or } x\ge 3\}$

**43.**

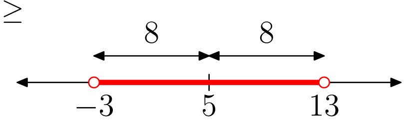


$(-\infty,-8)\cup(4,\infty)=\{x:\ x<-8 \text{ or } x>4\}$

**45.**

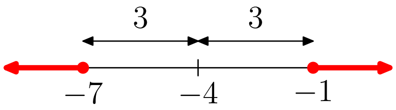
$[1/2,1]=\{x:\ 1/2\le x\le 1\}$

**47.**



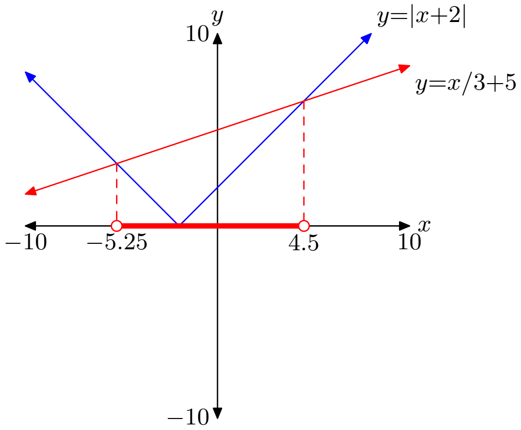
$(-3,13)=\{x:\ -3<x<13\}$

**49.**



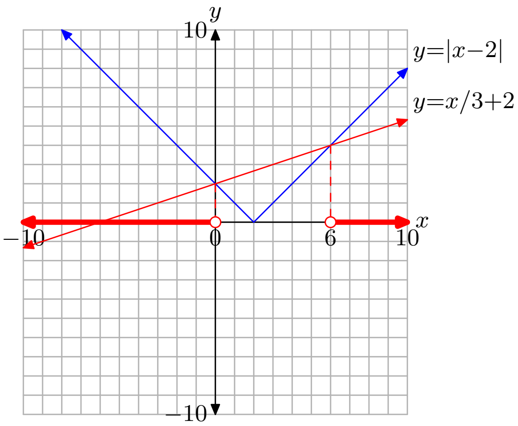
$(-\infty,-7]\cup[-1,\infty)=\{x:\ x\le -7 \text{ or } x\ge -1\}$

**51.**



$(-5.25,4.5)=\{x:\ -5.25<x<4.5\}$

**53.**

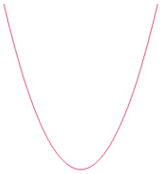


$(-\infty, 0) \cup (6, \infty) = \{x : x < 0 \text{ or } x > 6\}$

## 4.5 Index

# 5 Quadratic Functions

In this chapter we study one of the most famous of mathematical concepts–the *parabola*. The most basic parabola is shaped rather like a "U," as shown in the margin. Whereas the graphs of linear functions like $f(x) = mx + b$ are lines, the graphs of functions having the form



A parabola.

$$f(x) = ax^2 + bx + c, \tag{1}$$

where $a$, $b$, and $c$ are arbitrary numbers, are parabolas. These functions are called *quadratic functions*.

Apollonius (262 BC to 190 BC) wrote the quintessential text on the conic sections–of which the parabola is one–and is credited with giving the parabola its name.

In nature, approximations of parabolas are found in many diverse situations. Early in the 17th century, the parabolic trajectory of projectiles was discovered experimentally by Galileo (1564 to 1642), who performed experiments with balls rolling on inclined planes. The parabolic shape for projectiles was later proven mathematically by Isaac Newton (1643 to 1727). He found that, if we assume that there is no air resistance, parabolas can be used to model the trajectory of a body in motion under the influence of gravity (for instance, a rock flying through the air, neglecting air friction). We will study this application in detail in Section 5.5.

Parabolic arches in Las Vegas fountains.

Other applications of parabolas include the modeling of suspension bridges; the shapes of satellite dishes, heaters, and automobile headlights; braking distance and stopping distance of cars; and the path of water projected from a fountain, like at the water show at the Bellagio Hotel in Las Vegas.

## Table of Contents

# 5.1 The Parabola

In this section you will learn how to draw the graph of the quadratic function defined by the equation

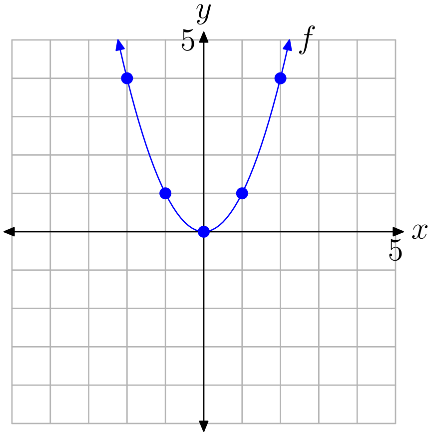
$$f(x) = a(x - h)^2 + k. \tag{1}$$

You will quickly learn that the graph of the quadratic function is shaped like a "U" and is called a *parabola*. The form of the quadratic function in **equation (1)** is called *vertex form*, so named because the form easily reveals the *vertex* or "turning point" of the parabola. Each of the constants in the vertex form of the quadratic function plays a role. As you will soon see, the constant $a$ controls the scaling (stretching or compressing of the parabola), the constant $h$ controls a horizontal shift and placement of the *axis of symmetry*, and the constant $k$ controls the vertical shift.

Let's begin by looking at the *scaling* of the quadratic.

## Scaling the Quadratic

The graph of the basic quadratic function $f(x) = x^2$ shown in **Figure 1**(a) is called a *parabola*. We say that the parabola in **Figure 1**(a) "opens upward." The point at $(0, 0)$, the "turning point" of the parabola, is called the *vertex* of the parabola. We've tabulated a few points for reference in the table in **Figure 1**(b) and then superimposed these points on the graph of $f(x) = x^2$ in **Figure 1**(a).



(a) A basic parabola.

| $x$ | $f(x) = x^2$ |
|---|---|
| $-2$ | 4 |
| $-1$ | 1 |
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |

(b) Table of $x$-values and function values satisfying $f(x) = x^2$.

**Figure 1.** The graph of the basic parabola is a fundamental starting point.

Now that we know the basic shape of the parabola determined by $f(x) = x^2$, let's see what happens when we scale the graph of $f(x) = x^2$ in the vertical direction. For

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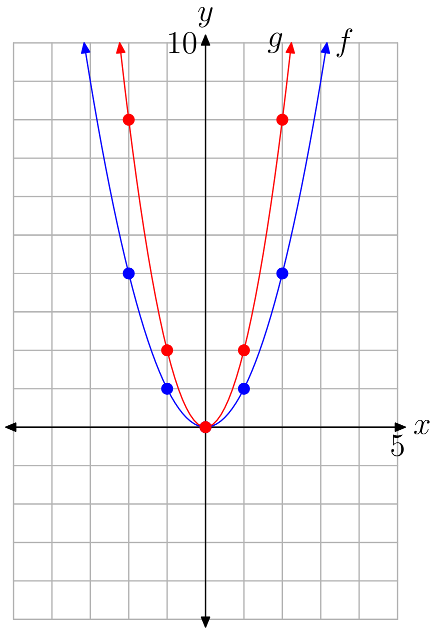
[1] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

example, let's investigate the graph of $g(x) = 2x^2$. The factor of 2 has a doubling effect. Note that each of the function values of $g$ is twice the corresponding function value of $f$ in the table in **Figure 2**(b).



(a) The graphs of $f$ and $g$.

| $x$ | $f(x) = x^2$ | $g(x) = 2x^2$ |
|---|---|---|
| $-2$ | 4 | 8 |
| $-1$ | 1 | 2 |
| 0 | 0 | 0 |
| 1 | 1 | 2 |
| 2 | 4 | 8 |

(b) Table of $x$-values and function values satisfying $f(x) = x^2$ and $g(x) = 2x^2$.

**Figure 2.** A stretch by a factor of 2 in the vertical direction.

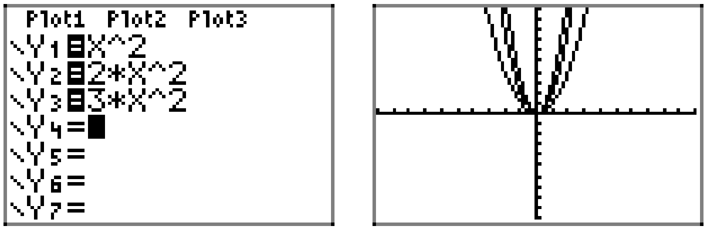
When the points in the table in **Figure 2**(b) are added to the coordinate system in **Figure 2**(a), the resulting graph of $g$ is stretched by a factor of two in the vertical direction. It's as if we had put the original graph of $f$ on a sheet of rubber graph paper, grabbed the top and bottom edges of the sheet, and then pulled each edge in the vertical direction to stretch the graph of $f$ by a factor of two. Consequently, the graph of $g(x) = 2x^2$ appears somewhat narrower in appearance, as seen in comparison to the graph of $f(x) = x^2$ in **Figure 2**(a). Note, however, that the vertex at the origin is unaffected by this scaling.

In like manner, to draw the graph of $h(x) = 3x^2$, take the graph of $f(x) = x^2$ and *stretch* the graph by a factor of three, tripling the $y$-value of each point on the original graph of $f$. This idea leads to the following result.

**Property 2.** If $a$ is a constant larger than 1, that is, if $a > 1$, then the graph of $g(x) = ax^2$, when compared with the graph of $f(x) = x^2$, is *stretched* by a factor of $a$.

▶ **Example 3.** *Compare the graphs of* $y = x^2$, $y = 2x^2$, *and* $y = 3x^2$ *on your graphing calculator.*

Load the functions $y = x^2$, $y = 2x^2$, and $y = 3x^2$ into the `Y=` menu, as shown in **Figure 3**(a). Push the `ZOOM` button and select `6:ZStandard` to produce the image shown in **Figure 3**(b).



```
Plot1 Plot2 Plot3
\Y1=X^2
\Y2=2*X^2
\Y3=3*X^2
\Y4=
\Y5=
\Y6=
\Y7=
```

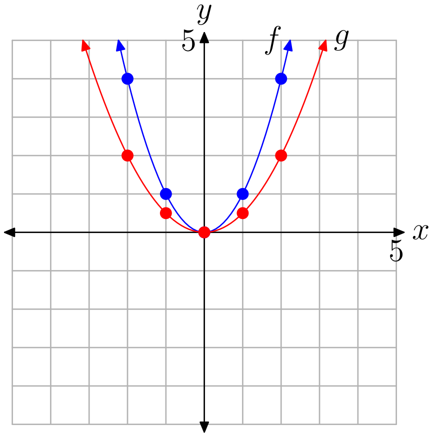
(a) (b)

**Figure 3.** Drawing $y = x^2$, $y = 2x^2$, and $y = 3x^2$ on the graphing calculator.

Note that as the "$a$" in $y = ax^2$ increases from 1 to 2 to 3, the graph of $y = ax^2$ stretches further and becomes, in a sense, narrower in appearance.

◇

Next, let's consider what happens when we scale by a number that is smaller than 1 (but greater than zero — we'll deal with the negative in a moment). For example, let's investigate the graph of $g(x) = (1/2)x^2$. The factor 1/2 has a halving effect. Note that each of the function values of $g$ is half the corresponding function value of $f$ in the table in **Figure 4**(b).



(a) The graphs of $f$ and $g$.

| $x$ | $f(x) = x^2$ | $g(x) = (1/2)x^2$ |
|---|---|---|
| −2 | 4 | 2 |
| −1 | 1 | 1/2 |
| 0 | 0 | 0 |
| 1 | 1 | 1/2 |
| 2 | 4 | 2 |

(a) Table of $x$-values and function values satisfying $f(x) = x^2$ and $g(x) = (1/2)x^2$.

**Figure 4.** A compression by a factor of 2 in the vertical direction.

When the points in the table in **Figure 4**(b) are added to the coordinate system in **Figure 4**(a), the resulting graph of $g$ is compressed by a factor of 2 in the vertical direction. It's as if we again placed the graph of $f(x) = x^2$ on a sheet of rubber graph paper, grabbed the top and bottom of the sheet, and then *squeezed* them together by a factor of two. Consequently, the graph of $g(x) = (1/2)x^2$ appears somewhat wider in appearance, as seen in comparison to the graph of $f(x) = x^2$ in **Figure 4**(a). Note again that the vertex at the origin is unaffected by this scaling.

> **Property 4.** If $a$ is a constant smaller than 1 (but larger than zero), that is, if $0 < a < 1$, then the graph of $g(x) = ax^2$, when compared with the graph of $f(x) = x^2$, is *compressed* by a factor of $1/a$.

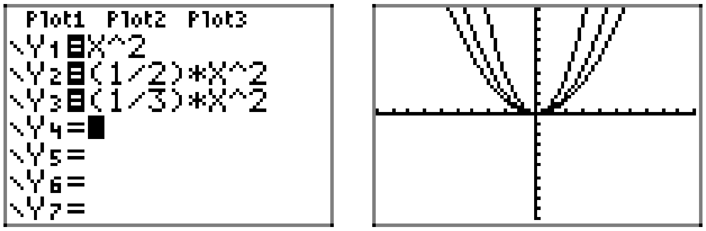
Some find **Property 4** somewhat counterintuitive. However, if you compare the function $g(x) = (1/2)x^2$ with the general form $g(x) = ax^2$, you see that $a = 1/2$. Property **4** states that the graph will be compressed by a factor of $1/a$. In this case, $a = 1/2$ and

$$\frac{1}{a} = \frac{1}{1/2} = 2.$$

Thus, **Property 4** states that the graph of $g(x) = (1/2)x^2$ should be compressed by a factor of $1/(1/2)$ or 2, which is seen to be the case in **Figure 4**(a).

► **Example 5.** *Compare the graphs of $y = x^2$, $y = (1/2)x^2$, and $y = (1/3)x^2$ on your graphing calculator.*

Load the equations $y = x^2$, $y = (1/2)x^2$, and $y = (1/3)x^2$ into the `Y=`, as shown in **Figure 5**(a). Push the `ZOOM` button and select `6:ZStandard` to produce the image shown in **Figure 5**(b).



```
Plot1 Plot2 Plot3
\Y1=X^2
\Y2=(1/2)*X^2
\Y3=(1/3)*X^2
\Y4=
\Y5=
\Y6=
\Y7=
```

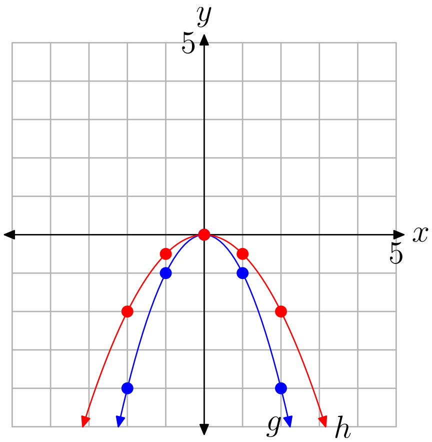
(a) (b)

**Figure 5.** Drawing $y = x^2$, $y = (1/2)x^2$, and $y = (1/3)x^2$ on the graphing calculator.

Note that as the "$a$" in $y = ax^2$ decreases from 1 to 1/2 to 1/3, the graph of $y = ax^2$ compresses further and becomes, in a sense, wider in appearance.

## Vertical Reflections

Let's consider the graph of $g(x) = ax^2$, when $a < 0$. For example, consider the graphs of $g(x) = -x^2$ and $h(x) = (-1/2)x^2$ in **Figure 6**.



(a) The graphs of $g$ and $h$.

| $x$ | $g(x) = -x^2$ | $h(x) = (-1/2)x^2$ |
|---|---|---|
| $-2$ | $-4$ | $-2$ |
| $-1$ | $-1$ | $-1/2$ |
| $0$ | $0$ | $0$ |
| $1$ | $-1$ | $-1/2$ |
| $2$ | $-4$ | $-2$ |

(b) Table of $x$-values and function values satisfying $g(x) = -x^2$ and $h(x) = (-1/2)x^2$.

**Figure 6.** A vertical reflection across the $x$-axis.

When the table in **Figure 6**(b) is compared with the table in **Figure 4**(b), it is easy to see that the numbers in the last two columns are the same, but they've been negated. The result is easy to see in **Figure 6**(a). The graphs have been reflected across the $x$-axis. Each of the parabolas now "opens downward."

However, it is encouraging to see that the scaling role of the constant $a$ in $g(x) = ax^2$ has not changed. In the case of $h(x) = (-1/2)x^2$, the $y$-values are still "compressed" by a factor of two, but the minus sign negates these values, causing the graph to reflect across the $x$-axis. Thus, for example, one would think that the graph of $y = -2x^2$ would be *stretched* by a factor of two, then reflected across the $x$-axis. Indeed, this is correct, and this discussion leads to the following property.

**Property 6.** If $-1 < a < 0$, then the graph of $g(x) = ax^2$, when compared with the graph of $f(x) = x^2$, is compressed by a factor of $1/|a|$, then reflected across the $x$-axis. Secondly, if $a < -1$, then the graph of $g(x) = ax^2$, when compared with the graph of $f(x) = x^2$, is stretched by a factor of $|a|$, then reflected across the $x$-axis.

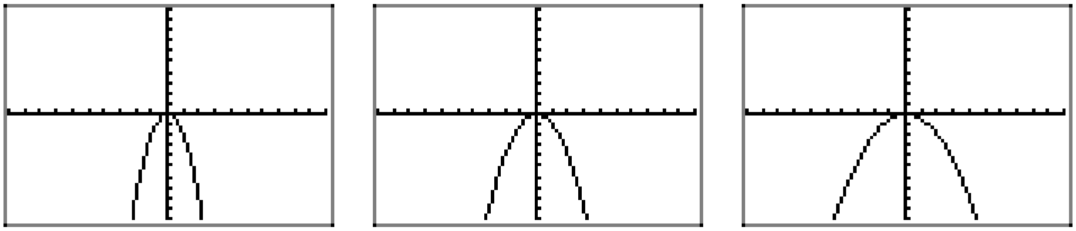
Again, some find **Property 6** confusing. However, if you compare $g(x) = (-1/2)x^2$ with the general form $g(x) = ax^2$, you see that $a = -1/2$. Note that in this case, $-1 < a < 0$. **Property 6** states that the graph will be compressed by a factor of $1/|a|$. In this case, $a = -1/2$ and

$$\frac{1}{|a|} = \frac{1}{|-1/2|} = 2.$$

That is, **Property 6** states that the graph of $g(x) = (-1/2)x^2$ is compressed by a factor of $1/(|-1/2|)$, or 2, then reflected across the $x$-axis, which is seen to be the case in **Figure 6**(a). Note again that the vertex at the origin is unaffected by this scaling and reflection.

▶ **Example 7.** *Sketch the graphs of* $y = -2x^2$, $y = -x^2$, *and* $y = (-1/2)x^2$ *on your graphing calculator.*

Each of the equations were loaded separately into `Y1` in the `Y=` menu. In each of the images in **Figure 7**, we selected `6:ZStandard` from the `ZOOM` menu to produce the image.



(a) $y = -2x^2$ (b) $y = -x^2$ (c) $y = (-1/2)x^2$

**Figure 7.**

In **Figure 7**(b), the graph of $y = -x^2$ is a reflection of the graph of $y = x^2$ across the $x$-axis and opens downward. In **Figure 7**(a), note that the graph of $y = -2x^2$ is stretched vertically by a factor of 2 (compare with the graph of $y = -x^2$ in **Figure 7**(b)) and reflected across the $x$-axis to open downward. In **Figure 7**(c), the graph of $(-1/2)x^2$ is compressed by a factor of 2, appears a bit wider, and is reflected across the $x$-axis to open downward.

◆

## *Horizontal Translations*

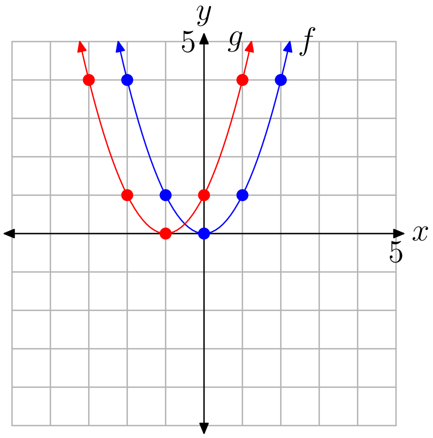
The graph of $g(x) = (x+1)^2$ in **Figure 8**(a) shows a basic parabola that is shifted one unit to the left. Examine the table in **Figure 8**(b) and note that the equation $g(x) = (x+1)^2$ produces the same $y$-values as does the equation $f(x) = x^2$, the only difference being that these $y$-values are calculated at $x$-values that are one unit less than those used for $f(x) = x^2$. Consequently, the graph of $g(x) = (x+1)^2$ must shift one unit to the left of the graph of $f(x) = x^2$, as is evidenced in **Figure 8**(a).

Note that this result is counterintuitive. One would think that replacing $x$ with $x+1$ would shift the graph one unit to the right, but the shift actually occurs in the opposite direction.

Finally, note that this time the vertex of the parabola has shifted 1 unit to the left and is now located at the point $(-1, 0)$.

We are led to the following conclusion.

**Property 8.** If $c > 0$, then the graph of $g(x) = (x+c)^2$ is shifted $c$ units to the left of the graph of $f(x) = x^2$.



(a) The graphs of $f$ and $g$.

| $x$ | $f(x) = x^2$ | $x$ | $g(x) = (x+1)^2$ |
|---|---|---|---|
| $-2$ | 4 | $-3$ | 4 |
| $-1$ | 1 | $-2$ | 1 |
| 0 | 0 | $-1$ | 0 |
| 1 | 1 | 0 | 1 |
| 2 | 4 | 1 | 4 |

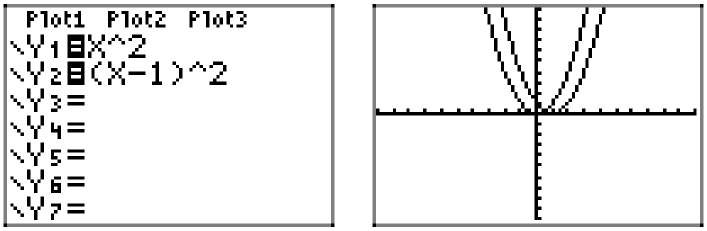
(a) Table of $x$-values and function values satisfying $f(x) = x^2$ and $g(x) = (x+1)^2$.

**Figure 8.** A horizontal shift or translation.

A similar thing happens when you replace $x$ with $x - 1$, only this time the graph is shifted one unit to the right.

▶ **Example 9.** *Sketch the graphs of $y = x^2$ and $y = (x-1)^2$ on your graphing calculator.*

Load the equations $y = x^2$ and $y = (x-1)^2$ into the `Y=` menu, as shown in **Figure 9**(a). Push the `ZOOM` button and select `6:ZStandard` to produce the image shown in **Figure 9**(b).



```
Plot1 Plot2 Plot3
\Y1=X^2
\Y2=(X-1)^2
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

(a) (b)

**Figure 9.** Drawing $y = x^2$ and $y = (x-1)^2$ on the graphing calculator.

Note that the graph of $y = (x-1)^2$ is shifted 1 unit to the right of the graph of $y = x^2$ and the vertex of the graph of $y = (x-1)^2$ is now located at the point $(1, 0)$.

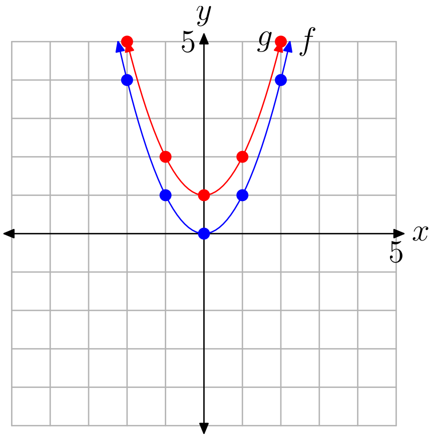
We are led to the following property.

**Property 10.** If $c > 0$, then the graph of $g(x) = (x-c)^2$ is shifted $c$ units to the right of the graph of $f(x) = x^2$.

## Vertical Translations

The graph of $g(x) = x^2 + 1$ in **Figure 10**(a) is shifted one unit upward from the graph of $f(x) = x^2$. This is easy to see as both equations use the same $x$-values in the table in **Figure 10**(b), but the function values of $g(x) = x^2 + 1$ are one unit larger than the corresponding function values of $f(x) = x^2$.

Note that the vertex of the graph of $g(x) = x^2 + 1$ has also shifted upward 1 unit and is now located at the point $(0, 1)$.



| $x$ | $f(x) = x^2$ | $g(x) = x^2 + 1$ |
|---|---|---|
| −2 | 4 | 5 |
| −1 | 1 | 2 |
| 0 | 0 | 1 |
| 1 | 1 | 2 |
| 2 | 4 | 5 |

**Figure 10.** A vertical shift or translation.

The above discussion leads to the following property.

**Property 11.** If $c > 0$, the graph of $g(x) = x^2 + c$ is shifted $c$ units upward from the graph of $f(x) = x^2$.

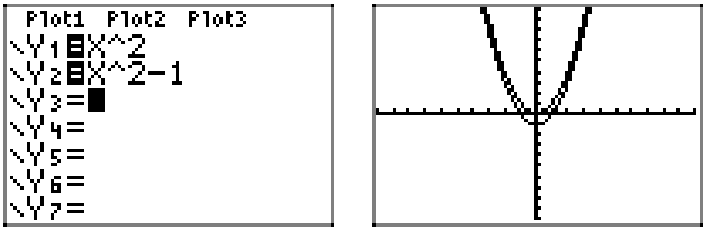
In a similar vein, the graph of $y = x^2 - 1$ is shifted downward one unit from the graph of $y = x^2$.

▶ **Example 12.** *Sketch the graphs of $y = x^2$ and $y = x^2 - 1$ on your graphing calculator.*

Load the equations $y = x^2$ and $y = x^2 - 1$ into the `Y=` menu, as shown in **Figure 11**(a). Push the `ZOOM` button and select `6:ZStandard` to produce the image shown in **Figure 11**(b).

Note that the graph of $y = x^2 - 1$ is shifted 1 unit downward from the graph of $y = x^2$ and the vertex of the graph of $y = x^2 - 1$ is now at the point $(0, -1)$.

◆



```
Plot1 Plot2 Plot3
\Y1=X^2
\Y2=X^2-1
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

(a) (b)

**Figure 11.** Drawing $y = x^2$ and $y = x^2 - 1$ on the graphing calculator.

The above discussion leads to the following property.

**Property 13.** If $c > 0$, the graph of $g(x) = x^2 - c$ is shifted $c$ units downward from the graph of $f(x) = x^2$.

## The Axis of Symmetry

In **Figure 1**, the graph of $y = x^2$ is symmetric with respect to the $y$-axis. One half of the parabola is a mirror image of the other with respect to the $y$-axis. We say the $y$-axis is acting as the *axis of symmetry*.

If the parabola is reflected across the $x$-axis, as in **Figure 6**, the axis of symmetry doesn't change. The graph is still symmetric with respect to the $y$-axis. Similar comments are in order for scalings and vertical translations. However, if the graph of $y = x^2$ is shifted right or left, then the axis of symmetry will change.

▶ **Example 14.** *Sketch the graph of* $y = -(x+2)^2 + 3$.

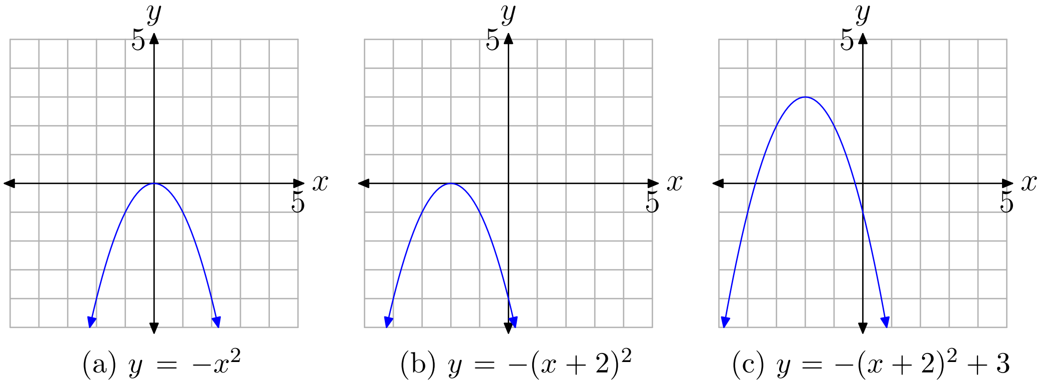
Although not required, this example is much simpler if you perform reflections before translations.

**Tip 15.** *If at all possible, perform scalings and reflections before translations.*

In the series shown in **Figure 12**, we first perform a reflection, then a horizontal translation, followed by a vertical translation.

- In **Figure 12**(a), the graph of $y = -x^2$ is a reflection of the graph of $y = x^2$ across the $x$-axis and opens downward. Note that the vertex is still at the origin.
- In **Figure 12**(b), we've replaced $x$ with $x+2$ in the equation $y = -x^2$ to obtain the equation $y = -(x+2)^2$. The effect is to shift the graph of $y = -x^2$ in **Figure 12**(a) 2 units to the left to obtain the graph of $y = -(x+2)^2$ in **Figure 12**(b). Note that the vertex is now at the point $(-2, 0)$.
- In **Figure 12**(c), we've added 3 to the equation $y = -(x+2)^2$ to obtain the equation $y = -(x+2)^2 + 3$. The effect is to shift the graph of $y = -(x+2)^2$ in **Figure 12**(b)

upward 3 units to obtain the graph of $y = -(x+2)^2 + 3$ in **Figure 12**(c). Note that the vertex is now at the point $(-2, 3)$.



(a) $y = -x^2$ (b) $y = -(x+2)^2$ (c) $y = -(x+2)^2 + 3$

**Figure 12.** Finding the graph of $y = -(x+2)^2 + 3$ through a series of transformations.

In practice, we can proceed more quickly. Analyze the equation $y = -(x+2)^2 + 3$. The minus sign tells us that the parabola "opens downward." The presence of $x + 2$ indicates a shift of 2 units to the left. Finally, adding the 3 will shift the graph 3 units upward. Thus, we have a parabola that "opens downward" with vertex at $(-2, 3)$. This is shown in **Figure 13**.

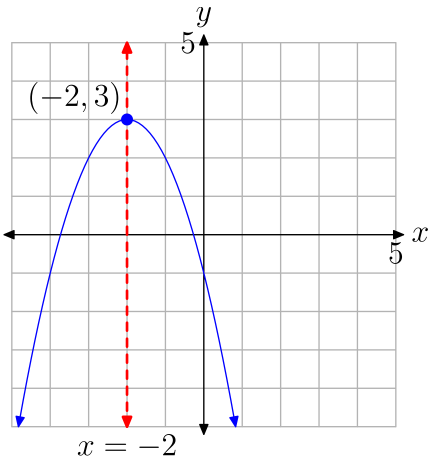


**Figure 13.** The axis of symmetry passes through the vertex.

The axis of symmetry passes through the vertex $(-2, 3)$ in **Figure 13** and has equation $x = -2$. Note that the right half of the parabola is a mirror image of its left half across this axis of symmetry. We can use the axis of symmetry to gain an accurate plot of the parabola with minimal plotting of points.

**Guidelines for Using the Axis of Symmetry.**

- Start by plotting the vertex and axis of symmetry as shown in **Figure 14**(a).
- Next, compute two points on either side of the axis of symmetry. We choose $x = -1$ and $x = 0$ and compute the corresponding $y$-values using the equation $y = -(x+2)^2 + 3$.

| $x$ | $y = -(x+2)^2 + 3$ |
|---|---|
| $-1$ | $2$ |
| $0$ | $-1$ |

  Plot the points from the table, as shown in **Figure 14**(b).
- Finally, plot the mirror images of these points across the axis of symmetry, as shown in **Figure 14**(c).

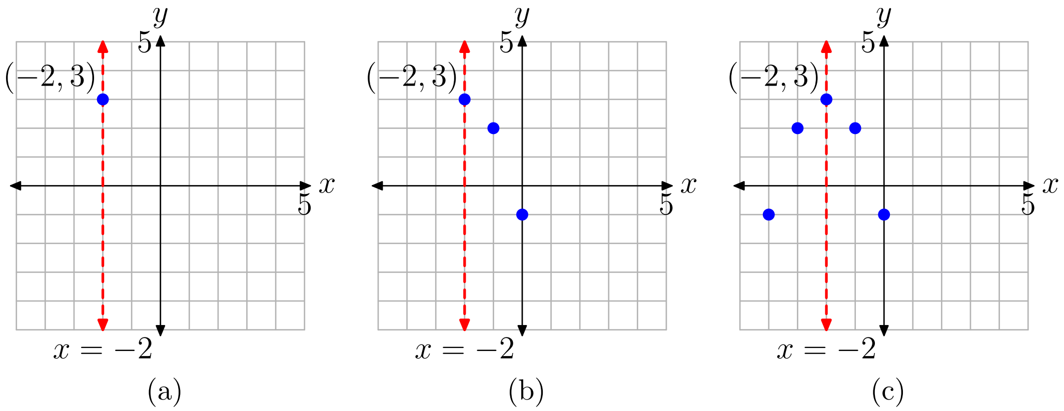


**Figure 14.** Using the axis of symmetry to establish accuracy.

The image in **Figure 14**(c) clearly contains enough information to complete the graph of the parabola having equation $y = -(x+2)^2 + 3$ in **Figure 15**.

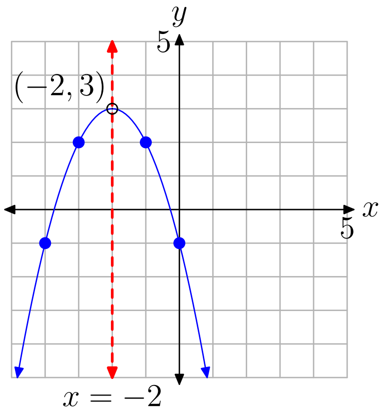


**Figure 15.** An accurate plot of $y = -(x+2)^2 + 3$.

Let's summarize what we've seen thus far.

**Summary 16.** *The form of the quadratic function*

$$f(x) = a(x-h)^2 + k$$

*is called* **vertex form**. *The graph of this quadratic function is a* **parabola**.

1. *The graph of the parabola opens* upward *if* $a > 0$, downward *if* $a < 0$.
2. *If the magnitude of* $a$ *is larger than 1, then the graph of the parabola is* stretched *by a factor of* $a$. *If the magnitude of* $a$ *is smaller than 1, then the graph of the parabola is* compressed *by a factor of* $1/a$.
3. *The parabola is translated* $h$ *units to the* right *if* $h > 0$, *and* $h$ *units to the* left *if* $h < 0$.
4. *The parabola is translated* $k$ *units* upward *if* $k > 0$, *and* $k$ *units* downward *if* $k < 0$.
5. *The coordinates of the* vertex *are* $(h, k)$.
6. *The* axis of symmetry *is a vertical line through the vertex whose equation is* $x = h$.

Let's look at one final example.

▶ **Example 17.** *Use the technique of* **Example 14** *to sketch the graph of* $f(x) = 2(x-2)^2 - 3$.

Compare $f(x) = 2(x-2)^2 - 3$ with $f(x) = a(x-h)^2 + k$ and note that $a = 2$. Hence, the parabola has been "stretched" by a factor of 2 and opens upward. The presence of $x - 2$ indicates a shift of 2 units to the right; and subtracting 3 shifts the parabola 3 units downward. Therefore, the vertex will be located at the point $(2, -3)$ and the axis of symmetry will be the vertical line having equation $x = 2$. This is shown in **Figure 16**(a).

**Note.** Some prefer a more strict comparison of $f(x) = 2(x-2)^2 - 3$ with the general vertex form $f(x) = a(x-h)^2 + k$, yielding $a = 2$, $h = 2$, and $k = -3$. This immediately identifies the vertex at $(h, k)$, or $(2, -3)$.

Next, evaluate the function $f(x) = 2(x-2)^2 - 3$ at two points lying to the right of the axis of symmetry (or to the left, if you prefer). Because the axis of symmetry is the vertical line $x = 2$, we choose to evaluate the function at $x = 3$ and 4.

$$f(3) = 2(3-2)^2 - 3 = -1$$
$$f(4) = 2(4-2)^2 - 3 = 5$$

This gives us two points to the right of the axis of symmetry, $(3, -1)$ and $(4, 5)$, which we plot in **Figure 16**(b).

Finally, we plot the mirror images of $(3, -1)$ and $(4, 5)$ across the axis of symmetry, which gives us the points $(1, -1)$ and $(0, 5)$, respectively. These are plotted in **Figure 16**(c). We then draw the parabola through these points.

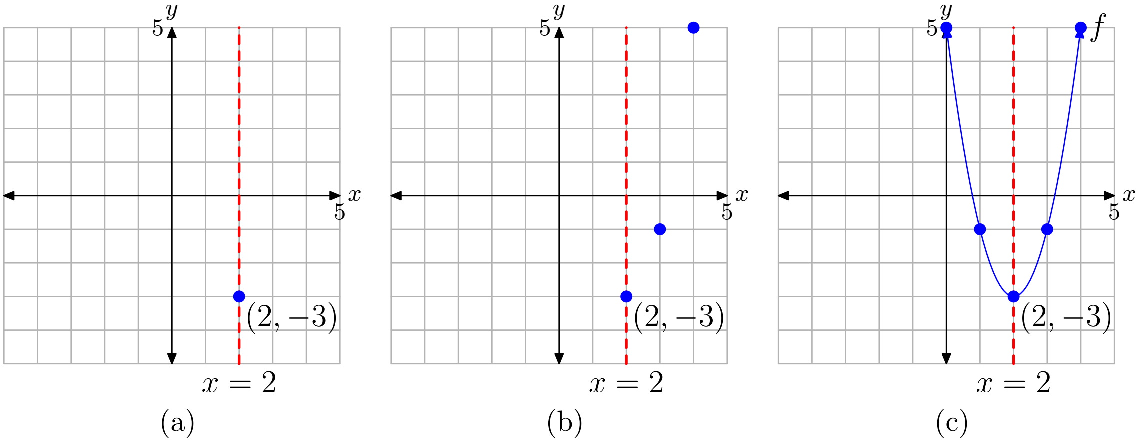


**Figure 16.** Creating the graph of $f(x) = 2(x-2)^2 - 3$.

Let's finish by describing the domain and range of the function defined by the rule $f(x) = 2(x-2)^2 - 3$. If you use the intuitive notion that the domain is the set of "permissible $x$-values," then one can substitute any number one wants into the equation $f(x) = 2(x-2)^2 - 3$. Therefore, the domain is all real numbers, which we can write as follows: Domain $= \mathbb{R}$ or Domain $= (-\infty, \infty)$.

You can also project each point on the graph of $f(x) = 2(x-2)^2 - 3$ onto the $x$-axis, as shown in **Figure 17**(a). If you do this, then the entire axis will "lie in shadow," so once again, the domain is all real numbers.

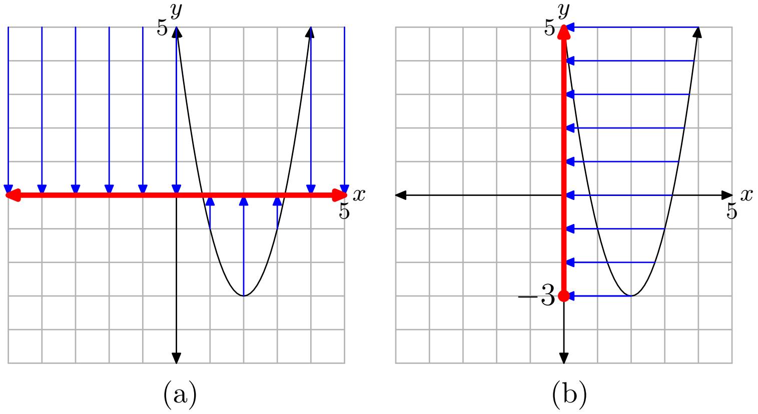


**Figure 17.** Projecting to find
(a) the domain and (b) the range.

To determine the range of the function $f(x) = 2(x-2)^2 - 3$, project each point on the graph of $f$ onto the $y$-axis, as shown in **Figure 17**(b). On the $y$-axis, all points greater than or equal to $-3$ "lie in shadow," so the range is described with Range $= \{y : y \geq -3\} = [-3, \infty)$.

◆

The following summarizes how one finds the domain and range of a quadratic function that is in vertex form.

**Summary 18.** *The domain of the quadratic function*

$$f(x) = a(x - h)^2 + k,$$

*regardless of the values of the parameters $a$, $h$, and $k$, is the set of all real numbers, easily described with $\mathbb{R}$ or $(-\infty, \infty)$. On the other hand, the range depends upon the values of $a$ and $k$.*

- *If $a > 0$, then the parabola opens upward and has vertex at $(h, k)$. Consequently, the range will be*

  $$[k, \infty) = \{y : y \geq k\}.$$

- *If $a < 0$, then the parabola opens downward and has vertex at $(h, k)$. Consequently, the range will be*

  $$(-\infty, k] = \{y : y \leq k\}.$$

## 5.1 Exercises

---

In **Exercises 1-6**, sketch the image of your calculator screen on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label each graph with its equation. *Remember to use a ruler to draw all lines, including axes.*

**1.** Use your graphing calculator to sketch the graphs of $f(x) = x^2$, $g(x) = 2x^2$, and $h(x) = 4x^2$ on one screen. Write a short sentence explaining what you learned in this exercise.

**2.** Use your graphing calculator to sketch the graphs of $f(x) = -x^2$, $g(x) = -2x^2$, and $h(x) = -4x^2$ on one screen. Write a short sentence explaining what you learned in this exercise.

**3.** Use your graphing calculator to sketch the graphs of $f(x) = x^2$, $g(x) = (x-2)^2$, and $h(x) = (x-4)^2$ on one screen. Write a short sentence explaining what you learned in this exercise.

**4.** Use your graphing calculator to sketch the graphs of $f(x) = x^2$, $g(x) = (x+2)^2$, and $h(x) = (x+4)^2$ on one screen. Write a short sentence explaining what you learned in this exercise.

**5.** Use your graphing calculator to sketch the graphs of $f(x) = x^2$, $g(x) = x^2 + 2$, and $h(x) = x^2 + 4$ on one screen. Write a short sentence explaining what you learned in this exercise.

**6.** Use your graphing calculator to sketch the graphs of $f(x) = x^2$, $g(x) = x^2 - 2$, and $h(x) = x^2 - 4$ on one screen. Write a short sentence explaining what you learned in this exercise.

In **Exercises 7-14**, write down the given quadratic function on your homework paper, then state the coordinates of the vertex.

**7.** $f(x) = -5(x-4)^2 - 5$

**8.** $f(x) = 5(x+3)^2 - 7$

**9.** $f(x) = 3(x+1)^2$

**10.** $f(x) = \dfrac{7}{5}\left(x + \dfrac{5}{9}\right)^2 - \dfrac{3}{4}$

**11.** $f(x) = -7(x-4)^2 + 6$

**12.** $f(x) = -\dfrac{1}{2}\left(x - \dfrac{8}{9}\right)^2 + \dfrac{2}{9}$

**13.** $f(x) = \dfrac{1}{6}\left(x + \dfrac{7}{3}\right)^2 + \dfrac{3}{8}$

**14.** $f(x) = -\dfrac{3}{2}\left(x + \dfrac{1}{2}\right)^2 - \dfrac{8}{9}$

In **Exercises 15-22**, state the equation of the axis of symmetry of the graph of the given quadratic function.

**15.** $f(x) = -7(x-3)^2 + 1$

**16.** $f(x) = -6(x+8)^2 + 1$

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[2] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

**17.** $f(x)=-\dfrac{7}{8}\left(x+\dfrac{1}{4}\right)^2+\dfrac{2}{3}$

**18.** $f(x)=-\dfrac{1}{2}\left(x-\dfrac{3}{8}\right)^2-\dfrac{5}{7}$

**19.** $f(x)=-\dfrac{2}{9}\left(x+\dfrac{2}{3}\right)^2-\dfrac{4}{5}$

**20.** $f(x)=-7(x+3)^2+9$

**21.** $f(x)=-\dfrac{8}{7}\left(x+\dfrac{2}{9}\right)^2+\dfrac{6}{5}$

**22.** $f(x)=3(x+3)^2+6$

In **Exercises 23-36**, perform each of the following tasks for the given quadratic function.

i. Set up a coordinate system on graph paper. Label and scale each axis.
ii. Plot the vertex of the parabola and label it with its coordinates.
iii. Draw the axis of symmetry and label it with its equation.
iv. Set up a table near your coordinate system that contains exact coordinates of two points on either side of the axis of symmetry. Plot them on your coordinate system and their “mirror images” across the axis of symmetry.
v. Sketch the parabola and label it with its equation.
vi. Use interval notation to describe both the domain and range of the quadratic function.

**23.** $f(x)=(x+2)^2-3$

**24.** $f(x)=(x-3)^2-4$

**25.** $f(x)=-(x-2)^2+5$

**26.** $f(x)=-(x+4)^2+4$

**27.** $f(x)=(x-3)^2$

**28.** $f(x)=-(x+2)^2$

**29.** $f(x)=-x^2+7$

**30.** $f(x)=-x^2+7$

**31.** $f(x)=2(x-1)^2-6$

**32.** $f(x)=-2(x+1)^2+5$

**33.** $f(x)=-\frac{1}{2}(x+1)^2+5$

**34.** $f(x)=\frac{1}{2}(x-3)^2-6$

**35.** $f(x)=2(x-5/2)^2-15/2$

**36.** $f(x)=-3(x+7/2)^2+15/4$

In **Exercises 37-44**, write the given quadratic function on your homework paper, then use set-builder and interval notation to describe the domain and the range of the function.

**37.** $f(x)=7(x+6)^2-6$

**38.** $f(x)=8(x+1)^2+7$

**39.** $f(x)=-3(x+4)^2-7$

**40.** $f(x)=-6(x-7)^2+9$

**41.** $f(x)=-7(x+5)^2-7$

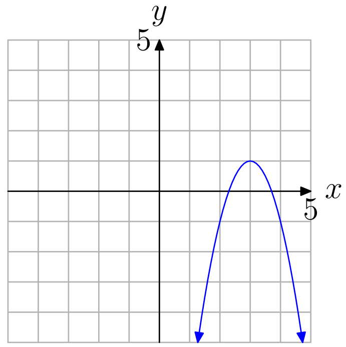
**42.** $f(x)=8(x-4)^2+3$

**43.** $f(x)=-4(x-1)^2+2$

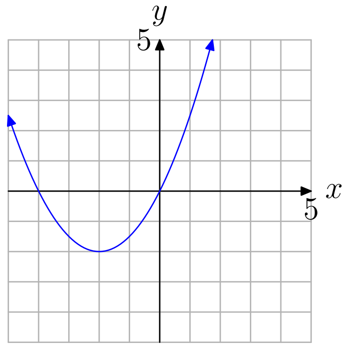
**44.** $f(x)=7(x-2)^2-3$

In **Exercises 45-52**, using the given value of $a$, find the specific quadratic function of the form $f(x) = a(x - h)^2 + k$ that has the graph shown. Note: $h$ and $k$ are integers. Check your solution with your graphing calculator.

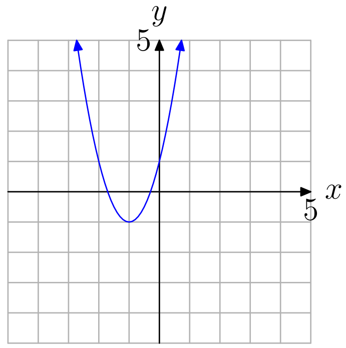
**45.** $a = -2$



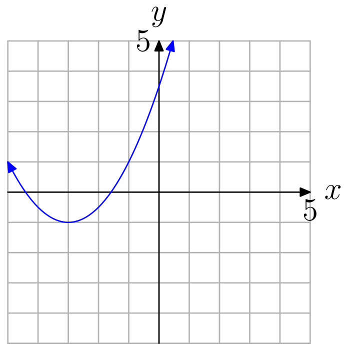
**46.** $a = 0.5$



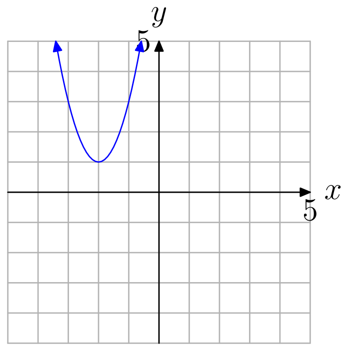
**47.** $a = 2$



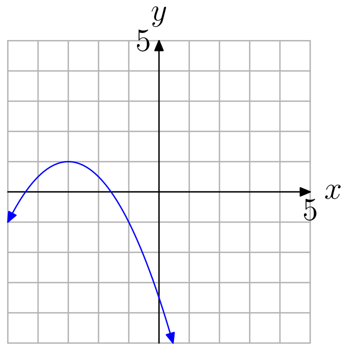
**48.** $a = 0.5$



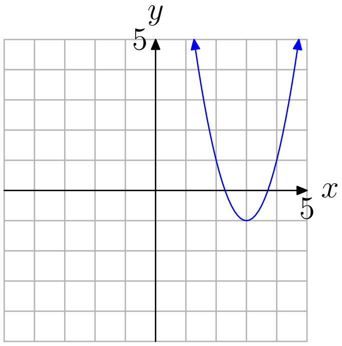
**49.** $a = 2$



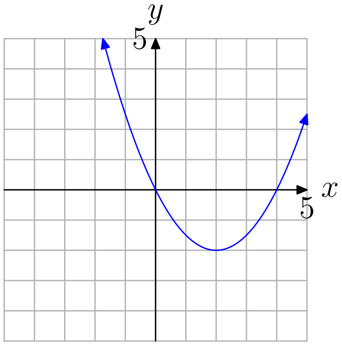
**50.** $a = -0.5$



**51.** $a = 2$



**52.** $a = 0.5$



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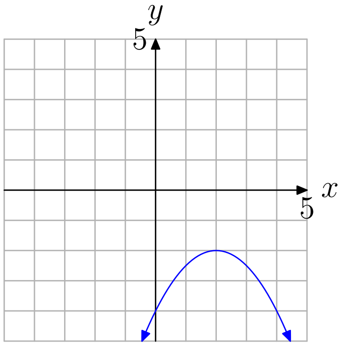
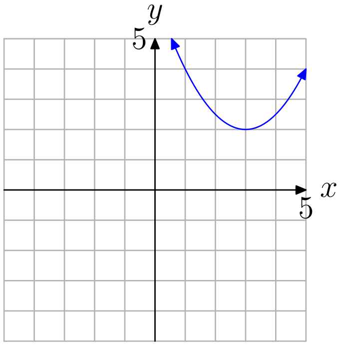
In **Exercises 53-54**, use the graph to determine the range of the function $f(x) = ax^2+bx+c$. The arrows on the graph are meant to indicate that the graph continues indefinitely in the continuing pattern and direction of each arrow. Describe your solution using interval notation.

**53.**



**54.**



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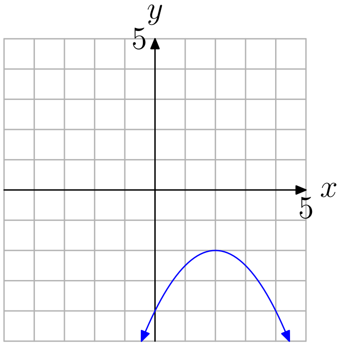
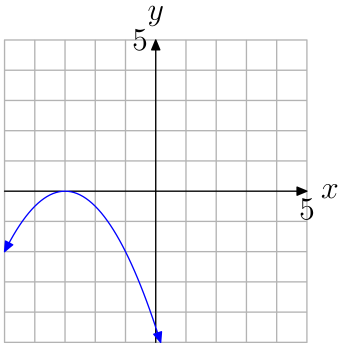
In **Exercises 55-56**, use the graph to determine the domain of the function $f(x) = ax^2+bx+c$. The arrows on the graph are meant to indicate that the graph continues indefinitely in the continuing pattern and direction of each arrow. Use interval notation to describe your solution.

**55.**

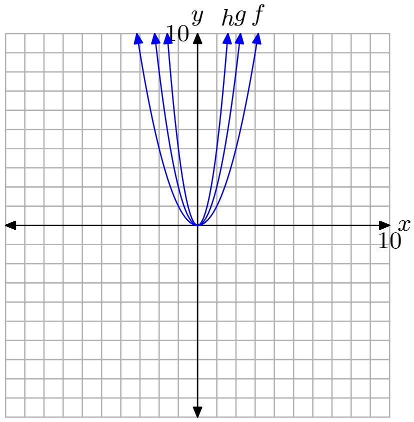


**56.**

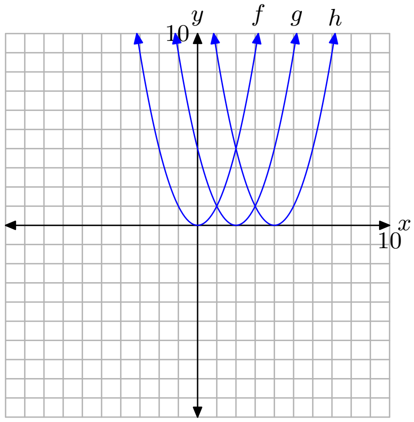


## 5.1 Answers

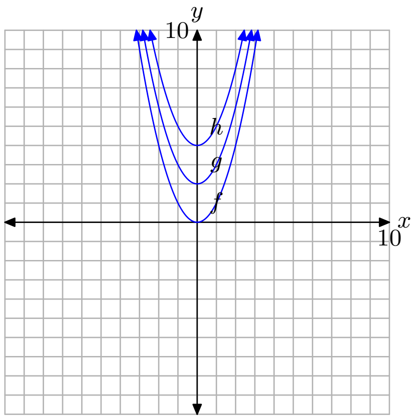
**1.** Multiplying by 2 scales vertically by a factor of 2. Multiplying by 4 scales vertically by a factor of 4.



**3.** The graph of $g(x) = (x-2)^2$ is shifted 2 units to the right of $f(x) = x^2$. The graph of $h(x) = (x-4)^2$ is shifted 4 units to the right of $f(x) = x^2$.



**5.** The graph of $g(x) = x^2 + 2$ is shifted 2 units to the upward from the graph of $f(x) = x^2$. The graph of $h(x) = x^2 + 4$ is shifted 4 units upward from the graph of $f(x) = x^2$.



**7.** $(4, -5)$

**9.** $(-1, 0)$

**11.** $(4, 6)$

**13.** $\left(-\frac{7}{3}, \frac{3}{8}\right)$

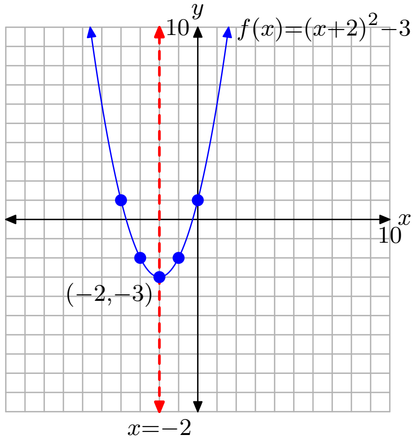
**15.** $x = 3$

**17.** $x = -\frac{1}{4}$

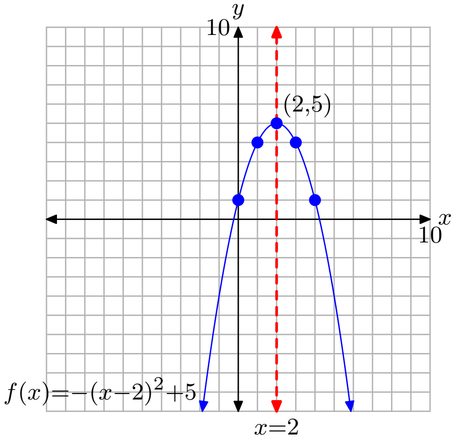
**19.** $x = -\frac{2}{3}$

**21.** $x = -\frac{2}{9}$

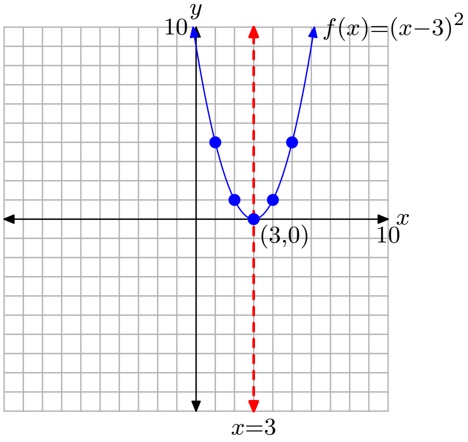
**23.** Domain$=(-\infty,\infty)$; Range$=[-3,\infty)$



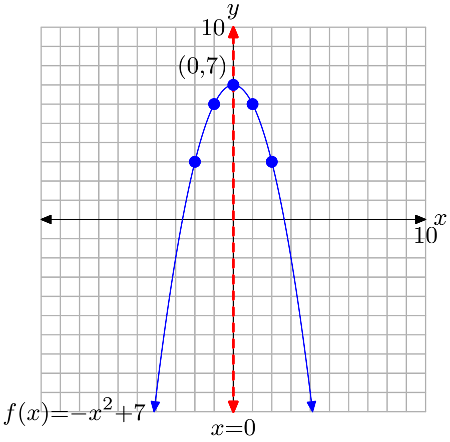
**25.** Domain$=(-\infty,\infty)$; Range$=(-\infty,5]$



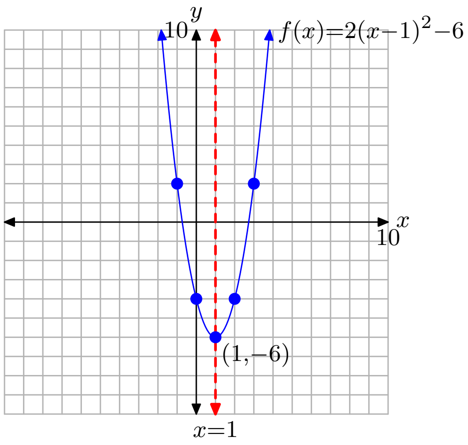
**27.** Domain$=(-\infty,\infty)$; Range$=[0,\infty)$



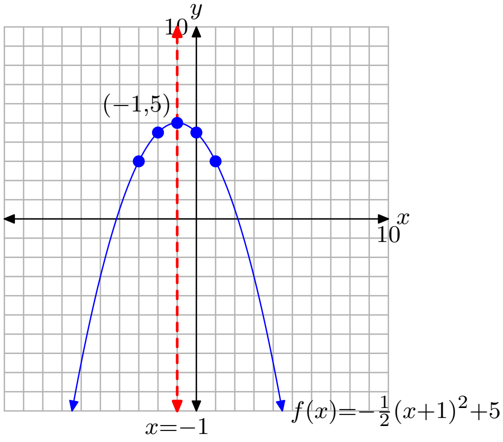
**29.** Domain$=(-\infty,\infty)$; Range$=(-\infty,7]$



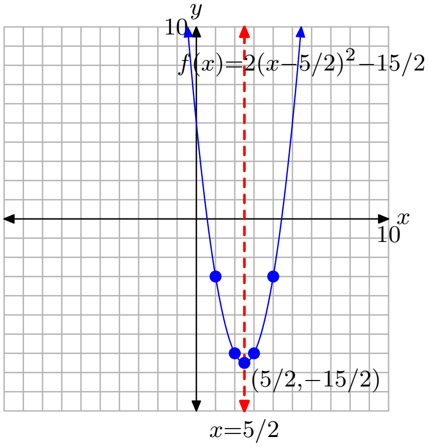
**31.** Domain$=(-\infty,\infty)$; Range$=[-6,\infty)$



**33.** Domain$=(-\infty,\infty)$; Range$=(-\infty,5]$



**35.** Domain$= (-\infty, \infty)$; Range$= [-15/2, \infty)$



**37.** Domain$= (-\infty, \infty)$; Range$= [-6, \infty) = \{y : y \geq -6\}$

**39.** Domain$= (-\infty, \infty)$; Range$= (-\infty, -7] = \{y : y \leq -7\}$

**41.** Domain$= (-\infty, \infty)$; Range$= (-\infty, -7] = \{y : y \leq -7\}$

**43.** Domain$= (-\infty, \infty)$; Range$= (-\infty, 2] = \{y : y \leq 2\}$

**45.** $f(x) = -2(x-3)^2 + 1$

**47.** $f(x) = 2(x+1)^2 - 1$

**49.** $f(x) = 2(x+2)^2 + 1$

**51.** $f(x) = 2(x-3)^2 - 1$

**53.** $(-\infty, -2]$

**55.** $(-\infty, \infty)$

# 5.2 Vertex Form

In the previous section, you learned that it is a simple task to sketch the graph of a quadratic function if it is presented in *vertex form*

$$f(x) = a(x - h)^2 + k. \tag{1}$$

The goal of the current section is to start with the most general form of the quadratic function, namely

$$f(x) = ax^2 + bx + c, \tag{2}$$

and manipulate the equation into *vertex form*. Once you have your quadratic function in vertex form, the technique of the previous section should allow you to construct the graph of the quadratic function.

However, before we turn our attention to the task of converting the general quadratic into vertex form, we need to review the necessary algebraic fundamentals. Let's begin with a review of an important algebraic shortcut called *squaring a binomial*.

## *Squaring a Binomial*

A *monomial* is a single algebraic term, usually constructed as a product of a number (called a *coefficient*) and one or more variables raised to nonnegative integral powers, such as $-3x^2$ or $14y^3z^5$. The key phrase here is "single term." A *binomial* is an algebraic sum or difference of two monomials (or terms), such as $x + 2y$ or $3ab^2 - 2c^3$. The key phrase here is "two terms."

To "square a binomial," start with an arbitrary binomial, such as $a+b$, then multiply it by itself to produce its square $(a + b)(a + b)$, or, more compactly, $(a + b)^2$. We can use the distributive property to expand the square of the binomial $a + b$.

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b) + b(a+b)\\ &= a^2 + ab + ba + b^2\end{aligned}$$

Because $ab = ba$, we can add the two middle terms to arrive at the following property.

**Property 3.** The square of the binomial $a + b$ is expanded as follows.

$$(a + b)^2 = a^2 + 2ab + b^2 \tag{4}$$

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▶ **Example 5.** *Expand* $(x+4)^2$.

We could proceed as follows.

$$\begin{aligned}(x+4)^2 &= (x+4)(x+4)\\ &= x(x+4)+4(x+4)\\ &= x^2+4x+4x+16\\ &= x^2+8x+16\end{aligned}$$

Although correct, this technique will not help us with our upcoming task. What we need to do is follow the algorithm suggested by **Property 3**.

**Algorithm for Squaring a Binomial.** To square the binomial $a+b$, proceed as follows:

1. Square the first term to get $a^2$.
2. Multiply the first and second terms together, and then multiply the result by two to get $2ab$.
3. Square the second term to get $b^2$.

Thus, to expand $(x+4)^2$, we should proceed as follows.

1. Square the first term to get $x^2$
2. Multiply the first and second terms together and then multiply by two to get $8x$.
3. Square the second term to get 16.

Proceeding in this manner allows us to perform the expansion mentally and simply write down the solution.

$$(x+4)^2 = x^2+2(x)(4)+4^2 = x^2+8x+16$$

◇

Here are a few more examples. In each, we've written an extra step to help clarify the procedure. In practice, you should simply write down the solution without any intermediate steps.

$$\begin{aligned}(x+3)^2 &= x^2+2(x)(3)+3^2 = x^2+6x+9\\ (x-5)^2 &= x^2+2(x)(-5)+(-5)^2 = x^2-10x+25\\ \left(x-\tfrac{1}{2}\right)^2 &= x^2+2(x)\left(-\tfrac{1}{2}\right)+\left(-\tfrac{1}{2}\right)^2 = x^2-x+\tfrac{1}{4}\end{aligned}$$

It is imperative that you master this shortcut before moving on to the rest of the material in this section.

## Perfect Square Trinomials

Once you've mastered squaring a binomial, as in

$$(a+b)^2 = a^2 + 2ab + b^2, \tag{6}$$

it's a simple matter to identify and factor trinomials (three terms) having the form $a^2 + 2ab + b^2$. You simply "undo" the multiplication. Whenever you spot a trinomial whose first and third terms are perfect squares, you should suspect that it factors as follows.

$$a^2 + 2ab + b^2 = (a+b)^2 \tag{7}$$

A trinomial that factors according to this rule or pattern is called a *perfect square trinomial*.

For example, the first and last terms of the following trinomial are perfect squares.

$$x^2 + 16x + 64$$

The square roots of the first and last terms are $x$ and 8, respectively. Hence, it makes sense to try the following.

$$x^2 + 16x + 64 = (x+8)^2$$

It is important that you check your result using multiplication. So, following the three-step algorithm for squaring a binomial:

1. Square $x$ to get $x^2$.
2. Multiply $x$ and 8 to get $8x$, then multiply this result by 2 to get $16x$.
3. Square 8 to get 64.

Hence, $x^2 + 16x + 64$ is a perfect square trinomial and factors as $(x+8)^2$.

As another example, consider $x^2 - 10x + 25$. The square roots of the first and last terms are $x$ and 5, respectively. Hence, it makes sense to try

$$x^2 - 10x + 25 = (x-5)^2.$$

Again, you should check this result. Note especially that twice the product of $x$ and $-5$ equals the middle term on the left, namely, $-10x$.

## Completing the Square

If a quadratic function is given in vertex form, it is a simple matter to sketch the parabola represented by the equation. For example, consider the quadratic function

$$f(x) = (x+2)^2 + 3,$$

which is in vertex form. The graph of this equation is a parabola that opens upward. It is translated 2 units to the left and 3 units upward. This is the advantage of vertex form. The transformations required to draw the graph of the function are easy to spot when the equation is written in vertex form.

It's a simple matter to transform the equation $f(x) = (x+2)^2 + 3$ into the general form of a quadratic function, $f(x) = ax^2 + bx + c$. We simply use the three-step algorithm to square the binomial; then we combine like terms.

$$\begin{aligned} f(x) &= (x+2)^2 + 3 \\ f(x) &= x^2 + 4x + 4 + 3 \\ f(x) &= x^2 + 4x + 7 \end{aligned}$$

Note, however, that the result of this manipulation, $f(x) = x^2 + 4x + 7$, is not as useful as vertex form, as it is difficult to identify the transformations required to draw the parabola represented by the equation $f(x) = x^2 + 4x + 7$.

It's really the *reverse* of the manipulation above that is needed. If we are presented with an equation in the form $f(x) = ax^2 + bx + c$, such as $f(x) = x^2 + 4x + 7$, then an algebraic method is needed to convert this equation to vertex form $f(x) = a(x-h)^2 + k$; or in this case, back to its original vertex form $f(x) = (x+2)^2 + 3$.

The procedure we seek is called *completing the square*. The name is derived from the fact that we need to "complete" the trinomial on the right side of $y = x^2 + 4x + 7$ so that it becomes a perfect square trinomial.

**Algorithm for Completing the Square** The procedure for *completing the square* involves three key steps.

1. Take half of the coefficient of $x$ and square the result.
2. Add and subtract the quantity from step one so that the right-hand side of the equation does not change.
3. Factor the resulting perfect square trinomial and combine constant terms.

Let's follow this procedure and place $f(x) = x^2 + 4x + 7$ in vertex form.

1. Take half of the coefficient of $x$. Thus, $(1/2)(4) = 2$. Square this result. Thus, $2^2 = 4$.

2. Add and subtract 4 on the right side of the equation $f(x) = x^2 + 4x + 7$.

$$f(x) = x^2 + 4x + 4 - 4 + 7$$

3. Group the first three terms on the right-hand side. These form a perfect square trinomial.

$$f(x) = (x^2 + 4x + 4) - 4 + 7$$

Now factor the perfect square trinomial and combine the constants at the end to get

$$f(x) = (x+2)^2 + 3.$$

That's it, we're done! We've returned the general quadratic $f(x) = x^2 + 4x + 7$ back to vertex form $f(x) = (x + 2)^2 + 3$.

Let's try that once more.

▶ **Example 8.** *Place the quadratic function* $f(x) = x^2 - 8x - 9$ *in vertex form.*

We follow the three-step algorithm for completing the square.

1. Take half of the coefficient of $x$ and square: i.e.,

$$[(1/2)(-8)]^2 = [-4]^2 = 16.$$

2. Add and subtract this amount to the right-hand side of the function.

$$f(x) = x^2 - 8x + 16 - 16 - 9$$

3. Group the first three terms on the right-hand side. These form a perfect square trinomial.

$$f(x) = (x^2 - 8x + 16) - 16 - 9$$

Factor the perfect square trinomial and combine the coefficients at the end.

$$f(x) = (x - 4)^2 - 25$$

◇

Now, let's see how we can use the technique of completing the square to help in drawing the graphs of general quadratic functions.

## Working with $f(x) = x^2 + bx + c$

The examples in this section will have the form $f(x) = x^2 + bx + c$. Note that the coefficient of $x^2$ is 1. In the next section, we will work with a harder form, $f(x) = ax^2 + bx + c$, where $a \neq 1$.

▶ **Example 9.** *Complete the square to place* $f(x) = x^2 + 6x + 2$ *in vertex form and sketch its graph.*

First, take half of the coefficient of $x$ and square; i.e., $[(1/2)(6)]^2 = 9$. On the right side of the equation, add and subtract this amount so as to not change the equation.

$$f(x) = x^2 + 6x + 9 - 9 + 2$$

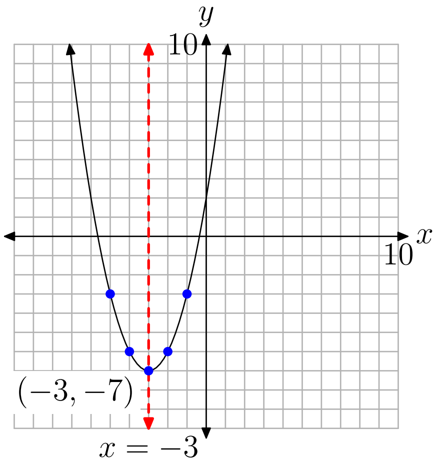
Group the first three terms on the right-hand side.

$$f(x) = (x^2 + 6x + 9) - 9 + 2$$

The first three terms on the right-hand side form a perfect square trinomial that is easily factored. Also, combine the constants at the end.

$$f(x) = (x + 3)^2 - 7$$

This is a parabola that opens upward. We need to shift the parabola 3 units to the left and then 7 units downward, placing the vertex at $(-3, -7)$ as shown in **Figure 1**(a). The axis of symmetry is the vertical line $x = -3$. The table in **Figure 1**(b) calculates two points to the right of the axis of symmetry, and mirror points on the left of the axis of symmetry make for an accurate plot of the parabola.



| $x$ | $y = (x+3)^2 - 7$ |
|---|---|
| $-2$ | $-6$ |
| $-1$ | $-3$ |

(a) (b)

**Figure 1.** Plotting the graph of the quadratic function $f(x) = (x+3)^2 - 7$.

◆

Let's look at another example.

▶ **Example 10.** *Complete the square to place* $f(x) = x^2 - 8x + 21$ *in vertex form and sketch its graph.*

First, take half of the coefficient of $x$ and square; i.e., $[(1/2)(-8)]^2 = 16$. On the right side of the equation, add and subtract this amount so as to not change the equation.

$$f(x) = x^2 - 8x + 16 - 16 + 21$$

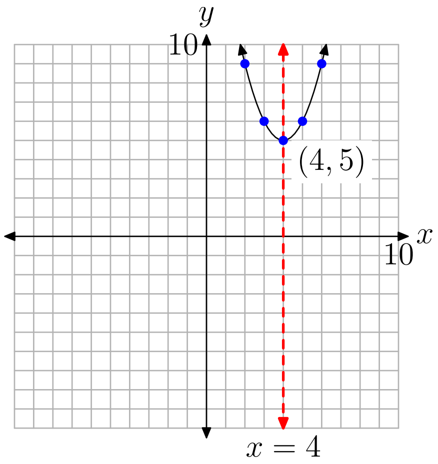
Group the first three terms on the right-hand side of the equation.

$$f(x) = (x^2 - 8x + 16) - 16 + 21$$

The first three terms form a perfect square trinomial that is easily factored. Also, combine constants at the end.

$$f(x) = (x-4)^2 + 5$$

This is a parabola that opens upward. We need to shift the parabola 4 units to the right and then 5 units upward, placing the vertex at $(4, 5)$, as shown in **Figure 2**(a). The table in **Figure 2**(b) calculates two points to the right of the axis of symmetry, and mirror points on the left of the axis of symmetry make for an accurate plot of the parabola.



(a)

| $x$ | $y = (x-4)^2 + 5$ |
|---|---|
| 5 | 6 |
| 6 | 9 |

(b)

**Figure 2.** Plotting the graph of the quadratic function $f(x) = (x-4)^2 + 5$.

◆

## *Working with $f(x) = ax^2 + bx + c$*

In the last two examples, the coefficient of $x^2$ was 1. In this section, we will learn how to complete the square when the coefficient of $x^2$ is some number other than 1.

▶ **Example 11.** *Complete the square to place $f(x) = 2x^2 + 4x - 4$ in vertex form and sketch its graph.*

In the last two examples, we gained some measure of success when the coefficient of $x^2$ was a 1. We were just getting comfortable with that situation and we'd like to continue to be comfortable, so let's start by factoring a 2 from each term on the right-hand side of the equation.

$$f(x) = 2\big[x^2 + 2x - 2\big]$$

If we ignore the factor of 2 out front, the coefficient of $x^2$ in the trinomial expression inside the parentheses is a 1. Ah, familiar ground! We will proceed as we did before, but we will carry the factor of 2 outside the parentheses in each step. Start by taking half of the coefficient of $x$ and squaring the result; i.e., $[(1/2)(2)]^2 = 1$.

Add and subtract this amount inside the parentheses so as to not change the equation.

$$f(x) = 2\big[x^2 + 2x + 1 - 1 - 2\big]$$

Group the first three terms inside the parentheses and combine constants.

$$f(x) = 2\big[(x^2 + 2x + 1) - 3\big]$$

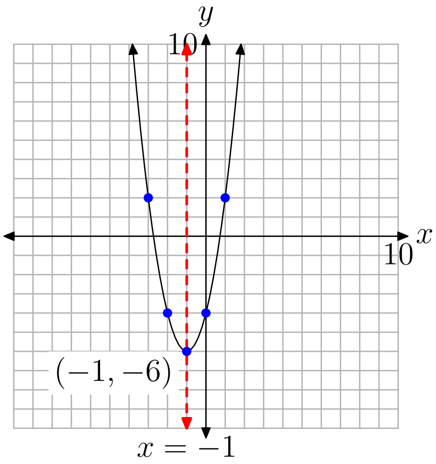
The grouped terms inside the parentheses form a perfect square trinomial that is easily factored.

$$f(x) = 2\left[(x+1)^2 - 3\right]$$

Finally, redistribute the 2.

$$f(x) = 2(x+1)^2 - 6$$

This is a parabola that opens upward. In addition, it is stretched by a factor of 2, so it will be somewhat narrower than our previous examples. The parabola is also shifted 1 unit to the left, then 6 units downward, placing the vertex at $(-1, -6)$, as shown in **Figure 3**(a). The table in **Figure 3**(b) calculates two points to the right of the axis of symmetry, and mirror points on the left of the axis of symmetry make for an accurate plot of the parabola.



(a)

| $x$ | $y = 2(x+1)^2 - 6$ |
|---|---|
| 0 | $-4$ |
| 1 | 2 |

(a)

**Figure 3.** Plotting the graph of the quadratic function $f(x) = 2x^2 + 4x - 4$.

Let's look at an example where the coefficient of $x^2$ is negative.

▶ **Example 12.** *Complete the square to place $f(x) = -x^2 + 6x - 2$ in vertex form and sketch its graph.*

In the last example, we factored out the coefficient of $x^2$. This left us with a trinomial having leading coefficient[4] 1, which enabled us to proceed much as we did before: halve the middle coefficient and square, add and subtract this amount, factor the resulting perfect square trinomial. Since we were successful with this technique in the last example, let's begin again by factoring out the leading coefficient, in this case a $-1$.

---

[4] The leading coefficient of a quadratic function is the coefficient of $x^2$. That is, if $f(x) = ax^2 + bx + c$, then the leading coefficient is "$a$." We'll have more to say about the leading coefficient in Chapter 6.

$$f(x) = -1\big[x^2 - 6x + 2\big]$$

If we ignore the factor of $-1$ out front, the coefficient of $x^2$ in the trinomial expression inside the parentheses is a 1. Again, familiar ground! We will proceed as we did before, but we will carry the factor of $-1$ outside the parentheses in each step. Start by taking half of the coefficient of $x$ and squaring the result; i.e., $[(1/2)(-6)]^2 = 9$.

Add and subtract this amount inside the parentheses so as to not change the equation.

$$f(x) = -1\big[x^2 - 6x + 9 - 9 + 2\big]$$

Group the first three terms inside the parentheses and combine constants.

$$f(x) = -1\big[(x^2 - 6x + 9) - 7\big]$$

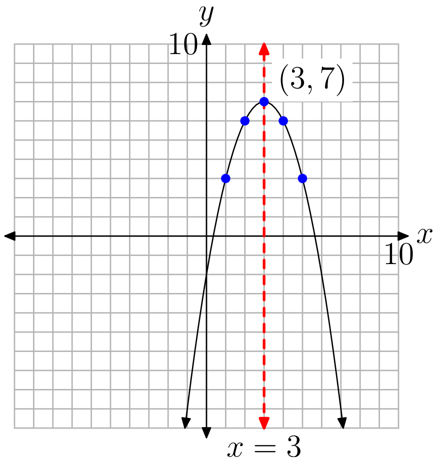
The grouped terms inside the parentheses form a perfect square trinomial that is easily factored.

$$f(x) = -1\big[(x - 3)^2 - 7\big]$$

Finally, redistribute the $-1$.

$$f(x) = -(x - 3)^2 + 7$$

This is a parabola that opens downward. The parabola is also shifted 3 units to the right, then 7 units upward, placing the vertex at $(3, 7)$, as shown in **Figure 4**(a). The table in **Figure 4**(b) calculates two points to the right of the axis of symmetry, and mirror points on the left of the axis of symmetry make for an accurate plot of the parabola.



(a)

| $x$ | $y = -(x-3)^2 + 7$ |
|---|---|
| 4 | 6 |
| 5 | 3 |

(b)

**Figure 4.** Plotting the graph of the quadratic function $f(x) = -(x - 3)^2 + 7$.

◆

Let's try one more example.

▶ **Example 13.** *Complete the square to place* $f(x) = 3x^2 + 4x - 8$ *in vertex form and sketch its graph.*

Let's begin again by factoring out the leading coefficient, in this case a 3.

$$f(x) = 3\left[x^2 + \tfrac{4}{3}x - \tfrac{8}{3}\right]$$

Fractions add a degree of difficulty, but, if you follow the same routine as in the previous examples, you should be able to get the needed result. Take half of the coefficient of $x$ and square the result; i.e., $[(1/2)(4/3)]^2 = [2/3]^2 = 4/9$.

Add and subtract this amount inside the parentheses so as to not change the equation.

$$f(x) = 3\left[x^2 + \tfrac{4}{3}x + \tfrac{4}{9} - \tfrac{4}{9} - \tfrac{8}{3}\right]$$

Group the first three terms inside the parentheses. You'll need a common denominator to combine constants.

$$f(x) = 3\left[\left(x^2 + \tfrac{4}{3}x + \tfrac{4}{9}\right) - \tfrac{4}{9} - \tfrac{24}{9}\right]$$

The grouped terms inside the parentheses form a perfect square trinomial that is easily factored.

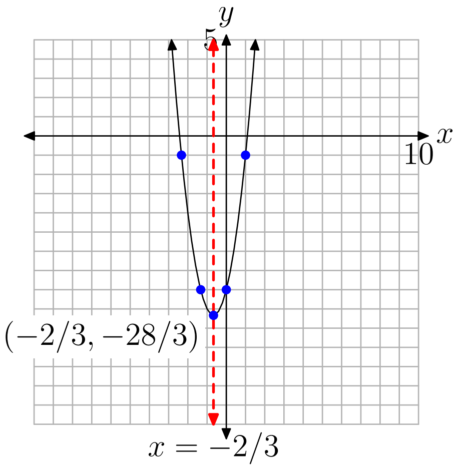
$$f(x) = 3\left[\left(x + \tfrac{2}{3}\right)^2 - \tfrac{28}{9}\right]$$

Finally, redistribute the 3.

$$f(x) = 3\left(x + \tfrac{2}{3}\right)^2 - \tfrac{28}{3}$$

This is a parabola that opens upward. It is also stretched by a factor of 3, so it will be narrower than all of our previous examples. The parabola is also shifted 2/3 units to the left, then 28/3 units downward, placing the vertex at (−2/3, −28/3), as shown in **Figure 5**(a). The table in **Figure 5**(b) calculates two points to the right of the axis of symmetry, and mirror points on the left of the axis of symmetry make for an accurate plot of the parabola.

◆



(a)

| $x$ | $y = 3(x+2/3)^2 - 28/3$ |
|---|---|
| 0 | $-8$ |
| 1 | $-1$ |

(b)

**Figure 5.** Plotting the graph of the quadratic function $f(x) = 3(x + 2/3)^2 - 28/3$.

## 5.2 *Exercises*

In **Exercises 1-8**, expand the binomial.

**1.** $\left(x+\frac{4}{5}\right)^2$

**2.** $\left(x-\frac{4}{5}\right)^2$

**3.** $(x+3)^2$

**4.** $(x+5)^2$

**5.** $(x-7)^2$

**6.** $\left(x-\frac{2}{5}\right)^2$

**7.** $(x-6)^2$

**8.** $\left(x-\frac{5}{2}\right)^2$

In **Exercises 9-16**, factor the perfect square trinomial.

**9.** $x^2-\frac{6}{5}x+\frac{9}{25}$

**10.** $x^2+5x+\frac{25}{4}$

**11.** $x^2-12x+36$

**12.** $x^2+3x+\frac{9}{4}$

**13.** $x^2+12x+36$

**14.** $x^2-\frac{3}{2}x+\frac{9}{16}$

**15.** $x^2+18x+81$

**16.** $x^2+10x+25$

In **Exercises 17-24**, transform the given quadratic function into vertex form $f(x) = (x-h)^2+k$ by completing the square.

**17.** $f(x)=x^2-x+8$

**18.** $f(x)=x^2+x-7$

**19.** $f(x)=x^2-5x-4$

**20.** $f(x)=x^2+7x-1$

**21.** $f(x)=x^2+2x-6$

**22.** $f(x)=x^2+4x+8$

**23.** $f(x)=x^2-9x+3$

**24.** $f(x)=x^2-7x+8$

In **Exercises 25-32**, transform the given quadratic function into vertex form $f(x) = a(x-h)^2+k$ by completing the square.

**25.** $f(x)=-2x^2-9x-3$

**26.** $f(x)=-4x^2-6x+1$

**27.** $f(x)=5x^2+5x+5$

**28.** $f(x)=3x^2-4x-6$

**29.** $f(x)=5x^2+7x-3$

**30.** $f(x)=5x^2+6x+4$

**31.** $f(x)=-x^2-x+4$

**32.** $f(x)=-3x^2-6x+4$

---

[5] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

In **Exercises 33-38**, find the vertex of the graph of the given quadratic function.

**33.** $f(x) = -2x^2 + 5x + 3$

**34.** $f(x) = x^2 + 5x + 8$

**35.** $f(x) = -4x^2 - 4x + 1$

**36.** $f(x) = 5x^2 + 7x + 8$

**37.** $f(x) = 4x^2 + 2x + 8$

**38.** $f(x) = x^2 + x - 7$

In **Exercises 39-44**, find the axis of symmetry of the graph of the given quadratic function.

**39.** $f(x) = -5x^2 - 7x - 8$

**40.** $f(x) = x^2 + 6x + 3$

**41.** $f(x) = -2x^2 - 5x - 8$

**42.** $f(x) = -x^2 - 6x + 2$

**43.** $f(x) = -5x^2 + x + 6$

**44.** $f(x) = x^2 - 9x - 6$

For each of the quadratic functions in **Exercises 45-66**, perform each of the following tasks.

i. Use the technique of completing the square to place the given quadratic function in vertex form.
ii. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.
iii. Draw the axis of symmetry and label it with its equation. Plot the vertex and label it with its coordinates.
iv. Set up a table near your coordinate system that calculates the coordinates of two points on either side of the axis of symmetry. Plot these points and their mirror images across the axis of symmetry. Draw the parabola and label it with its equation
v. Use the graph of the parabola to determine the domain and range of the quadratic function. Describe the domain and range using interval notation.

**45.** $f(x) = x^2 - 8x + 12$

**46.** $f(x) = x^2 + 4x - 1$

**47.** $f(x) = x^2 + 6x + 3$

**48.** $f(x) = x^2 - 4x + 1$

**49.** $f(x) = x^2 - 2x - 6$

**50.** $f(x) = x^2 + 10x + 23$

**51.** $f(x) = -x^2 + 6x - 4$

**52.** $f(x) = -x^2 - 6x - 3$

**53.** $f(x) = -x^2 - 10x - 21$

**54.** $f(x) = -x^2 + 12x - 33$

**55.** $f(x) = 2x^2 - 8x + 3$

**56.** $f(x) = 2x^2 + 8x + 4$

**57.** $f(x) = -2x^2 - 12x - 13$

**58.** $f(x) = -2x^2 + 24x - 70$

**59.** $f(x) = (1/2)x^2 - 4x + 5$

**60.** $f(x) = (1/2)x^2 + 4x + 6$

**61.** $f(x) = (-1/2)x^2 - 3x + 1/2$

**62.** $f(x) = (-1/2)x^2 + 4x - 2$

**63.** $f(x) = 2x^2 + 7x - 2$

**64.** $f(x) = -2x^2 - 5x - 4$

**65.** $f(x) = -3x^2 + 8x - 3$

**66.** $f(x) = 3x^2 + 4x - 6$

---

In **Exercises 67-72**, find the range of the given quadratic function. Express your answer in both interval and set notation.

**67.** $f(x) = -2x^2 + 4x + 3$

**68.** $f(x) = x^2 + 4x + 8$

**69.** $f(x) = 5x^2 + 4x + 4$

**70.** $f(x) = 3x^2 - 8x + 3$

**71.** $f(x) = -x^2 - 2x - 7$

**72.** $f(x) = x^2 + x + 9$

---

**Drill for Skill.** In **Exercises 73-76**, evaluate the function at the given value $b$.

**73.** $f(x) = 9x^2 - 9x + 4$; $b = -6$

**74.** $f(x) = -12x^2 + 5x + 2$; $b = -3$

**75.** $f(x) = 4x^2 - 6x - 4$; $b = 11$

**76.** $f(x) = -2x^2 - 11x - 10$; $b = -12$

---

**Drill for Skill.** In **Exercises 77-80**, evaluate the function at the given expression.

**77.** Evaluate $f(x+4)$ if $f(x) = -5x^2 + 4x + 2$.

**78.** Evaluate $f(-4x-5)$ if $f(x) = 4x^2 + x + 1$.

**79.** Evaluate $f(4x-1)$ if $f(x) = 4x^2 + 3x - 3$.

**80.** Evaluate $f(-5x-3)$ if $f(x) = -4x^2 + x + 4$.

## 5.2 *Answers*

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**1.** $x^2 + \frac{8}{5}x + \frac{16}{25}$

**3.** $x^2 + 6x + 9$

**5.** $x^2 - 14x + 49$

**7.** $x^2 - 12x + 36$

**9.** $\left(x - \frac{3}{5}\right)^2$

**11.** $(x-6)^2$

**13.** $(x+6)^2$

**15.** $(x+9)^2$

**17.** $\left(x - \frac{1}{2}\right)^2 + \frac{31}{4}$

**19.** $\left(x - \frac{5}{2}\right)^2 - \frac{41}{4}$

**21.** $(x+1)^2 - 7$

**23.** $\left(x - \frac{9}{2}\right)^2 - \frac{69}{4}$

**25.** $-2\left(x + \frac{9}{4}\right)^2 + \frac{57}{8}$

**27.** $5\left(x + \frac{1}{2}\right)^2 + \frac{15}{4}$

**29.** $5\left(x + \frac{7}{10}\right)^2 - \frac{109}{20}$

**31.** $-1\left(x + \frac{1}{2}\right)^2 + \frac{17}{4}$

**33.** $\left(\frac{5}{4}, \frac{49}{8}\right)$

**35.** $\left(-\frac{1}{2}, 2\right)$

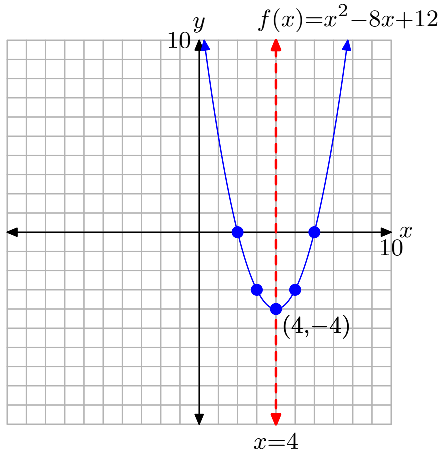
**37.** $\left(-\frac{1}{4}, \frac{31}{4}\right)$

**39.** $x = -\frac{7}{10}$

**41.** $x = -\frac{5}{4}$

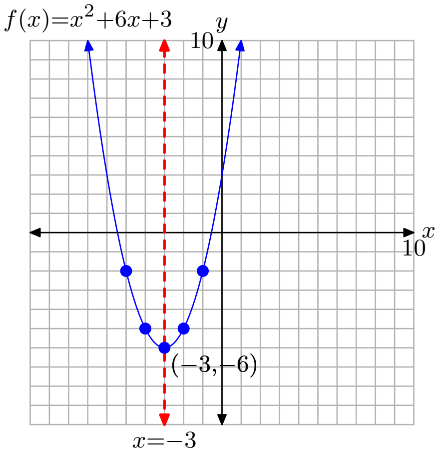
**43.** $x = \frac{1}{10}$

**45.** $f(x) = (x-4)^2 - 4$



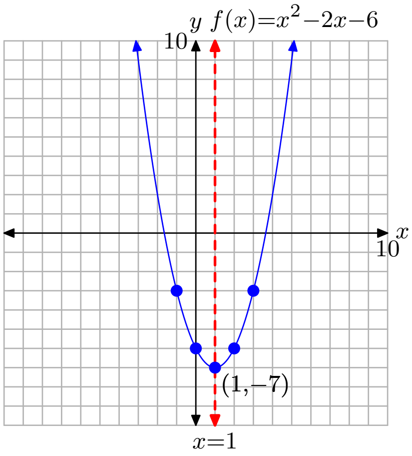
Domain = $\mathbb{R}$, Range = $[-4, \infty)$

**47.** $f(x) = (x+3)^2 - 6$



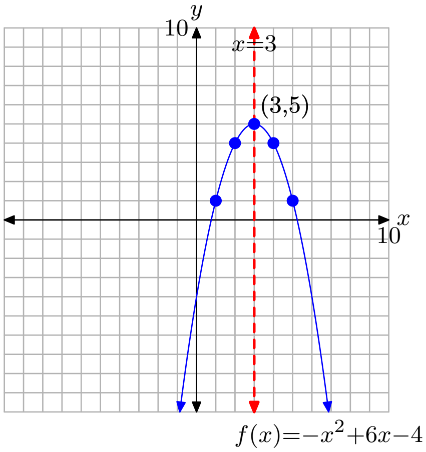
Domain $= \mathbb{R}$, Range $= [-6, \infty)$

**49.** $f(x) = (x-1)^2 - 7$



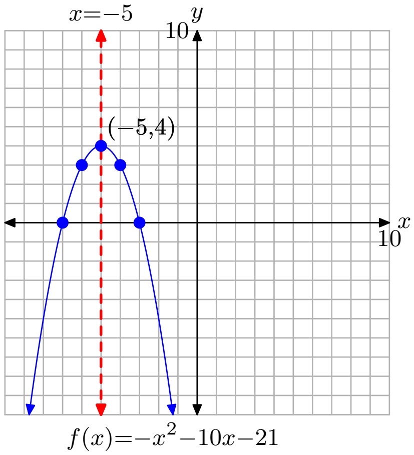
Domain $= \mathbb{R}$, Range $= [-7, \infty)$

**51.** $f(x) = -(x-3)^2 + 5$



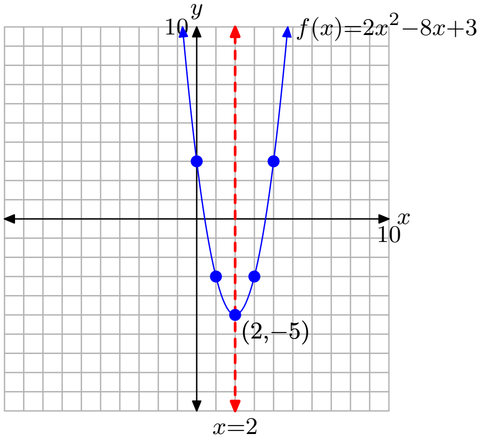
Domain $= \mathbb{R}$, Range $= (-\infty, 5]$

**53.** $f(x) = -(x+5)^2 + 4$



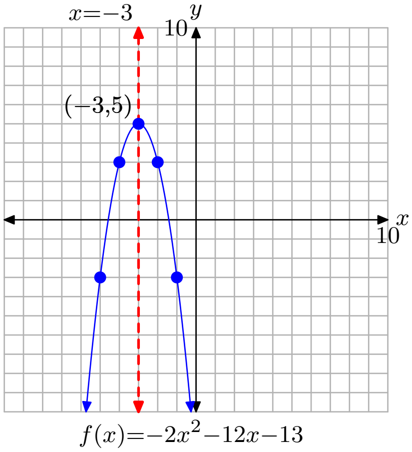
Domain $= \mathbb{R}$, Range $= (-\infty, 4]$

**55.** $f(x) = 2(x-2)^2 - 5$



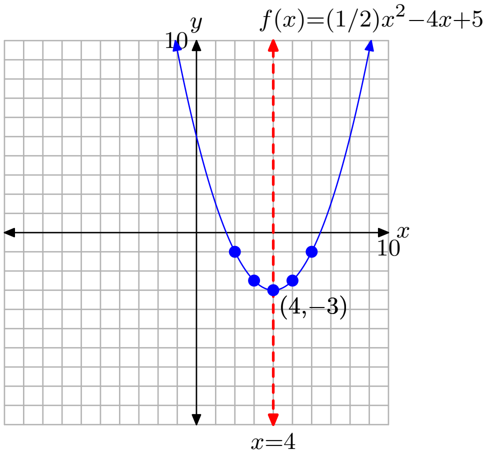
Domain = $\mathbb{R}$, Range = $[-5, \infty)$

**57.** $f(x) = -2(x+3)^2 + 5$



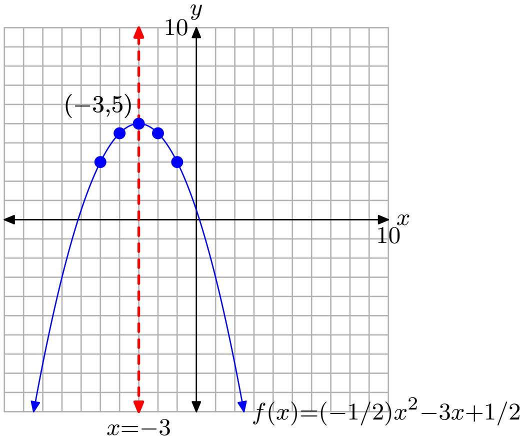
Domain = $\mathbb{R}$, Range = $(-\infty, 5]$

**59.** $f(x) = (1/2)(x-4)^2 - 3$



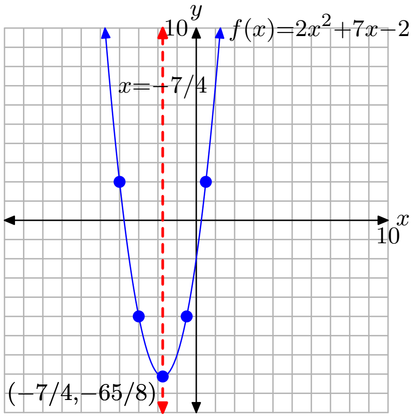
Domain = $\mathbb{R}$, Range = $[-3, \infty)$

**61.** $f(x) = (-1/2)(x+3)^2 + 5$



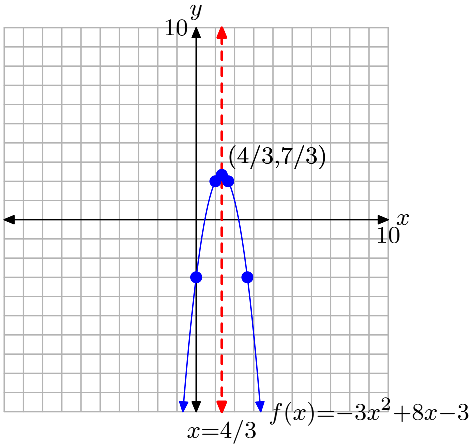
Domain = $\mathbb{R}$, Range = $(-\infty, 5])$

**63.** $f(x) = 2(x + 7/4)^2 - 65/8$



Domain $= \mathbb{R}$, Range $= [-65/8, \infty)$

**65.** $f(x) = -3(x - 4/3)^2 + 7/3$



Domain $= \mathbb{R}$, Range $= (-\infty, 7/3]$

**67.** $(-\infty, 5] = \{x \,|\, x \leq 5\}$

**69.** $\left[\frac{16}{5}, \infty\right) = \left\{x \,\middle|\, x \geq \frac{16}{5}\right\}$

**71.** $(-\infty, -6] = \{x \,|\, x \leq -6\}$

**73.** 382

**75.** 414

**77.** $-5x^2 - 36x - 62$

**79.** $64x^2 - 20x - 2$

# 5.3 Zeros of the Quadratic

We've seen how vertex form and intelligent use of the axis of symmetry can help to draw an accurate graph of the quadratic function defined by the equation $f(x) = ax^2+bx+c$. When drawing the graph of the parabola it is helpful to know where the graph of the parabola crosses the $x$-axis. That is the primary goal of this section, to find the zero crossings or $x$-intercepts of the parabola.

Before we begin, you'll need to review the techniques that will enable you to factor the quadratic expression $ax^2 + bx + c$.

## *Factoring $ax^2 + bx + c$ when $a = 1$*

Our intent in this section is to provide a quick review of techniques used to factor quadratic trinomials. We begin by showing how to factor trinomials having the form $ax^2 + bx + c$, where the leading coefficient is $a = 1$; that is, trinomials having the form $x^2+bx+c$. In the next section, we will address the technique used to factor $ax^2+bx+c$ when $a \neq 1$.

Let's begin with an example.

▶ **Example 1.** *Factor* $x^2 + 16x - 36$.

Note that the leading coefficient, the coefficient of $x^2$, is a 1. This is an important observation, because the technique presented here will not work when the leading coefficient does not equal 1.

Note the constant term of the trinomial $x^2 + 16x - 36$ is $-36$. List all integer pairs whose product equals $-36$.

| | | |
|---|---|---|
| 1, −36 | | −1, 36 |
| 2, −18 | | $\boxed{-2,\ 18}$ |
| 3, −12 | | −3, 12 |
| 4, −9 | | −4, 9 |
| 6, −6 | | −6, 6 |

Note that we've framed the pair −2, 18. We've done this because the sum of this pair of integers equals the coefficient of $x$ in the trinomial expression $x^2 + 16x - 36$. Use this framed pair to factor the trinomial.

$$x^2 + 16x - 36 = (x - 2)(x + 18)$$

It is important that you check your result. Use the distributive property to multiply.

$$\begin{aligned}(x - 2)(x + 18) &= x(x + 18) - 2(x + 18)\\ &= x^2 + 18x - 2x - 36\\ &= x^2 + 16x - 36\end{aligned}$$

---

[6] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

Thus, our factorization is correct.

◆

Let's summarize the technique.

**Algorithm.** To factor the quadratic $x^2 + bx + c$, proceed as follows:

1. List all the integer pairs whose product equals $c$.
2. Circle or frame the pair whose sum equals the coefficient of $x$, namely $b$. Use this pair to factor the trinomial.

Let's look at another example.

▶ **Example 2.** *Factor the trinomial* $x^2 - 25x - 84$.

List all the integer pairs whose product is $-84$.

| | | |
|---|---|---|
| $1, -84$ | | $-1, 84$ |
| $2, -42$ | | $-2, 42$ |
| $\boxed{3, -28}$ | | $-3, 28$ |
| $4, -21$ | | $-4, 21$ |
| $6, -14$ | | $-6, 14$ |
| $7, -12$ | | $-7, 12$ |

We've framed the pair whose sum equals the coefficient of $x$, namely $-25$. Use this pair to factor the trinomial.

$$x^2 - 25x - 84 = (x + 3)(x - 28)$$

Check.

$$\begin{aligned}(x + 3)(x - 28) &= x(x - 28) + 3(x - 28)\\ &= x^2 - 28x + 3x - 84\\ &= x^2 - 25x - 84\end{aligned}$$

◆

With experience, there are a number of ideas that will quicken the process.

- As you are listing the integer pairs, should you happen to note that the current pair has the appropriate sum, there is no need to list the remaining integer pairs. Simply halt the process of listing the integer pairs and use the current pair to factor the trinomial.
- Some students are perfectly happy being asked "Can you think of an integer pair whose product is $c$ and whose sum is $b$ (where $b$ and $c$ refer to the coefficients of $x^2 + bx + c$)?" If you can pick the pair "out of the air" like this, all is well and good.

Use the integer pair to factor the trinomial and don't bother listing any integer pairs.

Now, let's investigate how to proceed when the leading coefficient is not 1.

## *Factoring* $ax^2 + bx + c$ *when* $a \neq 1$

When $a \neq 1$, we use a technique called the *ac-test* to factor the trinomial $ax^2 + bx + c$. The process is best explained with an example.

► **Example 3.** *Factor* $2x^2 + 13x - 24$.

Note that the leading coefficient does not equal 1. Indeed, the coefficient of $x^2$ in this example is a 2. Therefore, the technique of the previous examples will not work. Thus, we turn to a similar technique called the *ac-test*.

First, compare

$$2x^2 + 13x - 24 \qquad \text{and} \qquad ax^2 + bx + c$$

and note that $a = 2$, $b = 13$, and $c = -24$. Compute the product of $a$ and $c$. This is how the technique earns its name "*ac*-test."

$$ac = (2)(-24) = -48$$

List all integer pairs whose product is $ac = -48$.

| | | |
|---|---|---|
| 1, −48 | | −1, 48 |
| 2, −24 | | −2, 24 |
| 3, −16 | | $\boxed{-3,\ 16}$ |
| 4, −12 | | −4, 12 |
| 6, −8 | | −6, 8 |

We've framed the pair whose sum is $b = 13$. The next step is to rewrite the trinomial $2x^2 + 13x - 24$, splitting the middle term into a sum, using our framed integer pair.

$$2x^2 + 13x - 24 = 2x^2 - 3x + 16x - 24$$

We factor an $x$ out of the first two terms, then an 8 out of the last two terms. This process is called *factoring by grouping*.

$$2x^2 - 3x + 16x - 24 = x(2x - 3) + 8(2x - 3)$$

We now factor out a common factor of $2x - 3$.

$$x(2x - 3) + 8(2x - 3) = (x + 8)(2x - 3)$$

It's helpful to see the complete process as a coherent unit.

$$\begin{aligned} 2x^2 + 13x - 24 &= 2x^2 - 3x + 16x - 24 \\ &= x(2x - 3) + 8(2x - 3) \\ &= (x + 8)(2x - 3) \end{aligned}$$

**Check**. Again, it is important to check the answer by multiplication.

$$\begin{aligned} (x + 8)(2x - 3) &= x(2x - 3) + 8(2x - 3) \\ &= 2x^2 - 3x + 16x - 24 \\ &= 2x^2 + 13x - 24 \end{aligned}$$

Because this is the original trinomial, our solution checks.[7]

◇

Let's summarize this process.

**Algorithm: ac-Test.** To factor the quadratic $ax^2 + bx + c$, proceed as follows:

1. List all integer pairs whose product equals $ac$.
2. Circle or frame the pair whose sum equals the coefficient of $x$, namely $b$.
3. Use the circled pair to express the middle term $bx$ as a sum.
4. Factor by "grouping."

Let's look at another example.

► **Example 4.** *Factor* $3x^2 + 34x - 24$.

Compare

$$3x^2 + 34x - 24 \quad \text{and} \quad ax^2 + bx + c$$

and note that $a = 3$, $b = 34$ and $c = -24$. List all integer pairs whose product equals $ac = (3)(-24) = -72$.

| | | |
|---|---|---|
| $1, -72$ | | $-1, 72$ |
| $2, -36$ | | $\boxed{-2, 36}$ |
| $3, -24$ | | $-3, 24$ |
| $4, -18$ | | $-4, 18$ |
| $6, -12$ | | $-6, 12$ |
| $8, -9$ | | $-8, 9$ |

We've framed the pair whose sum is the same as $b = 34$, the coefficient of $x$ in $3x^2 + 34x - 24$. Again, possible shortcuts are possible. If you can "think" of a pair whose product is $ac = -72$ and whose sum is $b = 34$, then it is not necessary to list any integer pairs. Alternatively, if you come across the needed pair as you are listing

---

[7] If you check a number of your results, it will soon become apparent why the $ac$-test works so well.

them, then you can halt the process. There is no need to list the remaining pairs if you have the one you need.

Use the framed pair to express the middle term as a sum, then factor by grouping.

$$\begin{aligned} 3x^2 + 34x - 24 &= 3x^2 - 2x + 36x - 24 \\ &= x(3x - 2) + 12(3x - 2) \\ &= (x + 12)(3x - 2) \end{aligned}$$

We leave it to the reader to check this result.

◆

## Intercepts

The points where the graph of a function crosses the $x$-axis are called the $x$-*intercepts* of graph of the function. Consider the graph of the quadratic function $f$ in **Figure 1**.

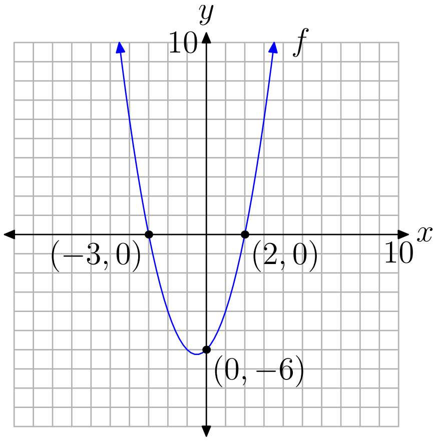


**Figure 1.** The $x$- and $y$-intercepts are key features of any graph.

Note that the graph of the $f$ crosses the $x$-axis at $(-3, 0)$ and $(2, 0)$. These are the $x$-intercepts of the parabola. Note that the $y$-coordinate of each $x$-intercept is zero.

In function notation, the solutions of $f(x) = 0$ (note the similarity to $y = 0$) are the $x$-coordinates of the points where the graph of $f$ crosses the $x$-axis. Analyzing the graph of $f$ in **Figure 1**, we see that both $-3$ and 2 are solutions of $f(x) = 0$.

Thus, the process for finding the $x$-intercepts is clear.

**Finding x-intercepts.** To find the $x$-intercepts of the graph of any function, set $y = 0$ and solve for $x$. Alternatively, if function notation is used, set $f(x) = 0$ and solve for $x$.

Let's look at an example.

▶ **Example 5.** *Find the $x$-intercepts of the graph of the quadratic function defined by $y = x^2 + 2x - 48$.*

To find the $x$-intercepts, first set $y = 0$.

$$0 = x^2 + 2x - 48$$

Next, factor the trinomial on the right. Note that the coefficient of $x^2$ is 1. We need only think of two integers whose product equals the constant term $-48$ and whose sum equals the coefficient of $x$, namely 2. The numbers 8 and $-6$ come to mind, so the trinomial factors as follows (readers should check this result).

$$0 = (x + 8)(x - 6)$$

To complete the solution, we need to use an important property of the real numbers called the *zero product property*.

**Zero Product Property.** If $a$ and $b$ are any real numbers such that

$$ab = 0,$$

then either $a = 0$ or $b = 0$.

In our case, we have $0 = (x + 8)(x - 6)$. Therefore, it must be the case that either

$$x + 8 = 0 \qquad \text{or} \qquad x - 6 = 0.$$

These equations can be solved independently to produce

$$x = -8 \qquad \text{or} \qquad x = 6.$$

Thus, the $x$-intercepts of the graph of $y = x^2 + 2x - 48$ are located at $(-8, 0)$ and $(6, 0)$.

◆

Let's look at another example.

▶ **Example 6.** *Find the $x$-intercepts of the graph of the quadratic function $f(x) = 2x^2 - 7x - 15$.*

To find the $x$-intercepts of the graph of the quadratic function $f$, we begin by setting

$$f(x) = 0.$$

Of course, $f(x) = 2x^2 - 7x - 15$, so we can substitute to obtain

$$2x^2 - 7x - 15 = 0.$$

We will now use the $ac$-test to factor the trinomial on the left. Note that $ac = (2)(-15) = -30$. List the integer pairs whose products equal $-30$.

| | | |
|---|---|---|
| 1, −30 | | −1, 20 |
| 2, −15 | | −2, 15 |
| $\boxed{3, -10}$ | | −3, 10 |
| 5, −6 | | −5, 6 |

Note that the framed pair sum to the coefficient of $x$ in $2x^2 - 7x - 15$. Use the framed pair to express the middle term as a sum, then factor by grouping.

$$\begin{aligned} 2x^2 - 7x - 15 &= 0 \\ 2x^2 + 3x - 10x - 15 &= 0 \\ x(2x+3) - 5(2x+3) &= 0 \\ (x-5)(2x+3) &= 0 \end{aligned}$$

Now we can use the zero product property. Either

$$x - 5 = 0 \qquad \text{or} \qquad 2x + 3 = 0.$$

Each of these can be solved independently to obtain

$$x = 5 \qquad \text{or} \qquad x = -3/2.$$

Thus, the $x$-intercepts of the graph of the quadratic function $f(x) = 2x^2 - 7x - 15$ are located at $(-3/2, 0)$ and $(5, 0)$.

One more definition is in order.

**Definition 7.** **Zeros of a Function.** *The solutions of $f(x) = 0$ are called the* **zeros** *of the function $f$.*

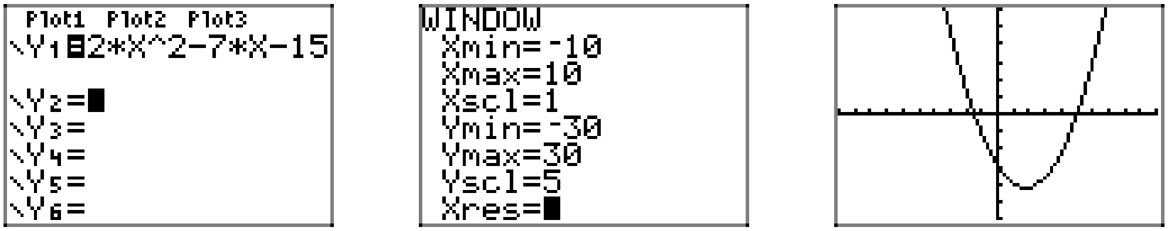
Thus, in the last example, both $-3/2$ and 5 are zeros of the quadratic function $f(x) = 2x^2 - 7x - 15$. Note the intimate relationship between the zeros of the quadratic function and the $x$-intercepts of the graph. Note that $-3/2$ is a zero and $(-3/2, 0)$ is an $x$-intercept. Similarly, 5 is a zero and $(5, 0)$ is an $x$-intercept.

The graphing calculator can be used to find the zeros of a function.

▶ **Example 8.** *Use the graphing calculator to find the zeros of the function $f(x) = 2x^2 - 7x - 15$.*

Enter the function $f(x) = 2x^2 - 7x - 15$ into `Y1` in the `Y=` menu; then adjust the window parameters as shown in **Figure 2**(b). Push the `GRAPH` button to produce the parabola shown in **Figure 2**(c).

To find a zero of the function, proceed as follows:



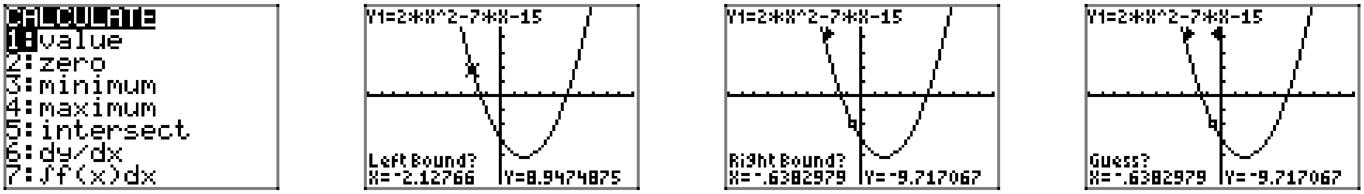
```
Plot1 Plot2 Plot3
\Y1■2*X^2-7*X-15
\Y2=■
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-30
 Ymax=30
 Yscl=5
 Xres=■
```

(a) (b) (c)

**Figure 2.** Plotting the quadratic function $f(x) = 2x^2 - 7x - 15$.

- Press `2nd` `TRACE` to open the `CALCULATE` window shown in **Figure 3**(a). From this menu, select `2:zero`.
- The calculator responds by asking for a "Left bound." Use the arrow keys to move the cursor slightly to the left of the leftmost $x$-intercept, as shown in **Figure 3**(b). Press the `ENTER` key.
- The calculator responds by asking for a "Right bound." Use the arrow keys to move the cursor slightly to the right of the leftmost $x$-intercept, as shown in **Figure 3**(c). Press the `ENTER` key.
- The calculator responds by asking for a "Guess." You may use the arrow keys to select a starting $x$-value any where between the left- and right-bounds you selected (note that the calculator marks these on the screen in **Figure 3**(d)). However, the cursor already lies between these marks, so we typically just hit `ENTER` at this point. We suggest you do so also.



```
Y1=2*X^2-7*X-15
Left Bound?
X=-2.12766    Y=8.9474875
```

```
Y1=2*X^2-7*X-15
Right Bound?
X=-.6382979   Y=-9.717067
```

```
Y1=2*X^2-7*X-15
Guess?
X=-.6382979   Y=-9.717067
```

(a) (b) (c) (d)

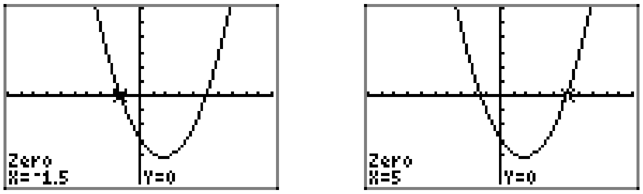
**Figure 3.** Using the `zero` utility to find an $x$-intercept.

The calculator responds by marking the $x$-intercept and reporting its $x$-value at the bottom of the screen, as shown in **Figure 4**(a). This is one of the zeros of the function. Note that this value of $-1.5$ agrees nicely with our hand calculated result $-3/2$ in **Example 6**. We followed precisely the same procedure outlined above to find the second $x$-intercept shown in **Figure 4**(b). Note that it also agrees with the hand calculated solution of **Example 6**.

◆

In a similar vein, the point where the graph of a function crosses the $y$-axis is called the *$y$-intercept* of the graph of the function. In **Figure 1** the $y$-intercept of the parabola is $(0, -6)$. Note that the $x$-coordinate of this $y$-intercept is zero.

Thus, the process for finding $y$-intercepts should be clear.



(a) (b)

**Figure 4.** The zeros of $f(x) - 2x^2 - 7x - 15$.

**Finding y-intercepts.** To find the $y$-intercepts of the graph of any function, set $x = 0$ and solve for $y$. Alternatively, if function notation is used, simply evaluate $f(0)$.

▶ **Example 9.** *Find the $y$-intercept of the quadratic function defined by $f(x) = x^2 - 3x - 11$.*

Evaluate the function at $x = 0$.

$$f(0) = (0)^2 - 3(0) - 11 = -11.$$

The coordinates of the $y$-intercept are $(0, -11)$.

◇

## *Putting it All Together*

We will find both $x$- and $y$-intercepts extremely useful when drawing the graph of a quadratic function.

▶ **Example 10.** *Place the quadratic function $y = x^2 + 2x - 24$ in vertex form. Plot the vertex and axis of symmetry and label them with their coordinates and equation, respectively. Find and plot the $x$- and $y$-intercepts of the parabola and label them with their coordinates.*

Take half of the coefficient of $x$, square, then add and subtract this amount to balance the equation. Factor and combine coefficients.

$$y = x^2 + 2x + 1 - 1 - 24$$
$$y = (x + 1)^2 - 25$$

The graph is a parabola that opens upward; it is shifted 1 unit to the left and 25 units downward. This information is enough to plot and label the vertex, then plot and label the axis of symmetry, as shown in **Figure 5**(a).

To find the $x$-intercepts, let $y = 0$ in $y = x^2 + 2x - 24$.

$$0 = x^2 + 2x - 24$$

The leading coefficient is a 1. The integer pair $-4$ and 6 has product $-24$ and sum 2. Thus, the right-hand side factors as follows.

$$0 = (x + 6)(x - 4)$$

In order that this product equals zero, either

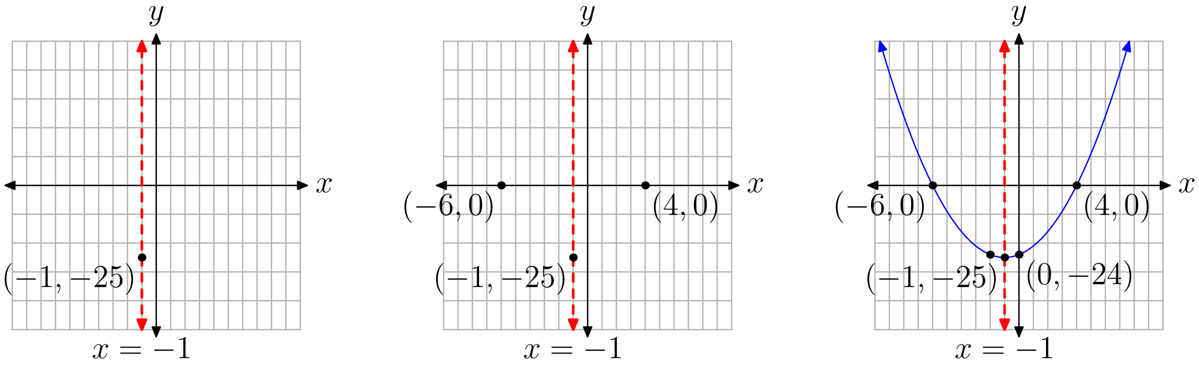
$$x + 6 = 0 \qquad \text{or} \qquad x - 4 = 0.$$

Solve each of these linear equations independently.

$$x = -6 \qquad \text{or} \qquad x = 4.$$

Recall that we let $y = 0$. We've found two solutions, $x = -6$ and $x = 4$. Thus, we have $x$-intercepts at $(-6, 0)$ and $(4, 0)$, as pictured in **Figure 5**(b).

Finally, to find the $y$-intercept, let $x = 0$ in $y = x^2+2x-24$. With this substitution, $y = -24$. Thus, the $y$-intercept is $(0, -24)$, as pictured in **Figure 5**(c). Note that we've also included the mirror image of the $y$-intercept across the axis of symmetry.



(a) Plotting the vertex and axis of symmetry.

(b) Adding the $x$-intercepts provides added accuracy.

(c) Adding the $y$-intercept and its mirror image provides an excellent final graph.

**Figure 5.**

◆

Let's look at one final example.

► **Example 11.** *Plot the parabola represented by the equation $f(x) = -2x^2-7x+15$. Plot and label the vertex, axis of symmetry, and the $x$- and $y$-intercepts.*

First, factor out a $-2$.

$$f(x) = -2\left[x^2 + \frac{7}{2}x - \frac{15}{2}\right]$$

Half[8] of 7/2 is 7/4. Squared, this amounts to 49/16. Add and subtract this last amount to keep the equation balanced.

$$f(x) = -2\left[x^2 + \frac{7}{2}x + \frac{49}{16} - \frac{49}{16} - \frac{15}{2}\right]$$

The first three terms inside the parentheses form a perfect square trinomial. The last two constants are combined with a common denominator.

$$\begin{aligned} f(x) &= -2\left[\left(x^2 + \frac{7}{2}x + \frac{49}{16}\right) - \frac{49}{16} - \frac{120}{16}\right] \\ f(x) &= -2\left[\left(x + \frac{7}{4}\right)^2 - \frac{169}{16}\right] \end{aligned}$$

Finally, redistribute the $-2$.

$$f(x) = -2\left(x + \frac{7}{4}\right)^2 + \frac{169}{8}$$

The graph of this last equation is a parabola that opens downward, translated 7/4 units to the left and 169/8 units upward. This is enough information to plot and label the vertex and axis of symmetry, as shown in **Figure 6**(a).

To find the $y$-intercepts, set $f(x) = 0$ in $f(x) = -2x^2 - 7x + 15$. We will also multiply both sides of the resulting equation by $-1$.

$$\begin{aligned} 0 &= -2x^2 - 7x + 15 \\ 0 &= 2x^2 + 7x - 15 \end{aligned}$$

After comparing $2x^2 + 7x - 15$ with $ax^2 + bx + c$, we note that the integer pair $-3$ and 10 have product equal to $ac = -30$ and sum equal to $b = 7$. Use this pair to express the middle term of $2x^2 + 7x - 15$ as a sum and then factor by grouping.

$$\begin{aligned} 0 &= 2x^2 - 3x + 10x - 15 \\ 0 &= x(2x - 3) + 5(2x - 3) \\ 0 &= (x + 5)(2x - 3) \end{aligned}$$

By the *zero product property*, either

$$x + 5 = 0 \qquad \text{or} \qquad 2x - 3 = 0.$$

Solve these linear equations independently.

$$x = -5 \qquad \text{or} \qquad x = \frac{3}{2}$$

These $x$-values are the zeros of $f$ (they make $f(x) = 0$), so we have $x$-intercepts at $(-5, 0)$ and $(3/2, 0)$, as shown in **Figure 6**(b).

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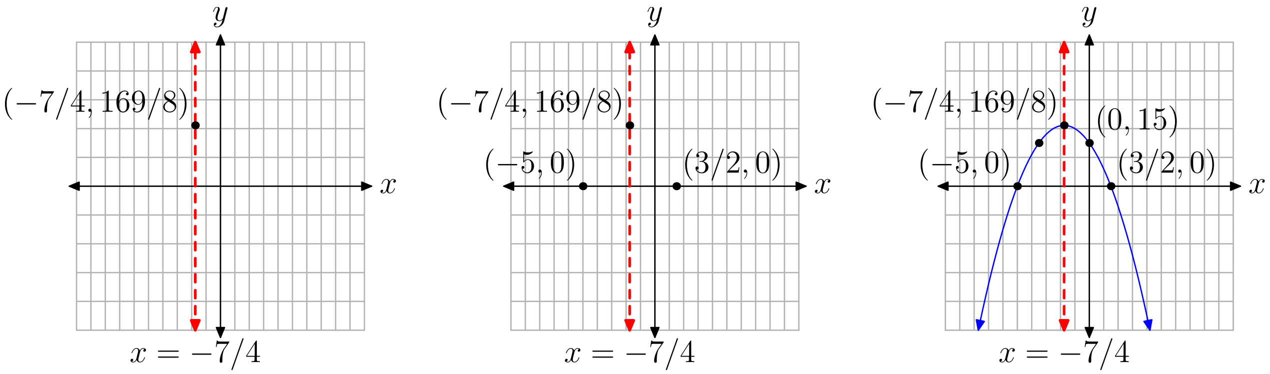
[8] $\frac{1}{2} \cdot \frac{7}{2} = \frac{7}{4}$



(a) Plotting the vertex and axis of symmetry.

(b) Adding the $x$-intercepts provides added accuracy.

(c) Adding the $y$-intercept and its mirror image provide an excellent final graph.

**Figure 6.**

Finally, to find the $y$-intercept, set $x = 0$ in $f(x) = -2x^2 - 7x + 15$ to get $f(0) = 15$. Note the positioning of the $y$-intercept $(0, 15)$ and its mirror image across the axis of symmetry in **Figure 6**(c).

◆

## 5.3 Exercises

In **Exercises 1-8**, factor the given quadratic polynomial.

**1.** $x^2 + 9x + 14$

**2.** $x^2 + 6x + 5$

**3.** $x^2 + 10x + 9$

**4.** $x^2 + 4x - 21$

**5.** $x^2 - 4x - 5$

**6.** $x^2 + 7x - 8$

**7.** $x^2 - 7x + 12$

**8.** $x^2 + 5x - 24$

In **Exercises 9-16**, find the zeros of the given quadratic function.

**9.** $f(x) = x^2 - 2x - 15$

**10.** $f(x) = x^2 + 4x - 32$

**11.** $f(x) = x^2 + 10x - 39$

**12.** $f(x) = x^2 + 4x - 45$

**13.** $f(x) = x^2 - 14x + 40$

**14.** $f(x) = x^2 - 5x - 14$

**15.** $f(x) = x^2 + 9x - 36$

**16.** $f(x) = x^2 + 11x - 26$

In **Exercises 17-22**, perform each of the following tasks for the quadratic functions.

i. Load the function into `Y1` of the `Y=` of your graphing calculator. Adjust the window parameters so that the vertex is visible in the viewing window.
ii. Set up a coordinate system on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Make a reasonable copy of the image in the viewing window of your calculator on this coordinate system and label it with its equation.
iii. Use the `zero` utility on your graphing calculator to find the zeros of the function. Use these results to plot the $x$-intercepts on your coordinate system and label them with their coordinates.
iv. Use a strictly algebraic technique (no calculator) to find the zeros of the given quadratic function. Show your work next to your coordinate system. Be stubborn! Work the problem until your algebraic and graphically zeros are a reasonable match.

**17.** $f(x) = x^2 + 5x - 14$

**18.** $f(x) = x^2 + x - 20$

**19.** $f(x) = -x^2 + 3x + 18$

**20.** $f(x) = -x^2 + 3x + 40$

**21.** $f(x) = x^2 - 16x - 36$

**22.** $f(x) = x^2 + 4x - 96$

---

[9] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

In **Exercises 23-30**, perform each of the following tasks for the given quadratic function.

i. Set up a coordinate system on graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Use the technique of completing the square to place the quadratic function in vertex form. Plot the vertex on your coordinate system and label it with its coordinates. Draw the axis of symmetry on your coordinate system and label it with its equation.
iii. Use a strictly algebraic technique (no calculators) to find the $x$-intercepts of the graph of the given quadratic function. Plot them on your coordinate system and label them with their coordinates.
iv. Find the $y$-intercept of the graph of the quadratic function. Plot the $y$-intercept on your coordinate system and its mirror image across the axis of symmetry, then label these points with their coordinates.
v. Using all the information plotted, draw the graph of the quadratic function and label it with the vertex form of its equation. Use interval notation to describe the domain and range of the quadratic function.

**23.** $f(x) = x^2 + 2x - 8$

**24.** $f(x) = x^2 - 6x + 8$

**25.** $f(x) = x^2 + 4x - 12$

**26.** $f(x) = x^2 + 8x + 12$

**27.** $f(x) = -x^2 - 2x + 8$

**28.** $f(x) = -x^2 - 2x + 24$

**29.** $f(x) = -x^2 - 8x + 48$

**30.** $f(x) = -x^2 - 8x + 20$

---

In **Exercises 31-38**, factor the given quadratic polynomial.

**31.** $42x^2 + 5x - 2$

**32.** $3x^2 + 7x - 20$

**33.** $5x^2 - 19x + 12$

**34.** $54x^2 - 3x - 1$

**35.** $-4x^2 + 9x - 5$

**36.** $3x^2 - 5x - 12$

**37.** $2x^2 - 3x - 35$

**38.** $-6x^2 + 25x + 9$

---

In **Exercises 39-46**, find the zeros of the given quadratic functions.

**39.** $f(x) = 2x^2 - 3x - 20$

**40.** $f(x) = 2x^2 - 7x - 30$

**41.** $f(x) = -2x^2 + x + 28$

**42.** $f(x) = -2x^2 + 15x - 22$

**43.** $f(x) = 3x^2 - 20x + 12$

**44.** $f(x) = 4x^2 + 11x - 20$

**45.** $f(x) = -4x^2 + 4x + 15$

**46.** $f(x) = -6x^2 - x + 12$

In **Exercises 47-52**, perform each of the following tasks for the given quadratic functions.

i. Load the function into `Y1` of the `Y=` of your graphing calculator. Adjust the window parameters so that the vertex is visible in the viewing window.
ii. Set up a coordinate system on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Make a reasonable copy of the image in the viewing window of your calculator on this coordinate system and label it with its equation.
iii. Use the `zero` utility on your graphing calculator to find the zeros of the function. Use these results to plot the $x$-intercepts on your coordinate system and label them with their coordinates.
iv. Use a strictly algebraic technique (no calculator) to find the zeros of the given quadratic function. Show your work next to your coordinate system. Be stubborn! Work the problem until your algebraic and graphically zeros are a reasonable match.

**47.** $f(x) = 2x^2 + 3x - 35$

**48.** $f(x) = 2x^2 - 5x - 42$

**49.** $f(x) = -2x^2 + 5x + 33$

**50.** $f(x) = -2x^2 - 5x + 52$

**51.** $f(x) = 4x^2 - 24x - 13$

**52.** $f(x) = 4x^2 + 24x - 45$

---

In **Exercises 53-60**, perform each of the following tasks for the given quadratic functions.

i. Set up a coordinate system on graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Use the technique of completing the square to place the quadratic function in vertex form. Plot the vertex on your coordinate system and label it with its coordinates. Draw the axis of symmetry on your coordinate system and label it with its equation.
iii. Use a strictly algebraic method (no calculators) to find the $x$-intercepts of the graph of the quadratic function. Plot them on your coordinate system and label them with their coordinates.
iv. Find the $y$-intercept of the graph of the quadratic function. Plot the $y$-intercept on your coordinate system and its mirror image across the axis of symmetry, then label these points with their coordinates.
v. Using all the information plotted, draw the graph of the quadratic function and label it with the vertex form of its equation. Use interval notation to describe the domain and range of the quadratic function.

**53.** $f(x) = 2x^2 - 8x - 24$

**54.** $f(x) = 2x^2 - 4x - 6$

**55.** $f(x) = -2x^2 - 4x + 16$

**56.** $f(x) = -2x^2 - 16x + 40$

**57.** $f(x) = 3x^2 + 18x - 48$

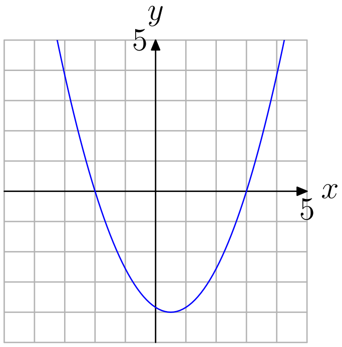
**58.** $f(x) = 3x^2 + 18x - 216$

**59.** $f(x) = 2x^2 + 10x - 48$

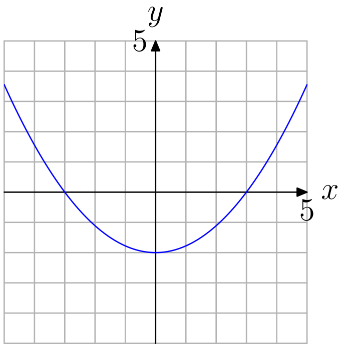
**60.** $f(x) = 2x^2 - 10x - 100$

In **Exercises 61-66**, Use the graph of $f(x) = ax^2 + bx + c$ shown to find all solutions of the equation $f(x) = 0$. (Note: Every solution is an integer.)

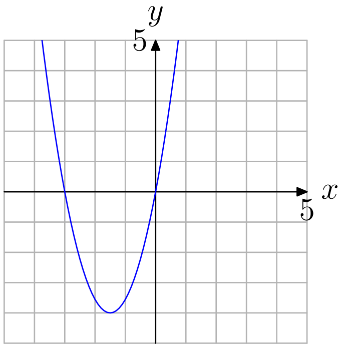
**61.**



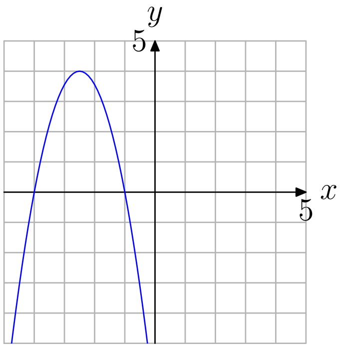
**62.**



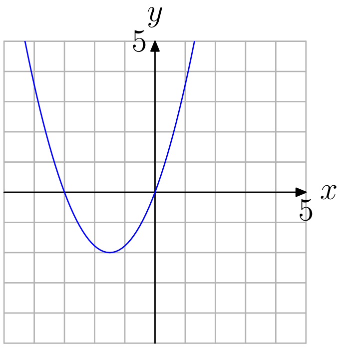
**63.**



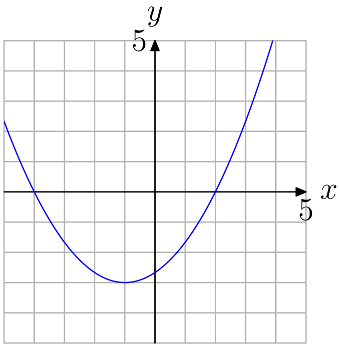
**64.**



**65.**



**66.**



## 5.3 Answers

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**1.** $(x+2)(x+7)$

**3.** $(x+9)(x+1)$

**5.** $(x-5)(x+1)$

**7.** $(x-4)(x-3)$

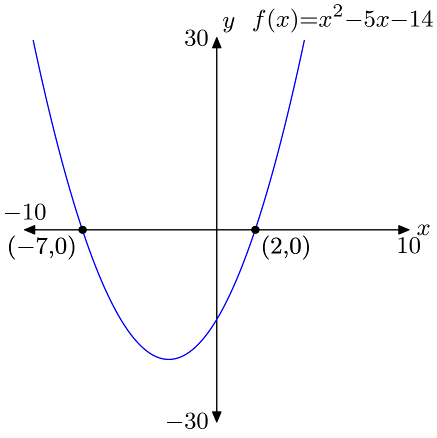
**9.** Zeros: $x=-3$, $x=5$

**11.** Zeros: $x=-13$, $x=3$

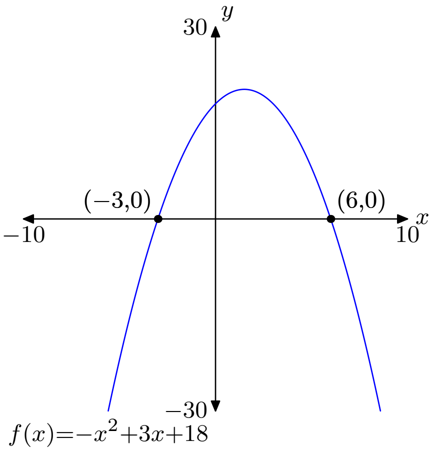
**13.** Zeros: $x=4$, $x=10$

**15.** Zeros: $x=-12$, $x=3$

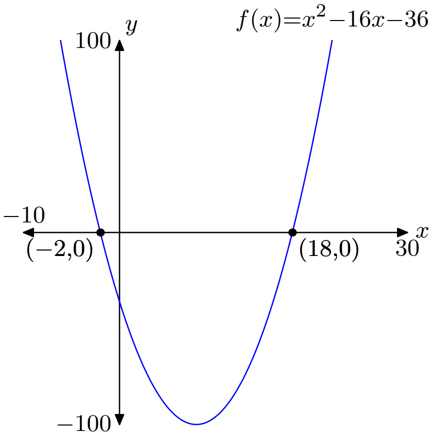
**17.**



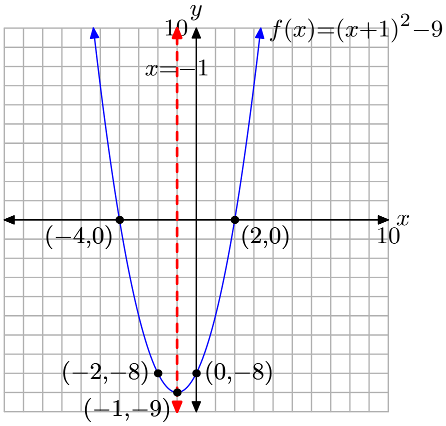
**19.**



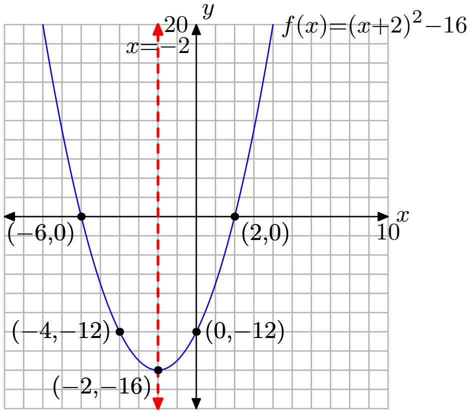
**21.**



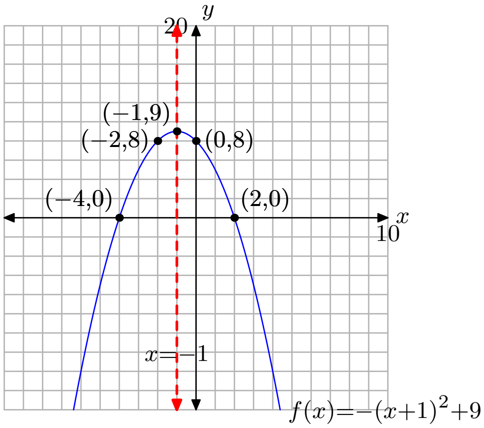
**23.** Domain = $(-\infty, \infty)$,
Range = $[-9, \infty)$



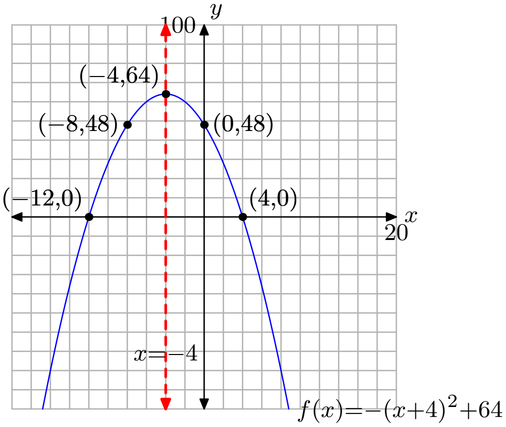
**25.** Domain = $(-\infty, \infty)$,
Range = $[-16, \infty)$



**27.** Domain = $(-\infty, \infty)$,
Range = $(-\infty, 9]$



**29.** Domain = $(-\infty, \infty)$,
Range = $(-\infty, 64]$



**31.** $(7x+2)(6x-1)$

**33.** $(x-3)(5x-4)$

**35.** $(4x-5)(-x+1)$

**37.** $(2x+7)(x-5)$

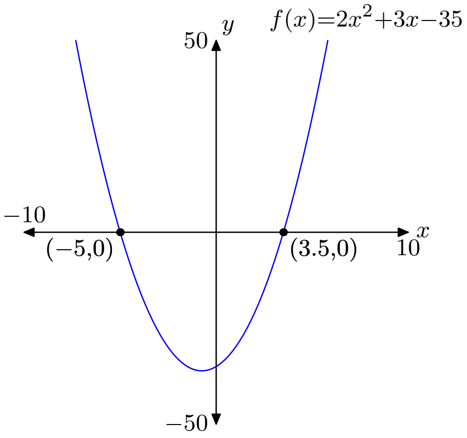
**39.** Zeros: $x=-5/2$, $x=4$

**41.** Zeros: $x=-7/2$, $x=4$

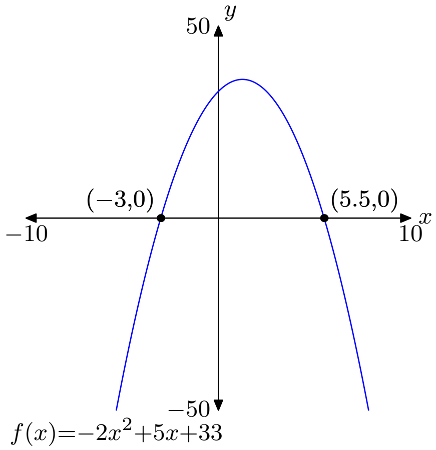
**43.** Zeros: $x=2/3$, $x=6$

**45.** Zeros: $x = -3/2$, $x = 5/2$

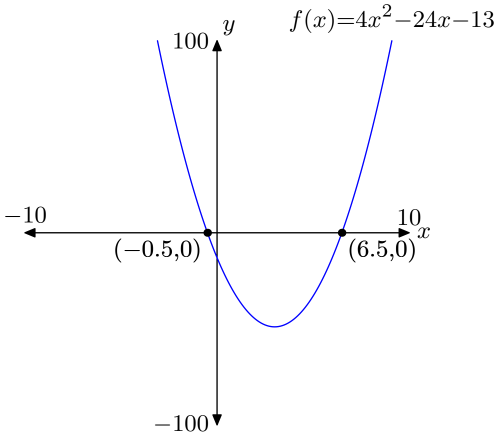
**47.**



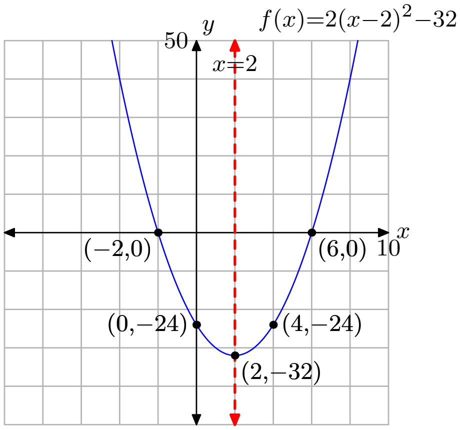
**49.**



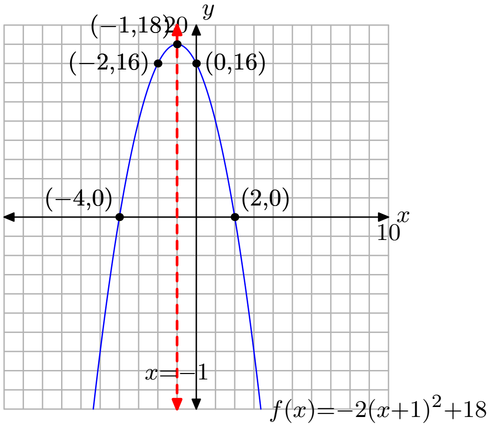
**51.**



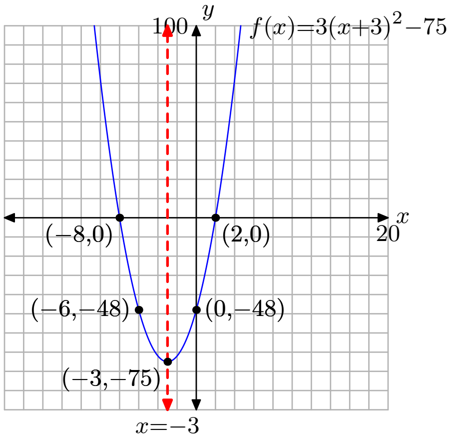
**53.** Domain $= (-\infty, \infty)$,
Range $= [-32, \infty)$



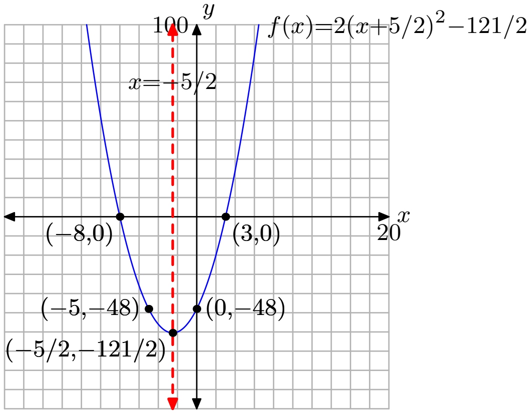
**55.** Domain $= (-\infty, \infty)$,
Range $= (-\infty, 18]$



**57.** Domain $= (-\infty, \infty)$,
Range $= [-75, \infty)$



**59.** Domain $= (-\infty, \infty)$,
Range $= [-121/2, \infty)$



**61.** $-2$, 3

**63.** $-3$, 0

**65.** $-3$, 0

# 5.4 The Quadratic Formula

Consider the general quadratic function

$$f(x) = ax^2 + bx + c.$$

In the previous section, we learned that we can find the zeros of this function by solving the equation

$$f(x) = 0.$$

If we substitute $f(x) = ax^2 + bx + c$, then the resulting equation

$$ax^2 + bx + c = 0 \tag{1}$$

is called a *quadratic equation*. In the previous section, we solved equations of this type by factoring and using the zero product property.

However, it is not always possible to factor the trinomial on the left-hand side of the quadratic **equation (1)** as a product of factors with integer coefficients. For example, consider the quadratic equation

$$2x^2 + 7x - 3 = 0. \tag{2}$$

Comparing $2x^2 + 7x - 3$ with $ax^2 + bx + c$, let's list all integer pairs whose product is $ac = (2)(-3) = -6$.

| | | |
|---|---|---|
| 1, −6 | | −1, 6 |
| 2, −3 | | −2, 3 |

Not a single one of these integer pairs adds to $b = 7$. Therefore, the quadratic trinomial $2x^2 + 7x - 3$ does not factor over the integers.[11] Consequently, we'll need another method to solve the quadratic **equation (2)**.

The purpose of this section is to develop a formula that will consistently provide solutions of the general quadratic **equation (1)**. However, before we can develop the "Quadratic Formula," we need to lay some groundwork involving the square roots of numbers.

## Square Roots

We begin our discussion of square roots by investigating the solutions of the equation $x^2 = a$. Consider the rather simple equation

$$x^2 = 25. \tag{3}$$

Because $(-5)^2 = 25$ and $(5)^2 = 25$, **equation (3)** has two solutions, $x = -5$ or $x = 5$. We usually denote these solutions simultaneously, using a "plus or minus" sign:

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[10] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

[11] This means that the trinomial $2x^2 + 7x - 3$ cannot be expressed as a product of factors with integral (integer) coefficients.

$$x = \pm 5$$

These solutions are called *square roots* of 25. Because there are two solutions, we need a different notation for each. We will denote the positive square root of 25 with the notation $\sqrt{25}$ and the negative square root of 25 with the notation $-\sqrt{25}$. Thus,

$$\sqrt{25} = 5 \qquad \text{and} \qquad -\sqrt{25} = -5.$$

In a similar vein, the equation $x^2 = 36$ has two solutions, $x = \pm\sqrt{36}$, or alternatively, $x = \pm 6$. The notation $\sqrt{36}$ calls for the positive square root, while the notation $-\sqrt{36}$ calls for the negative square root. That is,

$$\sqrt{36} = 6 \qquad \text{and} \qquad -\sqrt{36} = -6.$$

It is not necessary that the right-hand side of the equation $x^2 = a$ be a "perfect square." For example, the equation

$$x^2 = 7 \qquad \text{has solutions} \qquad x = \pm\sqrt{7}. \tag{4}$$

There is no *rational* square root of 7. That is, there is no way to express the square root of 7 in the form $p/q$, where $p$ and $q$ are integers. Therefore, $\sqrt{7}$ is an example of an irrational number. However, $\sqrt{7}$ is a perfectly valid real number and we're perfectly comfortable leaving our answer in the form shown in **equation (4)**.

However, if an approximation is needed for the square root of 7, we can reason that because 7 lies between 4 and 9, the square root of 7 will lie between 2 and 3. Because 7 is closer to 9 than 4, a reasonable approximation might be[12]

$$\sqrt{7} \approx 2.6.$$

A calculator can provide an even better approximation. For example, our TI83 reports

$$\sqrt{7} \approx 2.645751311.$$

There are two degenerate cases involving the equation $x^2 = a$ that demand our attention.

1. The equation $x^2 = 0$ has only one solution, namely $x = 0$. Thus, $\sqrt{0} = 0$.
2. The equation $x^2 = -4$ has no real solutions.[13] It is not possible to square a real number and get $-4$. In this situation, we will simply state that "the equation $x^2 = -4$ has no real solutions (no solutions that are real numbers)."

---

[12] The symbol $\approx$ means "approximately equal to."

[13] It is incorrect to state that the equation $x^2 = -4$ has "no solutions." If we introduce the set of *complex numbers* (a set of numbers introduced in college algebra and trigonometry), then the equation $x^2 = -4$ has two solutions, both of which are complex numbers.

▶ **Example 5.** *Find all real solutions of the equations* $x^2 = 30$, $x^2 = 0$, *and* $x^2 = -14$.

The solutions follow.

- The equation $x^2 = 30$ has two real solutions, namely $x = \pm\sqrt{30}$.
- The equation $x^2 = 0$ has one real solution, namely $x = 0$.
- The equation $x^2 = -14$ has no real solutions.

◇

Let's try additional examples.

▶ **Example 6.** *Find all real solutions of the equation* $(x + 2)^2 = 43$.

There are two possibilities for $x + 2$, namely

$$x + 2 = \pm\sqrt{43}.$$

To solve for $x$, subtract 2 from both sides of this last equation.

$$x = -2 \pm \sqrt{43}.$$

Although this last answer is usually the preferable form of the answer, there are some times when an approximation is needed. So, our TI83 gives the following approximations.

$$-2 - \sqrt{43} \approx -8.557438524 \qquad \text{and} \qquad -2 + \sqrt{43} \approx 4.557438524$$

◇

▶ **Example 7.** *Find all real solutions of the equation* $(x - 4)^2 = -15$.

If $x$ is a real number, then so is $x - 4$. It's not possible to square the real number $x - 4$ and get $-15$. Thus, this problem has no real solutions.[14]

◇

## *Development of the Quadratic Formula*

We now have all the groundwork in place to pursue a solution of the general quadratic equation

$$ax^2 + bx + c = 0. \tag{8}$$

We're going to use a form of "completing the square" to solve this equation for $x$. Let's begin by subtracting $c$ from both sides of the equation.

$$ax^2 + bx = -c$$

---

[14] Again, when you study the complex numbers, you will learn that this equation has two complex solutions. Hence, it is important that you do not say that "this problem has no solutions," as that is simply not true. You must say that "this problem has no real solutions."

Next, divide both sides of the equation by $a$.

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

Take half of the coefficient of $x$, as in $(1/2)(b/a)=b/(2a)$. Square this result to get $b^2/(4a^2)$. Add this amount to both sides of the equation.

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$$

On the left we factor the perfect square trinomial. On the right we get a common denominator and add the resulting equivalent fractions.

$$\begin{aligned}\left(x+\frac{b}{2a}\right)^2&=-\frac{4ac}{4a^2}+\frac{b^2}{4a^2}\\ \left(x+\frac{b}{2a}\right)^2&=\frac{b^2-4ac}{4a^2}\end{aligned}$$

Provided the right-hand side of this last equation is positive, we have two real solutions.

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

On the right, we take the square root of the top and the bottom of the fraction.[15]

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

To complete the solution, we need only subtract $b/(2a)$ from both sides of the equation.

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

Although this last answer is a perfectly good solution, we customarily rewrite the solution with a single common denominator.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{9}$$

This last result gives the solution to the general quadratic **equation (8)**. The solution (**9**) is called the *quadratic formula*.

---

[15] In a later section we will present a more formal approach to the symbolic manipulation of radicals. For now, you can compute $(2/3)^2$ with the calculation $(2/3)(2/3)=4/9$, or you can simply square numerator and denominator of the fraction, as in $(2/3)^2=(2^2/3^2)=4/9$. Conversely, one can take the square root of a fraction by taking the square root of the numerator divided by the square root of the denominator, as in $\sqrt{4/9}=\sqrt{4}/\sqrt{9}=2/3$.

**The Quadratic Formula.** The solutions to the *quadratic equation*

$$ax^2 + bx + c = 0 \tag{10}$$

are given by the *quadratic formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{11}$$

Although the development of the quadratic formula can be intimidating, in practice its application is quite simple. Let's look at some examples.

▶ **Example 12.** *Use the quadratic formula to solve the equation*

$$x^2 = 27 - 6x.$$

The first step is to place the equation in the form $ax^2 + bx + c = 0$ by moving every term to one side of the equation,[16] arranging the terms in descending powers of $x$.

$$x^2 + 6x - 27 = 0$$

Next, compare $x^2 + 6x - 27 = 0$ with the general form of the quadratic equation $ax^2 + bx + c = 0$ and note that $a = 1$, $b = 6$, and $c = -27$. Copy down the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute $a = 1$, $b = 6$, and $c = -27$ and simplify.

$$\begin{aligned} x &= \frac{-(6) \pm \sqrt{(6)^2 - 4(1)(-27)}}{2(1)} \\ x &= \frac{-6 \pm \sqrt{36 + 108}}{2} \\ x &= \frac{-6 \pm \sqrt{144}}{2} \end{aligned}$$

In this case, 144 is a perfect square. That is, $\sqrt{144} = 12$, so we can continue to simplify.

$$x = \frac{-6 \pm 12}{2}$$

It's important to note that there are *two* real answers, namely

$$x = \frac{-6 - 12}{2} \quad \text{or} \quad x = \frac{-6 + 12}{2}.$$

Simplifying,

$$x = -9 \quad \text{or} \quad x = 3.$$

---

[16] We like to say "Make one side equal to zero."

It's interesting to note that this problem could have been solved by factoring. Indeed,

$$\begin{aligned} x^2+6x-27&=0\\ (x-3)(x+9)&=0, \end{aligned}$$

so the zero product property requires that either $x-3=0$ or $x+9=0$, which leads to $x=3$ or $x=-9$, answers identical to those found by the quadratic formula.

◆

We'll have more to say about the "discriminant" soon, but it's no coincidence that the quadratic $x^2+6x-27$ factored. Here is the relevant fact.

**When the Discriminant is a Perfect Square**. In the quadratic formula,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a},$$

the number under the radical, $b^2-4ac$, is called the **discriminant**. When the discriminant is a perfect square, the quadratic function will always factor.

However, it is not always the case that we can factor the given quadratic. Let's look at another example.

▶ **Example 13.** *Given the quadratic function $f(x)=x^2-2x$, find all real solutions of $f(x)=2$.*

Because $f(x)=x^2-2x$, the equation $f(x)=2$ becomes

$$x^2-2x=2.$$

Set one side of the equation equal to zero by subtracting 2 from both sides of the equation.[17]

$$x^2-2x-2=0$$

Compare $x^2-2x-2=0$ with the general quadratic equation $ax^2+bx+c=0$ and note that $a=1$, $b=-2$ and $c=-2$. Write down the quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Next, substitute $a=1$, $b=-2$, and $c=-2$. Note the careful use of parentheses.[18]

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-2)}}{2(1)}$$

---

[17] Note that the quadratic expression on the left-hand side of the resulting equation does not factor over the integers. There are no integer pairs whose product is $-2$ that sum to $-2$.

[18] For example, without parentheses, $-2^2=-4$, whereas with parentheses $(-2)^2=4$.

Simplify.

$$x = \frac{2 \pm \sqrt{4+8}}{2}$$
$$x = \frac{2 \pm \sqrt{12}}{2}$$

In this case, 12 is not a perfect square, so we've simplified as much as is possible at this time.[19] However, we can approximate these solutions with the aid of a calculator.

$$x = \frac{2-\sqrt{12}}{2} \approx -0.7320508076 \qquad \text{and} \qquad x = \frac{2+\sqrt{12}}{2} \approx 2.732050808. \tag{14}$$

We will find these approximations useful in what follows.

◇

The equations in **Examples 12** and **13** represent a fundamental shift in our usual technique for solving equations. In the past, we've tried to "isolate" the terms containing $x$ (or whatever unknown we are solving for) on one side of the equation, and all other terms on the other side of the equation. Now, in **Examples 12** and **13**, we find ourselves moving everything to one side of the equation, making one side of the equation equal to zero. This bears some explanation.

**Linear or Nonlinear**. Let's assume that the unknown we are solving for is $x$.

- If the highest power of $x$ present in the equation is $x$ to the first power, then the equation is **linear**. Thus, for example, each of the equations
$$2x+3=7, \qquad 3-4x=5x+9, \qquad \text{and} \qquad ax+b=cx+d$$
is linear.
- If there are powers of $x$ higher than $x$ to the first power in the equation, then the equation is **nonlinear**. Thus, for example, each of the equations
$$x^2-4x=9, \qquad x^3=2x+3, \qquad \text{and} \qquad ax^2+bx=cx+d$$
is nonlinear.

The strategy for solving an equation will shift, depending on whether the equation is linear or nonlinear.

---

[19] In a later chapter on irrational functions, we will take up the topic of simplifying radical expressions. Until then, this form of the final answer will have to suffice.

**Solution Strategy—Linear Versus Nonlinear**. When solving equations, you must first ask if the equation is linear or nonlinear. Again, let's assume the unknown we wish to solve for is $x$.

- If the equation is linear, move all terms containing $x$ to one side of the equation, all the remaining terms to the other side of the equation.
- If the equation is nonlinear, move all terms to one side of the equation, making the other side of the equation zero.

Thus, because $ax + b = cx + d$ is linear in $x$, the first step in solving the equation would be to move all terms containing $x$ to one side of the equation, all other terms to the other side of the equation, as in

$$ax - cx = d - b.$$

On the other hand, the equation $ax^2 + bx = cx + d$ is nonlinear in $x$, so the first step would be to move all terms to one side of the equation, making the other side of the equation equal to zero, as in

$$ax^2 + bx - cx - d = 0.$$

In **Example 13**, the equation $x^2 - 2x = 2$ is nonlinear in $x$, so we moved everything to the left-hand side of the equation, making the right-hand side of the equation equal to zero, as in $x^2 - 2x - 2 = 0$. However, it doesn't matter which side you make equal to zero. Suppose instead that you move every term to the right-hand side of the equation, as in

$$0 = -x^2 + 2x + 2.$$

Comparing $0 = -x^2 + 2x + 2$ with general quadratic equation $0 = ax^2 + bx + c$, note that $a = -1$, $b = 2$, and $c = 2$. Write down the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Next, substitute $a = -1$, $b = 2$, and $c = 2$. Again, note the careful use of parentheses.

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(-1)(2)}}{2(-1)}$$

This leads to two solutions,

$$x = \frac{-2 \pm \sqrt{4 + 8}}{-2} = \frac{-2 \pm \sqrt{12}}{-2}.$$

In **Example 13**, we found the following solutions and their approximations.

$$x = \frac{2 - \sqrt{12}}{2} \approx -0.7320508076 \qquad \text{and} \qquad x = \frac{2 + \sqrt{12}}{2} \approx 2.732050808.$$

It is a fair question to ask if our solutions $x = (-2 \pm \sqrt{12})/(-2)$ are the same. One way to find out is to find decimal approximations of each on our calculator.

$$x = \frac{-2 - \sqrt{12}}{-2} \approx 2.732050808 \qquad \text{and} \qquad x = \frac{-2 + \sqrt{12}}{-2} \approx -0.7320508076.$$

The fact that we get the same decimal approximations should spark confidence that we have the same solutions. However, we can also manipulate the exact forms of our solutions to show that they match the previous forms found in **Example 13**.

Take the two solutions and multiply both numerator and denominator by minus one.

$$\frac{-2 - \sqrt{12}}{-2} = \frac{2 + \sqrt{12}}{2} \qquad \text{and} \qquad \frac{-2 + \sqrt{12}}{-2} = \frac{2 - \sqrt{12}}{2}$$

This shows that our solutions are identical to those found in **Example 13**.

We can do the same negation of numerator and denominator in compact form.

$$\frac{-2 \pm \sqrt{12}}{-2} = \frac{2 \mp \sqrt{12}}{2}$$

Note that this leads to the same two answers, $(2 - \sqrt{12})/2$ and $(2 + \sqrt{12})/2$.

Of the two methods (move all the terms to the left or all the terms to the right), we prefer the approach of **Example 13**. By moving the terms to the left-hand side of the equation, as in $x^2 - 2x - 2 = 0$, the coefficient of $x^2$ is positive $(a = 1)$ and we avoid the minus sign in the denominator produced by the quadratic formula.

## *Intercepts*

In **Example 13**, we used the quadratic formula to find the solutions of $x^2 - 2x - 2 = 0$. These solutions, and their approximations, are shown in **equation (14)**. It is important to make the connection that the solutions in **equation (14)** are the zeros of the quadratic function $g(x) = x^2 - 2x - 2$. The zeros also provide the $x$-coordinates of the $x$-intercepts of the graph of $g$ (a parabola). To emphasize this point, let's draw the graph of the parabola having the equation $g(x) = x^2 - 2x - 2$.

First, complete the square to place the quadratic function in vertex form. Take half the middle coefficient and square, as in $[(1/2)(-2)]^2 = 1$; then add and subtract this term so the equation remains balanced.

$$\begin{aligned} g(x) &= x^2 - 2x - 2 \\ g(x) &= x^2 - 2x + 1 - 1 - 2 \end{aligned}$$

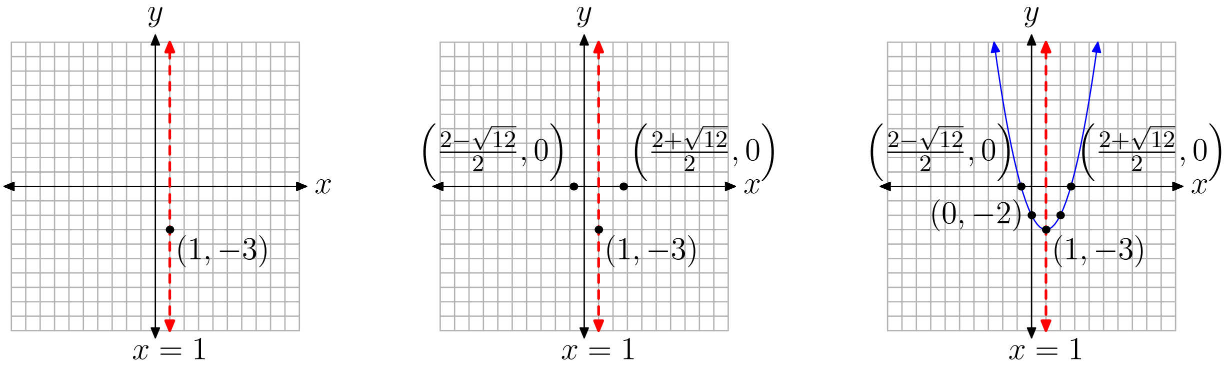
Factor the perfect square trinomial, then combine the constants at the end.

$$g(x) = (x - 1)^2 - 3$$

This is a parabola that opens upward. It is shifted to the right 1 unit and down 3 units. This makes it easy to identify the vertex and draw the axis of symmetry, as shown in **Figure 1**(a).

It will now be apparent why we used our calculator to approximate the solutions in (**14**). These are the $x$-coordinates of the $x$-intercepts. One $x$-intercept is located at approximately $(-0.73, 0)$, the other at approximately $(2.73, 0)$. These approximations are used to plot the location of the intercepts as shown in **Figure 1**(b). However, the actual values of the intercepts are $((2-\sqrt{12})/2, 0)$ and $((2+\sqrt{12})/2, 0)$, and these exact values should be used to annotate the intercepts, as shown in **Figure 1**(b).

Finally, to find the $y$-intercept, let $x = 0$ in $g(x) = x^2 - 2x - 2$. Thus, $g(0) = -2$ and the $y$-intercept is $(0, -2)$. The $y$-intercept and its mirror image across the axis of symmetry are both plotted in **Figure 1**(c), where the final graph of the parabola is also shown.



(a) Plotting the vertex and axis of symmetry.

(b) Adding the $x$-intercepts provides added accuracy.

(c) Adding the $y$-intercept and its mirror image provides an excellent final graph.

**Figure 1.**

We've made an important point and we pause to provide emphasis.

**Zeros and Intercepts**. Whenever you use the quadratic formula to solve the quadratic equation

$$ax^2 + bx + c = 0,$$

the solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

are the zeros of the quadratic function

$$f(x) = ax^2 + bx + c.$$

The solutions also provide the $x$-coordinates of the $x$-intercepts of the graph of $f$.

We need to discuss one final concept.

## The Discriminant

Consider again the quadratic equation $ax^2 + bx + c = 0$ and the solutions (zeros) provided by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The expression under the radical, $b^2 - 4ac$, is called the *discriminant*, which we denote by the letter $D$. That is, the formula for the discriminant is given by

$$D = b^2 - 4ac.$$

The discriminant is used to determine the nature and number of solutions to the quadratic equation $ax^2+bx+c = 0$. This is done without actually calculating the solutions.

Let's look at three key examples.

▶ **Example 15.** *Consider the quadratic equation*

$$x^2 - 4x - 4 = 0.$$

*Calculate the discriminant and use it to determine the nature and number of the solutions.*

Compare $x^2 - 4x - 4 = 0$ with $ax^2 + bx + c = 0$ and note that $a = 1$, $b = -4$, and $c = -4$. The discriminant is given by the calculation

$$D = b^2 - 4ac = (-4)^2 - 4(1)(-4) = 32.$$

Note that the discriminant $D$ is positive; i.e., $D > 0$.

Consider the quadratic function $f(x) = x^2 - 4x - 4$, which can be written in vertex form

$$f(x) = (x - 2)^2 - 8.$$

This is a parabola that opens upward. It is shifted to the right 2 units, then downward 8 units. Therefore, it will cross the $x$-axis in two locations. Hence, one would expect that the quadratic formula would provide two real solutions ($x$-intercepts). Indeed,

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-4)}}{2(1)} = \frac{4 \pm \sqrt{32}}{2}.$$

Note that the discriminant, $D = 32$ as calculated above, is the number under the square root. These solutions have approximations

$$x = \frac{4 - \sqrt{32}}{2} \approx -0.8284271247 \qquad \text{and} \qquad x = \frac{4 + \sqrt{32}}{2} \approx 4.828427125,$$

which aid in plotting an accurate graph of $f(x) = (x - 2)^2 - 8$, as shown in **Figure 2**.

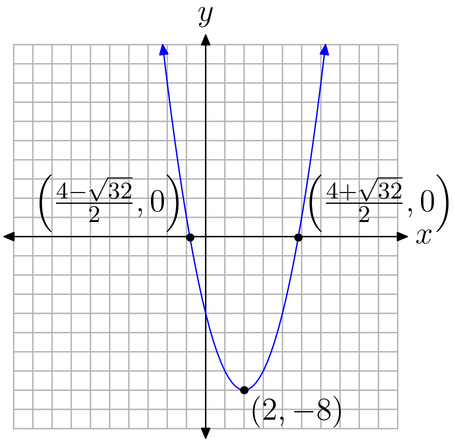


**Figure 2.** If the discriminant is positive, there are two real $x$-intercepts.

Thus, if the discriminant is positive, the parabola will have two real $x$-intercepts.

◇

Next, let's look at an example where the discriminant equals zero.

▶ **Example 16.** *Consider again the quadratic equation* $ax^2 + bx + c = 0$ *and the solutions (zeros) provided by the quadratic formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

*The expression under the radical,* $b^2 - 4ac$*, is called the* discriminant*, which we denote by the letter* $D$*. That is, the formula for the discriminant is given by*

$$D = b^2 - 4ac.$$

*The discriminant is used to determine the nature and number of solutions to the quadratic equation* $ax^2+bx+c = 0$*. This is done without actually calculating the solutions. Consider the quadratic equation*

$$x^2 - 4x + 4 = 0.$$

*Calculate the discriminant and use it to determine the nature and number of the solutions.*

Compare $x^2 - 4x + 4 = 0$ with $ax^2 + bx + c = 0$ and note that $a = 1$, $b = -4$, and $c = 4$. The discriminant is given by the calculation

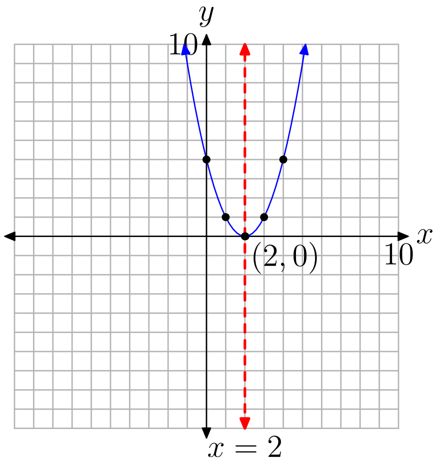
$$D = b^2 - 4ac = (-4)^2 - 4(1)(4) = 0.$$

Note that the discriminant equals zero.

Consider the quadratic function $f(x) = x^2 - 4x + 4$, which can be written in vertex form

$$f(x) = (x - 2)^2. \tag{17}$$

This is a parabola that opens upward and is shifted 2 units to the right. Note that there is no vertical shift, so the vertex of the parabola will rest on the $x$-axis, as shown in **Figure 3**. In this case, we found it necessary to plot two points to the right of the axis of symmetry, then mirror them across the axis of symmetry, in order to get an accurate plot of the parabola.



| $x$ | $f(x) = (x-2)^2$ |
|---|---|
| 3 | 1 |
| 4 | 4 |

**Figure 3.** At the right is a table of points satisfying $f(x) = (x-2)^2$. These points and their mirror images are seen as solid dots superimposed on the graph of $f(x) = (x-2)^2$ at the left.

Take a closer look at **equation (17)**. If we set $f(x) = 0$ in this equation, then we get $0 = (x-2)^2$. This could be written $0 = (x-2)(x-2)$ and we could say that the solutions are 2 and 2 again. However, mathematicians prefer to say that "2 is a solution of multiplicity 2" or "2 is a double solution."[20] Note how the parabola is tangent to the $x$-axis at the location of the "double solution." That is, the parabola comes down from positive infinity, touches (but does not cross) the $x$-axis at $x = 2$, then rises again to positive infinity. Of course, the situation would be reversed in the parabola opened downward, as in $g(x) = -(x-2)^2$, but the graph would still "kiss" the $x$-axis at the location of the "double solution."

Still, the key thing to note here is the fact that the discriminant $D = 0$ and the parabola has only one $x$-intercept. That is, the equation $x^2 - 4x + 4 = 0$ has a single real solution.

◆

Next, let's look what happens when the discriminant is negative.

▶ **Example 18.** *Consider the quadratic equation*

$$x^2 - 4x + 8 = 0.$$

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[20] Actually, mathematicians call these "double roots," but we prefer to postpone that language until the chapter on polynomial functions.

*Calculate the discriminant and use it to determine the nature and number of the solutions.*

Compare $x^2 - 4x + 8 = 0$ with $ax^2 + bx + c = 0$ and note that $a = 1$, $b = -4$, and $c = 8$. The discriminant is given by the calculation

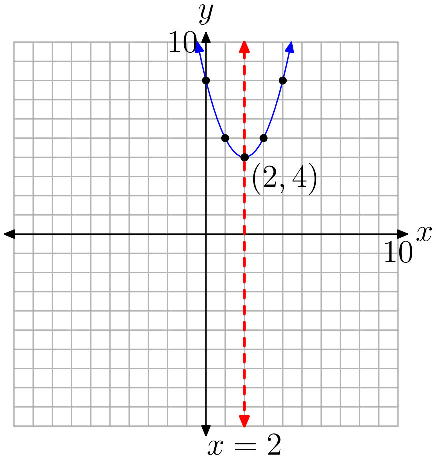
$$D = b^2 - 4ac = (-4)^2 - 4(1)(8) = -16.$$

Note that the discriminant is negative.

Consider the quadratic function $f(x) = x^2 - 4x + 8$, which can be written in vertex form

$$f(x) = (x - 2)^2 + 4.$$

This is a parabola that opens upward. Moreover, it has to be shifted 2 units to the right and 4 units upward, so there can be no $x$-intercepts, as shown in **Figure 4**. Again, we found it necessary in this example to plot two points to the right of the axis of symmetry, then mirror them, in order to get an accurate plot of the parabola.



| $x$ | $f(x) = (x-2)^2+4$ |
|---|---|
| 3 | 5 |
| 4 | 8 |

**Figure 4.** At the right is a table of points satisfying $f(x) = (x - 2)^2 + 4$. These points and their mirror images are seen as solid dots superimposed on the graph of $f(x) = (x - 2)^2 + 4$ at the left.

Once again, the key point in this example is the fact that the discriminant is negative and there are no real solutions of the quadratic equation (equivalently, there are no $x$-intercepts). Let's see what happens if we actually try to find the solutions of $x^2 - 4x + 8 = 0$ using the quadratic formula. Again, $a = 1$, $b = -4$, and $c = 8$, so

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(8)}}{2(1)}.$$

Simplifying,

$$x = \frac{4 \pm \sqrt{-16}}{2}.$$

Again, remember that the number under the square root is the discriminant. In this case the disriminant is $-16$. It is not possible to square a real number and get $-16$. Thus, the quadratic equation $x^2 - 4x + 8 = 0$ has no real solutions, as predicted.

◆

Let's summarize the findings in our last three examples.

**Summary 19.** *Consider the quadratic equation*

$$ax^2 + bx + c = 0.$$

*The discriminant is defined as*

$$D = b^2 - 4ac.$$

*There are three possibilities:*

1. *If $D > 0$, then the quadratic equation has two real solutions.*
2. *If $D = 0$, then the quadratic equation has one real solution.*
3. *If $D < 0$, then the quadratic equation has no real solutions.*

This key result is reflected in the graph of the quadratic function.

**Summary 20.** *Consider the quadratic function*

$$f(x) = ax^2 + bx + c.$$

*The graph of this function is a parabola. Three possibilities exist depending upon the value of the discriminant $D = b^2 - 4ac$.*

1. *If $D > 0$, the parabola has two $x$-intercepts.*
2. *If $D = 0$, the parabola has exactly one $x$-intercept.*
3. *If $D < 0$, the parabola has no $x$-intercepts.*

## 5.4 Exercises

In **Exercises 1-8**, find all real solutions of the given equation. Use a calculator to approximate the answers, correct to the nearest hundredth (two decimal places).

**1.** $x^2 = 36$

**2.** $x^2 = 81$

**3.** $x^2 = 17$

**4.** $x^2 = 13$

**5.** $x^2 = 0$

**6.** $x^2 = -18$

**7.** $x^2 = -12$

**8.** $x^2 = 3$

In **Exercises 9-16**, find all real solutions of the given equation. Use a calculator to approximate your answers to the nearest hundredth.

**9.** $(x-1)^2 = 25$

**10.** $(x+3)^2 = 9$

**11.** $(x+2)^2 = 0$

**12.** $(x-3)^2 = -9$

**13.** $(x+6)^2 = -81$

**14.** $(x+7)^2 = 10$

**15.** $(x-8)^2 = 15$

**16.** $(x+10)^2 = 37$

In **Exercises 17-28**, perform each of the following tasks for the given quadratic function.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Place the quadratic function in vertex form. Plot the vertex on your coordinate system and label it with its coordinates. Draw the axis of symmetry on your coordinate system and label it with its equation.
iii. Use the quadratic formula to find the $x$-intercepts of the parabola. Use a calculator to approximate each intercept, correct to the nearest tenth, and use these approximations to plot the $x$-intercepts on your coordinate system. However, label each $x$-intercept with its **exact** coordinates.
iv. Plot the $y$-intercept on your coordinate system and its mirror image across the axis of symmetry and label each with their coordinates.
v. Using all of the information on your coordinate system, draw the graph of the parabola, then label it with the vertex form of the function. Use interval notation to state the domain and range of the quadratic function.

**17.** $f(x) = x^2 - 4x - 8$

**18.** $f(x) = x^2 + 6x - 1$

**19.** $f(x) = x^2 + 6x - 3$

**20.** $f(x) = x^2 - 8x + 1$

**21.** $f(x) = -x^2 + 2x + 10$

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[21] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

**22.** $f(x) = -x^2 - 8x - 8$

**23.** $f(x) = -x^2 - 8x - 9$

**24.** $f(x) = -x^2 + 10x - 20$

**25.** $f(x) = 2x^2 - 20x + 40$

**26.** $f(x) = 2x^2 - 16x + 12$

**27.** $f(x) = -2x^2 + 16x + 8$

**28.** $f(x) = -2x^2 - 24x - 52$

---

In **Exercises 29-32**, perform each of the following tasks for the given quadratic equation.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Show that the discriminant is negative.
iii. Use the technique of completing the square to put the quadratic function in vertex form. Plot the vertex on your coordinate system and label it with its coordinates. Draw the axis of symmetry on your coordinate system and label it with its equation.
iv. Plot the $y$-intercept and its mirror image across the axis of symmetry on your coordinate system and label each with their coordinates.
v. Because the discriminant is negative (did you remember to show that?), there are no $x$-intercepts. Use the given equation to calculate one additional point, then plot the point and its mirror image across the axis of symmetry and label each with their coordinates.
vi. Using all of the information on your coordinate system, draw the graph of the parabola, then label it with the vertex form of function. Use interval notation to describe the domain and range of the quadratic function.

**29.** $f(x) = x^2 + 4x + 8$

**30.** $f(x) = x^2 - 4x + 9$

**31.** $f(x) = -x^2 + 6x - 11$

**32.** $f(x) = -x^2 - 8x - 20$

---

In **Exercises 33-36**, perform each of the following tasks for the given quadratic function.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Use the discriminant to help determine the value of $k$ so that the graph of the given quadratic function has exactly one $x$-intercept.
iii. Substitute this value of $k$ back into the given quadratic function, then use the technique of completing the square to put the quadratic function in vertex form. Plot the vertex on your coordinate system and label it with its coordinates. Draw the axis of symmetry on your coordinate system and label it with its equation.
iv. Plot the $y$-intercept and its mirror image across the axis of symmetry and label each with their coordinates.
v. Use the equation to calculate an additional point on either side of the axis of symmetry, then plot this point and its mirror image across the axis of symmetry and label each with their coordinates.
vi. Using all of the information on your coordinate system, draw the graph of the parabola, then label it with the vertex form of the function. Use

interval notation to describe the domain and range of the quadratic function.

**33.** $f(x) = x^2 - 4x + 4k$

**34.** $f(x) = x^2 + 6x + 3k$

**35.** $f(x) = kx^2 - 16x - 32$

**36.** $f(x) = kx^2 - 24x + 48$

---

**37.** Find all values of $k$ so that the graph of the quadratic function $f(x) = kx^2 - 3x + 5$ has exactly two $x$-intercepts.

**38.** Find all values of $k$ so that the graph of the quadratic function $f(x) = 2x^2 + 7x - 4k$ has exactly two $x$-intercepts.

**39.** Find all values of $k$ so that the graph of the quadratic function $f(x) = 2x^2 - x + 5k$ has no $x$-intercepts.

**40.** Find all values of $k$ so that the graph of the quadratic function $f(x) = kx^2 - 2x - 4$ has no $x$-intercepts.

---

In **Exercises 41-50**, find all real solutions, if any, of the equation $f(x) = b$.

**41.** $f(x) = 63x^2 + 74x - 1$; $b = 8$

**42.** $f(x) = 64x^2 + 128x + 64$; $b = 0$

**43.** $f(x) = x^2 - x - 5$; $b = 2$

**44.** $f(x) = 5x^2 - 5x$; $b = 3$

**45.** $f(x) = 4x^2 + 4x - 1$; $b = -2$

**46.** $f(x) = 2x^2 - 9x - 3$; $b = -1$

**47.** $f(x) = 2x^2 + 4x + 6$; $b = 0$

**48.** $f(x) = 24x^2 - 54x + 27$; $b = 0$

**49.** $f(x) = -3x^2 + 2x - 13$; $b = -5$

**50.** $f(x) = x^2 - 5x - 7$; $b = 0$

---

In **Exercises 51-60**, find all real solutions, if any, of the quadratic equation.

**51.** $-2x^2 + 7 = -3x$

**52.** $-x^2 = -9x + 7$

**53.** $x^2 - 2 = -3x$

**54.** $81x^2 = -162x - 81$

**55.** $9x^2 + 81 = -54x$

**56.** $-30x^2 - 28 = -62x$

**57.** $-x^2 + 6 = 7x$

**58.** $-8x^2 = 4x + 2$

**59.** $4x^2 + 3 = -x$

**60.** $27x^2 = -66x + 16$

---

In **Exercises 61-66**, find all of the $x$-intercepts, if any, of the given function.

**61.** $f(x) = -4x^2 - 4x - 5$

**62.** $f(x) = 49x^2 - 28x + 4$

**63.** $f(x) = -56x^2 + 47x + 18$

**64.** $f(x) = 24x^2 + 34x + 12$

**65.** $f(x) = 36x^2 + 96x + 64$

**66.** $f(x) = 5x^2 + 2x + 3$

---

In **Exercises 67-74**, determine the number of real solutions of the equation.

**67.** $9x^2 + 6x + 1 = 0$

**68.** $7x^2 - 12x + 7 = 0$

**69.** $-6x^2 + 4x - 7 = 0$

**70.** $-8x^2 + 11x - 4 = 0$

**71.** $-5x^2 - 10x - 5 = 0$

**72.** $6x^2 + 11x + 2 = 0$

**73.** $-7x^2 - 4x + 5 = 0$

**74.** $6x^2 + 10x + 4 = 0$

## 5.4 *Answers*

**1.** $x = \pm 6$

**3.** $x = \pm\sqrt{17} = \pm 4.12$

**5.** $x = 0$

**7.** No real solutions.

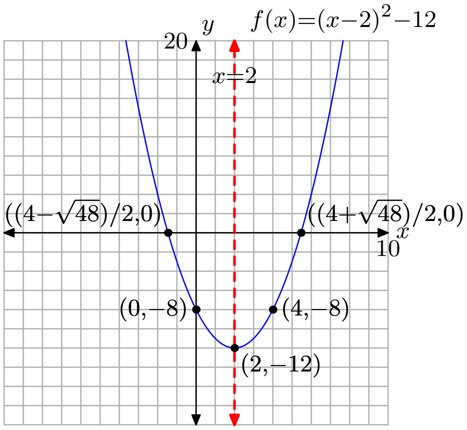
**9.** $x = -4$ or $x = 6$

**11.** $x = -2$

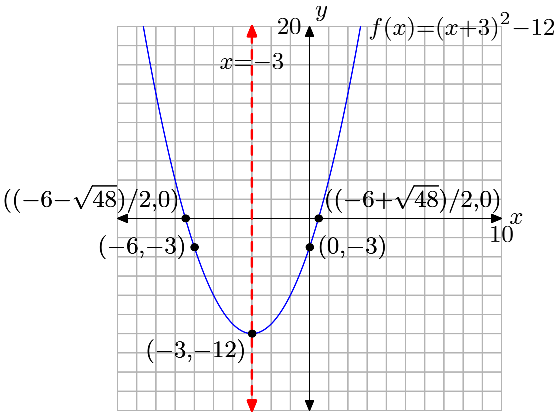
**13.** No real solutions.

**15.** $x = 8 \pm \sqrt{15} \approx 4.13,\ 11.87$

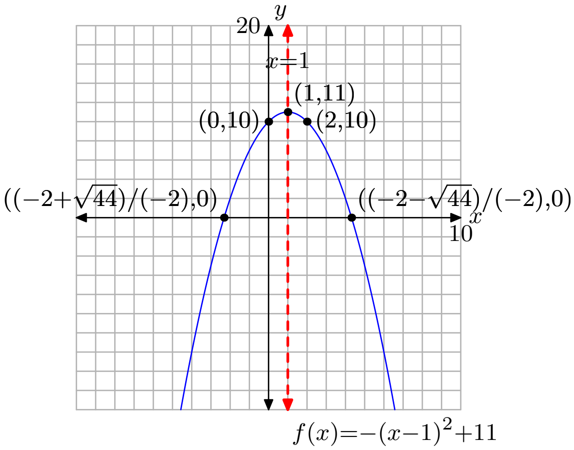
**17.** Domain $= (-\infty, \infty)$,
Range $= [-12, \infty)$



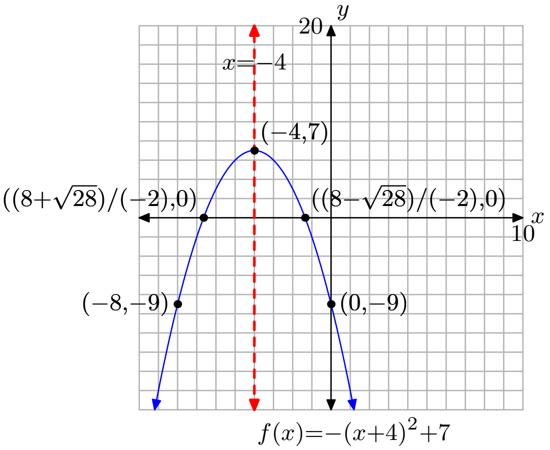
**19.** Domain $= (-\infty, \infty)$,
Range $= [-12, \infty)$



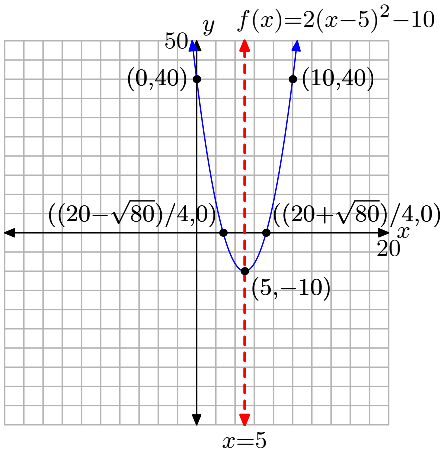
**21.** Domain $= (-\infty, \infty)$,
Range $= (-\infty, 11]$



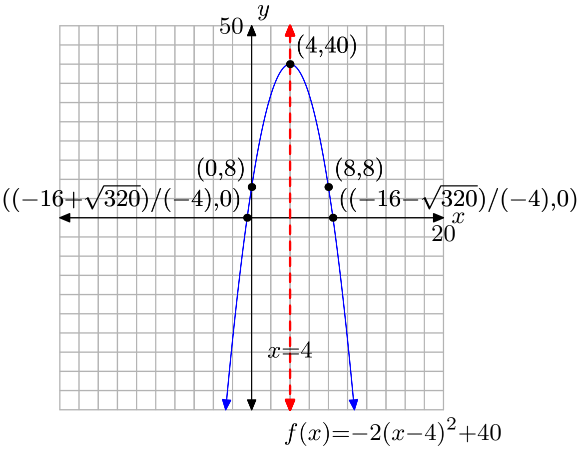
**23.** Domain = $(-\infty, \infty)$,
Range = $(-\infty, 7]$



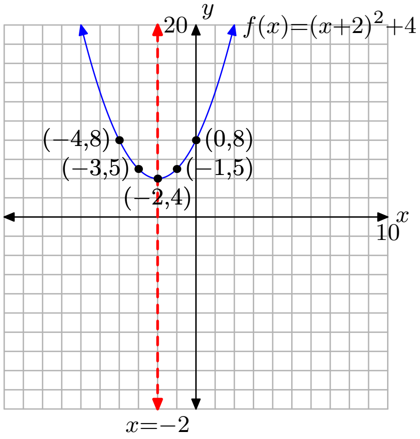
**25.** Domain = $(-\infty, \infty)$,
Range = $[-10, \infty)$



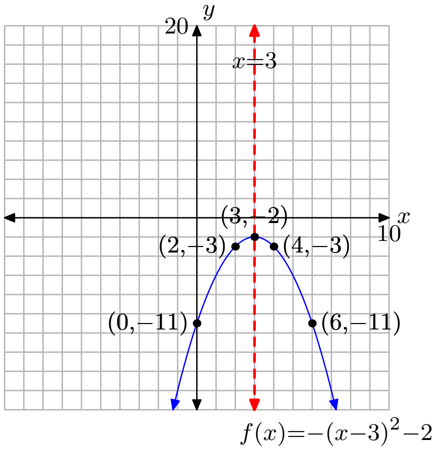
**27.** Domain = $(-\infty, \infty)$,
Range = $(-\infty, 40]$



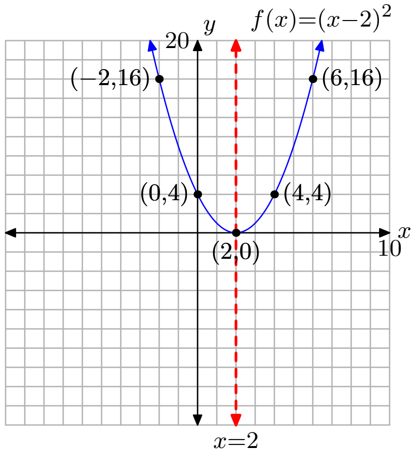
**29.** Domain = $(-\infty, \infty)$,
Range = $[4, \infty)$



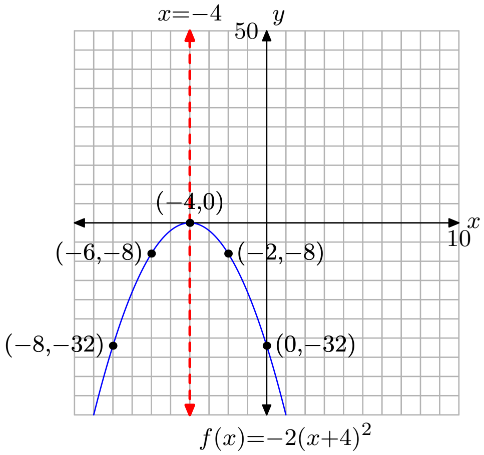
**31.** Domain $= (-\infty, \infty)$, Range $= (-\infty, -2]$



**33.** $k = 1$, Domain $= (-\infty, \infty)$, Range $= [0, \infty)$



**35.** $k = -2$, Domain $= (-\infty, \infty)$, Range $= (-\infty, 0]$



**37.** $\{k : k < 9/20\}$

**39.** $\{k : k > 1/40\}$

**41.** $-\frac{9}{7}, \frac{1}{9}$

**43.** $\frac{1+\sqrt{29}}{2}, \frac{1-\sqrt{29}}{2}$

**45.** $-\frac{1}{2}$

**47.** no real solutions

**49.** no real solutions

**51.** $\frac{3-\sqrt{65}}{4}, \frac{3+\sqrt{65}}{4}$

**53.** $-\frac{3-\sqrt{17}}{2}, -\frac{3+\sqrt{17}}{2}$

**55.** $-3$

**57.** $-\frac{7+\sqrt{73}}{2}, -\frac{7-\sqrt{73}}{2}$

**59.** no real solutions

**61.** no $x$-intercepts

**63.** $(\frac{9}{8}, 0)$, $(-\frac{2}{7}, 0)$

**65.** $(-\frac{4}{3}, 0)$

**67.** 1

**69.** 0

**71.** 1

**73.** 2

# 5.5 Motion

If a particle moves with uniform or constant acceleration, then it must behave according to certain standard laws of kinematics. In this section we will develop these laws of motion and apply them to a number of interesting applications.

## *Uniform Speed*

If an object travels with uniform (constant) speed $v$, then the distance $d$ traveled in time $t$ is given by the formula

$$d = vt, \tag{1}$$

or in words, “distance equals speed times time.” This concept is probably familiar to those of us who drive our cars on the highway. For example, if I drive my car at a constant speed of 50 miles per hour, in 3 hours I will travel 150 miles. That is,

$$150\,\text{mi} = 50\,\frac{\text{mi}}{\text{h}} \times 3\,\text{h}$$

Note that this computation has the form “distance equals speed times time.” It is important to note how the units balance on each side of this result. This is easily seen by canceling units much as you would cancel numbers with ordinary fractions.

$$150\,\text{mi} = 50\,\frac{\text{mi}}{\cancel{\text{h}}} \times 3\cancel{\text{h}}$$

In **Figure 1**(a) we’ve plotted the speed $v$ of the car versus time $t$. Because the speed is uniform (constant), the graph is a horizontal ray, starting at time $t = 0$ and moving to the right. In **Figure 1**(b), we’ve shaded the area under the constant speed ray over the time interval $[0, 3]$ hours. Note that the area of the shaded rectangular region has height equal to 50 miles per hour (50 mi/h) and width equal to 3 hours (3 h), so the area of this rectangle is

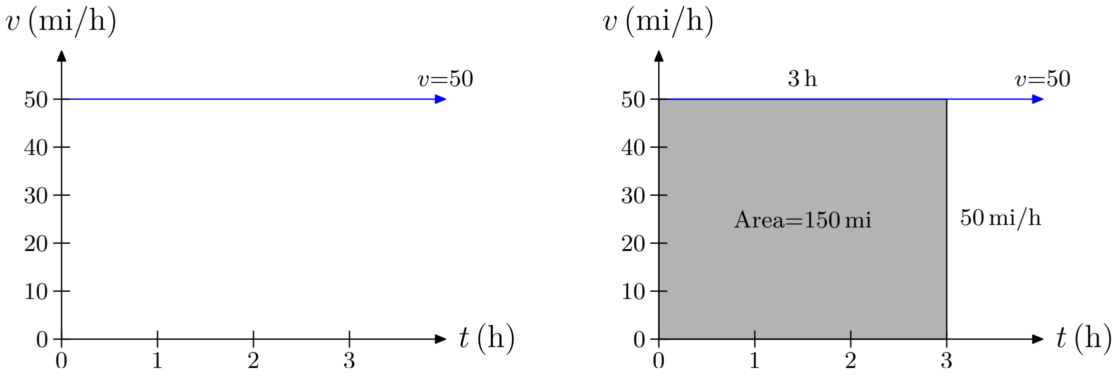
$$\text{Area} = \text{height} \times \text{width} = 50\,\frac{\text{mi}}{\text{h}} \times 3\,\text{h} = 150\,\text{mi}.$$

Note the units on the answer. The area under the constant speed ray is 150 miles. That is, the area under the speed curve is the distance traveled!

Our work has led us to the following result.

**Uniform Speed.** Suppose that an object travels with uniform (constant) speed $v$.

- The distance traveled $d$ is given by the formula $d = vt$, where $t$ is the time of travel.
- The graph of speed $v$ versus time $t$ will be a horizontal ray, starting at time $t = 0$ and moving to the right.
- The area of the rectangular region under the graph of $v$ over the time interval $[0, t]$ gives the distance traveled during that time period.



(a) Constant speed is represented by a horizontal ray.

(b) The area under the speed curve is the distance traveled.

**Figure 1.**

Let's look at another example.

▶ **Example 2.** *An object is traveling with uniform speed $v$. The graph of $v$ versus $t$ is shown in the graph that follows.*

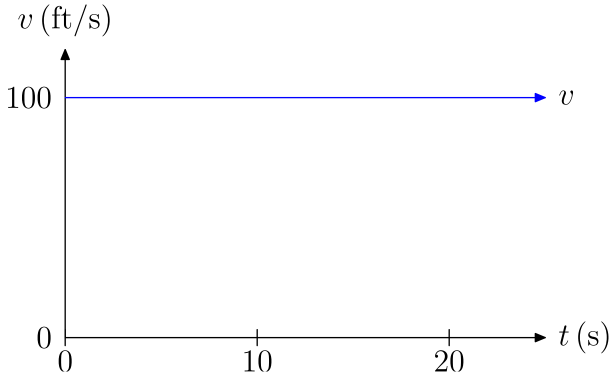


**Figure 2.** The graph of the speed of the object versus time.

*What is the speed of the object at any time $t$? How far will the object travel in 20 seconds?*

We read the speed from the graph. Note that the ray representing the speed is level (constant) at 100 feet per second (100 ft/s). Therefore, the speed at any time $t$ is $v = 100$. In function notation, we would write $v(t) = 100$, being mindful that the units are feet per second (ft/s).

To find the distance traveled in 20 seconds, we have two choices:

1. If we use the formula $d = vt$, then

$$\begin{aligned} d &= vt \\ d &= 100\,\frac{\text{ft}}{\cancel{\text{s}}} \times 20\,\cancel{\text{s}} \\ d &= 2000\,\text{ft} \end{aligned}$$

   That is, the object travels 2000 feet in the 20 seconds.

2. We can also find the distance traveled by shading the area under the uniform speed curve over the 20 second time interval.

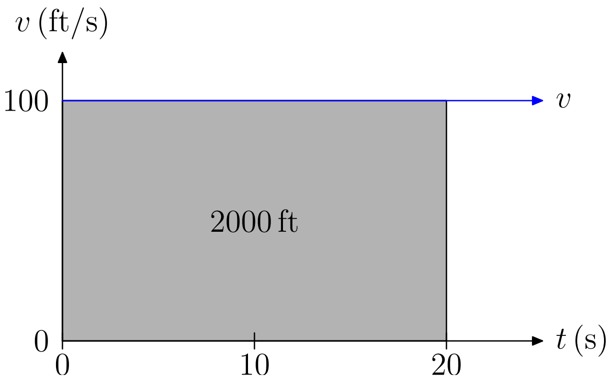


**Figure 3.** The area under the uniform speed curve represents the distance traveled.

Note that the height of the shaded rectangular region in **Figure 3** is 100 feet per second (100 ft/s) and the width is 20 seconds (20 s). Hence, the area of the shaded rectangular region is

$$\text{Area} = 100\,\frac{\text{ft}}{\cancel{\text{s}}} \times 20\,\cancel{\text{s}} = 2000\,\text{ft},$$

which is identical to the result found with the formula $d = vt$.

◆

## Uniform Acceleration

Let's get back in the car again and drive down the highway at a steady (constant) speed of $v = 30$ miles per hour. We decide to overtake a truck in front of us, so we step on the accelerator of the car, which increases the speed of the car, allowing us to pass the truck.

**Definition 3.** **Acceleration** *is the rate at which an object's speed is changing with respect to time.*

For example, suppose that when we step on the accelerator of the car, the speed of the car changes at a constant 20 miles per hour per hour. We would then say that the acceleration is *uniform* (constant) and would write

$$\text{Acceleration} = 20\ \frac{\text{mi/h}}{\text{h}},$$

or, more succinctly, as

$$\text{Acceleration} = 20\ \frac{\text{mi}}{\text{h}^2}.$$

The latter notation is preferred by scientists, but the notation $a = 20\,(\text{mi/h})/\text{h}$ is much easier to understand. That is, the speed is increasing at a constant rate of 20 miles per hour every hour.

- At the moment we step on the accelerator to pass the truck, the initial speed of the car is $v = 30$ miles per hour. If we maintain a constant acceleration of 20 miles per hour per hour, after 1 hour, the speed increases by 20 miles per hour, so the speed of the car at the end of 1 hour is

  $$v = 30 + 20(1),$$

  or $v = 50$ miles per hour.
- At the end of two hours, the speed of the car is

  $$v = 30 + 20(2),$$

  or $v = 70$ miles per hour.
- At the end of three hours, the speed of the car is

  $$v = 30 + 20(3),$$

  or $v = 90$ miles per hour.

Continuing in this manner, it's easy to see that the speed of the car at the end of $t$ hours will be given by the formula

$$v = 30 + 20t.$$

It's important to note that we are making an assumption that we keep our foot on that accelerator to maintain a uniform (constant) acceleration of 20 miles per hour per hour. *Granted, this is a pretty silly example with very low acceleration (is that Fred Flintstone's car?), but it does allow us to concentrate on the concept without having to deal with messy units.*

If we follow the argument above, it's not hard to develop the first equation of motion.

**First Equation of Motion.** If an object having initial speed $v_0$ experiences a constant acceleration $a$, then its speed at time $t$ is given by the formula

$$v = v_0 + at.$$

We follow the scientific practice of denoting the initial speed by $v_0$, the speed at time $t = 0$. That's why we subscript $v$ with zero.

Of course, the first equation of motion is valid only if each quantity possesses the proper units.

▶ **Example 4.** *Suppose that a particle has an initial speed of 20 feet per second (20 ft/s) and is given a constant acceleration of 4 feet per second per second ($4\,\mathrm{ft/s}^2$). What will be the speed of the particle after 3 minutes (3 min)?*

It is tempting to start with the formula

$$v = v_0 + at$$

and substitute $v_0 = 20\,\mathrm{ft/s}$, $a = 4\,\mathrm{ft/s}^2$, and $t = 3\,\mathrm{min}$.

$$v = 20\,\frac{\mathrm{ft}}{\mathrm{s}} + 4\,\frac{\mathrm{ft}}{\mathrm{s}^2} \times 3\ \mathrm{min}$$

However, note that the units will not cancel because the time is measured in minutes. What we need to do is change the time to seconds with the conversion[22]

$$t = 3\ \cancel{\mathrm{min}} \times 60\,\frac{\mathrm{s}}{\cancel{\mathrm{min}}} = 180\ \mathrm{s}.$$

Now the units should be correct. We substitute the time in seconds into the formula $v = v_0 + at$ and obtain

$$\begin{aligned} v &= 20\,\frac{\mathrm{ft}}{\mathrm{s}} + 4\,\frac{\mathrm{ft/s}}{\cancel{\mathrm{s}}} \times 180\ \cancel{\mathrm{s}} \\ &= 20\,\frac{\mathrm{ft}}{\mathrm{s}} + 720\,\frac{\mathrm{ft}}{\mathrm{s}} \\ &= 740\,\frac{\mathrm{ft}}{\mathrm{s}} \end{aligned}$$

Hence, the speed of the particle at three minutes is $v = 740\,\mathrm{ft/s}$.

◆

Let's look at another example.

▶ **Example 5.** *A ball is thrown into the air with an initial velocity of 180 feet per second (180 ft/s). It immediately begins to* decelerate *at a constant rate of 32 feet per second per second ($32\,\mathrm{ft/s}^2$). At what time will the ball reach its maximum height?*

When the ball reaches its maximum height, its velocity will equal zero. That is, at the exact moment when the ball is at its maximum height, it will stop before it returns to the ground. Thus, to find the time when the ball is at its maximum height, substitute $v = 0$ in the formula $v = v_0 + at$ and solve for $t$.

---

[22] There are 60 seconds in 1 minute.

$$\begin{aligned} 0 &= v_0 + at \\ at &= -v_0 \\ t &= -\frac{v_0}{a} \end{aligned} \tag{6}$$

When we say that the ball *decelerates* at a constant rate of 32 ft/s every second, we are implying that the ball *loses* speed at a rate of 32 ft/s every second. Thus, the acceleration is *negative* in this case and we write $a = -32\,\text{ft/s}^2$.

Finally, we need only substitute the initial speed ($v_0 = 180\,\text{ft/s}$) and the acceleration ($a = -32\,\text{ft/s}^2$) into **equation (6)** and simplify.

$$t = -\frac{180\,\text{ft/s}}{-32\,\text{ft/s}^2}$$

An analysis of the units is a good check that we are doing things correctly. Note that

$$\frac{\text{ft/s}}{\text{ft/s}^2} = \frac{\text{ft}}{\text{s}} \times \frac{\text{s}^2}{\text{ft}} = \text{s}$$

Thus, the time for the ball to reach its maximum height is

$$t = 5\,\text{s}.$$

◆

## Area is Distance Traveled

If we plot the graph of the speed $v$ versus the time $t$, note that the equation $v = v_0 + at$ has the form $y = mx + b$, particularly if we arrange the equation in the order $v = at + v_0$. It is then easily seen that the graph will be a line with intercept equaling the initial velocity $v_0$ and slope equaling the acceleration $a$. The graph of $v = v_0 + at$ is shown in **Figure 4**(a).

In **Figure 4**(b), we've shaded the area under the graph of $v = v_0 + at$ over the time interval $[0, t]$. There is a natural question to ask. Will the area under the graph of $v = v_0 + at$ in **Figure 4**(b) represent the distance traveled during the time interval $[0, t]$?

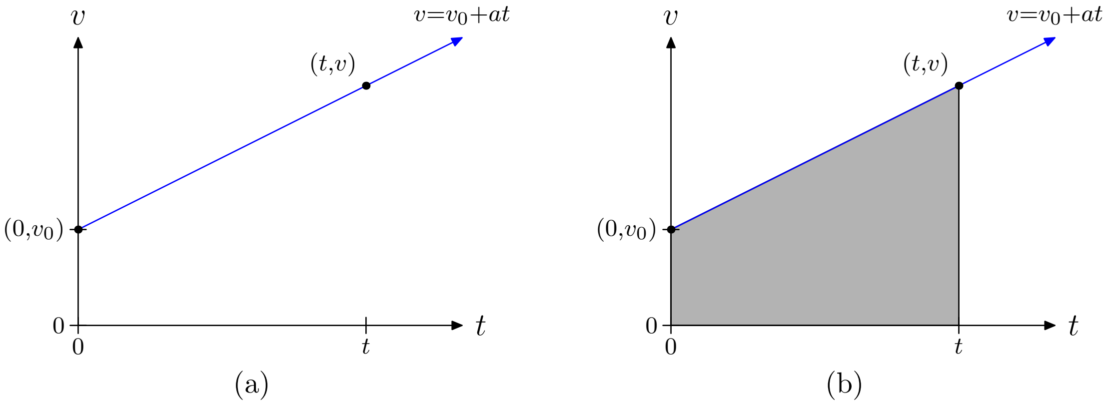


**Figure 4.** The graph of $v = v_0 + at$ is a line with intercept $v_0$ and slope $a$.

We know the area under a uniform (constant) speed ray will equal the distance traveled. Can we use this fact to answer our question on the shaded triangular region in **Figure 4**(b)?

Let's take the time interval $[0, t]$ in **Figure 4**(b) and divide it up into 4 equal subintervals of time, as shown in **Figure 5**(a).

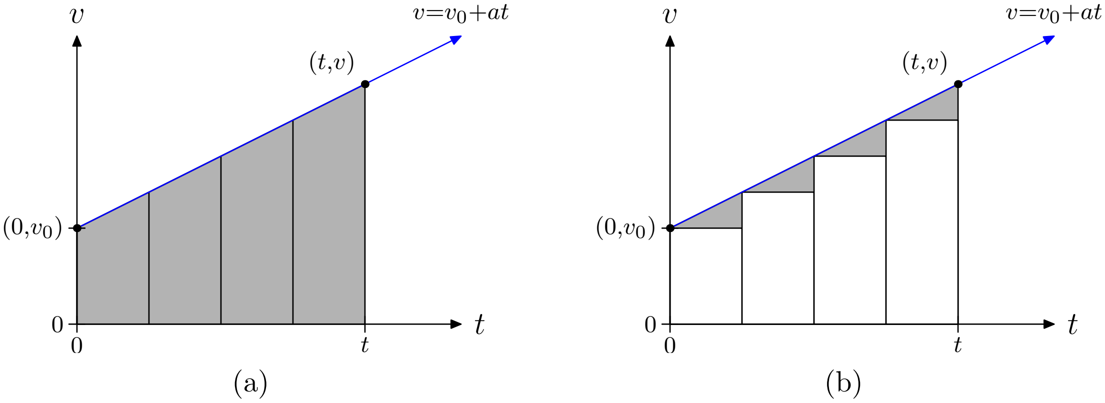


**Figure 5.** Subdividing the area into rectangles.

Next, use the left endpoint of each subinterval of time to draw four rectangles and fill them with the color white, as shown in **Figure 5**(b). The top of each rectangle is horizontal, so we know that this represents uniform (constant) speed. Therefore, the area of each white rectangle represents the distance traveled during that subinterval of time. If we sum the areas of all four rectangles, then we get the total distance traveled during the time span $[0, t]$, with, of course, the assumption that the speed is constant during each of the subintervals of time.

However, the speed is not constant during each subinterval of time, so the sum of the areas of the rectangles only approximates the total distance traveled on the time interval $[0, t]$.

The key idea is to draw more rectangles. In **Figure 6**(a), we've divided the time interval $[0, t]$ into 8 equal subintervals of time. In **Figure 6**(b), we again use the left endpoints of each subinterval of time to draw rectangles and we fill them with the color white.

Again, the top of each white rectangle is horizontal, which represents a uniform (constant) speed on that subinterval of time. Therefore, the area of each white rectangle again represents the distance traveled during that subinterval of time. The sum of all 8 rectangles represents the distance traveled during the time interval $[0, t]$, assuming that the speed is constant during each of the subintervals of time.

However, the speed is not constant on the time interval $[0, t]$, so the sum of the eight rectangles only offers an approximation of the distance traveled during the time interval $[0, t]$, albeit a better approximation than that offered by the sum of the areas of only four rectangles in **Figure 5**(b).

As we subdivide the time interval $[0, t]$ further, two things will happen.

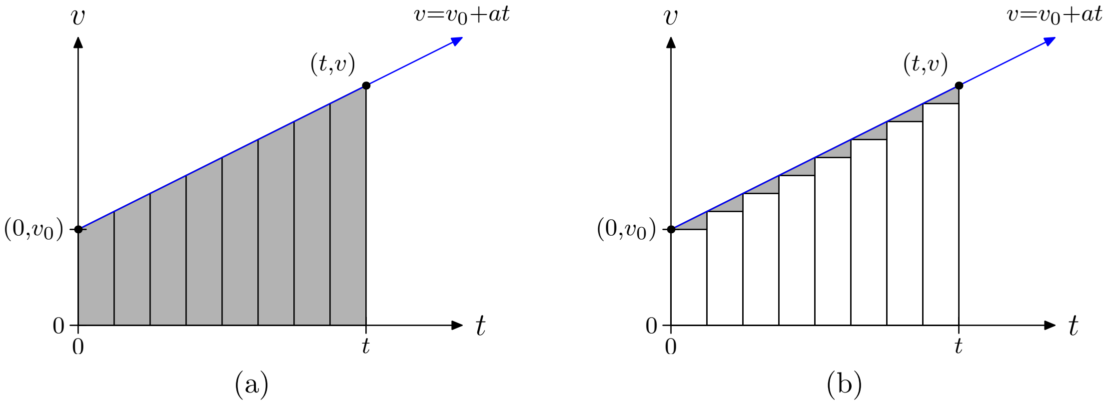


**Figure 6.** Subdividing the area into rectangles.

1. The subintervals of time will become smaller (in fact, infinitesimally small). When that happens, it becomes more and more reasonable to assume that the speed is constant during that subinterval of time. Therefore, in the limit, the sum of the areas of the rectangles will represent the total distance traveled over the time interval $[0, t]$.

2. The sum of the areas of the rectangles converges to the area of the shaded region under the speed curve in **Figure 4**(a).

This argument leads to one compelling conclusion.

**Area Equals Distance Traveled.** The area under the speed curve $v = v_0 + at$ over the time interval $[0, t]$ represents the distance traveled during the time interval $[0, t]$.

In **Figure 7**, the shaded region under $v = v_0 + at$ is a trapezoid. To find the area of this trapezoid, we add the bases (parallel sides) together, multiply by the height, then take half of the result.

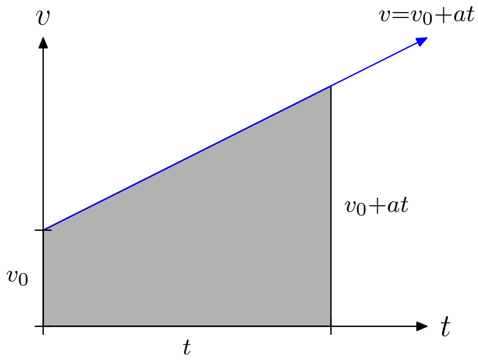


**Figure 7.** The area of the shaded trapezoidal region represents the distance traveled.

Thus, the area of the shaded region in **Figure 7** is given by the formula

$$\text{Area} = \frac{1}{2}\big[v_0 + (v_0 + at)\big]t.$$

Sum the quantity inside the parentheses, then distribute the 1/2 and the $t$ to obtain

$$\text{Area} = v_0\,t + \frac{1}{2}\,at^2. \tag{7}$$

## Motion in One Dimension

Suppose that a particle is constrained to move along the real line. In addition, suppose that at time $t = 0$, the initial position of the particle is at $x_0$ and the particle has initial speed $v_0$ and is moving to the right (as shown in **Figure 8**). Let's assume that the particle experiences uniform acceleration $a$ that is positive so that the particle continues to move to the right with increasing speed. At time $t$, let the particle's position be denoted by $x$ and its speed by $v$ (also shown in **Figure 8**).

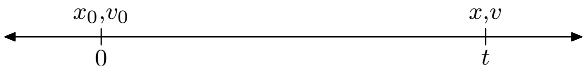


**Figure 8.** A particle moves on the line with uniform acceleration.

Because we've assumed that the particle moves to the right with increasing speed, the distance traveled by the particle is given by the expression $x - x_0$. However, we've also learned that the distance traveled is the area under the graph of the velocity (shown in **Figure 7**), which we calculated in **equation (7)** to be $v_0\,t + (1/2)at^2$. We conclude that

$$x - x_0 = v_0\,t + \frac{1}{2}\,at^2, \tag{8}$$

which leads to the *second equation of motion*.

**Second Equation of Motion.** Suppose that a particle moves on the real line with uniform acceleration $a$. Moreover, assume that the particle's position and speed at time $t = 0$ are given by $x_0$ and $v_0$, respectively. Let $x$ represent the particle's position at time $t$. Then, the particle's position at time $t$ is given by the formula

$$x = x_0 + v_0\,t + \frac{1}{2}\,at^2. \tag{9}$$

In developing the equation of motion **equation (9)**, we've avoided the notion of *velocity*. However, if a particle is constrained to move along the real line, it can move to the right or it can move to the left. This adds another dimension to speed.

Let's draw a number line (as shown in **Figure 9**), locate the origin, and agree that positive displacements are to the right and negative displacements are to the left.

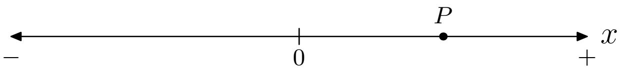


**Figure 9.** Orienting the real line.

Next, we define what is meant by *velocity*.

**Definition 10.** **Velocity** *is the rate at which an object's position is changing with respect to time.*

For example, suppose that displacements on the oriented line in **Figure 9** are measured in meters. Furthermore, suppose that the particle at point $P$ in **Figure 9** has velocity $v = 20$ meters per second. This would mean that the position of the particle is changing by a positive 20 meters each second. Because of the way we've oriented the line **Figure 9**, this means that the particle is moving to the right at a rate of 20 meters per second.

On the other hand, if the velocity of the particle at point $P$ was $v = -20$ meters per second, this would mean that the position of the particle is changing by a negative 20 meters each second. Because of the orientation we've chosen in **Figure 9**, this would mean that the particle is moving to the left at a rate of 20 meters per second.

Note that in each case (positive or negative velocity) the speed is 20 meters per second. What velocity brings to the table is an additional attribute of orientation. The sign of the velocity indicates a direction, while the magnitude of the velocity indicates a speed.

It is important for us to state that the equations of motion apply equally well when we introduce the notion of velocity. Thus, we can summarize as follows.

**The Equations of Motion.** Suppose that a particle moves on an oriented real line with uniform acceleration $a$. Further, let $x_0$ and $v_0$ represent the initial position and velocity of the particle at time $t = 0$.

- The velocity $v$ of the particle at time $t$ is given by the formula
$$v = v_0 + at.$$
- The position $x$ of the particle at time $t$ is given by the formula
$$x = x_0 + v_0\,t + \frac{1}{2}\,at^2.$$

Let's look at some applications of these *Equations of Motion*.

▶ **Example 11.** *Orient the real line as in* **Figure 9**. *Suppose that at time $t = 0$ the particle is located 2 meters to the right of the origin and is moving at a rate of 3 meters per second. Further, suppose that particle is moving with a uniform acceleration of $1.5\,\mathrm{m/s^2}$. Find the speed and position of the particle at the end of 10 seconds.*

We're given that $v_0 = 3\,\mathrm{m/s}$ and $a = 1.5\,\mathrm{m/s^2}$. Thus, after $t = 10$ seconds,

$$\begin{aligned}
v &= v_0 + at \\
v &= 3\,\frac{\mathrm{m}}{\mathrm{s}} + 1.5\,\frac{\mathrm{m/s}}{\cancel{\mathrm{s}}} \times 10\,\cancel{\mathrm{s}} \\
v &= 3\,\frac{\mathrm{m}}{\mathrm{s}} + 15\,\frac{\mathrm{m}}{\mathrm{s}} \\
v &= 18\,\frac{\mathrm{m}}{\mathrm{s}}.
\end{aligned}$$

We're also given that $x_0 = 2\,\mathrm{m}$. Thus, after $t = 10$ seconds,

$$\begin{aligned}
x &= x_0 + v_0\,t + \frac{1}{2}at^2 \\
x &= 2\,\mathrm{m} + \left(3\,\frac{\mathrm{m}}{\mathrm{s}}\right)(10\,\mathrm{s}) + \frac{1}{2}\left(1.5\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)(10\,\mathrm{s})^2 \\
x &= 2\,\mathrm{m} + \left(3\,\frac{\mathrm{m}}{\cancel{\mathrm{s}}}\right)(10\cancel{\mathrm{s}}) + \frac{1}{2}\left(1.5\,\frac{\mathrm{m}}{\cancel{\mathrm{s}^2}}\right)(100\cancel{\mathrm{s}^2}) \\
x &= 2\,\mathrm{m} + 30\,\mathrm{m} + 75\,\mathrm{m} \\
x &= 107\,\mathrm{m}.
\end{aligned}$$

Thus, at the end of $t = 10$ seconds, the particle is located 107 meters to the right of the origin and has velocity 18 meters per second (it is moving to the right with speed 18 meters per second).

◆

Let's look at another application of the *Equations of Motion*.

▶ **Example 12.** *A car is traveling down the highway at a speed of 60 miles per hour. Suddenly, a deer appears in the road ahead and the driver applies the brakes, decelerating the car at a constant rate of 12.9 feet per second every second. How long does it take the car to stop and how far does it travel during this time?*

The velocity of the car is given by the formula $v = v_0 + at$. The car will stop when $v = 0$. Therefore, substitute $v = 0$ in the formula and solve for $t$.

$$\begin{aligned}
v &= v_0 + at \\
0 &= v_0 + at \\
t &= -\frac{v_0}{a}
\end{aligned} \tag{13}$$

At time $t = 0$, the car's initial velocity is $v_0 = 60\,\mathrm{mi/h}$. The car is decelerating so it is losing speed at the given rate of 12.9 feet per second every second; i.e., $a = -12.9\,\mathrm{ft/s^2}$. We could try substituting these numbers into our last result.

$$t = -\frac{60\,\text{mi/h}}{-12.9\,\text{ft/s}^2}$$

The problem is immediately apparent: the units will not cancel. We have two choices; we can either (1) change the initial velocity into feet per second, or (2) change the acceleration into miles per hour per hour. We will do the former with the following calculation.

$$v_0 = \frac{60\,\cancel{\text{mi}}}{\cancel{\text{h}}} \times \frac{5280\,\text{ft}}{\cancel{\text{mi}}} \times \frac{1\,\cancel{\text{h}}}{60\,\cancel{\text{min}}} \times \frac{1\,\cancel{\text{min}}}{60\,\text{s}} = 88\,\text{ft/s}$$

We'll substitute this number in **equation (13)**.

$$\begin{aligned} t &= -\frac{v_0}{a} \\ t &= -\frac{88\,\text{ft/s}}{-12.9\,\text{ft/s}^2} \\ t &\approx 6.8\,\text{s} \end{aligned}$$

Again, it is important to check the units. Note that

$$\frac{\text{ft/s}}{\text{ft/s}^2} = \frac{\text{ft}}{\text{s}} \times \frac{\text{s}^2}{\text{ft}} = \text{s},$$

which is the correct unit for time.

We'll now find the stopping distance by letting the initial position of the car be $x_0 = 0$ feet. Thus, $x = x_0 + v_0\,t + (1/2)at^2$ becomes

$$x = v_0\,t + \frac{1}{2}\,at^2,$$

and $x$ will represent the stopping distance.

Now, substitute the initial speed $v_0 = 88$ feet per second, the acceleration $a = -12.9$ feet per second each second, and the stopping time $t = 6.8$ seconds. Thus,

$$\begin{aligned} x &= v_0\,t + \frac{1}{2}\,at^2 \\ x &= \left(\frac{88\,\text{ft}}{\text{s}}\right)(6.8\,\text{s}) + \frac{1}{2}\left(\frac{-12.9\,\text{ft/s}}{\text{s}}\right)(6.8\,\text{s})^2 \\ x &= \left(\frac{88\,\text{ft}}{\cancel{\text{s}}}\right)(6.8\cancel{\text{s}}) + \frac{1}{2}\left(\frac{-12.9\,\text{ft}}{\cancel{\text{s}^2}}\right)(46.24\cancel{\text{s}^2}) \\ x &= 598.4\,\text{ft} - 298.248\,\text{ft} \\ x &\approx 300\,\text{ft}, \end{aligned}$$

where we've rounded the stopping distance to the nearest foot.

◆

## The Acceleration Due to Gravity

If we neglect air resistance, then a body will fall to the surface of the earth with uniform acceleration. Physicists use the letter $g$ to represent the acceleration due to gravity. Near the surface of the earth, this acceleration is given by $g = 32\,\text{ft/s}^2$ or, in the metric system, $g = 9.8\,\text{m/s}^2$.

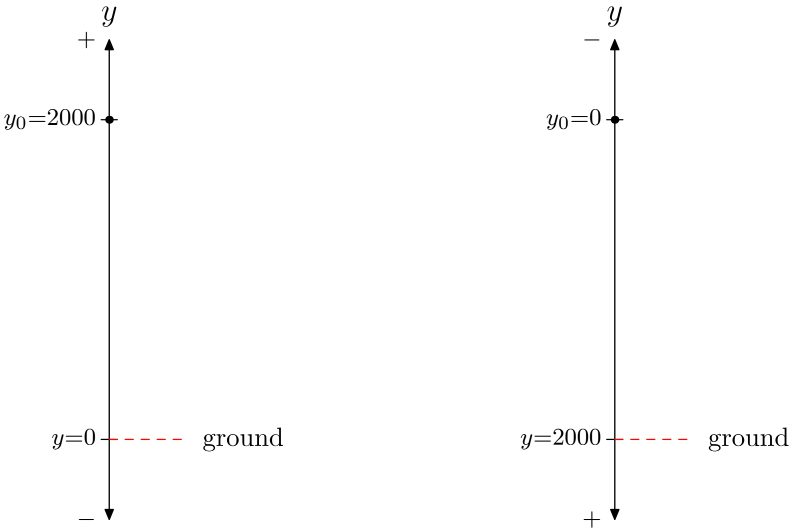
Remember, acceleration is the rate at which a body's velocity is changing with respect to time. Consequently, if we drop a body from rest at a very large height, after 1 second, its velocity will be 32 feet per second. After 2 seconds, its speed will be 64 feet per second. After 3 seconds, its speed will be 96 feet per second. Note how the speed is changing at a rate of 32 feet per second every second of time.

Gravity always attracts an object to the center of the earth, so we have to keep this in mind when using the *Equations of Motion*.

Let's look at an example.

▶ **Example 14.** *A ball is released from rest from a hot-air balloon that is hovering at a distance of 2000 feet above the surface of the earth. How long will it take until the ball strikes the ground?*

In this exercise, we'll rotate the real line so that it is vertical, as shown in **Figure 10**(a). We'll set the origin at ground level and let the positive $y$-direction point upward (indicated by the + sign at the top of the line in **Figure 10**(a).



(a) The origin at ground level.

(b) The origin at the release point.

**Figure 10.**

We'll start with the equation $y = y_0 + v_0\,t + (1/2)\,at^2$, and then note that the initial velocity is $v_0 = 0$ feet per second (the ball is released from rest), so the equation becomes

$$y = y_0 + \frac{1}{2}\,at^2.$$

We're asked to find when the ball hits the ground, so that means we're asked to find when $y = 0$ (see **Figure 10**(a). Set $y = 0$ in the last equation and solve for $t$.[23]

$$\begin{aligned} 0 &= y_0 + \frac{1}{2}\,at^2 \\ t^2 &= -\frac{2y_0}{a} \\ t &= \sqrt{-\frac{2y_0}{a}} \end{aligned}$$

Note that positive displacements are upward (see **Figure 10**(a)). If the velocity is positive, the ball is moving upward. In our case, the ball is moving downward, so the velocity is negative. As the ball moves downward, its speed becomes greater, so the velocity becomes more and more negative. Hence, the acceleration must be negative; i.e., $a = -32\,\text{ft/s}^2$. Substitute this acceleration and the initial position $y_0 = 2000\,\text{ft}$ into the last result and simplify.

$$\begin{aligned} t &= \sqrt{-\frac{2(2000\,\text{ft})}{-32\,\text{ft/s}^2}} \\ t &\approx 11.2\,\text{s} \end{aligned}$$

We've rounded the result to the nearest tenth of a second. Again, checking the units is important. In this case,

$$\sqrt{\frac{\text{ft}}{\text{ft/s}^2}} = \sqrt{\text{ft} \times \frac{\text{s}^2}{\text{ft}}} = \sqrt{\text{s}^2} = \text{s}.$$

Alternatively, we could set up the real line as shown in **Figure 10**(b), where we've placed the origin at the point of release and reversed the orientation (the positive $y$-direction is now downward). Thus, the initial position is $y_0 = 0$ feet and the initial velocity is $v_0 = 0$ feet per second (the ball is released from rest). Set these values in the equation $y = y_0 + v_0\,t + (1/2)\,at^2$ and solve for $t$.

$$\begin{aligned} y &= \frac{1}{2}\,at^2 \\ t^2 &= \frac{2y}{a} \\ t &= \sqrt{\frac{2y}{a}} \end{aligned}$$

Positive displacements are in the downward direction (note the reversal of orientation in **Figure 10**(b)). This means that when the velocity is positive, the ball is moving downward. When we release the ball, it is going to pick up more speed, so the

---

[23] There are actually two answers for $t$, namely $t = \pm\sqrt{-2y_0/a}$, but only the positive time makes sense in this problem situation.

velocity becomes more and more positive. Hence, the acceleration is positive in this orientation; i.e., $a = 32\,\text{ft/s}^2$.

When the ball hits ground level, the position is $y = 2000\,\text{ft}$. Substitute this value of $y$ and the acceleration in the last result and simplify (check the units).

$$t = \sqrt{\frac{2(2000\,\text{ft})}{32\,\text{ft/s}^2}}$$

Note that this will give the same result as before; i.e., $t \approx 11.2$ seconds.

◆

▶ **Example 15.** *A ball is thrown into the air from shoulder height (about 5 feet) with an initial upward velocity of 100 feet per second. Find the time it takes the ball to return to the ground.*

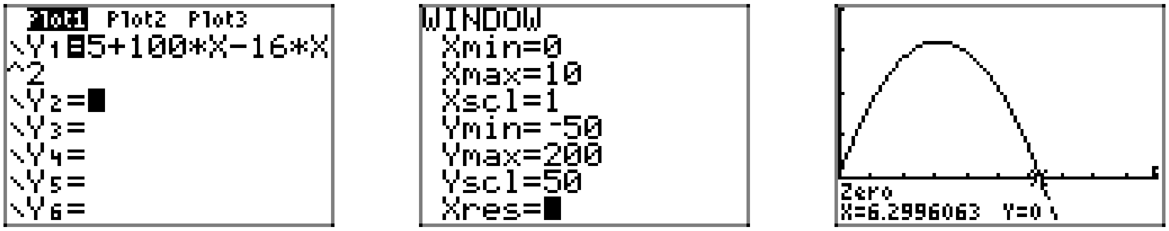
Let's find a solution using the graphing calculator. Using the orientation of **Figure 10**(a), start with the equation

$$y = y_0 + v_0 t + \frac{1}{2}at^2$$

and note that the initial position is $y_0 = 5$ feet, the initial velocity is $v_0 = 100$ feet per second, and the acceleration is $a = -32$ feet per second per second. Substitute these numbers into the previous equation to obtain

$$y = 5 + 100t + \frac{1}{2}(-32)t^2$$
$$y = 5 + 100t - 16t^2.$$

Enter this equation into the `Y=` menu as shown in **Figure 11**(a). Adjust the window parameters as shown in **Figure 11**(b) to produce the image shown in **Figure 11**(c).[24]



```
Plot1 Plot2 Plot3
\Y1■5+100*X-16*X
^2
\Y2=■
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
 Xmin=0
 Xmax=10
 Xscl=1
 Ymin=-50
 Ymax=200
 Yscl=50
 Xres=■
```

```
Zero
X=6.2996063  Y=0
```

(a) (b) (c)

**Figure 11.** Using the graphing calculator to determine the time to return to the ground.

To determine the time it takes the ball to return to the ground, we must locate where the height of the ball is $y = 0$ feet. Because the graph in **Figure 11**(c) is a plot of height or position (on the vertical axis) versus time (on the horizontal axis), this

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[24] It is important to understand that the curve in **Figure 11**(c) is not the actual flight path of the ball. Indeed, the ball goes straight up, then straight down, so all of its motion is constrained to a vertical line. Rather, the graph in **Figure 11**(c) is a graph of the height or position of the ball versus time.

occurs when the graph in **Figure 11**(c) crosses the horizontal axis; that is, at a zero of the function defined by $y = 5 + 100t - 16t^2$. To determine this time, use the utility `2:zero` in the `CALC` menu to determine the zero. The result is shown in **Figure 11**(c), where we determine it takes approximately $t \approx 6.29$ seconds for the ball to return to the ground.

Alternatively, we can set $y = 0$ in the equation $y = y_0 + v_0\,t + (1/2)\,at^2$ and use the quadratic formula to solve for the time $t$.

$$\begin{aligned}
0 &= y_0 + v_0 t + \frac{1}{2}\,at^2 \\
t &= \frac{-v_0 \pm \sqrt{v_0^2 - 4\left(\frac{1}{2}a\right)(y_0)}}{2\left(\frac{1}{2}\,a\right)} \\
t &= \frac{-v_0 \pm \sqrt{v_0^2 - 2ay_0}}{a}
\end{aligned}$$

We can now insert $y_0 = 5\,\text{ft}$, $v_0 = 100\,\text{ft/s}$, and $a = -32\,\text{ft/s}^2$, and then use a calculator to obtain

$$\begin{aligned}
t &= \frac{-100\,\text{ft/s} \pm \sqrt{(100\,\text{ft/s})^2 - 2(-32\,\text{ft/s}^2)(5\,\text{ft})}}{-32\,\text{ft/s}^2}. \\
t &\approx -0.05,\ 6.29\,\text{s}
\end{aligned}$$

The negative answer does not apply in this situation, so we keep the solution $t \approx 6.29$ seconds. Note how this agrees with the solution found on the graphing calculator.

Again, it is important to make sure the units check. Underneath the radical, both terms have units $\text{ft}^2/\text{s}^2$. When the square root is taken, these units become ft/s. Thus, both terms in the numerator are in ft/s, but the denominator has units $\text{ft/s}^2$. When you invert and multiply, as we saw in **Example 12**, the units simplify to seconds.

◆

## 5.5 *Exercises*

In **Exercises 1-12**, write down the formula $d = vt$ and solve for the unknown quantity in the problem. Once that is completed, substitute the known quantities in the result and simplify. Make sure to check that your units cancel and provide the appropriate units for your solution.

**1.** If Martha maintains a constant speed of 30 miles per hour, how far will she travel in 5 hours?

**2.** If Jamal maintains a constant speed of 25 miles per hour, how far will he travel in 5 hours?

**3.** If Arturo maintains a constant speed of 30 miles per hour, how long will it take him to travel 120 miles?

**4.** If Mei maintains a constant speed of 25 miles per hour, how long will it take her to travel 150 miles?

**5.** If Allen maintains a constant speed and travels 250 miles in 5 hours, what is is his constant speed?

**6.** If Jane maintains a constant speed and travels 300 miles in 6 hours, what is is her constant speed?

**7.** If Jose maintains a constant speed of 15 feet per second, how far will he travel in 5 minutes?

**8.** If Tami maintains a constant speed of 1.5 feet per second, how far will she travel in 4 minutes?

**9.** If Carmen maintains a constant speed of 80 meters per minute, how far will she travel in 600 seconds?

**10.** If Alphonso maintains a constant speed of 15 feet per second, how long will it take him to travel 1 mile? *Note: 1 mile equals 5280 feet.*

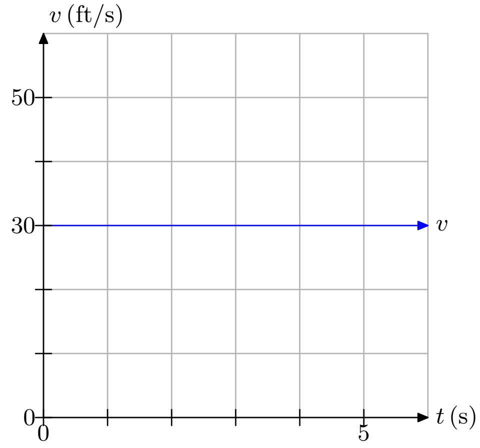
**11.** If Hoshi maintains a constant speed of 200 centimeters per second, how long will it take her to travel 20 meters? *Note: 100 centimeters equals 1 meter.*

**12.** If Maeko maintains a constant speed and travels 5 miles in 12 minutes, what is her speed in miles per hour?

In **Exercises 13-18**, a plot of speed $v$ versus time $t$ is presented.

i. Make an accurate duplication of the plot on graph paper. Label and scale each axis. Mark the units on each axis.
ii. Use the graph to determine the distance traveled over the time period $[0, 5]$, using the time units given on the graph.

**13.**



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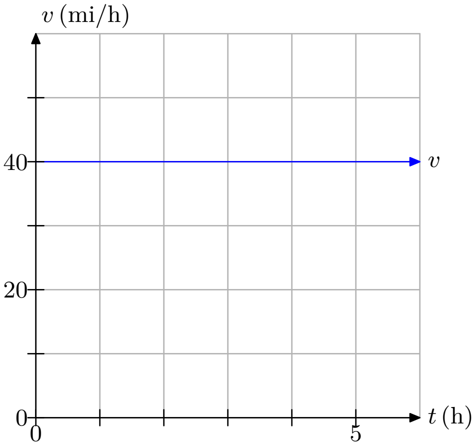
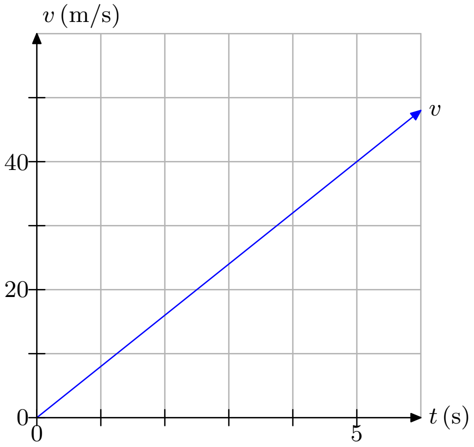
[25] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

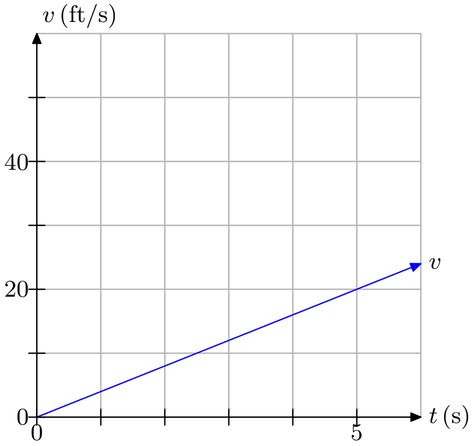
**14.**



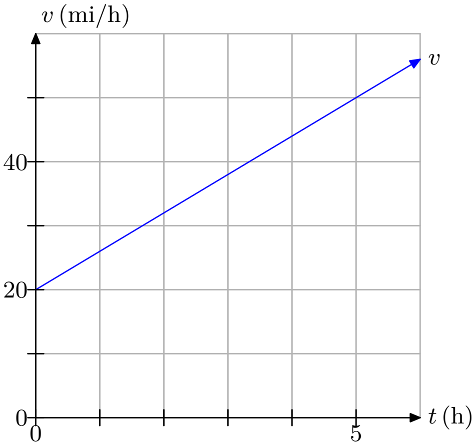
**15.**



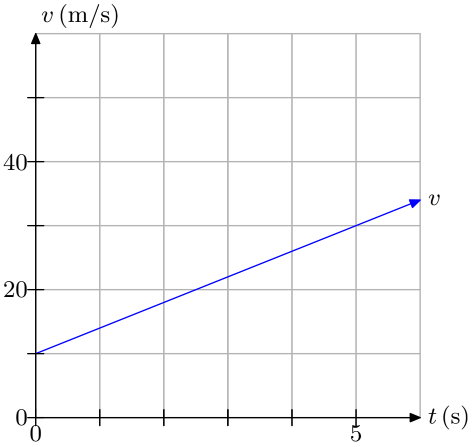
**16.**



**17.**



**18.**



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**19.** You're told that a car moves with a constant acceleration of 7.5 ft/s$^2$. In your own words, explain what this means.

**20.** You're told that an object will fall on a distant planet with constant acceleration 6.5 m/s$^2$. In your own words, explain what this means.

**21.** You're told that the acceleration of a car is $-18$ ft/s$^2$. In your own words, explain what this means.

**22.** An observer on a distant planet throws an object into the air and as it moves upward he reports that the object has a constant acceleration of $-4.5\,\mathrm{m/s^2}$. In your own words, explain what this means.

In **Exercises 23-28**, perform each of the following tasks.

i. Solve the equation $v = v_0 + at$ for the unknown quantity.
ii. Substitute the known quantities (with units) into your result, then simplify. Make sure the units cancel and provide appropriate units for your solution.

**23.** A rocket accelerates from rest with constant acceleration $15.8\,\mathrm{m/s^2}$. What will be the speed of the rocket after 3 minutes?

**24.** A stone is dropped from rest on a distant planet and it accelerates towards the ground with constant acceleration $3.8\,\mathrm{ft/s^2}$. What will be the speed of the stone after 2 minutes?

**25.** A stone is thrown downward on a distant planet with an initial speed of $20\,\mathrm{ft/s}$. If the stone experiences constant acceleration of $32\,\mathrm{ft/s^2}$, what will be the speed of the stone after 1 minute?

**26.** A ball is hurled upward with an initial speed of $80\,\mathrm{m/s}$. If the ball experiences a constant acceleration of $-9.8\,\mathrm{m/s^2}$, what will be the speed of the ball at the end of 5 seconds?

**27.** An object is shot into the air with an initial speed of $100\,\mathrm{m/s}$. If the object experiences constant deceleration of $9.8\,\mathrm{m/s^2}$, how long will it take the ball to reach its maximum height?

**28.** An object is released from rest on a distant planet and after 5 seconds, its speed is $98\,\mathrm{m/s}$. If the object falls with constant acceleration, determine the acceleration of the object.

In **Exercises 29-42**, use the appropriate equation of motion, either $v = v_0 + at$ or $x = x_0 + v_0 t + (1/2)at^2$ or both, to solve the question posed in the exercise.

i. Select the appropriate equation of motion and solve for the unknown quantity.
ii. Substitute the known quantities (with their units) into your result and simplify. Check that cancellation of units provide units appropriate for your solution.
iii. Find a decimal approximation for your answer.

**29.** A rocket with initial velocity $30\,\mathrm{m/s}$ moves along a straight line with constant acceleration $2.5\,\mathrm{m/s^2}$. Find the velocity and the distance traveled by the rocket at the end of 10 seconds.

**30.** A car is traveling at $88\,\mathrm{ft/s}$ when it applies the brakes and begins to slow with constant deceleration of $5\,\mathrm{ft/s^2}$. What is its speed and how far has it traveled at the end of 5 seconds?

**31.** A car is traveling at $88\,\mathrm{ft/s}$ when it applies the brakes and slows to $58\,\mathrm{ft/s}$ in 10 seconds. Assuming constant deceleration, find the deceleration and the distance traveled by the car in the 10 second time interval. *Hint: Compute the deceleration first.*

**32.** A stone is hurled downward from above the surface of a distant planet with initial speed 45 m/s. At then end of 10 seconds, the velocity of the stone is 145 m/s. Assuming constant acceleration, find the acceleration of the stone and the distance traveled in the 10 second time period.

**33.** An object is shot into the air from the surface of the earth with an initial velocity of 180 ft/s. Find the maximum height of the object and the time it takes the object to reach that maximum height. *Hint: The acceleration due to gravity near the surface of the earth is well known.*

**34.** An object is shot into the air from the surface of a distant planet with an initial velocity of 180 m/s. Find the maximum height of the object and the time it takes the object to reach that maximum height. Assume that the acceleration due to gravity on this distant planet is 5.8 $m/s^2$. *Hint: Calculate the time to the maximum height first.*

**35.** A car is traveling down the highway at 55 mi/h when the driver spots a slide of rocks covering the road ahead and hits the brakes, providing a constant deceleration of 12 $ft/s^2$. How long does it take the car to come to a halt and how far does it travel during this time period?

**36.** A car is traveling down the highway in Germany at 81 km/h when the driver spots that traffic is stopped in the road ahead and hits the brakes, providing a constant deceleration of 2.3 $m/s^2$. How long does it take the car to come to a halt and how far does it travel during this time period? *Note: 1 kilometer equals 1000 meters.*

**37.** An object is released from rest at some distance over the surface of the earth. How far (in meters) will the object fall in 5 seconds and what will be its velocity at the end of this 5 second time period? *Hint: You should know the acceleration due to gravity near the surface of the earth.*

**38.** An object is released from rest at some distance over the surface of a distant planet. How far (in meters) will the object fall in 5 seconds and what will be its velocity at the end of this 5 second time period? Assume the acceleration due to gravity on the distant planet is 13.5 $m/s^2$.

**39.** An object is released from rest at a distance of 352 feet over the surface of the earth. How long will it take the object to impact the ground?

**40.** An object is released from rest at a distance of 400 meters over the surface of a distant planet. How long will it take the object to impact the ground? Assume that the acceleration due to gravity on the distant planet equals 5.3 $m/s^2$.

**41.** On earth, a ball is thrown upward from an initial height of 5 meters with an initial velocity of 100 m/s. How long will it take the ball to return to the ground?

**42.** On earth, a ball is thrown upward from an initial height of 5 feet with an initial velocity of 100 ft/s. How long will it take the ball to return to the ground?

A ball is thrown into the air near the surface of the earth. In **Exercises 43-46**, the initial height of the ball and the initial velocity of the ball are given. Complete the following tasks.

i. Use $y = y_0 + v_0 t + (1/2)at^2$ to set up a formula for the height $y$ of the ball as a function of time $t$. Use the appropriate constant for the acceleration due to gravity near the surface of the earth.
ii. Load the equation from the previous part into `Y1` in your graphing calculator. Adjust your viewing window so that both the vertex and the time when the ball returns to the ground are visible. Copy the image onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax.
iii. Use the `zero` utility in the `CALC` menu of your graphing calculator to determine the time when the ball returns to the ground. Record this answer in the appropriate location on your graph.
iv. Use the quadratic formula to determine the time the ball returns to the ground. Use your calculator to find a decimal approximation of your solution. It should agree with that found using the `zero` utility on your graphing calculator. Be stubborn! Check your work until the answers agree.

**43.** $y_0 = 50\,\text{ft}$, $v_0 = 120\,\text{ft/s}$.

**44.** $y_0 = 30\,\text{m}$, $v_0 = 100\,\text{m/s}$.

**45.** $y_0 = 20\,\text{m}$, $v_0 = 110\,\text{m/s}$.

**46.** $y_0 = 100\,\text{ft}$, $v_0 = 200\,\text{ft/s}$.

---

**47.** A rock is thrown upward at an initial speed of 64 ft/s. How many seconds will it take the rock to rise 61 feet? Round your answer to the nearest hundredth of a second.

**48.** A penny is thrown downward from the top of a tree at an initial speed of 28 ft/s. How many seconds will it take the penny to fall 289 feet? Round your answer to the nearest hundredth of a second.

**49.** A water balloon is thrown downward from the roof of a building at an initial speed of 24 ft/s. The building is 169 feet tall. How many seconds will it take the water balloon to hit the ground? Round your answer to the nearest hundredth of a second.

**50.** A rock is thrown upward at an initial speed of 60 ft/s. How many seconds will it take the rock to rise 51 feet? Round your answer to the nearest hundredth of a second.

**51.** A ball is thrown upward from a height of 42 feet at an initial speed of 63 ft/s. How many seconds will it take the ball to hit the ground? Round your answer to the nearest hundredth of a second.

**52.** A rock is thrown upward from a height of 32 feet at an initial speed of 25 ft/s. How many seconds will it take the rock to hit the ground? Round your answer to the nearest hundredth of a second.

**53.** A penny is thrown downward from the top of a tree at an initial speed of 16 ft/s. The tree is 68 feet tall. How many seconds will it take the penny to hit the ground? Round your answer to the nearest hundredth of a second.

**54.** A penny is thrown downward off of the edge of a cliff at an initial speed of 32 ft/s. How many seconds will it take the penny to fall 210 feet? Round your answer to the nearest hundredth of a second.

## 5.5 *Answers*

**1.** 150 miles

**3.** 4 hours

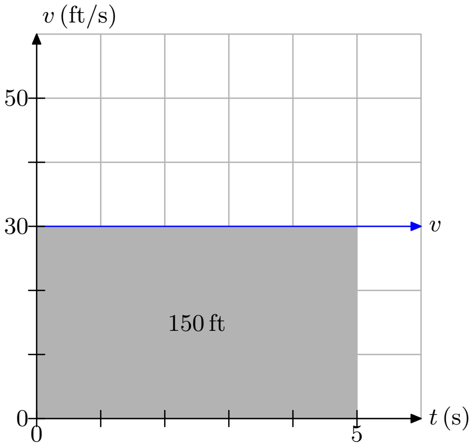
**5.** 50 miles per hour

**7.** 4500 feet

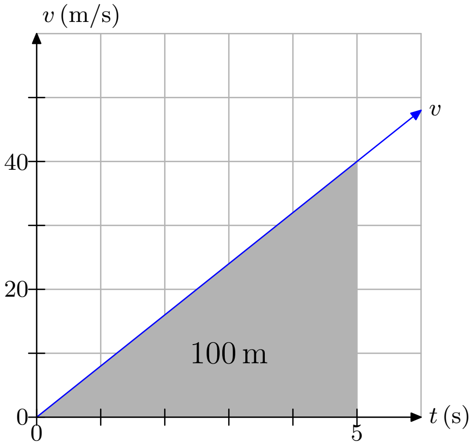
**9.** 800 meters

**11.** 10 seconds

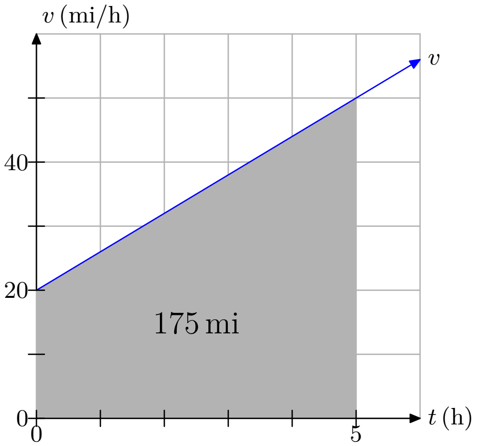
**13.** The distance traveled is 150 feet.



**15.** The distance traveled is 100 meters.



**17.** The distance traveled is 175 miles.



**19.** It means that the velocity of the car increases at a rate of 7.5 feet per second every second.

**21.** It means that the velocity of the car is decreasing at a rate of 18 feet per second every second.

**23.** $2,844\,\mathrm{m/s}$

**25.** $1,940\,\mathrm{ft/s}$

**27.** Approximately 10.2 seconds.

**29.** Velocity $= 55\,\mathrm{m/s}$,
Distance traveled $= 425\,\mathrm{m}$.

**31.** Acceleration $= -3\,\mathrm{ft/s^2}$,
Distance traveled $= 730\,\mathrm{ft}$.

**33.** Time to max height $= 5.625\,\mathrm{s}$,
Max height $= 506.25\,\mathrm{ft}$.

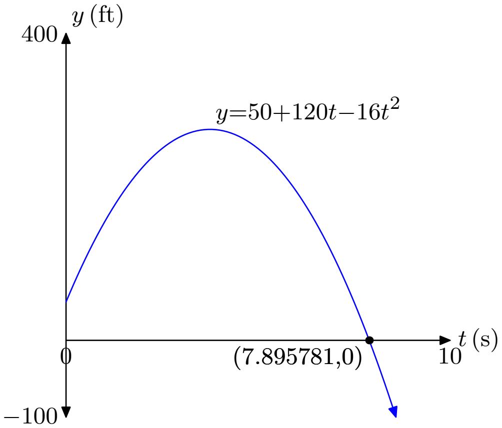
**35.** Time to stop $\approx 6.72\,\mathrm{s}$,
Distance traveled $\approx 271\,\mathrm{ft}$.

**37.** Distance = 122.5 m,
Velocity = −49 m/s.

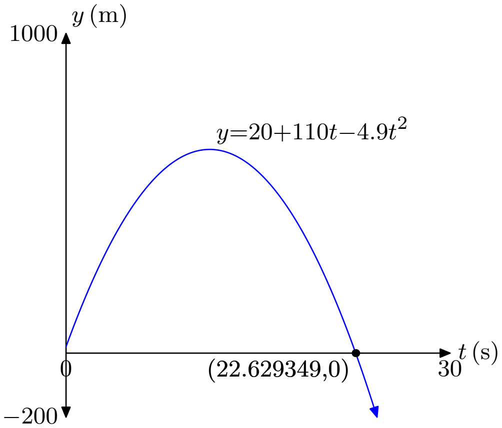
**39.** Time ≈ 4.69 s

**41.** Time ≈ 20.5 s

**43.**



**45.**



**47.** 1.57 seconds

**49.** 2.59 seconds

**51.** 4.52 seconds

**53.** 1.62 seconds

# 5.6 Optimization

In this section we will explore the science of optimization. Suppose that you are trying to find a pair of numbers with a fixed sum so that the product of the two numbers is a maximum. This is an example of an optimization problem. However, optimization is not limited to finding a maximum. For example, consider the manufacturer who would like to minimize his costs based on certain criteria. This is another example of an optimization problem. As you can see, optimization can encompass finding either a maximum or a minimum.

Optimization can be applied to a broad family of different functions. However, in this section, we will concentrate on finding the maximums and minimums of quadratic functions. There is a large body of real-life applications that can be modeled by quadratic functions, so we will find that this is an excellent entry point into the study of optimization.

## *Finding the Maximum or Minimum of a Quadratic Function*

Consider the quadratic function

$$f(x) = -x^2 + 4x + 2.$$

Let's complete the square to place this quadratic function in vertex form. First, factor out a minus sign.

$$f(x) = -\left[x^2 - 4x - 2\right]$$

Take half of the coefficient of $x$ and square, as in $[(1/2)(-4)]^2 = 4$. Add and subtract this amount to keep the equation balanced.

$$f(x) = -\left[x^2 - 4x + 4 - 4 - 2\right]$$

Factor the perfect square trinomial, combine the constants at the end, and then redistribute the minus sign to place the quadratic function in vertex form.

$$\begin{aligned} f(x) &= -\left[(x-2)^2 - 6\right] \\ f(x) &= -(x-2)^2 + 6 \end{aligned}$$

This is a parabola that opens downward, has been shifted 2 units to the right and 6 units upward. This places the vertex of the parabola at $(2, 6)$, as shown in **Figure 1**. Note that the maximum function value ($y$-value) occurs at the vertex of the parabola. A mathematician would say that the function "attains a maximum value of 6 at $x$ equals 2."

Note that 6 is greater than or equal to any other $y$-value (function value) that occurs on the parabola. This gives rise to the following definition.

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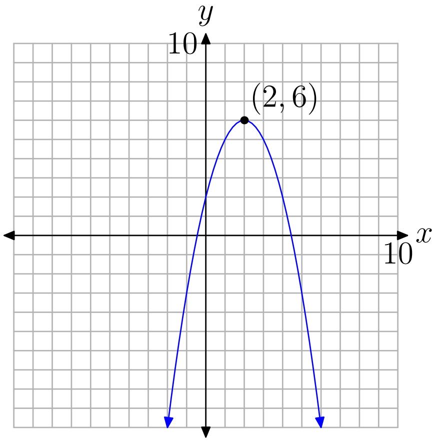

**Figure 1.** The maximum value of the function, 6, occurs at the vertex of the parabola, $(2, 6)$.

**Definition 1.** *Let $c$ be in the domain of $f$. The function $f$ is said to achieve a maximum at $x = c$ if $f(c) \geq f(x)$ for all $x$ in the domain of $f$.*

Next, let's look at a quadratic function that attains a minimum on its domain.

▶ **Example 2.** *Find the minimum value of the quadratic function defined by the equation*

$$f(x) = 2x^2 + 12x + 12.$$

Factor out a 2.

$$f(x) = 2\left[x^2 + 6x + 6\right] \tag{3}$$

Take half of the coefficient of $x$ and square, as in $[(1/2)(6)]^2 = 9$. Add and subtract this amount to keep the equation balanced.

$$f(x) = 2\left[x^2 + 6x + 9 - 9 + 6\right]$$

Factor the trinomial and combine the constants, and then redistribute the 2 in the next step.

$$\begin{aligned} f(x) &= 2\left[(x+3)^2 - 3\right] \\ f(x) &= 2(x+3)^2 - 6 \end{aligned}$$

The graph is a parabola that opens upward, shifted 3 units to the left and 6 units downward. This places the vertex at $(-3, -6)$, as shown in **Figure 2**. Note that the minimum function value ($y$-value) occurs at the vertex of the parabola. A mathematician would say that the function "attains a minimum value of $-6$ at $x$ equals $-3$.

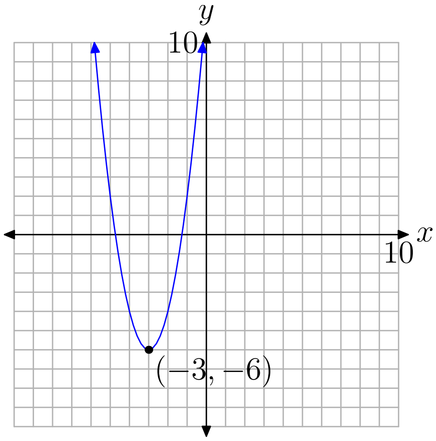

**Figure 2.** The minimum value of the function, -6, occurs at the vertex of the parabola, $(-3, -6)$.

Note that $-6$ is less than or equal to any other $y$-value (function value) that occurs on the parabola.

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This last example gives rise to the following definition.

**Definition 4.** *Let $c$ be in the domain of $f$. The function $f$ is said to achieve a minimum at $x = c$ if $f(c) \leq f(x)$ for all $x$ in the domain of $f$.*

## *A Shortcut for the Vertex*

It should now be clear that the vertex of the parabola plays a crucial role when optimizing a quadratic function. We also know that we can complete the square to find the coordinates of the vertex. However, it would be nice if we had a quicker way of finding the coordinates of the vertex. Let's look at the general quadratic function

$$y = ax^2 + bx + c$$

and complete the square to find the coordinates of the vertex. First, factor out the $a$.

$$y = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

Take half of the coefficient of $x$ and square, as in $[(1/2)(b/a)]^2 = [b/(2a)]^2 = b^2/(4a^2)$. Add and subtract this amount to keep the equation balanced.

$$y = a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

Factor the perfect square trinomial and make equivalent fractions for the constant terms with a common denominator.

$$y = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{4ac}{4a^2}\right]$$

$$y = a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$$

Finally, redistribute that $a$. Note how multiplying by $a$ cancels one $a$ in the denominator of the constant term.

$$y = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$$

Now, here's the key idea. The results depend upon the values of $a$, $b$, and $c$, but it should be clear that the coordinates of the vertex are

$$\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right).$$

The $y$-value of the vertex is a bit hard to memorize, but the $x$-value of the vertex is easy to memorize.

**Vertex Shortcut.** Given the parabola represented by the quadratic function

$$y = ax^2 + bx + c,$$

the $x$-coordinate of the vertex is given by the formula

$$x_{\text{vertex}} = -\frac{b}{2a}.$$

Let's test this with the quadratic function given in **Example 2**

▶ **Example 5.** *Use the formula $x_{\text{vertex}} = -b/(2a)$ to find the $x$-coordinate of the vertex of the parabola represented by the quadratic function in* **Example 2**.

In **Example 2**, the quadratic function was represented by the equation

$$f(x) = 2x^2 + 12x + 12.$$

In vertex form

$$f(x) = 2(x+3)^2 - 6,$$

the coordinates of the vertex were easily seen to be $(-3, -6)$ (see **Figure 2**). Let's see what the new formula for the $x$-coordinate of the vertex reveals.

As usual, compare $f(x) = 2x^2 + 12x + 12$ with $f(x) = ax^2 + bx + c$ and note that $a = 2$, $b = 12$ and $c = 12$. Thus, the $x$-coordinate of the vertex is given by

$$x_{\text{vertex}} = -\frac{b}{2a} = -\frac{12}{2(2)} = -3.$$

Note that this agrees with the previous result (see **Figure 2**). We could find the $y$-coordinate of the vertex with

$$y_{\text{vertex}} = \frac{4ac - b^2}{4a} = \frac{4(2)(12) - (12)^2}{4(2)} = \frac{-48}{8} = -6,$$

but we find this formula for the $y$-coordinate of the vertex a bit hard to memorize. We find it easier to do the following. Since we know the $x$-coordinate of the vertex is $x = -3$, we can find the $y$-coordinate of the vertex by simply substituting $x = -3$ in the equation of the parabola. That is, with $f(x) = 2x^2 + 12x - 12$,

$$f(-3) = 2(-3)^2 + 12(-3) + 12 = -6.$$

◇

Let's highlight this last technique.

**Finding the $y$-coordinate of the Vertex.** Given the parabola represented by the quadratic function

$$f(x) = ax^2 + bx + c,$$

we've seen that the $x$-coordinate of the vertex is given by $x = -b/(2a)$. To find the $y$-coordinate of the vertex, it is probably easiest to evaluate the function at $x = -b/(2a)$. That is, the $y$-coordinate of the vertex is given by

$$y_{\text{vertex}} = f\left(-\frac{b}{2a}\right).$$

Let's look at another example.

▶ **Example 6.** *Consider the parabola having equation*

$$f(x) = -2x^2 + 3x - 8.$$

*Find the coordinates of the vertex.*

First, use the new formula to find the $x$-coordinate of the vertex.

$$x_{\text{vertex}} = -\frac{b}{2a} = -\frac{3}{2(-2)} = \frac{3}{4}.$$

Next, substitute $x = 3/4$ to find the corresponding $y$-coordinate.

$$\begin{aligned} f\left(\frac{3}{4}\right) &= -2\left(\frac{3}{4}\right)^2 + 3\left(\frac{3}{4}\right) - 8 \\ &= -2\left(\frac{9}{16}\right) + \frac{9}{4} - 8 \\ &= -\frac{9}{8} + \frac{18}{8} - \frac{64}{8} \\ &= -\frac{55}{8} \end{aligned}$$

Thus, the coordinates of the vertex are $(3/4, -55/8)$.

◆

## Applications

We're now in a position to do some applications of optimization. Let's start with an easy example.

▶ **Example 7.** *Find two real numbers $x$ and $y$ that sum to 50 and that have a product that is a maximum.*

Before we apply the theory of the previous examples, let's just play with the numbers a bit to get a feel for what we are being asked to do. We need to find two numbers that sum to 50, so let's start with $x = 5$ and $y = 45$. Clearly, the sum of these two numbers is 50. On the other hand, their product is $xy = (5)(45) = 225$. Let's place this result in a table.

| $x$ | $y$ | $xy$ |
|---|---|---|
| 5 | 45 | 225 |

For a second guess, select $x = 10$ and $y = 40$. The sum of these two numbers is 50 and their product is $xy = 400$. For a third guess, select $x = 20$ and $y = 30$. The sum of these two numbers is 50 and their product is $xy = 600$. Let's add these results to our table.

| $x$ | $y$ | $xy$ |
|---|---|---|
| 5 | 45 | 225 |
| 10 | 40 | 400 |
| 20 | 30 | 600 |

Thus far, the best pair is $x = 20$ and $y = 30$, because their product is the maximum in the table above. But is there another pair with a larger product? Remember our goal is to find a pair with a product that is a maximum. That is, our pair must have a product larger than any other pair. Can you find a pair that has a product larger than 600?

Now that we have a feel for what we are being asked to do (find two numbers that sum to 50 and that have a product that is a maximum), let's try an approach that is more abstract than the "guess and check" approach of our tables. Our first constraint is the fact that the sum of the numbers $x$ and $y$ must be 50. We can model this constraint with the equation

$$x + y = 50. \tag{8}$$

We're being asked to maximize the product. Thus, you want to find a formula for the product. Let's let $P$ represent the product of $x$ and $y$ and write

$$P = xy. \tag{9}$$

Note that $P$ is a function of **two** variables $x$ and $y$. However, all of our functions in this course have thus far been a function of a single variable. So, how can we get rid of one of the variables? Simple, first solve **equation (8)** for $y$.

$$\begin{aligned} x + y &= 50 \\ y &= 50 - x \end{aligned} \tag{10}$$

Now, substitute **equation (10)** into the product in **equation (9)**.

$$P = x(50 - x),$$

or, equivalently,

$$P = -x^2 + 50x. \tag{11}$$

Note that $P$ is now a function of a single variable $x$. Note further that the function defined by **equation (11)** is quadratic. If we compare $P = -x^2 + 50x$ with the general form $P = ax^2 + bx + c$, note that $a = -1$ and $b = 50$ (we have no need of the fact that $c = 0$). Therefore, if we plot $P$ versus $x$, the graph is a parabola that opens downward (see **Figure 3**) and the maximum value of $P$ will occur at the vertex. The $x$-coordinate of the vertex is found with

$$x_{\text{vertex}} = -\frac{b}{2a} = -\frac{50}{2(-1)} = 25.$$

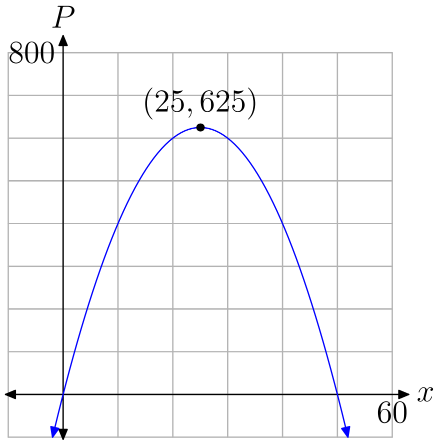

**Figure 3.** The maximum product $P$=625 occurs at the vertex of the parabola, $(25, 625)$.

Thus, our first number is $x = 25$. We can find the second number $y$ by substituting $x = 25$ in **equation (10)**.

$$y = 50 - x = 50 - 25 = 25.$$

Note that the sum of $x$ and $y$ is $x+y=25+25=50$. There are two ways that we can find their product. Since we now know the numbers $x$ and $y$, we can multiply to find $P=xy=(25)(25)=625$. Alternatively, we could substitute $x=25$ in **equation (11)** to get

$$P=-x^2+50x=-(25)^2+50(25)=-625+1250=625.$$

When you compare this result with our experimental tables, things come together. We've found two numbers $x$ and $y$ that sum to 50 with a product that is a maximum. No other numbers that sum to 50 have a larger product.

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Our little formula $x_{\text{vertex}}=-b/(2a)$ has proven to be a powerful ally. Let's try another example.

▶ **Example 12.** *Find two real numbers with a difference of 8 such that the sum of the squares of the two numbers is a minimum.*

Let's begin by letting $x$ and $y$ represent the numbers we seek. Next, let's play a bit as we did in the previous example. Try $x=9$ and $y=1$. The difference of these two numbers is certainly 8. The sum of the squares of these two numbers is $S=9^2+1^2=82$. Let's put this result in tabular form.

| $x$ | $y$ | $S=x^2+y^2$ |
|---|---|---|
| 9 | 1 | 82 |

For a second guess, select $x=8$ and $y=0$. The difference is $x-y=8-0=8$, but this time the sum of the squares is $S=8^2+0^2=64$. For a third guess, try $x=7$ and $y=-1$. Again, the difference is $x-y=7-(-1)=8$, but the sum of the squares is now $S=7^2+(-1)^2=50$. Let's add these results to our table.

| $x$ | $y$ | $S=x^2+y^2$ |
|---|---|---|
| 9 | 1 | 82 |
| 8 | 0 | 64 |
| 7 | $-1$ | 50 |

Thus far, the pair that minimizes the sum of the squares is $x=7$ and $y=-1$. However, could there be another pair with a difference of 8 and the sum of the squares is smaller than 50? Experiment further to see if you can best the current minimum of 50.

Let's try an analytical approach. Our first constraint is the fact that the difference of the two numbers must equal 8. This is easily expressed as

$$x-y=8. \tag{13}$$

Next, we're asked to minimize the sum of the squares of the two numbers. This requires that we find a formula for the sum of the squares. Let $S$ represent the sum of the squares of $x$ and $y$. Thus,

$$S = x^2 + y^2. \tag{14}$$

Note that $S$ is a function of two variables. We can eliminate one of the variables by solving **equation (13)** for $x$,

$$x = y + 8, \tag{15}$$

then substituting this result in **equation (14)**.

$$S = (y + 8)^2 + y^2.$$

Expand and simplify.

$$S = 2y^2 + 16y + 64 \tag{16}$$

Compare $S = 2y^2 + 16y + 64$ with the general quadratic $S = ay^2 + by + c$ and note that $a = 2$ and $b = 16$. Thus, the plot of $S$ versus $y$ will be a parabola that opens upward (see **Figure 4**) and the minimum value of $S$ will occur at the vertex. The $y$-coordinate of the vertex[27] is found with

$$y_{\text{vertex}} = -\frac{b}{2a} = -\frac{16}{2(2)} = -4.$$

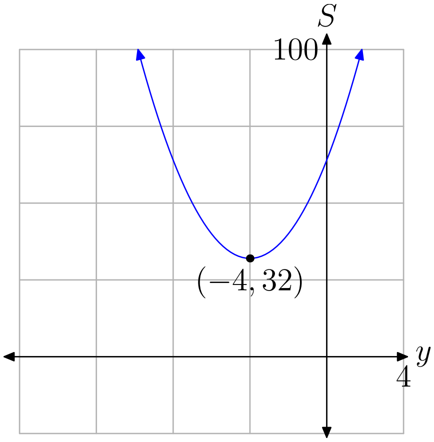

**Figure 4.** Plotting the sum of the squares $S$ versus $y$. The minimum $S$, 32, occurs at the vertex, $(-4, 32)$.

Thus, the first number we seek is $y = -4$. We can find the second number by substituting $y = -4$ in **equation (15)**.

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[27] Because we've plotted $S$ versus $y$, the horizontal axis is labeled $y$. Thus, $y$ has taken the usual role of $x$. That's why we write $y_{\text{vertex}} = -b/(2a)$ instead of $x_{\text{vertex}} = -b/(2a)$ in this example.

$$x = y + 8 = (-4) + 8 = 4.$$

Hence, the numbers we seek are $x = 4$ and $y = -4$. Note that the difference of these two numbers is $x - y = 4 - (-4) = 8$ and the sum of their squares is $S = (4)^2 + (-4)^2 = 32$, which is smaller than the best result found in our tabular experiment above. Indeed, our work show that this is the smallest possible value of $S$.

Alternatively, you can find $S$ by substituting $y = -4$ in **equation (16)**. We'll leave it to our readers to verify that this also gives a minimum value of $S = 32$.

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Let's look at another application.

► **Example 17.** *Mary wants to fence a rectangular garden to keep the deer from eating her fruit and vegetables. One side of her garden abuts her shed wall so she will not need to fence that side. However, she also wants to use material to separate the rectangular garden in two sections (see* **Figure 5***). She can afford to buy 80 total feet of fencing to use for the perimeter and the section dividing the rectangular garden. What dimensions will maximize the total area of the rectangular garden?*

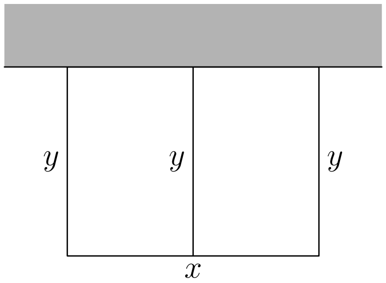

**Figure 5.** Mary's rectangular garden needs fencing on three sides and also for the fence to divide the garden.

Again, before we take an algebraic approach, let's just experiment. Note that we've labeled the width with the letter $x$ and the height with the letter $y$ in our sketch of the garden in **Figure 5**.

There is a total of 80 feet of fence material. Suppose that we let $y = 5$ ft. Because there are three sides of length $y = 5$ ft, we've used 15 feet of material. That leaves 65 feet of material which will be used to fence the width of the garden. That is, the width is $x = 65$ ft. Thus, the dimensions of the garden are $x = 65$ ft by $y = 5$ ft. The area equals the product of these two measures, so $A = 325\,\text{ft}^2$. Let's put this result into a table.

| $x$ | $y$ | $A = xy$ |
|---|---|---|
| 65 ft | 5 ft | $325\,\text{ft}^2$ |

Suppose instead that we let the height be $y = 10$ ft. Again, there are three sections with this length, so this will take 30 ft of material. That leaves 50 ft of material, so the width $x = 50\,\mathrm{ft}$. The area is the product of these two measures, so $A = 500\,\mathrm{ft}^2$. As a third experiment, let the height $y = 15\,\mathrm{ft}$. Subtracting three of these lengths from $80\,\mathrm{ft}$, we see that the width $x = 35\,\mathrm{ft}$. The area is the product of these measures, so $A = 525\,\mathrm{ft}^2$. Let's add these last two number experiments to our table.

| $x$ | $y$ | $A = xy$ |
|---|---|---|
| 65 ft | 5 ft | 325 ft$^2$ |
| 50 ft | 10 ft | 500 ft$^2$ |
| 35 ft | 15 ft | 525 ft$^2$ |

At this point, the last set of dimensions yields the maximum area, but is it possible that another choice of $x$ and $y$ will yield a larger area? Experiment further with numbers of your choice to see if you can find dimensions that will yield an area larger than the current maximum in the table, namely $525\,\mathrm{ft}^2$.

Let's now call on what we've learned in this section to attack this model. First, we're constrained by the amount of material we have for the job, a total of 80 ft of fencing. This constraint requires that 3 times the height of the garden, added to the width of the garden, should equal the available amount of fencing material. In symbols,

$$x + 3y = 80. \tag{18}$$

We're asked to maximize the area, so we focus our efforts on finding a formula for the area of the rectangular garden. Because the area $A$ of the rectangular garden is the product of the width and the height,

$$A = xy. \tag{19}$$

We now have a formula for the area of the rectangular garden, but unfortunately we have the area $A$ as a function of **two** variables. We need to eliminate one or the other of these variables. This is easily done by solving **equation (18)** for $x$.

$$x = 80 - 3y \tag{20}$$

Next, substitute this result in **equation (19)** to get

$$A = (80 - 3y)y,$$

or, equivalently,

$$A = -3y^2 + 80y. \tag{21}$$

Note that we have expressed the area $A$ as a function of a single variable $y$. Also, the function defined by **equation (21)** is quadratic. Compare $A = -3y^2 + 80y$ with the general form $A = ay^2 + by + c$ and note that $a = -3$ and $b = 80$ (we have no need of the fact that $c = 0$). Therefore, if we plot $A$ versus $y$, the graph is a parabola that opens downward (see **Figure 6**), so the maximum value of $A$ will occur at the vertex. The $y$-coordinate of the vertex[28] is found with

$$y_{\text{vertex}} = -\frac{b}{2a} = -\frac{80}{2(-3)} = \frac{80}{6} = \frac{40}{3}.$$

To find the width of the rectangular garden, substitute $y = 40/3$ into **equation (20)** and solve for $x$.

$$x = 80 - 3y = 80 - 3\left(\frac{40}{3}\right) = 80 - 40 = 40. \tag{22}$$

Thus, the width of the rectangular garden is 40 ft. We can find the area of the garden by multiplying the width and the height.

$$A = xy = (40)\left(\frac{40}{3}\right) = \frac{1600}{3} = 533\,\tfrac{1}{3}$$

Note that the resulting area, $A = 533\,\frac{1}{3}\,\text{ft}^2$, is only slightly bigger the last tabular entry found with our numerical experiments.

You can also find the area of the rectangular region by substituting $y = 40/3$ into **equation (21)**. We'll leave it to our readers to check that this provides the same measure for the area. You will also notice that the second coordinate of the vertex in **Figure 6** is the maximum area $A = 1600/3\,\text{ft}^2$.

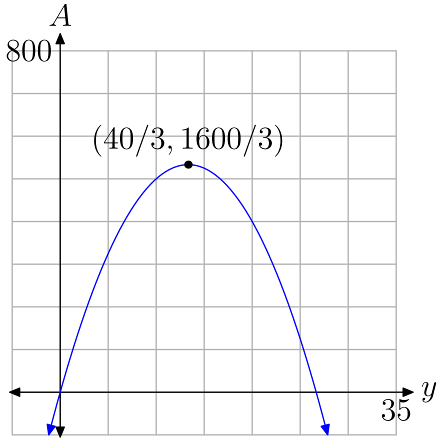

**Figure 6.** The maximum area, $A = 1600/3\,\text{ft}^2$, occurs at the vertex of the parabola, $(40/3, 1600/3)$.

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[28] Because we've plotted $A$ versus $y$, the horizontal axis is labeled $y$. Thus, $y$ has taken the usual role of $x$ because the horizontal axis represents the height $y$ of the rectangular garden. That's why we write $y_{\text{vertex}} = -b/(2a)$ instead of $x_{\text{vertex}} = -b/(2a)$ in this example.

## 5.6 *Exercises*

**1.** Find the exact maximum value of the function $f(x) = -x^2 - 3x$.

**2.** Find the exact maximum value of the function $f(x) = -x^2 - 5x - 2$.

**3.** Find the vertex of the graph of the function $f(x) = -3x^2 - x - 6$.

**4.** Find the range of the function $f(x) = -2x^2 - 9x + 2$.

**5.** Find the exact maximum value of the function $f(x) = -3x^2 - 9x - 4$.

**6.** Find the equation of the axis of symmetry of the graph of the function $f(x) = -x^2 - 5x - 9$.

**7.** Find the vertex of the graph of the function $f(x) = 3x^2 + 3x + 9$.

**8.** Find the exact minimum value of the function $f(x) = x^2 + x + 1$.

**9.** Find the exact minimum value of the function $f(x) = x^2 + 9x$.

**10.** Find the range of the function $f(x) = 5x^2 - 3x - 4$.

**11.** Find the range of the function $f(x) = -3x^2 + 8x - 2$.

**12.** Find the exact minimum value of the function $f(x) = 2x^2 + 5x - 6$.

**13.** Find the range of the function $f(x) = 4x^2 + 9x - 8$.

**14.** Find the exact maximum value of the function $f(x) = -3x^2 - 8x - 1$.

**15.** Find the equation of the axis of symmetry of the graph of the function $f(x) = -4x^2 - 2x + 9$.

**16.** Find the exact minimum value of the function $f(x) = 5x^2 + 2x - 3$.

**17.** A ball is thrown upward at a speed of 8 ft/s from the top of a 182 foot high building. How many seconds does it take for the ball to reach its maximum height? Round your answer to the nearest hundredth of a second.

**18.** A ball is thrown upward at a speed of 9 ft/s from the top of a 143 foot high building. How many seconds does it take for the ball to reach its maximum height? Round your answer to the nearest hundredth of a second.

**19.** A ball is thrown upward at a speed of 52 ft/s from the top of a 293 foot high building. What is the maximum height of the ball? Round your answer to the nearest hundredth of a foot.

**20.** A ball is thrown upward at a speed of 23 ft/s from the top of a 71 foot high building. What is the maximum height of the ball? Round your answer to the nearest hundredth of a foot.

**21.** Find two numbers whose sum is 20 and whose product is a maximum.

**22.** Find two numbers whose sum is 36 and whose product is a maximum.

**23.** Find two numbers whose difference is 12 and whose product is a minimum.

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[29] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

**24.** Find two numbers whose difference is 24 and whose product is a minimum.

**25.** One number is 3 larger than twice a second number. Find two such numbers so that their product is a minimum.

**26.** One number is 2 larger than 5 times a second number. Find two such numbers so that their product is a minimum.

**27.** Among all pairs of numbers whose sum is $-10$, find the pair such that the sum of their squares is the smallest possible.

**28.** Among all pairs of numbers whose sum is $-24$, find the pair such that the sum of their squares is the smallest possible.

**29.** Among all pairs of numbers whose sum is 14, find the pair such that the sum of their squares is the smallest possible.

**30.** Among all pairs of numbers whose sum is 12, find the pair such that the sum of their squares is the smallest possible.

**31.** Among all rectangles having perimeter 40 feet, find the dimensions (length and width) of the one with the greatest area.

**32.** Among all rectangles having perimeter 100 feet, find the dimensions (length and width) of the one with the greatest area.

**33.** A farmer with 1700 meters of fencing wants to enclose a rectangular plot that borders on a river. If no fence is required along the river, what is the largest area that can be enclosed?

**34.** A rancher with 1500 meters of fencing wants to enclose a rectangular plot that borders on a river. If no fence is required along the river, and the side parallel to the river is $x$ meters long, find the value of $x$ which will give the largest area of the rectangle.

**35.** A park ranger with 400 meters of fencing wants to enclose a rectangular plot that borders on a river. If no fence is required along the river, and the side parallel to the river is $x$ meters long, find the value of $x$ which will give the largest area of the rectangle.

**36.** A rancher with 1000 meters of fencing wants to enclose a rectangular plot that borders on a river. If no fence is required along the river, what is the largest area that can be enclosed?

**37.** Let $x$ represent the demand (the number the public will buy) for an object and let $p$ represent the object's unit price (in dollars). Suppose that the unit price and the demand are linearly related by the equation $p = (-1/3)x + 40$.

**a)** Express the revenue $R$ (the amount earned by selling the objects) as a function of the demand $x$.

**b)** Find the demand that will maximize the revenue.

**c)** Find the unit price that will maximize the revenue.

**d)** What is the maximum revenue?

**38.** Let $x$ represent the demand (the number the public will buy) for an object and let $p$ represent the object's unit price (in dollars). Suppose that the unit price and the demand are linearly related by the equation $p = (-1/5)x + 200$.

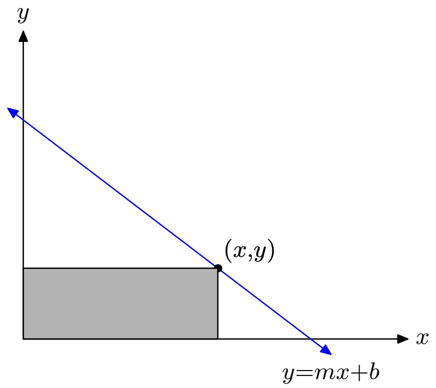
**a)** Express the revenue $R$ (the amount earned by selling the objects) as a function of the demand $x$.

**b)** Find the demand that will maximize the revenue.

**c)** Find the unit price that will maximize the revenue.

**d)** What is the maximum revenue?

**39.** A point from the first quadrant is selected on the line $y = mx + b$. Lines are drawn from this point parallel to the axes to form a rectangle under the line in the first quadrant. Among all such rectangles, find the dimensions of the rectangle with maximum area. What is the maximum area? *Assume* $m < 0$.



**40.** A rancher wishes to fence a rectangular area. The east-west sides of the rectangle will require stronger support due to prevailing east-west storm winds. Consequently, the cost of fencing for the east-west sides of the rectangular area is \$18 per foot. The cost for fencing the north-south sides of the rectangular area is \$12 per foot. Find the dimension of the largest possible rectangular area that can be fenced for \$7200.

## 5.6 *Answers*

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**1.** $\frac{9}{4}$

**3.** $\left(-\frac{1}{6}, -\frac{71}{12}\right)$

**5.** $\frac{11}{4}$

**7.** $\left(-\frac{1}{2}, \frac{33}{4}\right)$

**9.** $-\frac{81}{4}$

**11.** $\left(-\infty, \frac{10}{3}\right] = \left\{x \,\middle|\, x \leq \frac{10}{3}\right\}$

**13.** $\left[-\frac{209}{16}, \infty\right) = \left\{x \,\middle|\, x \geq -\frac{209}{16}\right\}$

**15.** $x = -\frac{1}{4}$

**17.** 0.25

**19.** 335.25

**21.** 10 and 10

**23.** 6 and $-6$

**25.** $3/2$ and $-3/4$

**27.** $-5$, $-5$

**29.** 7, 7

**31.** 10 feet by 10 feet

**33.** 361250 square meters

**35.** 200

**37.**

**a)** $R = (-1/3)x^2 + 40x$

**b)** $x = 60$ objects

**c)** $p = 20$ dollars

**d)** $R = \$1200$

**39.** $x = -b/(2m)$, $y = b/2$, $A = -b^2/(4m)$

## 5.7 Index

# 6 Polynomial Functions

Polynomial functions have the form $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and play an important role in mathematics, science, and engineering. To model curves needed in construction, engineers will piece special cubic (third degree) polynomials together to form a construct called a *spline*. Statisticians will often model data with polynomials and use the polynomial to make predictions between data points, a technique called *interpolation*. Mathematicians will approximate complicated functions with polynomials.

In this chapter, we introduce the polynomial function, discuss its end-behavior, its zeros, and the extreme values of its graph, and then apply what we've learned to several interesting applications.

## Table of Contents

# 6.1 Polynomial Functions

We've seen in previous sections that a *monomial* is the product of a number and one or more variable factors, each raised to a positive integral power, as in $-3x^2$ or $4x^3y^4$. We've also seen that a *binomial* is the sum or difference of two monomial terms, as in $3x+5$, $x^2+4$, or $3xy^2-2x^2y$. We've also seen that a *trinomial* is the sum or difference of three monomial terms, as in $x^2-2x-3$ or $x^2-4xy+5y^2$.

The root word "poly" means "many," as in polygon (many sides) or polyglot (speaking many languages—multilingual). In algebra, the word *polynomial* means "many terms," where the phrase "many terms" can be construed to mean anywhere from one to an arbitrary, but finite, number of terms. Consequently, a monomial could be considered a polynomial, as could binomials and trinomials.

In our work, we will concentrate for the most part on polynomials of a single variable. What follows is a more formal definition of a polynomial in a single variable $x$.

**Definition 1.** *The function $p$, defined by*

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \tag{2}$$

*is called a polynomial in $x$.*

There are several important points to be made about this definition.

1. The polynomial in our definition is arranged in *ascending* powers of $x$. We could just as easily arrange our polynomial in *descending* powers of $x$, as in

$$p(x) = a_nx^n + \cdots + a_2x^2 + a_1x + a_0.$$

2. The numbers $a_0$, $a_1$, $a_2$, ..., $a_n$ are called the *coefficients* of the polynomial $p$.
   - If all of the coefficients are integers, then we say that "$p$ is a polynomial with integer coefficients."
   - If all of the coefficients are rational numbers, then we say that "$p$ is a polynomial with rational coefficients."
   - If all of the coefficients are real numbers, then we say that "$p$ is a polynomial with real coefficients."
3. The *degree* of the polynomial $p$ is $n$, the highest power of $x$.
4. The *leading term* of the polynomial $p$ is the term with the highest power of $x$. In the case of **equation (2)**, the leading term is $a_nx^n$.

Let's look at an example.

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[1] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

▶ **Example 3.** *Consider the polynomial*

$$p(x) = 3 - 4x^2 + 5x^3 - 6x. \tag{4}$$

*Find the degree, the leading term, and make a statement about the coefficients of $p$.*

First, put the polynomial terms in order. Whether you use ascending or descending powers of $x$ makes no difference. Choose one or the other. In descending powers of $x$,

$$p(x) = 5x^3 - 4x^2 - 6x + 3, \tag{5}$$

but in ascending powers of $x$,

$$p(x) = 3 - 6x - 4x^2 + 5x^3. \tag{6}$$

In either case, **equation (5)** or **equation (6)**, the degree of the polynomial is 3. Also, in either case, the leading term[2] of the polynomial is $5x^3$. Because all coefficients of this polynomial are integers, we say that "$p$ is a polynomial with integer coefficients." However, all the coefficients are also rational numbers, so we could say that $p$ is a polynomial with rational coefficients. For that matter, all of the coefficients of $p$ are real numbers, so we could also say that $p$ is a polynomial with real coefficients.

◆

Let's look at another example.

▶ **Example 7.** *Consider the polynomial*

$$p(x) = 3 - \tfrac{4}{3}x + \tfrac{2}{5}x^2 - 9x^3 + 12x^4.$$

*Find the degree, the leading term, and make a statement about the coefficients of $p$.*

Fortunately, the polynomial $p$ is already arranged in ascending powers of $x$. The degree of $p$ is 4 and the leading term is $12x^4$. Not all of the coefficients are integers, so we cannot say that "$p$ is a polynomial with integer coefficients." However, all of the coefficients are rational numbers, so we can say that "$p$ is a polynomial with rational coefficients." Because all of the coefficients of $p$ are real numbers, we could also say that "$p$ is a polynomial with real coefficients."

◆

▶ **Example 8.** *Consider the polynomial*

$$p(x) = 3 - \tfrac{4}{3}x + \sqrt{2}x^2 - 9x^3 + \pi x^5.$$

*Find the degree, the leading term, and make a statement about the coefficients of $p$.*

Fortunately, the polynomial $p$ is already arranged in ascending powers of $x$. The degree of $p$ is 5 and the leading term is $\pi x^5$. Not all of the coefficients are integers, so we cannot say that "$p$ is a polynomial with integer coefficients." Not all of the

---

[2] Note that in the ascending case, the phrase "leading term" is somewhat of a misnomer, as the term with the highest power of $x$ comes last. Unfortunately, we'll have to live with this terminology.

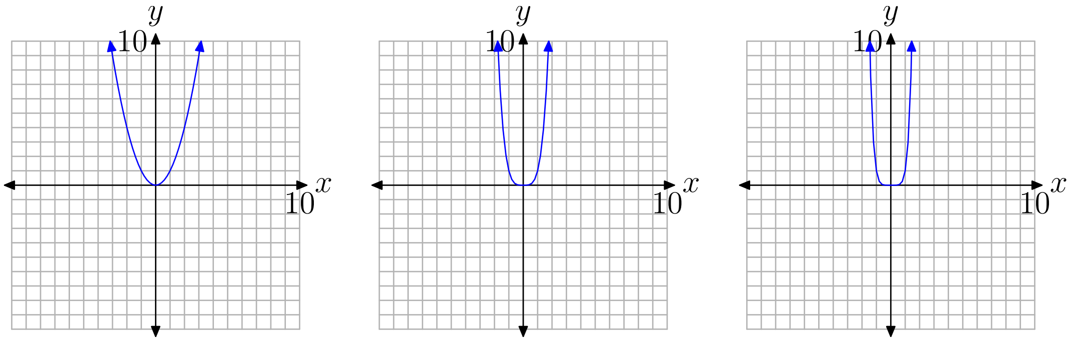
coefficients are rational numbers, so we cannot say that "$p$ is a polynomial with rational coefficients." However, because all of the coefficients of $p$ are real numbers, we can say that "$p$ is a polynomial with real coefficients."

◆

## The Graph of $y = x^n$

The primary goal in this section is to discuss the *end-behavior* of arbitrary polynomials. By "end-behavior," we mean the behavior of the polynomial for very small values of $x$ (like $-1\,000$, $-10\,000$, $-100\,000$, etc.) or very large values of $x$ (like $1\,000$, $10\,000$, $100\,000$, etc.). Before we can explore the end-behavior of arbitrary polynomials, we must first examine the end-behavior of some very basic monomials. Specifically, we need to investigate the end-behavior of the graphs of $y = x^n$, where $n = 1, 2, 3, \ldots$.

Let's first examine the graph of $y = x^n$, when $n$ is even. The graphs are simple enough to draw, either by creating a table of points or by using your graphing calculator. In **Figure 1**(a), (b), and (c), we've drawn the graphs of $y = x^2$, $y = x^4$, and $y = x^6$, respectively.



(a) The graph of $y = x^2$. (b) The graph of $y = x^4$. (c) The graph of $y = x^6$.

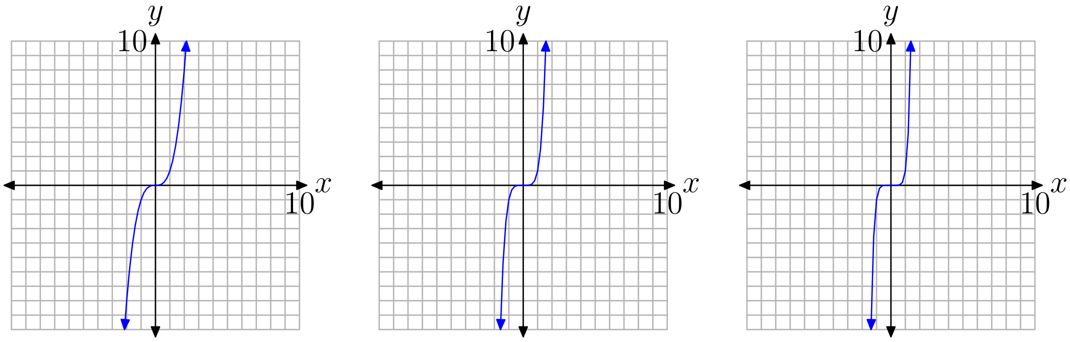
**Figure 1.** Examples of the graph of $y = x^n$, when $n$ is an even integer.

The graphs in **Figure 1** share an important trait. As you sweep your eyes from left to right, each graph falls from positive infinity, wiggles through the origin, then rises back to positive infinity.

Next, let's examine the graph of $y = x^n$, when $n$ is odd. Again, a table of points or a graphing calculator will help produce the graphs of $y = x^3$, $y = x^5$, and $y = x^7$, as shown in **Figure 2**(a), (b), and (c), respectively.

The graphs in **Figure 2** share an important trait. As you sweep your eyes from left to right, each graph rises from negative infinity, wiggles through the origin, then rises up to positive infinity.

The behavior shown in **Figure 1** and **Figure 2** is typical.

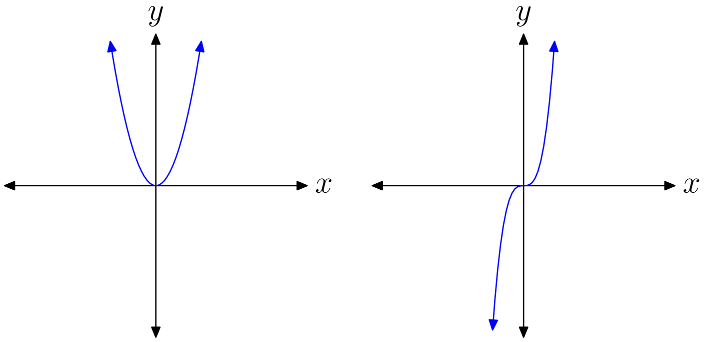


(a) The graph of $y = x^3$. (b) The graph of $y = x^5$. (c) The graph of $y = x^7$.

**Figure 2.** Examples of the graph of $y = x^n$, when $n$ is an odd integer.

**Property 9.** When $n$ is an even natural number, the graph of $y = x^n$ will look like that shown in **Figure 3**(a). If $n$ is an odd natural number, then the graph of $y = x^n$ will be similar to that shown in **Figure 3**(b).

1. When $n$ is even, as you sweep your eyes from left to right, the graph of $y = x^n$ falls from positive infinity, wiggles through the origin, then rises back to positive infinity.
2. If $n$ is odd, as you sweep your eyes from left to right, the graph of $y = x^n$ rises from negative infinity, wiggles through the origin, then rises to positive infinity.



(a) The graph of $y = x^n$, $n$ even.

(b) The graph of $y = x^n$, $n$ odd.

**Figure 3.**

## The Graph of $y = ax^n$

Now that we know the general shape of the graph of $y = x^n$, let's scale this function by multiplying by a constant, as in $y = ax^n$.

In our study of the parabola, we learned that if we multiply by a factor of $a$, where $a > 1$, then we will stretch the graph in the vertical direction by a factor of $a$. Conversely, if we multiply the graph by a factor of $a$, where $0 < a < 1$, then we will compress the graph in the vertical direction by a factor of $1/a$. If $a < 0$, then not only will we scale the graph, but multiplying by this factor will also reflect the graph across the horizontal axis.

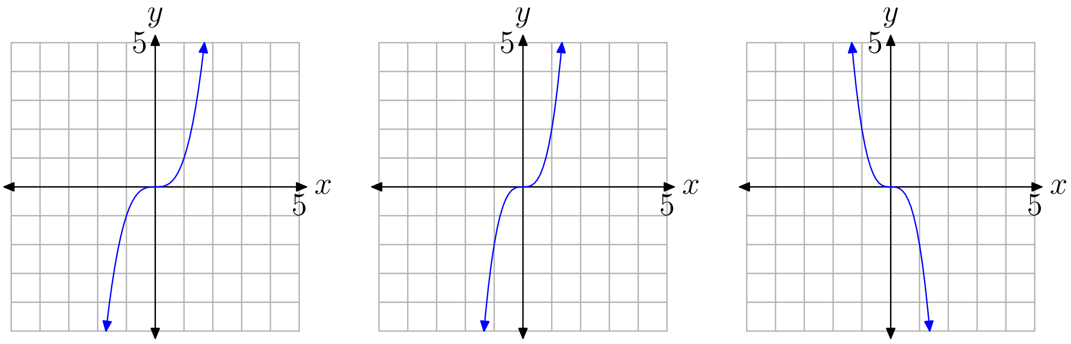
Let's look at a few examples.

▶ **Example 10.** *Sketch the graph of* $y = -2x^3$.

We know what the graph of $y = x^3$ looks like. As we sweep our eyes from left to right, the graph rises from negative infinity, wiggles through the origin, then rises to positive infinity. This behavior is shown in **Figure 4**(a).

If we multiply by a factor of 2, then we stretch the original graph by a factor of 2 in the vertical direction. The graph of $y = 2x^3$ is shown in **Figure 4**(b). Note the stretching in the vertical direction.

Finally, if we negate by multiplying by $-2$, this will stretch the graph by a factor of 2, as in **Figure 4**(b), but it will also reflect the graph across the $x$-axis. The graph of $y = -2x^3$ is shown in **Figure 4**(c).



(a) The graph of $y = x^3$. (b) The graph of $y = 2x^3$. (c) The graph of $y = -2x^3$.

**Figure 4.** Scaling by $-2$ stretches vertically by a factor of 2, then reflects the graph across the $x$-axis.

◆

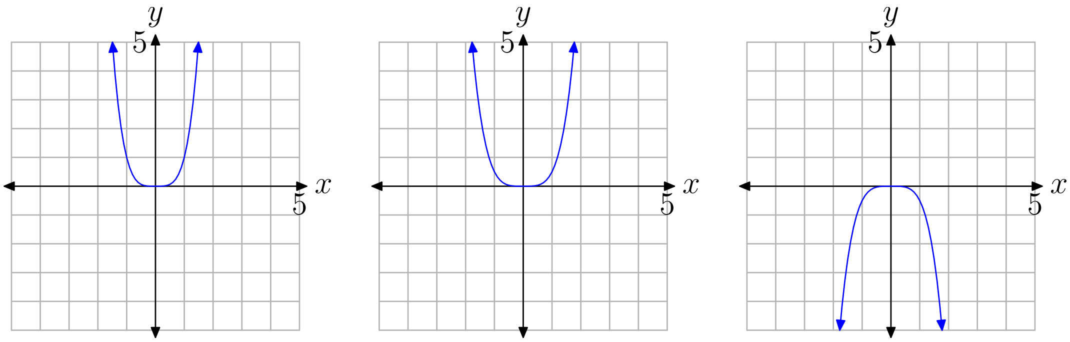
Let's look at another example.

▶ **Example 11.** *Sketch the graph of* $y = -\frac{1}{2}x^4$.

We know what the graph of $y = x^4$ looks like. As we sweep our eyes from left to right, the graph falls from positive infinity, wiggles through the origin, then rises back to positive infinity. This behavior is shown in **Figure 5**(a).

If we multiply by $1/2$, then we will compress the graph by a factor of 2. Note that the graph of $y = \frac{1}{2}x^4$ in **Figure 5**(b) is compressed by a factor of 2 in the vertical direction.

Finally, if we multiply by $-1/2$, not only will we compress the graph by a factor of 2, we will also reflect the graph across the $x$-axis. The graph of $y = -\frac{1}{2}x^4$ is shown in **Figure 5**(c).



(a) The graph of $y = x^4$. (b) The graph of $y = \frac{1}{2}x^4$. (c) The graph of $y = -\frac{1}{2}x^4$.

**Figure 5.** Scaling by $-1/2$ compresses vertically by a factor of 2, then reflects the graph across the $x$-axis.

◆

Hopefully, at this point you can now sketch the graph of $y = ax^n$ for any real number $a$ and any natural number $n$, either even or odd, without the use of a calculator.

Let's put this new-found knowledge to use in investigating the end-behavior of polynomials.

## *End Behavior*

Consider the polynomial

$$p(x) = x^3 - 7x^2 + 7x + 15. \tag{12}$$

Here's a key fact that we will use to determine the end-behavior of any polynomial.

**Property 13.** A polynomial's end-behavior is completely determined by its leading term. That is, the end-behavior of the graph of the polynomial will match the end-behavior of the graph of its leading term.

In a moment, we will show why this property is true. In the meantime, let's accept the veracity of this statement and apply it to the polynomial defined by **equation (12)**. The leading term of the polynomial $p(x) = x^3 - 7x^2 + 7x + 15$ is $x^3$. We know the end behavior of graph of $y = x^3$. As we sweep our eyes from left to right, the graph of $y = x^3$ will rise from negative infinity, wiggle through the origin, then continue to rise to positive infinity. We pictured this behavior earlier in **Figure 4**(a).

**Property 13** tells us that the graph of the polynomial $p(x) = x^3 - 7x^2 + 7x + 15$ will exhibit the same end-behavior as the graph of its leading term, $y = x^3$. We can predict that, as we sweep our eyes from left to right, the graph of the polynomial $p(x) = x^3 - 7x^2 + 7x + 15$ will rise from negative infinity, wiggle a bit, then rise to positive infinity. We don't know what happens in-between,[3] but we do know what happens at far left- and right-hand ends.

Our conjecture is verified by drawing the graph (use a graphing calculator). The graph of the polynomial $p(x) = x^3 - 7x^2 + 7x + 15$ is shown in **Figure 6**. Sure enough, as we sweep our eyes from left to right, the graph in **Figure 6** rises from negative infinity as predicted, wiggles a bit, then continues its rise to positive infinity.

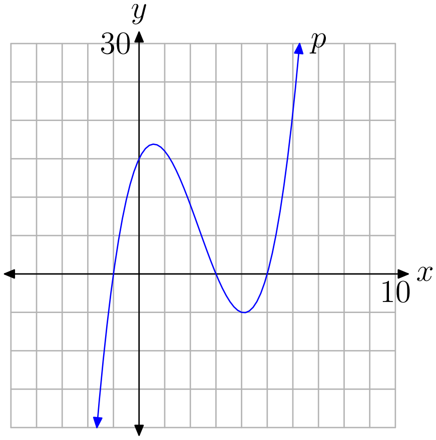


**Figure 6.** Note that this graph of $p(x) = x^3 - 7x^2 + 7x + 15$ has the same end-behavior as the graph of $y = x^3$.

**Why Does it Work?** Why does **Property 13** predict so accurately the end-behavior of this polynomial?

$$p(x) = x^3 - 7x^2 + 7x + 15$$

We can demonstrate why by first factoring out the leading term.

$$p(x) = x^3 \left(1 - \frac{7}{x} + \frac{7}{x^2} + \frac{15}{x^3}\right) \tag{14}$$

Now, ask the following question. What happens to the polynomial as we move to the right end? That is, what happens to the polynomial as we use large values of $x$, such as 1 000, 10 000, or even 100 000?

Consider the fraction $7/x$. Because the numerator is fixed at 7, and the denominator is getting bigger and bigger (growing without bound), the fraction is getting closer and closer to zero. Calculus students would use the notation

$$\lim_{x\to\infty} \frac{7}{x} = 0.$$

Don't be put off by the notation. We're using sophisticated mathematical notation for a very simple idea that says "As $x$ approaches infinity, the fraction $7/x$ approaches zero."

---

[3] We'll investigate what happens in-between in the next section.

Using similar reasoning, each of the fractions in **equation (14)** go to zero as $x$ goes to infinity (increases without bound). Thus, as $x$ gets larger and larger (as we move further and further to the right),

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} x^3\left(1 - \frac{7}{x} + \frac{7}{x^2} + \frac{15}{x^3}\right) \approx x^3(1 - 0 + 0 + 0) \approx x^3. \tag{15}$$

That is, as $x$ increases without bound, the graph of $p(x) = x^3 - 7x^2 + 7x + 15$ should approximate the graph of $y = x^3$.

Using similar reasoning, each of the fractions in **equation (14)** go to zero as $x$ goes to minus infinity. That is, if you are putting in numbers for $x$ such as $-1\,000$, $-10\,000$, $-100\,000$, and the like, the fractions in **equation (14)** will go to zero. Hence, the polynomial $p(x)$ must still approach its leading term $x^3$ for very small values of $x$ (as $x$ approaches $-\infty$).

If you superimpose the graph of $y = x^3$ on the graph of $p(x) = x^3 - 7x^2 + 7x + 15$, as in **Figure 7**, it's clear that the polynomial $p$ has the same end-behavior as the graph of its leading term $y = x^3$.

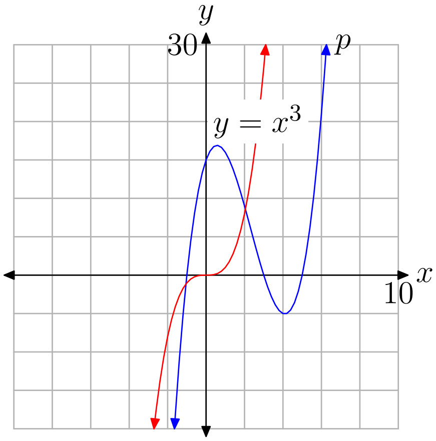
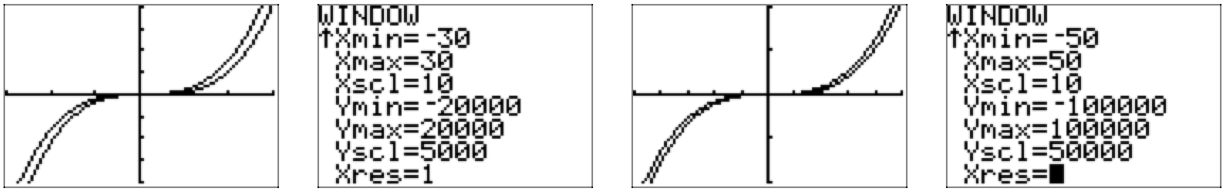


**Figure 7.** As you move to the extreme left or right, the graph of $p(x) = x^3 - 7x^2 + 7x + 15$ approaches the graph of its leading term $y = x^3$.

You can provide a more striking demonstration of the validity of the claim in **equation (15)** by plotting both the polynomial $p$ and its leading term $y = x^3$ on your calculator, then zooming out by adjusting the window parameters as shown in **Figure 8**(b). Note how the graph of $p(x) = x^3 - 7x^2 + 7x + 15$ more closely resembles the graph of its leading term $y = x^3$, at least at the right and left edges of the viewing window. When we zoom further out, by adjusting the window parameters as shown in **Figure 8**(d), note how that graph of $p$ approaches the graph of its leading term $y = x^3$ even more closely at each edge of the viewing window.



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WINDOW
↑Xmin=-30
 Xmax=30
 Xscl=10
 Ymin=-20000
 Ymax=20000
 Yscl=5000
 Xres=1
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WINDOW
↑Xmin=-50
 Xmax=50
 Xscl=10
 Ymin=-100000
 Ymax=100000
 Yscl=50000
 Xres=█
```

(a) (b) (c) (d)

**Figure 8.** Zooming out clearly demonstrates that the end-behavior of $p(x) = x^3 - 7x^2 + 7x + 15$ matches that of its leading term $y = x^3$.

Let's look at another example.

► **Example 16.** *Consider the polynomial*

$$p(x) = -x^4 + 37x^2 + 24x - 180.$$

*Comment on the end-behavior of p and use your graphing calculator to sketch its graph.*

The leading term of $p(x) = -x^4 + 37x^2 + 24x - 180$ is $y = -x^4$. We know the end-behavior of the graph of the leading term. As we sweep our eyes from left to right, the graph of $y = -x^4$ rises from negative infinity, wiggles through the origin, then falls back to minus infinity. The graph of $p$ should exhibit the same end-behavior. Indeed, in **Figure 9**, note that the graph of $y = -x^4$ and $y = -x^4 + 37x^2 + 24x - 180$ both share the same end-behavior.

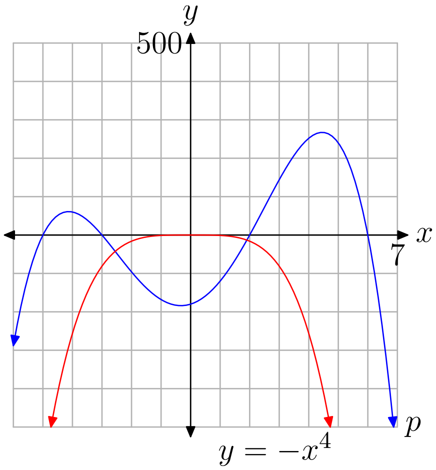


**Figure 9.** The polynomial $p(x) = -x^4 + 37x^2 + 24x - 180$ has the same end-behavior as the graph of its leading term $y = -x^4$.

## 6.1 Exercises

In **Exercises 1-8**, arrange each polynomial in descending powers of $x$, state the degree of the polynomial, identify the leading term, then make a statement about the coefficients of the given polynomial.

**1.** $p(x) = 3x - x^2 + 4 - x^3$

**2.** $p(x) = 4 + 3x^2 - 5x + x^3$

**3.** $p(x) = 3x^2 + x^4 - x - 4$

**4.** $p(x) = -3 + x^2 - x^3 + 5x^4$

**5.** $p(x) = 5x - \frac{3}{2}x^3 + 4 - \frac{2}{3}x^5$

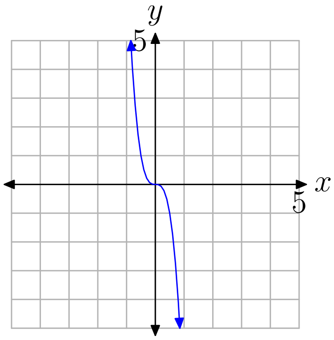
**6.** $p(x) = -\frac{3}{2}x + 5 - \frac{7}{3}x^5 + \frac{4}{3}x^3$

**7.** $p(x) = -x + \frac{2}{3}x^3 - \sqrt{2}x^2 + \pi x^6$

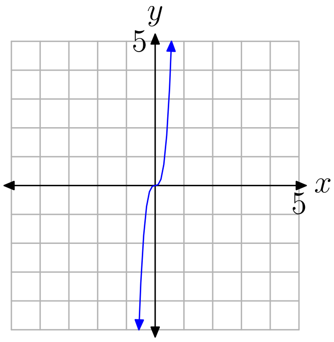
**8.** $p(x) = 3 + \sqrt{2}x^4 + \sqrt{3}x - 2x^2 + \sqrt{5}x^6$

In **Exercises 9-14**, you are presented with the graph of $y = ax^n$. In each case, state whether the degree is even or odd, then state whether $a$ is a positive or negative number.

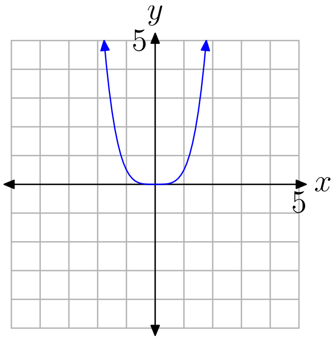
**9.**



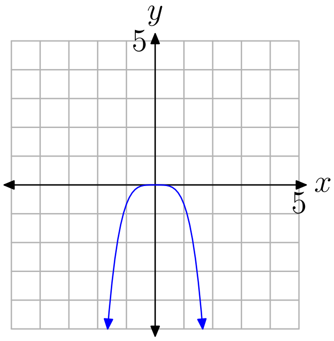
**10.**



**11.**



**12.**



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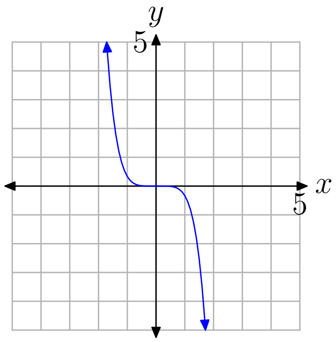
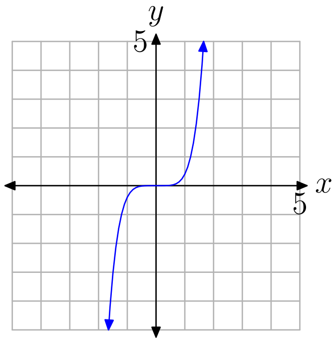
**13.**

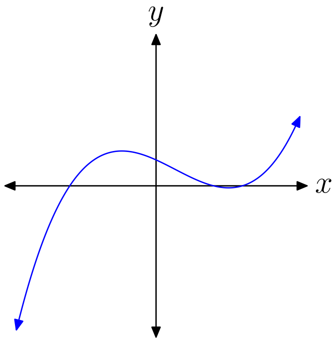


**14.**

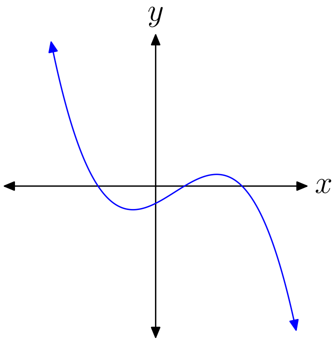


In **Exercises 15-20**, you are presented with the graph of the polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$. In each case, state whether the degree of the polynomial is even or odd, then state whether the leading coefficient $a_n$ is positive or negative.

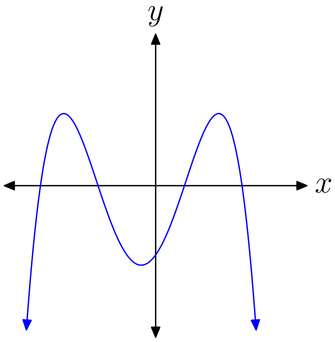
**15.**



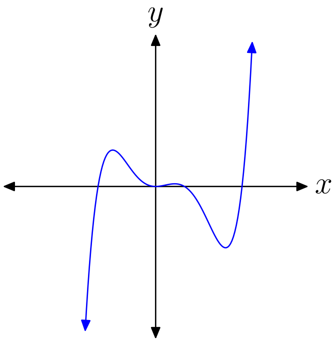
**16.**



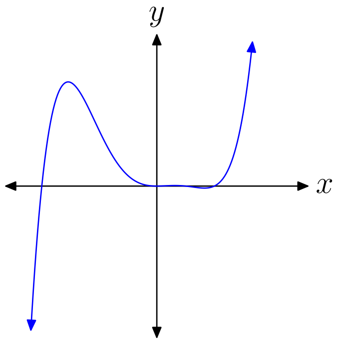
**17.**



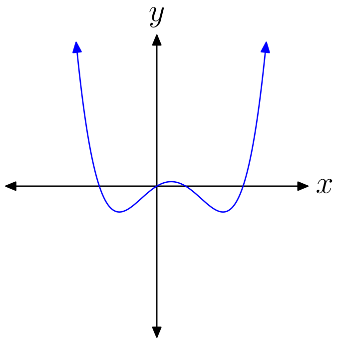
**18.**



**19.**



**20.**



For each polynomial in **Exercises 21-30**, perform each of the following tasks.

i. Predict the end-behavior of the polynomial by drawing a very rough sketch of the polynomial. Do this without the assistance of a calculator. The only concern here is that your graph show the correct end-behavior.
ii. Draw the graph on your calculator, adjust the viewing window so that all "turning points" of the polynomial are visible in the viewing window, and copy the result onto your homework paper. As usual, label and scale each axis with xmin, xmax, ymin, and ymax. Does the actual end-behavior agree with your predicted end-behavior?

**21.** $p(x) = -3x^3 + 2x^2 + 8x - 4$

**22.** $p(x) = 2x^3 - 3x^2 + 4x - 8$

**23.** $p(x) = x^3 + x^2 - 17x + 15$

**24.** $p(x) = -x^4 + 2x^2 + 29x - 30$

**25.** $p(x) = x^4 - 3x^2 + 4$

**26.** $p(x) = -x^4 + 8x^2 - 12$

**27.** $p(x) = -x^5 + 3x^4 - x^3 + 2x$

**28.** $p(x) = 2x^4 - 3x^3 + x - 10$

**29.** $p(x) = -x^6 - 4x^5 + 27x^4 + 78x^3 + 4x^2 + 376x - 480$

**30.** $p(x) = x^5 - 27x^3 + 30x^2 - 124x + 120$

## 6.1 Answers

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**1.** $p(x) = -x^3 - x^2 + 3x + 4$, degree $= 3$, leading term $= -x^3$, "$p$ is a polynomial with integer coefficients," "$p$ is a polynomial with rational coefficients," or "$p$ is a polynomial with real coefficients."

**3.** $p(x) = x^4 + 3x^2 - x - 4$, degree $= 4$, leading term $= x^4$, "$p$ is a polynomial with integer coefficients," "$p$ is a polynomial with rational coefficients," or "$p$ is a polynomial with real coefficients."

**5.** $p(x) = -\frac{2}{3}x^5 - \frac{3}{2}x^3 + 5x + 4$, degree $=$ 5, leading term $= -\frac{2}{3}x^5$, "$p$ is a polynomial with rational coefficients," or "$p$ is a polynomial with real coefficients."

**7.** $p(x) = \pi x^6 + \frac{2}{3}x^3 - \sqrt{2}x^2 - x$, degree $=$ 6, leading term $= \pi x^6$, "$p$ is a polynomial with real coefficients."

**9.** $y = ax^n$, $n$ odd, $a < 0$.

**11.** $y = ax^n$, $n$ even, $a > 0$.

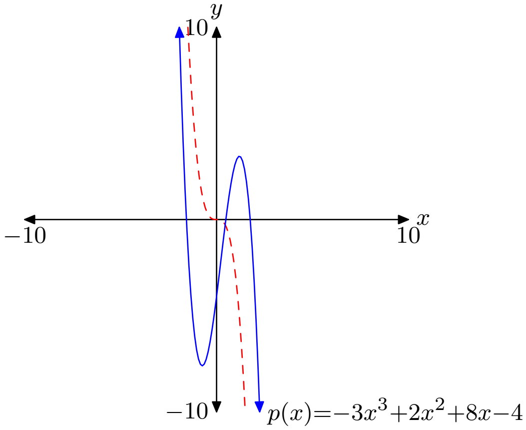
**13.** $y = ax^n$, $n$ odd, $a < 0$.

**15.** odd, positive

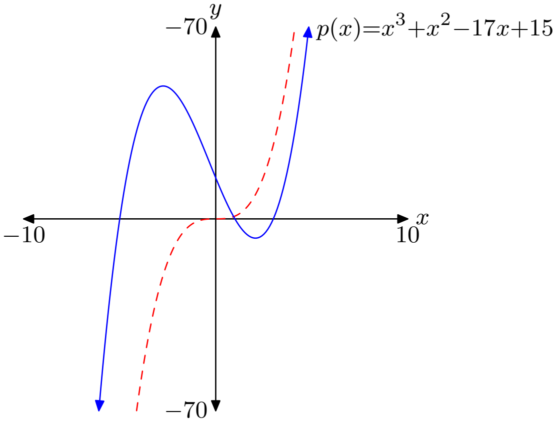
**17.** even, negative

**19.** odd, positive

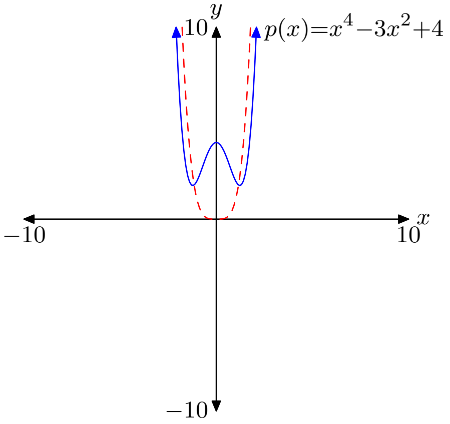
**21.** Note that the leading term $-3x^3$ (dashed) has the same end-behavior as the polynomial $p$.



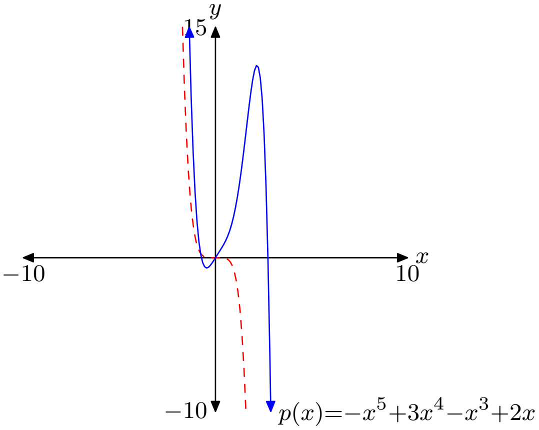
**23.** Note that the leading term $x^3$ (dashed) has the same end-behavior as the polynomial $p$.



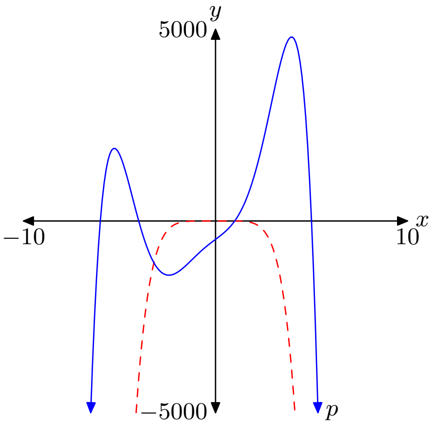
**25.** Note that the leading term $x^4$ (dashed) has the same end-behavior as the polynomial $p$.



**27.** Note that the leading term $-x^5$ (dashed) has the same end-behavior as the polynomial $p$.



**29.** Note that the leading term $-x^6$ (dashed) has the same end-behavior as the polynomial $p$.



# 6.2 Zeros of Polynomials

In the previous section we studied the end-behavior of polynomials. We know that a polynomial's end-behavior is identical to the end-behavior of its leading term. Our focus was concentrated on the far right- and left-ends of the graph and not upon what happens in-between.

In this section, our focus shifts to the interior. There are two important areas of concentration: the local maxima and minima of the polynomial, and the location of the $x$-intercepts or zeros of the polynomial. In this section we concentrate on finding the zeros of the polynomial.

## Zeros

Let's begin with a formal definition of the zeros of a polynomial.

**Definition 1.** *Let* $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ *be a polynomial with real coefficients. We say that* $a$ *is a zero of the polynomial if and only if* $p(a) = 0$.

The definition also holds if the coefficients are complex, but that's a topic for a more advanced course.

For example, $-5$ is a zero of the polynomial $p(x) = x^2 + 3x - 10$ because

$$\begin{aligned} p(-5) &= (-5)^2 + 3(-5) - 10 \\ &= 25 - 15 - 10 \\ &= 0. \end{aligned}$$

Similarly, $-1$ is a zero of the polynomial $p(x) = x^3 + 3x^2 - x - 3$ because

$$\begin{aligned} p(-1) &= (-1)^3 + 3(-1)^2 - (-1) - 3 \\ &= -1 + 3 + 1 - 3 \\ &= 0. \end{aligned}$$

Let's look at a more extensive example.

► **Example 2.** *Find the zeros of the polynomial defined by*

$$p(x) = (x + 3)(x - 2)(x - 5). \tag{3}$$

At first glance, the function does not appear to have the form of a polynomial. However, two applications of the distributive property provide the product of the last two factors.

$$\begin{aligned} p(x) &= (x + 3)(x(x - 5) - 2(x - 5)) \\ &= (x + 3)(x^2 - 5x - 2x + 10) \\ &= (x + 3)(x^2 - 7x + 10) \end{aligned}$$

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[5] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

A third and fourth application of the distributive property reveals the nature of our function.

$$\begin{aligned} p(x) &= x(x^2 - 7x + 10) + 3(x^2 - 7x + 10) \\ &= x^3 - 7x^2 + 10x + 3x^2 - 21x + 30 \qquad (4) \\ &= x^3 - 4x^2 - 11x + 30 \end{aligned}$$

Hence, $p$ is clearly a polynomial. However, the original factored form provides quicker access to the zeros of this polynomial. Using **Definition 1**, we need to find values of $x$ that make $p(x) = 0$. That is, we need to solve the equation

$$p(x) = 0.$$

Of course, $p(x) = (x + 3)(x - 2)(x - 5)$, so, equivalently, we need to solve the equation

$$(x + 3)(x - 2)(x - 5) = 0.$$

By the *zero product property*, either

$$x + 3 = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{or} \quad x - 5 = 0.$$

These are linear (first degree) equations, each of which can be solved independently. Thus, either

$$x = -3 \quad \text{or} \quad x = 2 \quad \text{or} \quad x = 5.$$

Hence, the zeros of the polynomial $p$ are $-3$, 2, and 5.

Let's use **equation (4)** to check that $-3$ is a zero of the polynomial $p$. Substitute $-3$ for $x$ in $p(x) = x^3 - 4x^2 - 11x + 30$.

$$\begin{aligned} p(-3) &= (-3)^3 - 4(-3)^2 - 11(-3) + 30 \\ &= -27 - 36 + 33 + 30 \\ &= 0 \end{aligned}$$

This calculation verifies that $-3$ is a zero of the polynomial $p$. However, it is much easier to check that $-3$ is a zero of the polynomial using **equation (3)**. Substitute $-3$ for $x$ in $p(x) = (x + 3)(x - 2)(x - 5)$.

$$\begin{aligned} p(-3) &= (-3 + 3)(-3 - 2)(-3 - 5) \\ &= (0)(-5)(-8) \\ &= 0 \end{aligned}$$

We'll leave it to our readers to check that 2 and 5 are also zeros of the polynomial $p$.

It's very important to note that once you know the linear (first degree) factors of a polynomial, the zeros follow with ease. In the last example, $p(x) = (x+3)(x-2)(x-5)$, so the linear factors are $x + 3$, $x - 2$, and $x - 5$. Consequently, the zeros are $-3$, 2, and 5.

◆

Before continuing, we take a moment to review an important multiplication pattern.

## The Difference of Two Squares

A special multiplication pattern that appears frequently in this text is called the *difference of two squares*. Use the distributive property to expand $(a+b)(a-b)$.

$$\begin{aligned}(a+b)(a-b) &= a(a-b)+b(a-b)\\ &= a^2-ab+ba-b^2\end{aligned}$$

Since $ab=ba$, we have the following result.

**Property 5.** The **Difference of Two Squares** Pattern:

$$(a+b)(a-b)=a^2-b^2$$

Thus, if you have two binomials with identical first and second terms, but the terms of one are separated by a plus sign, while the terms of the second are separated by a minus sign, then you multiply by squaring the first and second terms and separating these squares with a minus sign. Hence the name, the "difference of two squares."

For example,

$$(2x+3)(2x-3)=(2x)^2-(3)^2=4x^2-9.$$

Note how we simply squared the matching first and second terms and then separated our squares with a minus sign. In similar fashion,

$$\begin{aligned}(x+5)(x-5) &= x^2-25\\ (5x+4)(5x-4) &= 25x^2-16\\ (3x-7)(3x+7) &= 9x^2-49.\end{aligned}$$

In each case, note how we squared the matching first and second terms, then separated the squares with a minus sign.

Once you've mastered multiplication using the "Difference of Squares" pattern, it is easy to factor using the same pattern. You simply reverse the procedure. For example,

$$4x^2-9=(2x+3)(2x-3).$$

We start by taking the square root of the two squares. Thus, the square root of $4x^2$ is $2x$ and the square root of 9 is 3. We then form two binomials with the results $2x$ and 3 as matching first and second terms, separating one pair with a plus sign, the other pair with a minus sign.

In similar fashion,

$$9x^2-49=(3x+7)(3x-7).$$

Again, note how we take the square root of each term, form two binomials with the results, then separate one pair with a plus, the other with a minus.

We'll find the "Difference of Squares" pattern handy in what follows.

## *Finding Zeros by Factoring*

We will now explore how we can find the zeros of a polynomial by factoring, followed by the application of the *zero product property*. It is important to understand that the polynomials of this section have been carefully selected so that you will be able to factor them using the various techniques that follow.

Let's explore *factoring by grouping.*

▶ **Example 6.** *Find the zeros of the polynomial*

$$p(x) = x^3 + 2x^2 - 25x - 50.$$

In **Example 2** we learned that it is easy to spot the zeros of a polynomial if the polynomial is expressed as a product of linear (first degree) factors. In this example, the polynomial is not factored, so it would appear that the first thing we'll have to do is factor our polynomial.

Whenever you are presented with a four term expression, one thing you can try is factoring by grouping.[6] So, with this thought in mind, let's factor an $x$ out of the first two terms, then a $-25$ out of the second two terms.

$$\begin{aligned} p(x) &= x^3 + 2x^2 - 25x - 50 \\ &= x^2(x+2) - 25(x+2) \end{aligned}$$

Note that this last result is the difference of two terms. The polynomial is not yet fully factored as it is not yet a product of two or more factors. However, note that each of the two terms has a common factor of $x + 2$. Let's factor out this common factor.

$$p(x) = (x^2 - 25)(x + 2)$$

We've still not completely factored our polynomial. The first factor is the difference of two squares and can be factored further.

$$p(x) = (x + 5)(x - 5)(x + 2)$$

The polynomial $p$ is now fully factored. To find the zeros of the polynomial $p$, we need to solve the equation

$$p(x) = 0.$$

However, $p(x) = (x + 5)(x - 5)(x + 2)$, so equivalently, we need to solve the equation

$$(x + 5)(x - 5)(x + 2) = 0.$$

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[6] It's important to understand that the polynomial presented in **Example 6** has been specially selected so that factoring by grouping will work. This method of grouping will not always be successful. For example, the technique will not work on the polynomial $p(x) = x^3 - 6x^2 - x + 30$. To factor this polynomial, you need more advanced theory typically taught in a college algebra course.

We can use the zero product property. Either

$$x + 5 = 0 \qquad \text{or} \qquad x - 5 = 0 \qquad \text{or} \qquad x + 2 = 0.$$

Again, each of these linear (first degree) equations can be solved independently. Either

$$x = -5 \qquad \text{or} \qquad x = 5 \qquad \text{or} \qquad x = -2.$$

Thus, the zeros of the polynomial $p$ are $-5$, 5, and $-2$. We'll leave it to our readers to check these results.

Again, it is very important to realize that once the linear (first degree) factors are determined, the zeros of the polynomial follow. In this example, the linear factors are $x+5$, $x-5$, and $x+2$. It immediately follows that the zeros of the polynomial are $-5$, 5, and $-2$.

◆

In the next example, we will see that sometimes the first step is to factor out the greatest common factor.

► **Example 7.** *Find the zeros of the polynomial*

$$p(x) = x^4 + 2x^3 - 16x^2 - 32x \tag{8}$$

To find the zeros of the polynomial, we need to solve the equation

$$p(x) = 0.$$

Equivalently, because $p(x) = x^4 + 2x^3 - 16x^2 - 32x$, we need to solve the equation

$$x^4 + 2x^3 - 16x^2 - 32x = 0.$$

Note that each term on the left-hand side has a common factor of $x$. Thus, our first step is to factor out this common factor of $x$.

$$x[x^3 + 2x^2 - 16x - 32] = 0$$

The four-term expression inside the brackets looks familiar. Let's try factoring by grouping. Factor an $x^2$ out of the first two terms, then a $-16$ from the third and fourth terms.

$$x\left[x^2(x+2) - 16(x+2)\right] = 0$$

We now have a common factor of $x+2$, so we factor it out.

$$x\left[(x^2 - 16)(x+2)\right] = 0$$

The brackets are no longer needed (multiplication is associative) so we leave them off, then use the difference of squares pattern to factor $x^2 - 16$.

$$x(x+4)(x-4)(x+2) = 0$$

The zero product property tells us that either

$$x = 0 \qquad \text{or} \qquad x + 4 = 0 \qquad \text{or} \qquad x - 4 = 0 \qquad \text{or} \qquad x + 2 = 0.$$

Each of these linear (first degree) factors can be solved independently. Either

$$x = 0 \qquad \text{or} \qquad x = -4 \qquad \text{or} \qquad x = 4 \qquad \text{or} \qquad x = -2.$$

Thus, the zeros of the polynomial $p$ are 0, $-4$, 4, and $-2$. We'll leave it to our readers to check these results.

Again, it is very important to note that once you've determined the linear (first degree) factors of a polynomial, then you know the zeros. In this case, the linear factors are $x$, $x + 4$, $x - 4$, and $x + 2$. Therefore, the zeros are 0, $-4$, 4, and $-2$, respectively. This discussion leads to a result called the *Factor Theorem*.

◇

**Factor Theorem.** Let $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be a polynomial with real coefficients. If $x - a$ is a factor of the polynomial $p(x)$, then $a$ is a zero of the polynomial. That is, if $x - a$ is a factor of the polynomial $p(x)$, then $p(a) = 0$.

The upshot of all of these remarks is the fact that, if you know the linear factors of the polynomial, then you know the zeros. The converse is also true, but we will not need it in this course.[7]

Let's examine the connection between the zeros of the polynomial and the $x$-intercepts of the graph of the polynomial.

## *The $x$-intercepts and the Zeros of a Polynomial*

For the discussion that follows, let's assume that the independent variable is $x$ and the dependent variable is $y$. Corresponding to these assignments, we will also assume that we've labeled the horizontal axis with $x$ and the vertical axis with $y$, as shown in **Figure 1**.

The key fact for the remainder of this section is that a function is zero at the points where its graph crosses the $x$-axis. The phrases "function values" and "$y$-values" are equivalent (provided your dependent variable is $y$), so when you are asked where your function value is equal to zero, you are actually being asked "where is your $y$-value equal to zero?" Of course, $y = 0$ where the graph of the function crosses the horizontal axis (again, providing you are using the letter $y$ for your dependent variable—labeling the vertical axis with $y$).

A polynomial is a function, so, like any function, a polynomial is zero where its graph crosses the horizontal axis. As you can see in **Figure 1**, the graph of the polynomial crosses the horizontal axis at $x = -6$, $x = 1$, and $x = 5$. Note that at each of these intercepts, the $y$-value (function value) equals zero. The zeros of the polynomial are

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[7] The converse is the statement "If $a$ is a zero of a polynomial $p(x)$, then $x - a$ is a factor of $p(x)$." You'll encounter this converse in a college algebra course.

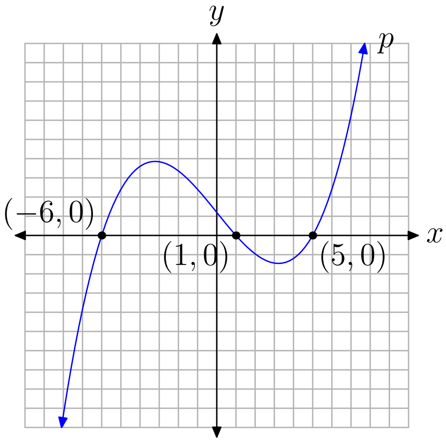

**Figure 1.** A function is zero where its graph crosses the horizontal axis.

−6, 1, and 5. Therefore the $x$-intercepts of the graph of the polynomial are located at $(-6,0)$, $(1,0)$, and $(5,0)$.

Let's use these ideas to plot the graphs of several polynomials.

▶ **Example 9.** *Sketch the graph of the polynomial in* **Example 6***.*

In **Example 6**, the polynomial $p(x) = x^3 + 2x^2 - 25x - 50$ factored into linear factors

$$p(x) = (x+5)(x-5)(x+2).$$

Consequently, the zeros of the polynomial were −5, 5, and −2. Thus, the $x$-intercepts of the graph of the polynomial are located at $(-5,0)$, $(5,0)$, and $(-2,0)$.

The polynomial $p(x) = x^3+2x^2-25x-50$ has leading term $x^3$. Consequently, as we swing our eyes from left to right, the graph of the polynomial $p$ must rise from negative infinity, wiggle through its $x$-intercepts, then continue to rise to positive infinity. We have no choice but to sketch a graph similar to that in **Figure 2**.

Note that there are two "turning points" of the polynomial in **Figure 2**. You might ask how we knew where to put these "turning points" of the polynomial. The answer is "we didn't know where to put them." We know they have to be there, but we don't know their precise location.[8] That's why we haven't scaled the vertical axis, because without the aid of a calculator, it's hard to determine the precise location of the turning points shown in **Figure 2**.

However, note that knowledge of the end-behavior and the zeros of the polynomial allows us to construct a reasonable facsimile of the actual graph. If we want more accuracy than a rough approximation provides, such as the accuracy displayed in **Figure 2**, we'll have to use our graphing calculator, as demonstrated in **Figure 3**.

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[8] Finding these "turning points" or local extrema is an exercise that calculus students regularly perform.

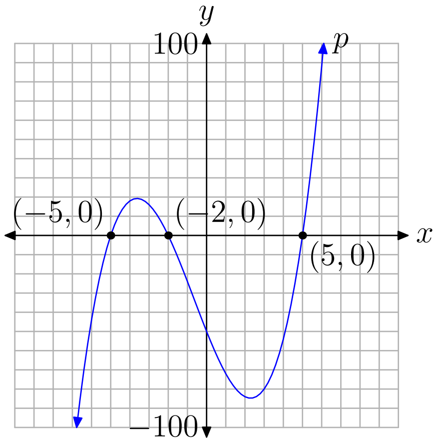
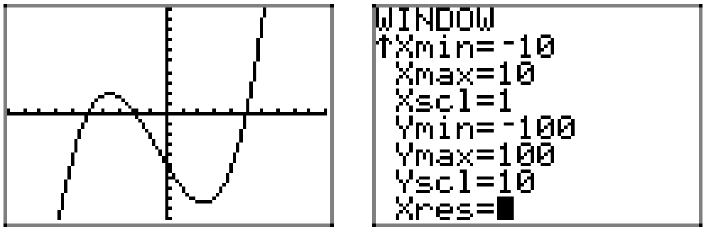


**Figure 2.** The graph rises from negative infinity, wiggles through its $x$-intercepts, then rises to positive infinity.



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WINDOW
↑Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-100
 Ymax=100
 Yscl=10
 Xres=█
```

**Figure 3.** The graph of $p(x) = x^3 + 2x^2 - 25x - 50$ and the window settings used.

We'll have more to say about the "turning points" (relative extrema) in the next section. For now, let's continue to focus on the end-behavior and the zeros.

◆

Let's look at another example.

▶ **Example 10.** *Sketch the graph of the polynomial in* **Example 7**.

In **Example 7**, the polynomial $p(x) = x^4 + 2x^3 - 16x^2 - 32x$ factored into a product of linear factors

$$p(x) = x(x + 4)(x - 4)(x + 2).$$

Consequently, the zeros of the polynomial are 0, −4, 4, and −2. Thus, the $x$-intercepts of the graph of the polynomial are located at $(0, 0)$, $(-4, 0)$, $(4, 0)$ and $(-2, 0)$.

The polynomial $p(x) = x^4 + 2x^3 - 16x^2 - 32x$ has leading term $x^4$. Consequently, as we swing our eyes from left to right, the graph of the polynomial $p$ must fall from positive infinity, wiggle through its $x$-intercepts, then rise back to positive infinity. We have no choice but to sketch a graph similar to that in **Figure 4**.

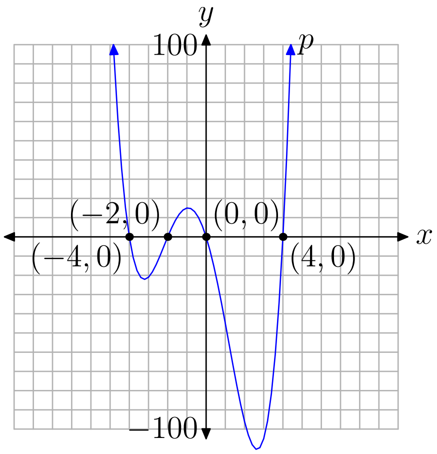
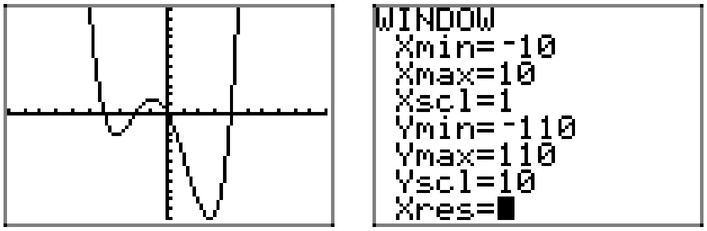


**Figure 4.** The graph falls from positive infinity, wiggles through its $x$-intercepts, then rises back to positive infinity.

Again, we can draw a sketch of the graph without the use of the calculator, using only the end-behavior and zeros of the polynomial. However, if we want the accuracy depicted in **Figure 4**, particularly finding correct locations of the "turning points," we'll have to resort to the use of a graphing calculator. This is shown in **Figure 5**.



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WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-110
 Ymax=110
 Yscl=10
 Xres=█
```

**Figure 5.** The graph of $p(x) = x^4 - 20x^2 + 64$ and the window settings used.

◆

Let's look at a final example that requires factoring out a greatest common factor followed by the *ac*-test.

▶ **Example 11.** *Find the zeros of the polynomial*

$$p(x) = 4x^3 - 2x^2 - 30x.$$

First, notice that each term of this trinomial is divisible by $2x$. This is the greatest common divisor, or equivalently, the greatest common factor. You should always look to factor out the greatest common factor in your first step.

$$\begin{aligned} p(x) &= 4x^3 - 2x^2 - 30x \\ &= 2x[2x^2 - x - 15] \end{aligned}$$

Next, compare the trinomial $2x^2 - x - 15$ with $ax^2 + bx + c$ and note that $ac = -30$. The integer pair $\{5, -6\}$ has product $-30$ and sum $-1$. Rewrite the middle term of $2x^2 - x - 15$ in terms of this pair and factor by grouping.

$$\begin{aligned} p(x) &= 2x[2x^2 + 5x - 6x - 15] \\ &= 2x[x(2x+5) - 3(2x+5)] \\ &= 2x(x-3)(2x+5) \end{aligned} \tag{12}$$

To find the zeros, we need to solve the polynomial equation $p(x) = 0$, or equivalently,

$$2x(x-3)(2x+5) = 0.$$

Using the zero product property, either

$$2x = 0, \qquad \text{or} \qquad x - 3 = 0, \qquad \text{or} \qquad 2x + 5 = 0.$$

Each of these linear factors can be solved independently. Thus, either

$$x = 0, \qquad \text{or} \qquad x = 3, \qquad \text{or} \qquad x = -\frac{5}{2}.$$

Thus, the zeros of the polynomial are 0, 3, and $-5/2$.

Alternatively, one can factor out a 2 from the third factor in **equation (12)**.

$$\begin{aligned} p(x) &= 2x(x-3)(2)\left(x + \frac{5}{2}\right) \\ &= 4x(x-3)\left(x + \frac{5}{2}\right) \end{aligned}$$

In this form,

- $x$ is a factor, so $x = 0$ is a zero,
- $x - 3$ is a factor, so $x = 3$ is a zero, and
- $x + 5/2$ is a factor, so $x = -5/2$ is a zero.

The leading term of $p(x) = 4x^3 - 2x^2 - 30x$ is $4x^3$, so as our eyes swing from left to right, the graph of the polynomial must rise from negative infinity, wiggle through its zeros, then rise to positive infinity. The graph must therefore be similar to that shown in **Figure 6**.

Again, the intercepts and end-behavior provide ample clues to the shape of the graph, but, if we want the accuracy portrayed in **Figure 6**, then we must rely on the graphing calculator. The graph and window settings used are shown in **Figure 7**.

◆

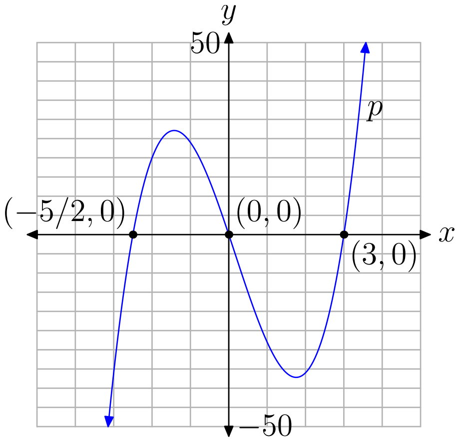
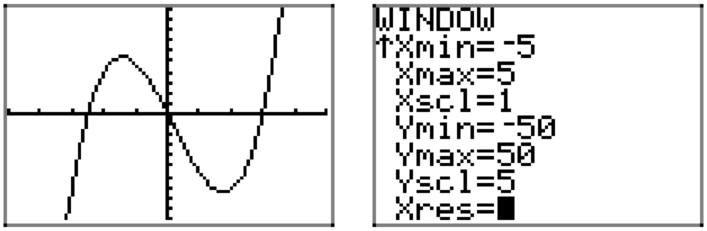


**Figure 6.** The graph rises from negative infinity, wiggles through its zeros, then rises to positive infinity.



```
WINDOW
↑Xmin=-5
 Xmax=5
 Xscl=1
 Ymin=-50
 Ymax=50
 Yscl=5
 Xres=█
```

**Figure 7.** The graph of $p(x) = 4x^3 - 2x^2 - 30x$ and the window settings used.

## 6.2 Exercises

In **Exercises 1-6**, use direct substitution to show that the given value is a zero of the given polynomial.

**1.** $p(x) = x^3 - 3x^2 - 13x + 15$, $x = -3$

**2.** $p(x) = x^3 - 2x^2 - 13x - 10$, $x = -2$

**3.** $p(x) = x^4 - x^3 - 12x^2$, $x = 4$

**4.** $p(x) = x^4 - 2x^3 - 3x^2$, $x = -1$

**5.** $p(x) = x^4 + x^2 - 20$, $x = -2$

**6.** $p(x) = x^4 + x^3 - 19x^2 + 11x + 30$, $x = -1$

In **Exercises 7-28**, identify all of the zeros of the given polynomial without the aid of a calculator. Use an algebraic technique and show all work (factor when necessary) needed to obtain the zeros.

**7.** $p(x) = (x - 2)(x + 4)(x - 5)$

**8.** $p(x) = (x - 1)(x - 3)(x + 8)$

**9.** $p(x) = -2(x - 3)(x + 4)(x - 2)$

**10.** $p(x) = -3(x + 1)(x - 1)(x - 8)$

**11.** $p(x) = x(x - 3)(2x + 1)$

**12.** $p(x) = -3x(x + 5)(3x - 2)$

**13.** $p(x) = -2(x + 3)(3x - 5)(2x + 1)$

**14.** $p(x) = 3(x - 2)(2x + 5)(3x - 4)$

**15.** $p(x) = 3x^3 + 5x^2 - 12x - 20$

**16.** $p(x) = 3x^3 + x^2 - 12x - 4$

**17.** $p(x) = 2x^3 + 5x^2 - 2x - 5$

**18.** $p(x) = 2x^3 - 5x^2 - 18x + 45$

**19.** $p(x) = x^4 + 4x^3 - 9x^2 - 36x$

**20.** $p(x) = -x^4 + 4x^3 + x^2 - 4x$

**21.** $p(x) = -2x^4 - 10x^3 + 8x^2 + 40x$

**22.** $p(x) = 3x^4 + 6x^3 - 75x^2 - 150x$

**23.** $p(x) = 2x^3 - 7x^2 - 15x$

**24.** $p(x) = 2x^3 - x^2 - 10x$

**25.** $p(x) = -6x^3 + 4x^2 + 16x$

**26.** $p(x) = 9x^3 + 3x^2 - 30x$

**27.** $p(x) = -2x^7 - 10x^6 + 8x^5 + 40x^4$

**28.** $p(x) = 6x^5 - 21x^4 - 45x^3$

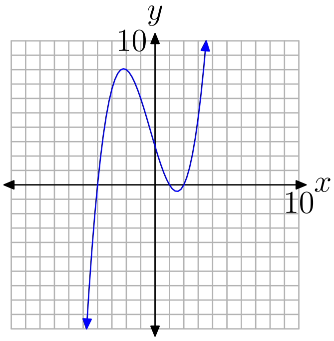
---

[9] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

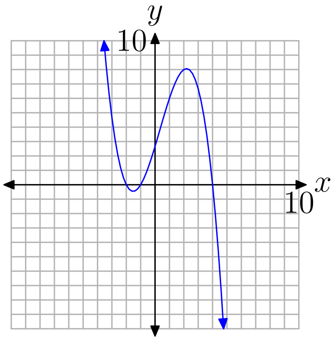
In **Exercises 29-34**, the graph of a polynomial is given. Perform each of the following tasks.

i. Copy the image onto your homework paper. Label and scale your axes, then label each $x$-intercept with its coordinates.
ii. Identify the zeros of the polynomial.

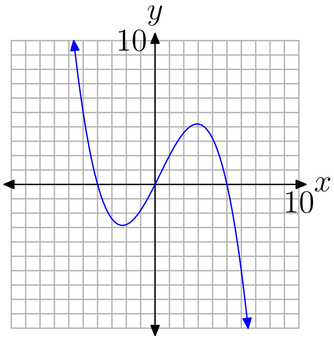
**29.**



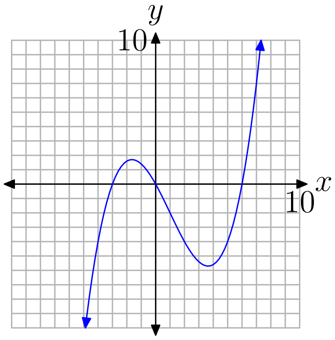
**30.**



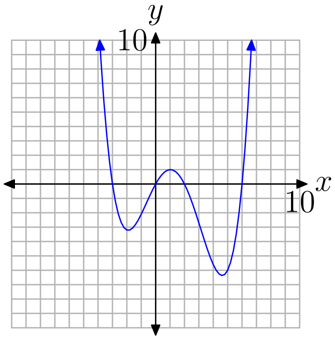
**31.**



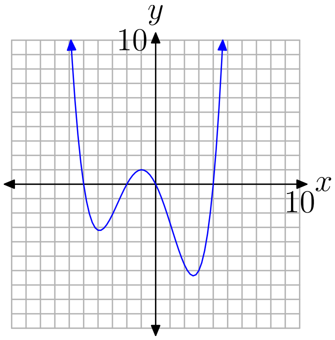
**32.**



**33.**



**34.**



For each of the polynomials in **Exercises 35-46**, perform each of the following tasks.

i. Factor the polynomial to obtain the zeros. Show your work.
ii. Set up a coordinate system on graph paper. Label and scale the horizontal axis. Use the zeros and end-behavior to help sketch the graph of the polynomial without the use of a calculator.
iii. Verify your result with a graphing calculator.

**35.** $p(x) = 5x^3 + x^2 - 45x - 9$

**36.** $p(x) = 4x^3 + 3x^2 - 64x - 48$

**37.** $p(x) = 4x^3 - 12x^2 - 9x + 27$

**38.** $p(x) = x^3 + x^2 - 16x - 16$

**39.** $p(x) = x^4 + 2x^3 - 25x^2 - 50x$

**40.** $p(x) = -x^4 - 5x^3 + 4x^2 + 20x$

**41.** $p(x) = -3x^4 - 9x^3 + 3x^2 + 9x$

**42.** $p(x) = 4x^4 - 29x^2 + 25$

**43.** $p(x) = -x^3 - x^2 + 20x$

**44.** $p(x) = 2x^3 - 7x^2 - 30x$

**45.** $p(x) = 2x^3 + 3x^2 - 35x$

**46.** $p(x) = -2x^3 - 11x^2 + 21x$

## 6.2 *Answers*

---

**1.** $p(-3) = (-3)^3 - 3(-3)^2 - 13(-3) + 15 = 0$

**3.** $p(4) = 4^4 - 4^3 - 12(4)^2 = 0$

**5.** $p(-2) = (-2)^4 + (-2)^2 - 20 = 0$

**7.** $-4$, 2, and 5

**9.** $-4$, 2 and 3

**11.** $-1/2$, 0, and 3

**13.** $-3$, $-1/2$, and $5/3$,

**15.** $-2$, $-5/3$, and 2

**17.** $-5/2$, $-1$, and 1

**19.** 0, $-3$, 3, and $-4$

**21.** 0, $-2$, 2, and $-5$

**23.** $-3/2$, 0, and 5

**25.** $-4/3$, 0, and 2

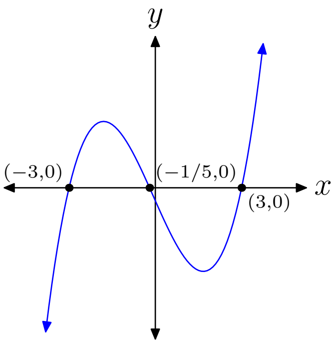
**27.** 0, $-2$, 2, and $-5$

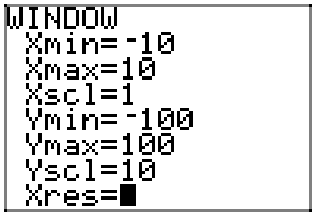
**29.** Zeros: $-4$, 1, and 2

**31.** Zeros: $-4$, 0, and 5

**33.** Zeros: 0, 6, $-3$, 2

**35.**





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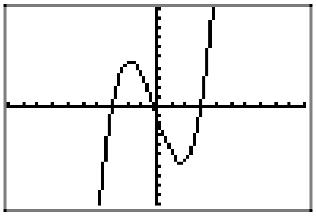

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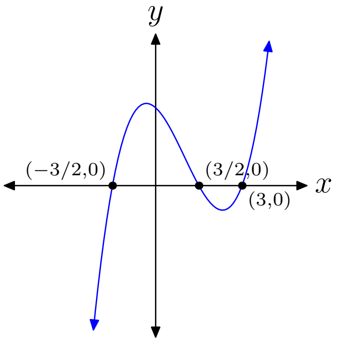

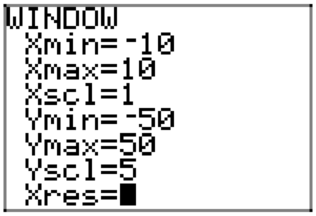

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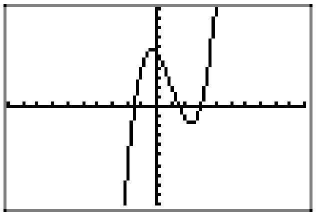

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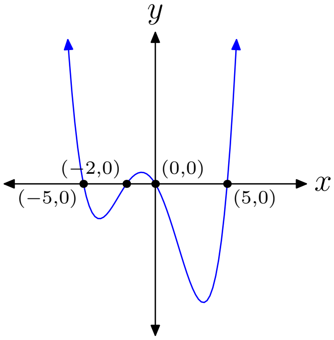

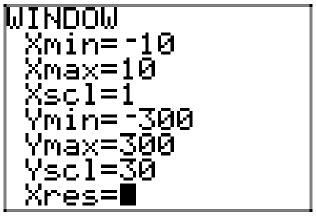

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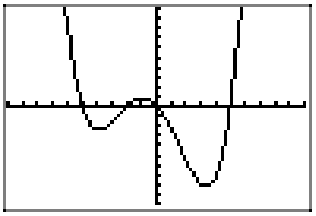

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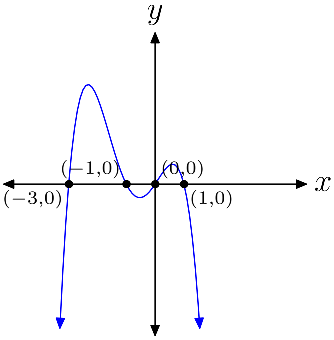

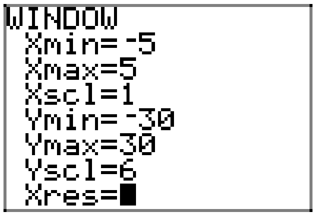

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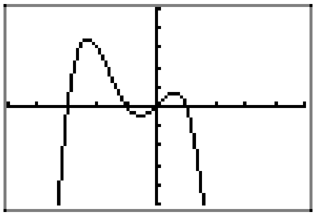
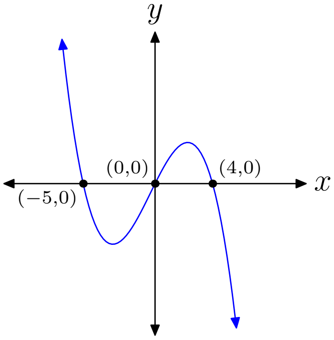
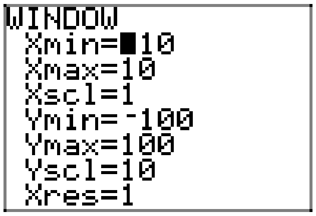


**43.**





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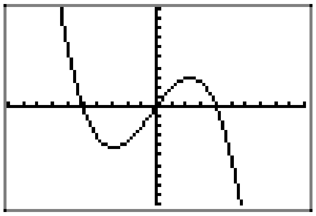
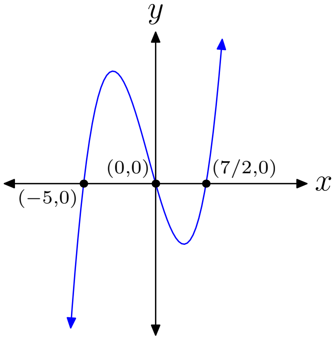
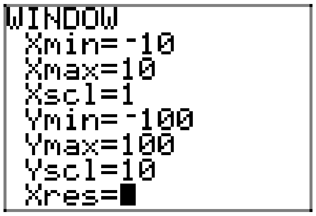


**45.**





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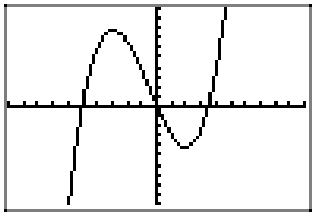

# 6.3 Extrema and Models

In the last section, we used end-behavior and zeros to sketch the graph of a given polynomial. We also mentioned that it takes a semester of calculus to learn an analytic technique used to calculate the "turning points" of the polynomial. That said, we'll still pursue the coordinates of the "turning points" in this section, but we will use the graphing calculator to assist us in this quest; and then we will use this technique with some applications.

## Extrema

Before we begin, we'd first like to differentiate between *local extrema* and *absolute extrema*.[11] This is best accomplished by means of an example. Consider, if you will, the graphs of three polynomial functions in **Figure 1**.

In the first figure, **Figure 1**(a), the point $A$ is the "absolute" lowest point on the graph. Therefore, the $y$-value of point $A$ is an *absolute minimum* value of the function.

In the second figure, **Figure 1**(b), there is no "absolute" highest point on the graph (the graph goes up to positive infinity), nor is there an "absolute" lowest point on the graph (the graph goes down to negative infinity). Therefore, this function has neither an absolute minimum nor an *absolute maximum*.

However, point $B$ in **Figure 1**(b) is the highest point in its immediate neighborhood. If you wander too far to the right, there are points on the graph higher than point $B$, but locally point $B$ is the highest point. Therefore, the $y$-value of point $B$ is called a *local maximum* value of the function.

Similarly, point $C$ in **Figure 1**(b) is the lowest point in its immediate neighborhood. If you wander too far to the left, there are points on the graph lower than point $C$, but, in its neighborhood, point $C$ is the lowest point. Therefore, the $y$-value of point $C$ is called a *local minimum* of the function.

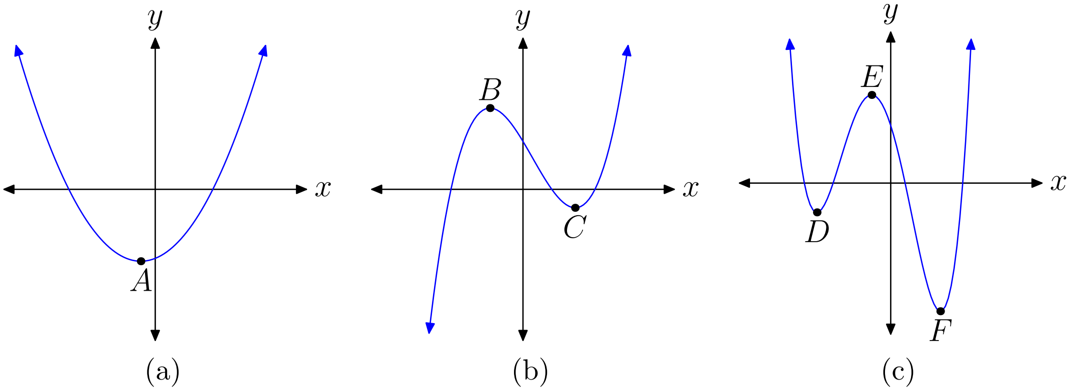

**Figure 1.** Differentiating between local and absolute extrema.

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[11] The term *extrema*, the plural of *extremum*, is a mathematical term that is used to refer to absolute or local maxima or minima of a function. *Note: Some mathematicians prefer the words* ***global*** *and* ***relative*** *to the words absolute and local. They are equivalent.*

Finally, take a look at the graph in **Figure 1**(c). Point $F$ is the "absolute" lowest point on the graph, so the $y$-value of point $F$ is an absolute minimum of the function. On the other hand, there is no highest point on the graph in **Figure 1**(c), as each end of the graph escapes to positive infinity. Hence, the function has no absolute maximum.

Locally, point $D$ in **Figure 1**(c) is the lowest point, so the $y$-value of point $D$ is a local minimum of the function. Similarly, in its immediate neighborhood, point $E$ is the highest point, so the $y$-value of point $E$ is a local maximum.

We now present the formal definitions.

**Definition 1.** *Suppose that $c$ is in the domain of a function $f$ and $f(c) \geq f(x)$ for all $x$ in the domain of $f$. Then we say that $f(c)$ is an absolute maximum of the function $f$. Similarly, if $f(c) \leq f(x)$ for all $x$ in the domain of $f$, then $f(c)$ is an absolute minimum of the function $f$.*

The definition of local extrema is less restrictive.

**Definition 2.** *Let $c$ be in the domain of $f$. If $f(c) \geq f(x)$ for all $x$ in a neighborhood containing $c$, then we say that $f(c)$ is a local maximum of the function $f$. On the other hand, if $f(c) \leq f(x)$ for all $x$ in a neighborhood containing $c$, then we say that $f(c)$ is a local minimum of the function $f$.*

When mathematicians say "a neighborhood containing $c$," they usually mean a small open interval $(a, b)$ that contains $c$.

Let's explore the use of the graphing calculator in finding extrema.

► **Example 3.** *Consider the polynomial function defined by the equation*

$$p(x) = 2(x-6)(x+2)(x+4). \tag{4}$$

*Use the graphing calculator to help find and classify all extrema of this function.*

First, how much of the graph can you draw without the use of a calculator? The linear factors of $p(x)$ are $x-6$, $x+2$, and $x+4$, so the zeros are 6, $-2$, and $-4$, respectively. So we now know where the graph of $p(x)$ crosses the $x$-axis.

To determine the end-behavior of $p(x)$, we need to determine the leading term. It is not necessary to completely expand the polynomial with the distributive property.[12] A little thought quickly reveals that if we were to do just that, the leading term in this case would be $2x^3$. Consequently, as we sweep our eyes from left to right, the end-behavior of the polynomial should match that of its leading term $2x^3$, rising from negative infinity, wiggling through its $x$-intercepts, then rising to positive infinity. The only choice is a graph similar to that in **Figure 2**.

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[12] However, the careful reader will quietly use the distributive property to expand **equation (4)** to see that this is so.

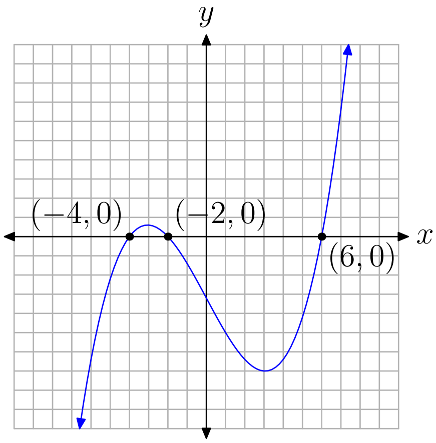
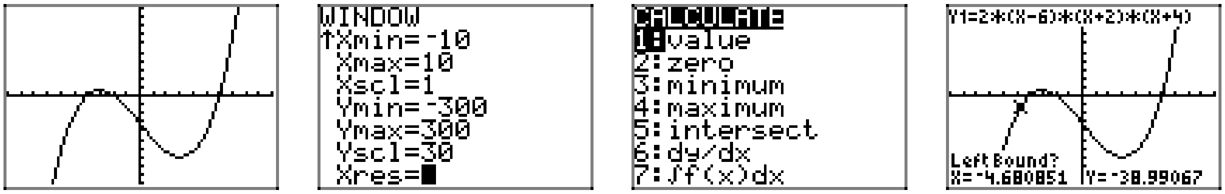


**Figure 2.** The graph of $p(x) = 2(x - 6)(x + 2)(x + 4)$ rises from negative infinity, wiggles through its $x$-intercepts, then continues to rise to positive infinity.

Note that the graph achieves a local maximum somewhere near $x = -3$ and a local minimum at approximately $x = 3$. We can find better approximations of the local extrema by using the `maximum` and `minimum` utilities in the `CALC` menu of the graphing calculator.

- First, plot the graph of the polynomial $p(x) = 2(x - 6)(x + 2)(x + 4)$, as shown in **Figure 3**(a), using the window parameters shown in **Figure 3**(b).
- Open the `CALCULATE` menu by pressing `2nd CALC`. This reveals a menu of choices as shown in **Figure 3**(c). To start the utility to help find the local maximum near $x = -3$ (see **Figure 2**), press `4:maximum` on the menu.
- The utility responds by asking for a "Left Bound." Use the arrow keys to move the cursor slightly left of the local maximum near $x = -3$, as shown in **Figure 3**(d), then press the `ENTER` key.



```
WINDOW
↑Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-300
 Ymax=300
 Yscl=30
 Xres=█
```

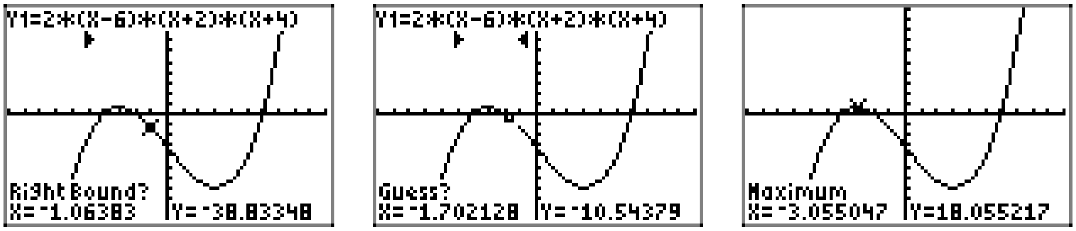
```
Y1=2*(X-6)*(X+2)*(X+4)
Left Bound?
X=-4.680851  Y=-38.99067
```

(a) (b) (c) (d)

**Figure 3.** Using the graphing calculator to determine the local maximum.

- The utility responds by asking for a "Right Bound." Use the arrow keys to move the cursor slightly to the right of the local maximum near $x = -3$, as shown in **Figure 4**(e), then press the `ENTER` key.
- The utility responds by asking for a "Guess." Move the cursor so that it lies between the "Left Bound" and "Right Bound" made earlier, as shown in **Figure 4**(f), then

press the `ENTER` key. Anywhere between the left bound and right bound (note marks at top of screen in **Figure 4**(f)) will do.



(e) (f) (g)

**Figure 4.** Using the graphing calculator to determine the local maximum (continued).

- The calculator responds by placing the cursor at the point where the local maximum occurs and reports its coordinates at the bottom of the screen, as shown in **Figure 4**(g).

The coordinates of the point where the local maximum occurs are approximately

$$(-3.055047, 18.055217).$$

We say that the function achieves a local maximum value of 18.055217 and that maximum occurs at $x \approx -3.055047$.

In a similar manner, one can use the `minimum` utility in the `CALC` menu to find the local minimum that occurs near $x = 3$, as shown in **Figure 5**.

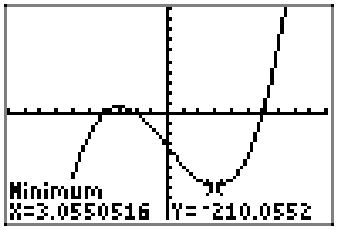


**Figure 5.** Use the `minimum` utility to find the local minimum.

The coordinates of the point where the local minimum occurs are approximately

$$(3.0550516, -210.0552).$$

We say that the function achieves a local minimum value of $-210.0552$ and that minimum occurs at $x \approx 3.0550516$.

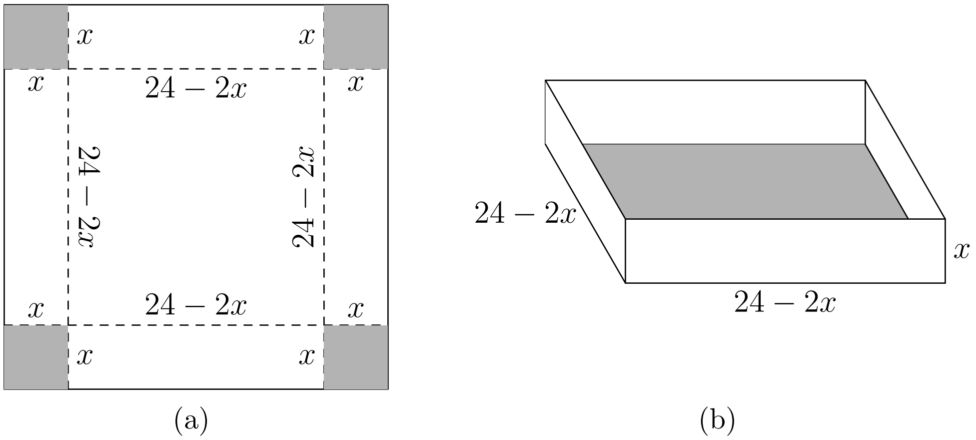
◆

## Applications

In this section we will look at some applications that are modeled by polynomials.

▶ **Example 5.** *A square piece of cardboard measures 24 inches per side. John cuts four smaller squares from each corner of the cardboard, tossing the material aside. He then bends up the sides of the remaining cardboard to form an open box with no top. Find the dimensions of the squares cut from each corner of the original piece of cardboard so that John maximizes the resulting volume of the box.*

Let $x$ represent the length of the side of the square cut from each corner of the larger square (see **Figure 6**(a)). Because each side of the original square measures 24 inches, and we're cutting two lengths of $x$ inches off each end, the resulting length and width of the box is $24 - 2x$ inches (see **Figure 6**(a) and/or (b)). When we toss away the square corners, then fold up the sides, we get a box with the dimensions shown in **Figure 6**(b).



(a) (b)

**Figure 6.** Material and resulting box.

Because the volume of a box is computed by taking the product of the length and width of the base, multiplied by the height of the box, the volume of the box is given by the formula

$$V = x(24 - 2x)(24 - 2x). \tag{6}$$

We can simplify **equation (6)** somewhat. Take a factor of 2 from each factor of $24-2x$, as in

$$V = x(2)(12 - x)(2)(12 - x),$$

then combine factors to write

$$V = 4x(12 - x)^2. \tag{7}$$

We see that $x$ and $12 - x$ are linear factors of $V$. Hence, the zeros of $V$ are 0 and 12, respectively. Because $12 - x$ is used as a factor twice, 2 is a "double root," so the graph should be tangent[13] to the $x$-axis at $x = 2$.

If we were to expand **equation (7)** completely, we would get a polynomial with leading term $4x^3$. Hence, the end-behavior of our volume polynomial should match the end-behavior of its leading term, rising from negative infinity, wiggling through it zeros, then rising to positive infinity. However, because we have a "double root" at $x = 2$, we expect the graph to "kiss" the horizontal axis at this zero rather than pass through this zero.

Thus, the only possible shape the volume polynomial can assume is that shown in **Figure 7**.

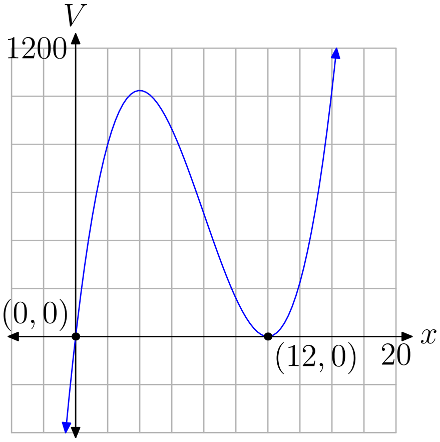


**Figure 7.** The volume of the box as a function of the edge length of the removed squares.

The domain of the polynomial defined by **equation (7)** is the set of all real numbers, or, in interval notation, $(-\infty, \infty)$. In **Figure 7**, if you project all the points on the graph onto the $x$-axis, the entire $x$-axis would be shaded, further indicating that the domain of the volume function is all real numbers.

However, this mathematical domain $(-\infty, \infty)$ ignores the fact that $x$ represents the length of the square cut from each corner of the original square of cardboard (see **Figure 6**(a)). You cannot cut a square having a side of negative length. Upon further inspection, the largest square that could be cut from each corner would have an edge measuring 12 inches. Remember, you have to cut four squares, one from each corner, and the edge of the original square piece of cardboard measures just 24 inches. Thus, the problem constrains $x$ to the interval $[0, 12]$. This domain is called the *empirical domain*, or, if you will, the *practical domain*.

---

[13] This is similar to the graph of $y = (x - 2)^2$. The graph is a parabola that "kisses" the $x$-axis at its vertex $(2, 0)$. This point of tangency is typical of all "double roots."

**Definition 8.** *The empirical domain of a function is a subset of the mathematical domain, restricted so as to satisfy the constraints of the model.*

Thus, only a portion of the graph in **Figure 7** makes sense for this application—the part that is drawn over the empirical domain $[0, 12]$, as shown in **Figure 8**.

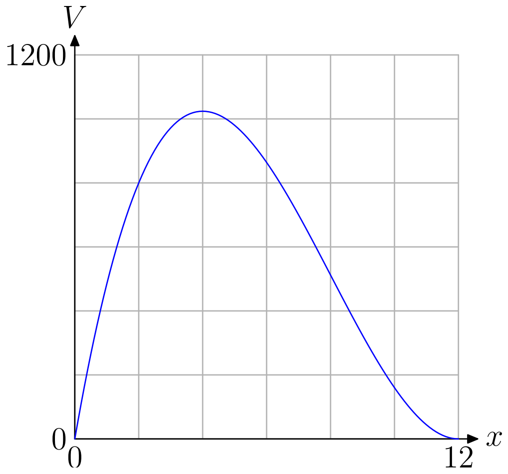
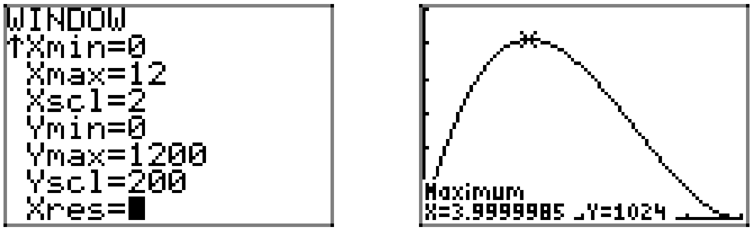


**Figure 8.** Sketching the volume function over the empirical domain $[0, 12]$.

Remember, the original goal was to find the value of $x$ that would maximize the volume of the box. A quick glance at the graph in **Figure 8** shows that there is an absolute maximum (at least on the empirical domain $[0, 12]$) near $x = 4$. To obtain a better approximation, use the `maximum` utility in the `CALC` menu on your calculator, as we did to obtain the approximation shown in **Figure 9**(b).



```
WINDOW
↑Xmin=0
 Xmax=12
 Xscl=2
 Ymin=0
 Ymax=1200
 Yscl=200
 Xres=█
```

```
Maximum
X=3.9999985 Y=1024
```

(a) (b)

**Figure 9.** Finding the length of the edge that maximizes the volume of the box.

Indeed, it would appear that a maximum volume of 1024 cubic inches ($V = 1024\,\text{in}^3$) is attained at $x \approx 3.9999985$. It's probably safe to say that the maximum volume occurs if squares having sides of length 4 inches are cut from the corners of the original piece of cardboard. The 3.9999985 probably contains a bit of error due to roundoff error on the calculator. Indeed, it is highly likely that some readers will get exactly $x = 4$ when they use the `maximum` utility, depending on the bounds and initial guess used, so don't be worried if your calculator approximation differs slightly from ours.

◆

Let's look at another example.

▶ **Example 9.** *Find the dimensions of the rectangle of largest area that has its base on the $x$-axis and its other two vertices above the $x$-axis and lying on the graph of the parabola $y = 4 - x^2$.*

The graph of $y = 4 - x^2$ is a parabola that opens downward and is shifted upward 4 units. The right side of this equation factors

$$y = (2 + x)(2 - x),$$

so the zeros of this function are $-2$ and 2. Because the rectangle has its base on the $x$-axis and its other vertices are on the parabola lying above the $x$-axis, we need only sketch the parabola on the domain $[-2, 2]$ (see **Figure 10**).

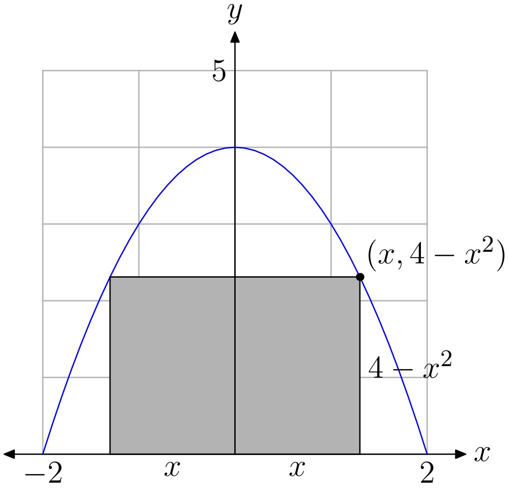


**Figure 10.** A rectangle inscribed under a parabola.

Because of symmetry, we can restrict $x$ to the empirical domain $[0, 2]$. In **Figure 10**, note that we've selected a value of $x$ from $[0, 2]$, then plotted the point having this $x$-value on the parabola. Of course, the $y$-value of this point is $y = 4 - x^2$. Thus, the height of the rectangle is $4 - x^2$ and the base (or width) of the rectangle is twice $x$, or $2x$. The area of the rectangle is given by

$$A = \text{width} \cdot \text{height}.$$

Hence, the area $A$ as a function of $x$ is given by the polynomial

$$A = 2x(4 - x^2). \tag{10}$$

Note that **equation (10)** is a third degree polynomial having leading term $-2x^3$. Thus, the graph of the polynomial, as we sweep our eyes from left to right, must fall from positive infinity, wiggle through its $x$-intercepts, then continue falling to negative infinity.

We can factor **equation (10)** to obtain

$$A = 2x(2 + x)(2 - x).$$

Therefore, the zeros of the polynomial are 0, −2, and 2, respectively. Thus, the polynomial must have shape similar to that shown in **Figure 11**. Note that the graph has $x$-intercepts at $(-2, 0)$, $(0, 0)$, and $(2, 0)$.

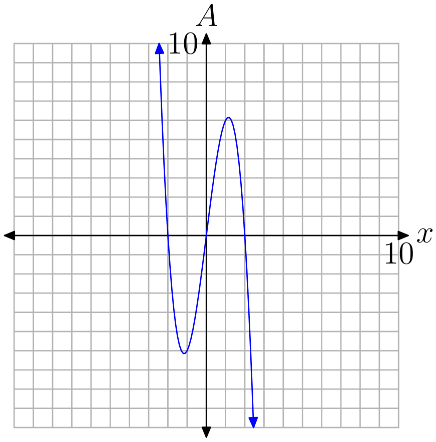


**Figure 11.** The graph of the polynomial $A = 2x(2 + x)(2 - x)$.

Because of the practical nature of this problem, we need to restrict $x$ to the empirical domain $[0, 2]$, as discussed above (see **Figure 10**). The graph of $A = 2x(2+x)(2-x)$, restricted to the domain $[0, 2]$, is shown in **Figure 12**.

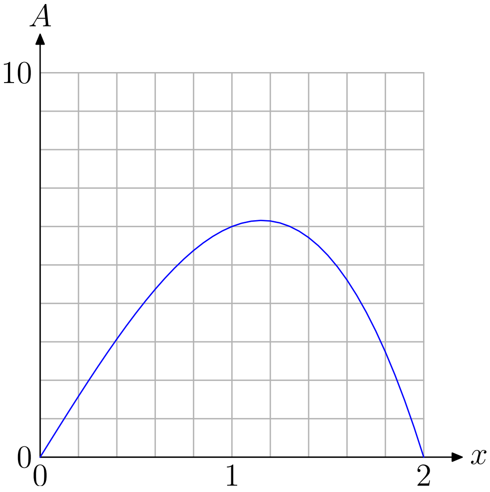
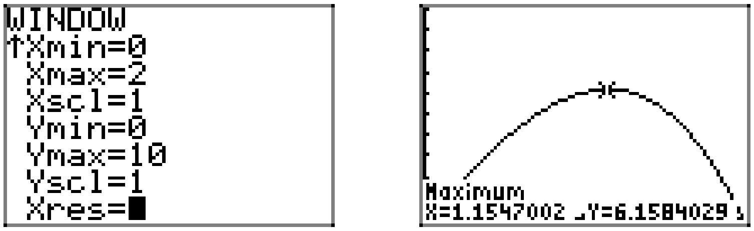


**Figure 12.** The graph of the polynomial $A = 2x(2 + x)(2 - x)$ restricted to the empirical domain $[0, 2]$.

It appears (see **Figure 12**) that $A$ achieves an absolute maximum (at least on the empirical domain $[0, 2]$) near $x \approx 1.2$. To obtain a better approximation, use the `maximum` utility in the `CALC` menu of the graphing calculator, as we did to obtain the approximation shown in **Figure 13**(b).



```
WINDOW
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 Xmax=2
 Xscl=1
 Ymin=0
 Ymax=10
 Yscl=1
 Xres=█
```

```
Maximum
X=1.1547002  Y=6.1584029
```

(a) (b)

**Figure 13.** Use the `maximum` utility to obtain an approximation of the maximum value of $A$.

The result in **Figure 13**(b) shows that we achieve a rectangle of maximum area $A \approx 6.1584029$ if we choose $x \approx 1.1547002$. Remember, your answers may differ slightly according to the left and right bounds you select, your guess, and also due to the inherent roundoff error in all calculators.

◆

## 6.3 Exercises

In **Exercises 1-8**, perform each of the following tasks for the given polynomial.

i. Without the aid of a calculator, use an algebraic technique to identify the zeros of the given polynomial. Factor if necessary.
ii. On graph paper, set up a coordinate system. Label each axis, but scale only the $x$-axis. Use the zeros and the end-behavior to draw a "rough graph" of the given polynomial without the aid of a calculator.
iii. Classify each local extrema as a *relative minimum* or *relative maximum*. *Note: It is not necessary to find the coordinates of the relative extrema. Indeed, this would be difficult without a calculator. All that is required is that you label each extrema as a relative maximum or minimum.*

**1.** $p(x) = (x+6)(x-1)(x-5)$

**2.** $p(x) = (x+2)(x-4)(x-7)$

**3.** $p(x) = x^3 - 6x^2 - 4x + 24$

**4.** $p(x) = x^3 + x^2 - 36x - 36$

**5.** $p(x) = 2x^3 + 5x^2 - 42x$

**6.** $p(x) = 2x^3 - 3x^2 - 44x$

**7.** $p(x) = -2x^3 + 4x^2 + 70x$

**8.** $p(x) = -6x^3 - 21x^2 + 90x$

In **Exercises 9-16**, perform each of the following tasks for the given polynomial.

i. Use a graphing calculator to draw the graph of the polynomial. Adjust the viewing window so that the extrema or "turning points" of the polynomial are visible in the viewing window. Copy the resulting image onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax.
ii. Use the `maximum` and/or `minimum` utility in your calculator's `CALC` menu to find the coordinates of the extrema. Label each extremum on your homework copy with its coordinates and state whether the extremum is a relative or absolute maximum or minimum.

**9.** $p(x) = x^3 - 8x^2 - 5x + 84$

**10.** $p(x) = x^3 + 3x^2 - 33x - 35$

**11.** $p(x) = -x^3 + 21x - 20$

**12.** $p(x) = -x^3 + 5x^2 + 12x - 36$

**13.** $p(x) = x^4 - 50x^2 + 49$

**14.** $p(x) = x^4 - 29x^2 + 100$

**15.** $p(x) = x^4 - 2x^3 - 39x^2 + 72x + 108$

**16.** $p(x) = x^4 - 3x^3 - 31x^2 + 63x + 90$

**17.** A square piece of cardboard measures 12 inches per side. Cherie cuts four smaller squares from each corner of the cardboard square, tossing the material aside. She then bends up the sides of the remaining cardboard to form an open box with no top. Find the dimensions of the squares cut from each corner of the original piece of cardboard so that

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[14] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

Cherie maximizes the volume of the resulting box. Perform each of the following steps in your analysis.

**a)** Set up an equation that determines the volume of the box as a function of $x$, the length of the edge of each square cut from the four corners of the cardboard. Include any pictures used to determine this volume function.

**b)** State the empirical domain of the function created in part (**a**). Use your calculator to sketch the graph of the function over this empirical domain. Adjust the viewing window so that all extrema are visible in the viewing window.

**c)** Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Use the maximum utility to find the coordinates of the absolute maximum on the function's empirical domain.

**d)** What are the measures of the four squares cut from each corner of the original cardboard? What is the maximum volume of the box?

**18.** A rectangular piece of cardboard measures 8 inches by 12 inches. Schuyler cuts four smaller squares from each corner of the cardboard square, tossing the material aside. He then bends up the sides of the remaining cardboard to form an open box with no top. Find the dimensions of the squares cut from each corner of the original piece of cardboard so that Schuyler maximizes the volume of the resulting box. Perform each of the following steps in your analysis.

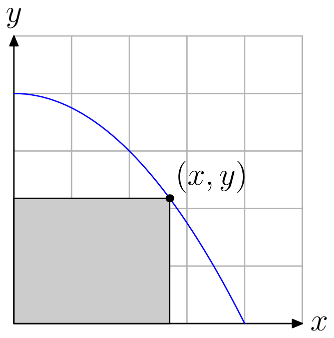
**a)** Set up an equation that determines the volume of the box as a function of $x$, the length of the edge of each square cut from the four corners of the cardboard. Include any pictures used to determine this volume function.

**b)** State the empirical domain of the function created in part (**a**). Use your calculator to sketch the graph of the function over this empirical domain. Adjust the viewing window so that all extrema are visible in the viewing window.

**c)** Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Use the maximum utility to find the coordinates of the absolute maximum on the function's empirical domain.

**d)** What are the measures of the four squares cut from each corner of the original cardboard? What is the maximum volume of the box?

**19.** Restrict the graph of the parabola $y = 4 - x^2/4$ to the first quadrant, then inscribe a rectangle inside the parabola, as shown in the figure that follows.



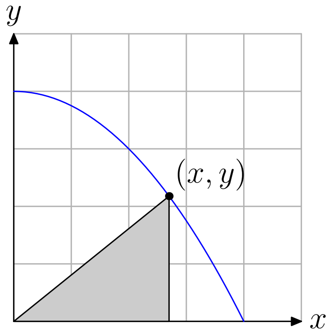
**a)** Express the area of the inscribed rectangle as a function of $x$.

**b)** State the empirical domain of the function defined in part (**a**). Use your calculator to graph the area function over its empirical domain. Adjust the window parameters so that all extrema are visible in the viewing window.

**c)** Copy the image in your viewing window to your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Use the `maximum` utility to find the coordinates of the absolute maximum on the function's empirical domain. Label your graph with this result.

**d)** What are the dimensions of the rectangle of maximum area?

**20.** Restrict the graph of the parabola $y = 4 - x^2/4$ to the first quadrant, then inscribe a triangle inside the parabola, as shown in the figure that follows.



**a)** Express the area of the inscribed triangle as a function of $x$.

**b)** State the empirical domain of the function defined in part (**a**). Use your calculator to graph the area function over its empirical domain. Adjust the window parameters so that all extrema are visible in the viewing window.

**c)** Copy the image in your viewing window to your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Use the `maximum` utility to find the coordinates of the absolute maximum on the function's empirical domain. Label your graph with this result.

**d)** What are the length of the base and height of the triangle of maximum area?

## 6.3 Answers

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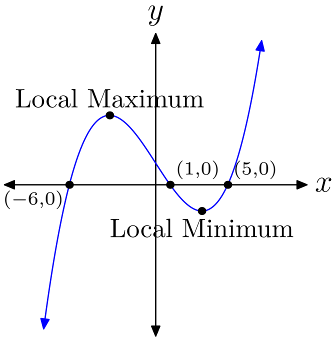
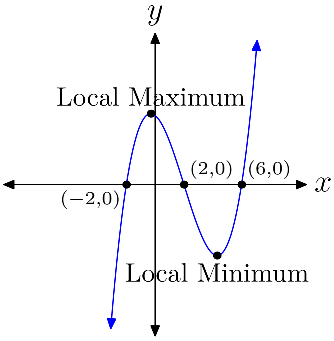
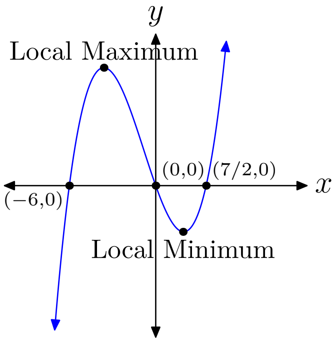
**1.**



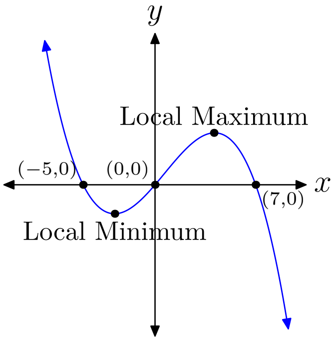
**3.**



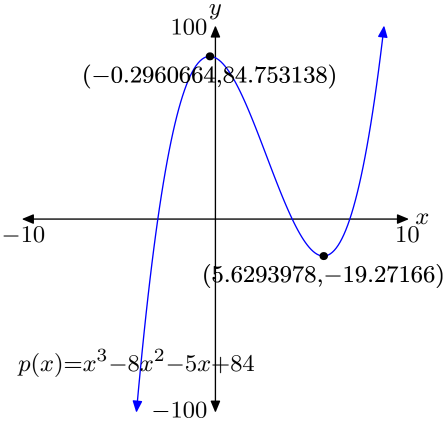
**5.**



**7.**

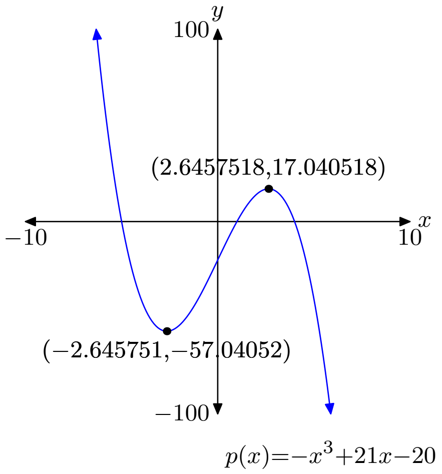


**9.** Relative max: $(-0.2960664, 84.753138)$
Relative min: $(5.6293978, -19.27166)$
Answers may differ slightly due to round-off error.

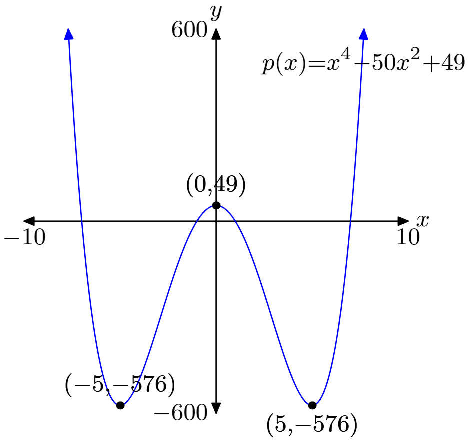


**11.** Relative min: $(-2.645751, -57.04052)$

Relative max: $(2.6457518, 17.040518)$
Answers may differ slightly due to round-off error.

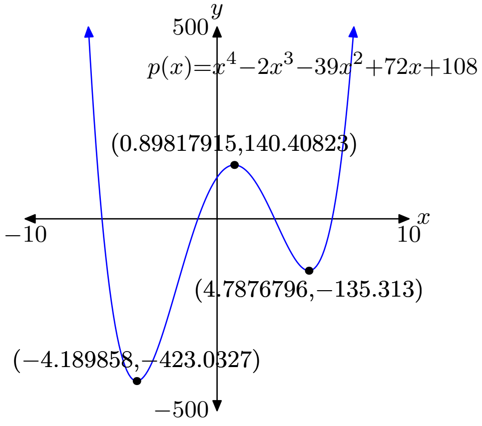


**13.** Absolute min: $(-5, -576)$
Relative max: $(0, 49)$
Absolute min: $(5, -576)$
Answers may differ slightly due to round-off error.



**15.** Absolute min: $(-4.189858, -423.0327)$

Relative max: $(0.89817915, 140.40823)$
Relative min: $(4.7876796, -135.313)$
Answers may differ slightly due to round-off error.

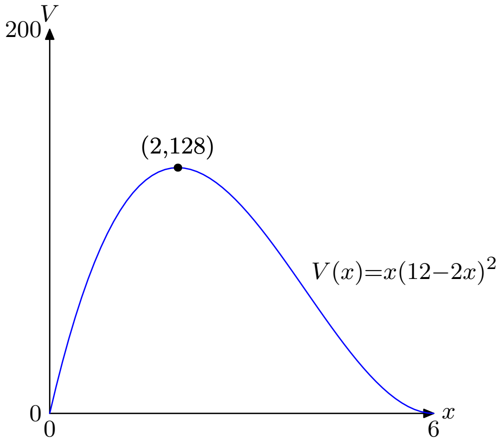


**17.**

**a)** $V = x(12-2x)^2$

**b)** $[0, 6]$

**c)** Absolute max: $(2, 128)$



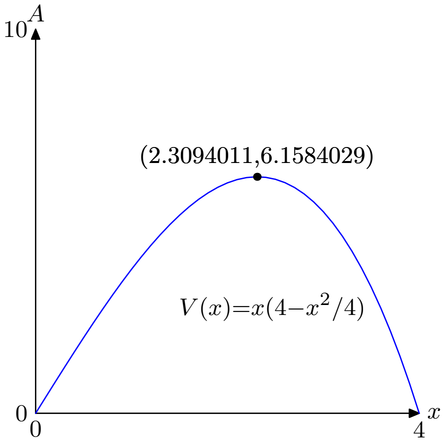
**d)** Cut square 2 inches on a side to produce a box having value 128 in$^3$.

**19.**

**a)** $A = x(4 - x^2/4)$

**b)** $[0, 4]$

**c)** Absolute max: $(2.3094011, 6.1584029)$



**d)** $x = 2.3094011$, $y = 2.6666666$

## 6.4 Index

# 7 Rational Functions

In this chapter, we begin our study of *rational functions* — functions of the form $p(x)/q(x)$, where $p$ and $q$ are both polynomials. Rational functions are similar in structure to rational numbers (commonly thought of as fractions), and they are studied and used extensively in mathematics, engineering, and science.

We will learn how to manipulate these functions, and discover the myriad algebraic tricks and pitfalls that accompany them. We will also see some of the ways that they can be applied to everyday situations, such as modeling the length of time it takes a group of people to complete a task, or calculating the distance traveled by an object.

In more advanced mathematics courses, such as college algebra and calculus, you will learn even more about the intricate nature of rational functions. In many science and engineering courses, you will use rational functions to model what you are studying. In your everyday life, you can use rational functions for a number of useful calculations, such as the amount of time or work that a given task might require. For these reasons, along with the fact that learning how to manipulate rational functions will further your understanding of mathematics, this chapter warrants a good deal of attention.

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## 7.1 Introducing Rational Functions

In the previous chapter, we studied polynomials, functions having equation form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n. \tag{1}$$

Even though this polynomial is presented in *ascending* powers of $x$, the leading term of the polynomial is still $a_nx^n$, the term with the highest power of $x$. The degree of the polynomial is the highest power of $x$ present, so in this case, the degree of the polynomial is $n$.

In this section, our study will lead us to the *rational* functions. Note the root word "ratio" in the term "rational." Does it remind you of the word "fraction"? It should, as rational functions are functions in a very specific fractional form.

**Definition 2.** *A rational function is a function that can be written as a quotient of two polynomial functions. In symbols, the function*

$$f(x) = \frac{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n}{b_0 + b_1x + b_2x^2 + \cdots + b_mx^m} \tag{3}$$

*is called a* rational function*.*

For example,

$$f(x) = \frac{1+x}{x+2}, \quad g(x) = \frac{x^2-2x-3}{x+4}, \quad \text{and} \quad h(x) = \frac{3-2x-x^2}{x^3+2x^2-3x-5} \tag{4}$$

are rational functions, while

$$f(x) = \frac{1+\sqrt{x}}{x^2+1}, \quad g(x) = \frac{x^2+2x-3}{1+x^{1/2}-3x^2}, \quad \text{and} \quad h(x) = \sqrt{\frac{x^2-2x-3}{x^2+4x-12}} \tag{5}$$

are **not** rational functions.

Each of the functions in **equation (4)** are rational functions, because in each case, the numerator and denominator of the given expression is a valid polynomial.

However, in **equation (5)**, the numerator of $f(x)$ is not a polynomial (polynomials do not allow the square root of the independent variable). Therefore, $f$ is not a rational function.

Similarly, the denominator of $g(x)$ in **equation (5)** is not a polynomial. Fractions are not allowed as exponents in polynomials. Thus, $g$ is not a rational function.

Finally, in the case of function $h$ in **equation (5)**, although the radicand (the expression inside the radical) is a rational function, the square root prevents $h$ from being a rational function.

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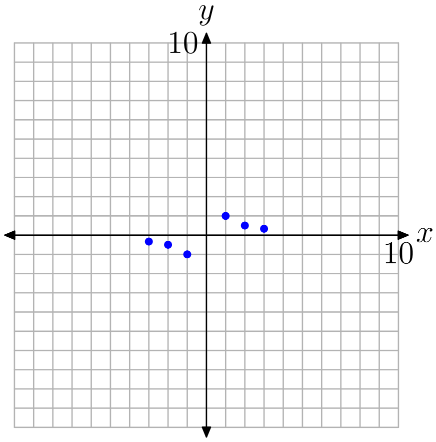
An important skill to develop is the ability to draw the graph of a rational function. Let's begin by drawing the graph of one of the simplest (but most fundamental) rational functions.

## *The Graph of $y = 1/x$*

In all new situations, when we are presented with an equation whose graph we've not considered or do not recognize, we begin the process of drawing the graph by creating a table of points that satisfy the equation. It's important to remember that the graph of an equation is the set of all points that satisfy the equation. We note that zero is not in the domain of $y = 1/x$ (division by zero makes no sense and is not defined), and create a table of points satisfying the equation shown in **Figure 1**.



| $x$ | $y = 1/x$ |
|---|---|
| $-3$ | $-1/3$ |
| $-2$ | $-1/2$ |
| $-1$ | $-1$ |
| 1 | 1 |
| 2 | $1/2$ |
| 3 | $1/3$ |

**Figure 1.** At the right is a table of points satisfying the equation $y = 1/x$. These points are plotted as solid dots on the graph at the left.

At this point (see **Figure 1**), it's pretty clear what the graph is doing between $x = -3$ and $x = -1$. Likewise, it's clear what is happening between $x = 1$ and $x = 3$. However, there are some open areas of concern.

1. What happens to the graph as $x$ increases without bound? That is, what happens to the graph as $x$ moves toward $\infty$?
2. What happens to the graph as $x$ decreases without bound? That is, what happens to the graph as $x$ moves toward $-\infty$?
3. What happens to the graph as $x$ approaches zero from the right?
4. What happens to the graph as $x$ approaches zero from the left?

Let's answer each of these questions in turn. We'll begin by discussing the "end-behavior" of the rational function defined by $y = 1/x$. First, the right end. What happens as $x$ increases without bound? That is, what happens as $x$ increases toward $\infty$? In **Table 1**(a), we computed $y = 1/x$ for $x$ equalling 100, 1 000, and 10 000. Note how the $y$-values in **Table 1**(a) are all positive and approach zero.

Students in calculus use the following notation for this idea.

$$\lim_{x \to \infty} y = \lim_{x \to \infty} \frac{1}{x} = 0 \qquad (6)$$

They say "the limit of $y$ as $x$ approaches infinity is zero." That is, as $x$ approaches infinity, $y$ approaches zero.

| $x$ | $y = 1/x$ |
|---|---|
| 100 | 0.01 |
| 1 000 | 0.001 |
| 10 000 | 0.0001 |

(a)

| $x$ | $y = 1/x$ |
|---|---|
| −100 | −0.01 |
| −1 000 | −0.001 |
| −10 000 | −0.0001 |

(b)

**Table 1.** Examining the end-behavior of $y = 1/x$.

A completely similar event happens at the left end. As $x$ decreases without bound, that is, as $x$ decreases toward $-\infty$, note that the $y$-values in **Table 1**(b) are all negative and approach zero. Calculus students have a similar notation for this idea.

$$\lim_{x\to-\infty} y = \lim_{x\to-\infty} \frac{1}{x} = 0. \tag{7}$$

They say "the limit of $y$ as $x$ approaches negative infinity is zero." That is, as $x$ approaches negative infinity, $y$ approaches zero.

These numbers in **Tables 1**(a) and **1**(b), and the ideas described above, predict the correct end-behavior of the graph of $y = 1/x$. At each end of the $x$-axis, the $y$-values must approach zero. This means that the graph of $y = 1/x$ must approach the $x$-axis for $x$-values at the far right- and left-ends of the graph. In this case, we say that the $x$-axis acts as a *horizontal asymptote* for the graph of $y = 1/x$. As $x$ approaches either positive or negative infinity, the graph of $y = 1/x$ approaches the $x$-axis. This behavior is shown in **Figure 2**.

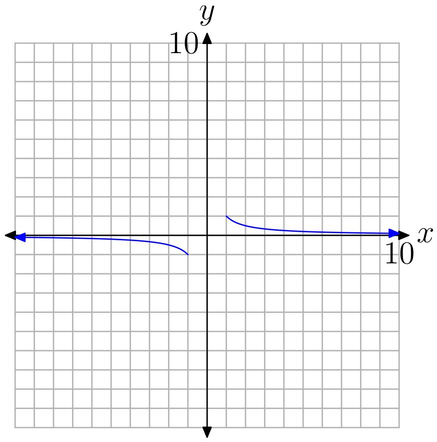
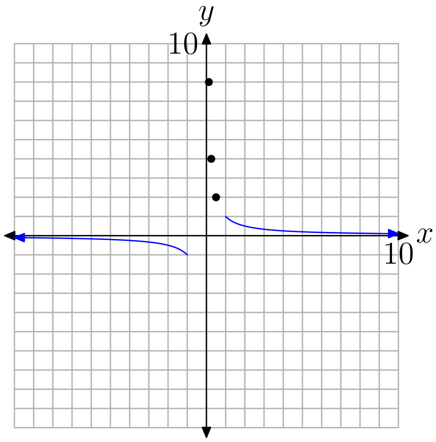


**Figure 2.** The graph of $1/x$ approaches the $x$-axis as $x$ increases or decreases without bound.

Our last investigation will be on the interval from $x = -1$ to $x = 1$. Readers are again reminded that the function $y = 1/x$ is undefined at $x = 0$. Consequently, we will break this region in half, first investigating what happens on the region between $x = 0$

and $x = 1$. We evaluate $y = 1/x$ at $x = 1/2$, $x = 1/4$, and $x = 1/8$, as shown in the table in **Figure 3**, then plot the resulting points.



| $x$ | $y = 1/x$ |
|---|---|
| $1/2$ | $2$ |
| $1/4$ | $4$ |
| $1/8$ | $8$ |

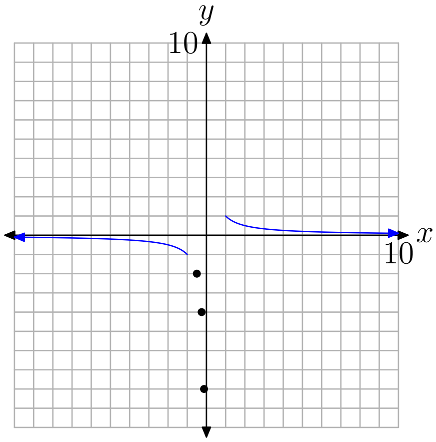
**Figure 3.** At the right is a table of points satisfying the equation $y = 1/x$. These points are plotted as solid dots on the graph at the left.

Note that the $x$-values in the table in **Figure 3** approach zero from the right, then note that the corresponding $y$-values are getting larger and larger. We could continue in this vein, adding points. For example, if $x = 1/16$, then $y = 16$. If $x = 1/32$, then $y = 32$. If $x = 1/64$, then $y = 64$. Each time we halve our value of $x$, the resulting value of $x$ is closer to zero, and the corresponding $y$-value doubles in size. Calculus students describe this behavior with the notation

$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} \frac{1}{x} = \infty. \tag{8}$$

That is, as "$x$ approaches zero from the right, the value of $y$ grows to infinity." This is evident in the graph in **Figure 3**, where we see the plotted points move closer to the vertical axis while at the same time moving upward without bound.

A similar thing happens on the other side of the vertical axis, as shown in **Figure 4**.



| $x$ | $y = 1/x$ |
|---|---|
| $-1/2$ | $-2$ |
| $-1/4$ | $-4$ |
| $-1/8$ | $-8$ |

**Figure 4.** At the right is a table of points satisfying the equation $y = 1/x$. These points are plotted as solid dots on the graph at the left.

Again, calculus students would write

$$\lim_{x\to 0^-} y = \lim_{x\to 0^-} \frac{1}{x} = -\infty. \tag{9}$$

That is, "as $x$ approaches zero from the left, the values of $y$ decrease to negative infinity." In **Figure 4**, it is clear that as points move closer to the vertical axis (as $x$ approaches zero) from the left, the graph decreases without bound.

The evidence gathered to this point indicates that the vertical axis is acting as a *vertical asymptote*. As $x$ approaches zero from either side, the graph approaches the vertical axis, either rising to infinity, or falling to negative infinity. The graph cannot cross the vertical axis because the function is undefined there. The completed graph is shown in **Figure 5**.

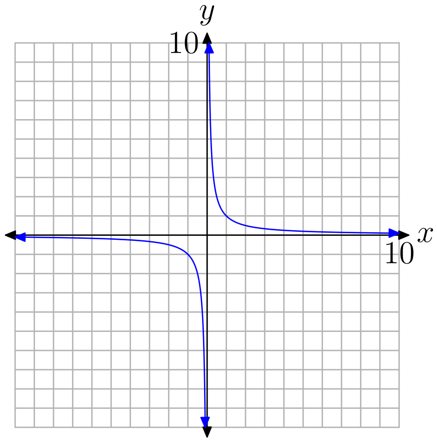


**Figure 5.** The completed graph of $y = 1/x$. Note how the $x$-axis acts as a horizontal asymptote, while the $y$-axis acts as a vertical asymptote.

The complete graph of $y = 1/x$ in **Figure 5** is called a *hyperbola* and serves as a fundamental starting point for all subsequent discussion in this section.

We noted earlier that the domain of the function defined by the equation $y = 1/x$ is the set $D = \{x : x \neq 0\}$. Zero is excluded from the domain because division by zero is undefined. It's no coincidence that the graph has a vertical asymptote at $x = 0$. We'll see this relationship reinforced in further examples.

## Translations

In this section, we will translate the graph of $y = 1/x$ in both the horizontal and vertical directions.

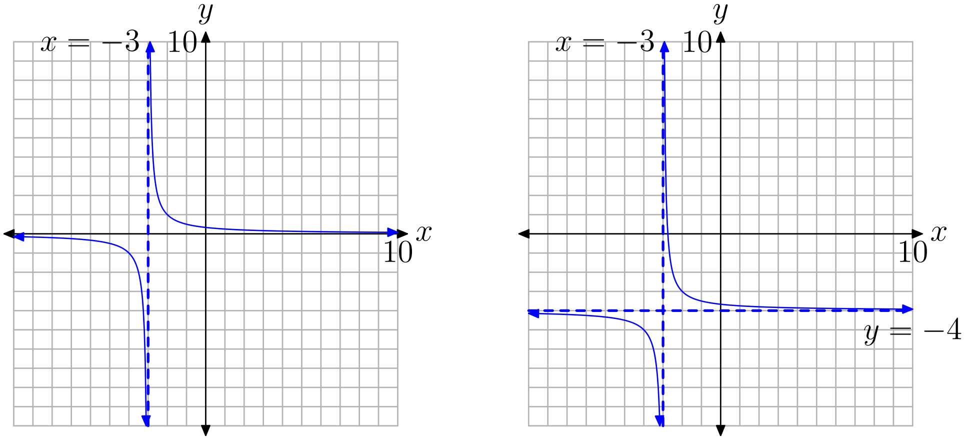
▶ **Example 10.** *Sketch the graph of*

$$y = \frac{1}{x+3} - 4. \tag{11}$$

Technically, the function defined by $y = 1/(x + 3) - 4$ does not have the general form (**3**) of a rational function. However, in later chapters we will show how $y = 1/(x + 3) - 4$ can be manipulated into the general form of a rational function.

We know what the graph of $y = 1/x$ looks like. If we replace $x$ with $x + 3$, this will shift the graph of $y = 1/x$ three units to the left, as shown in **Figure 6**(a). Note that the vertical asymptote has also shifted 3 units to the left of its original position (the $y$-axis) and now has equation $x = -3$. By tradition, we draw the vertical asymptote as a dashed line.

If we subtract 4 from the result in **Figure 6**(a), this will shift the graph in **Figure 6**(a) four units downward to produce the graph shown in **Figure 6**(b). Note that the horizontal asymptote also shifted 4 units downward from its original position (the $x$-axis) and now has equation $y = -4$.



(a) $y = 1/(x + 3)$ (b) $y = 1/(x + 3) - 4$

**Figure 6.** Shifting the graph of $y = 1/x$.

If you examine **equation (11)**, you note that you cannot use $x = -3$ as this will make the denominator of **equation (11)** equal to zero. In **Figure 6**(b), note that there is a vertical asymptote in the graph of **equation (11)** at $x = -3$. This is a common occurrence, which will be a central theme of this chapter.

◇

Let's ask another key question.

▶ **Example 12.** *What are the domain and range of the rational function presented in* **Example 10***?*

You can glance at the equation

$$y = \frac{1}{x+3} - 4$$

of **Example 10** and note that $x = -3$ makes the denominator zero and must be excluded from the domain. Hence, the domain of this function is $D = \{x : x \neq -3\}$.

However, you can also determine the domain by examining the graph of the function in **Figure 6**(b). Note that the graph extends indefinitely to the left and right. One might first guess that the domain is all real numbers if it were not for the vertical asymptote at $x = -3$ interrupting the continuity of the graph. Because the graph of the function gets arbitrarily close to this vertical asymptote (on either side) without actually touching the asymptote, the graph does not contain a point having an $x$-value equaling $-3$. Hence, the domain is as above, $D = \{x : x \neq -3\}$. This is comforting that the graphical analysis agrees with our earlier analytical determination of the domain.

The graph is especially helpful in determining the range of the function. Note that the graph rises to positive infinity and falls to negative infinity. One would first guess that the range is all real numbers if it were not for the horizontal asymptote at $y = -4$ interrupting the continuity of the graph. Because the graph gets arbitrarily close to the horizontal asymptote (on either side) without actually touching the asymptote, the graph does not contain a point having a $y$-value equaling $-4$. Hence, $-4$ is excluded from the range. That is, $R = \{y : y \neq -4\}$.

◇

## *Scaling and Reflection*

In this section, we will both scale and reflect the graph of $y = 1/x$. For extra measure, we also throw in translations in the horizontal and vertical directions.

▶ **Example 13.** *Sketch the graph of*

$$y = -\frac{2}{x-4} + 3. \tag{14}$$

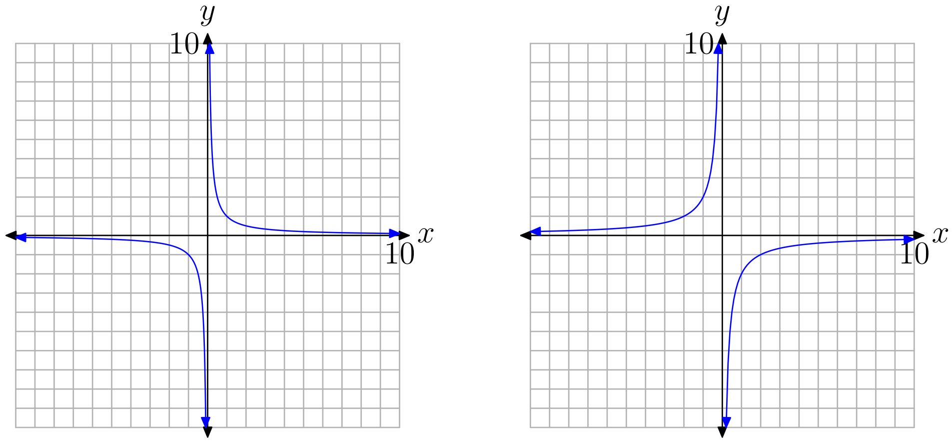
First, we multiply the equation $y = 1/x$ by $-2$ to get

$$y = -\frac{2}{x}.$$

Multiplying by 2 should stretch the graph in the vertical directions (both positive and negative) by a factor of 2. Note that points that are very near the $x$-axis, when doubled, are not going to stray too far from the $x$-axis, so the horizontal asymptote will remain the same. Finally, multiplying by $-2$ will not only stretch the graph, it will also reflect the graph across the $x$-axis, as shown in **Figure 7**(b).[2]

---

[2] Recall that we saw similar behavior when studying the parabola. The graph of $y = -2x^2$ stretched (vertically) the graph of the equation $y = x^2$ by a factor of 2, then reflected the result across the $x$-axis.



(a) $y = 1/x$ (b) $y = -2/x$

**Figure 7.** Scaling and reflecting the graph of $y = 1/x$.

Replacing $x$ with $x - 4$ will shift the graph 4 units to the right, then adding 3 will shift the graph 3 units up, as shown in **Figure 8**. Note again that $x = 4$ makes the denominator of $y = -2/(x - 4) + 3$ equal to zero and there is a vertical asymptote at $x = 4$. The domain of this function is $D = \{x : x \neq 4\}$.

As $x$ approaches positive or negative infinity, points on the graph of $y = -2/(x-4)+3$ get arbitrarily close to the horizontal asymptote $y = 3$ but never touch it. Therefore, there is no point on the graph that has a $y$-value of 3. Thus, the range of the function is the set $R = \{y : y \neq 3\}$.

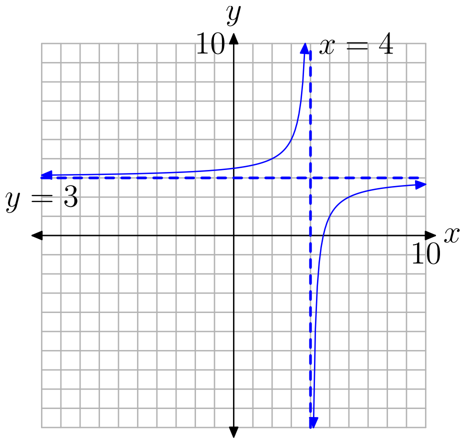


**Figure 8.** The graph of $y = -2/(x - 4) + 3$ is shifted 4 units right and 3 units up.

◆

## *Difficulties with the Graphing Calculator*

The graphing calculator does a very good job drawing the graphs of "continuous functions."

> A continuous function is one that can be drawn in one continuous stroke, never lifting pen or pencil from the paper during the drawing.

Polynomials, such as the one in **Figure 9**, are continuous functions.

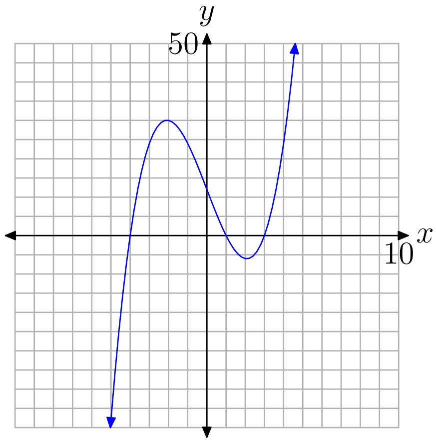


**Figure 9.** A polynomial is a continuous function.

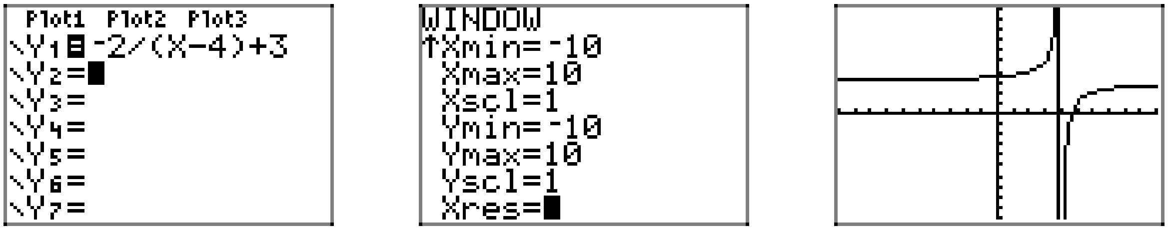
Unfortunately, a rational function with vertical asymptote(s) is not a continuous function. First, you have to lift your pen at points where the denominator is zero, because the function is undefined at these points. Secondly, it's not uncommon to have to jump from positive infinity to negative infinity (or vice-versa) when crossing a vertical asymptote. When this happens, we have to lift our pen and shift it before continuing with our drawing.

However, the graphing calculator does not know how to do this "lifting" of the pen near vertical asymptotes. The graphing calculator only knows one technique, plot a point, then connect it with a segment to the last point plotted, move an incremental distance and repeat. Consequently, when the graphing calculator crosses a vertical asymptote where there is a shift from one type of infinity to another (e.g., from positive to negative), the calculator draws a "false line" of connection, one that it should not draw. Let's demonstrate this aberration with an example.

▶ **Example 15.** *Use a graphing calculator to draw the graph of the rational function in* **Example 13**.

Load the equation into your calculator, as shown in **Figure 10**(a). Set the window as shown in **Figure 10**(b), then push the `GRAPH` button to draw the graph shown in **Figure 10**(c). Results may differ on some calculators, but in our case, note the "false line" drawn from the top of the screen to the bottom, attempting to "connect" the two branches of the hyperbola.

Some might rejoice and claim, "Hey, my graphing calculator draws vertical asymptotes." However, before you get too excited, note that in **Figure 8** the vertical asymptote should occur at *exactly* $x = 4$. If you look very carefully at the "vertical line" in **Figure 10**(c), you'll note that it just misses the tick mark at $x = 4$. This "vertical line" is a line that the calculator should not draw. The calculator is attempting to draw a continuous function where one doesn't exist.



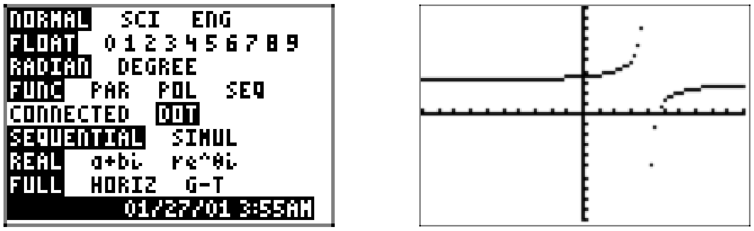
```
Plot1 Plot2 Plot3
\Y1=-2/(X-4)+3
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
WINDOW
↑Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=
```

(a) (b) (c)

**Figure 10.** The calculator attempts to draw a continuous function when it shouldn't.

One possible workaround[3] is to press the `MODE` button on your keyboard, which opens the menu shown in **Figure 11**(a). Use the arrow keys to highlight `DOT` instead of `CONNECTED` and press the `ENTER` key to make the selection permanent. Press the `GRAPH` button to draw the graph in **Figure 11**(b).



```
NORMAL  SCI  ENG
FLOAT  0123456789
RADIAN  DEGREE
FUNC  PAR  POL  SEQ
CONNECTED  DOT
SEQUENTIAL  SIMUL
REAL  a+bi  re^θi
FULL  HORIZ  G-T
          01/27/01 3:55AM
```

(a) (b)

**Figure 11.** The same graph in "dot mode."

This "dot mode" on your calculator calculates the next point on the graph and plots the point, but it does not connect it with a line segment to the previously plotted point. This mode is useful in demonstrating that the vertical line in **Figure 10**(c) is not really part of the graph, but we lose some parts of the graph we'd really like to see. Compromise is in order.

This example clearly shows that intelligent use of the calculator is a required component of this course. The calculator is not simply a "black box" that automatically does what you want it to do. In particular, when you are drawing rational functions, it helps to know ahead of time the placement of the vertical asymptotes. Knowledge

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[3] Instructors might discuss a number of alternative strategies to represent rational functions on the graphing calculator. What we present here is only one of a number of approaches.

of the asymptotes, coupled with what you see on your calculator screen, should enable you to draw a graph as accurate as that shown in **Figure 8**.

◆

**Gentle reminder**. You'll want to set your calculator back in "connected mode." To do this, press the `MODE` button on your keyboard to open the menu in **Figure 10**(a) once again. Use your arrow keys to highlight `CONNECTED`, then press the `ENTER` key to make the selection permanent.

## 7.1 Exercises

In **Exercises 1-14**, perform each of the following tasks for the given rational function.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.
ii. Use geometric transformations as in Examples 10, 12, and 13 to draw the graphs of each of the following rational functions. Draw the vertical and horizontal asymptotes as dashed lines and label each with its equation. You may use your calculator to **check** your solution, but you should be able to draw the rational function without the use of a calculator.
iii. Use set-builder notation to describe the domain and range of the given rational function.

**1.** $f(x) = -2/x$

**2.** $f(x) = 3/x$

**3.** $f(x) = 1/(x-4)$

**4.** $f(x) = 1/(x+3)$

**5.** $f(x) = 2/(x-5)$

**6.** $f(x) = -3/(x+6)$

**7.** $f(x) = 1/x - 2$

**8.** $f(x) = -1/x + 4$

**9.** $f(x) = -2/x - 5$

**10.** $f(x) = 3/x - 5$

**11.** $f(x) = 1/(x-2) - 3$

**12.** $f(x) = -1/(x+1) + 5$

**13.** $f(x) = -2/(x-3) - 4$

**14.** $f(x) = 3/(x+5) - 2$

In **Exercises 15-22**, find all vertical asymptotes, if any, of the graph of the given function.

**15.** $f(x) = -\dfrac{5}{x+1} - 3$

**16.** $f(x) = \dfrac{6}{x+8} + 2$

**17.** $f(x) = -\dfrac{9}{x+2} - 6$

**18.** $f(x) = -\dfrac{8}{x-4} - 5$

**19.** $f(x) = \dfrac{2}{x+5} + 1$

**20.** $f(x) = -\dfrac{3}{x+9} + 2$

**21.** $f(x) = \dfrac{7}{x+8} - 9$

**22.** $f(x) = \dfrac{6}{x-5} - 8$

In **Exercises 23-30**, find all horizontal asymptotes, if any, of the graph of the given function.

**23.** $f(x) = \dfrac{5}{x+7} + 9$

**24.** $f(x) = -\dfrac{8}{x+7} - 4$

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[4] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

**25.** $f(x) = \dfrac{8}{x+5} - 1$

**26.** $f(x) = -\dfrac{2}{x+3} + 8$

**27.** $f(x) = \dfrac{7}{x+1} - 9$

**28.** $f(x) = -\dfrac{2}{x-1} + 5$

**29.** $f(x) = \dfrac{5}{x+2} - 4$

**30.** $f(x) = -\dfrac{6}{x-1} - 2$

In **Exercises 31-38**, state the domain of the given rational function using set-builder notation.

**31.** $f(x) = \dfrac{4}{x+5} + 5$

**32.** $f(x) = -\dfrac{7}{x-6} + 1$

**33.** $f(x) = \dfrac{6}{x-5} + 1$

**34.** $f(x) = -\dfrac{5}{x-3} - 9$

**35.** $f(x) = \dfrac{1}{x+7} + 2$

**36.** $f(x) = -\dfrac{2}{x-5} + 4$

**37.** $f(x) = -\dfrac{4}{x+2} + 2$

**38.** $f(x) = \dfrac{2}{x+6} + 9$

In **Exercises 39-46**, find the range of the given function, and express your answer in set notation.

**39.** $f(x) = \dfrac{2}{x-3} + 8$

**40.** $f(x) = \dfrac{4}{x-3} + 5$

**41.** $f(x) = -\dfrac{5}{x-8} - 5$

**42.** $f(x) = -\dfrac{2}{x+1} + 6$

**43.** $f(x) = \dfrac{7}{x+7} + 5$

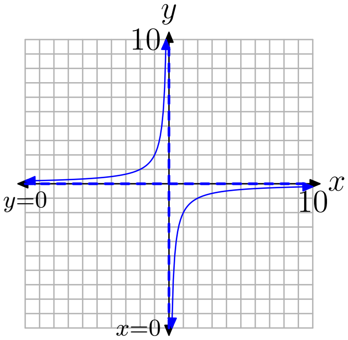
**44.** $f(x) = -\dfrac{8}{x+3} + 9$

**45.** $f(x) = \dfrac{4}{x+3} - 2$

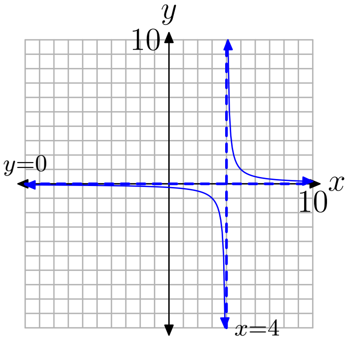
**46.** $f(x) = -\dfrac{5}{x-4} + 9$

## 7.1 Answers

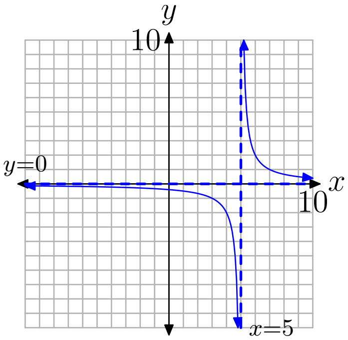
**1.** $D = \{x : x \neq 0\}$, $R = \{y : y \neq 0\}$



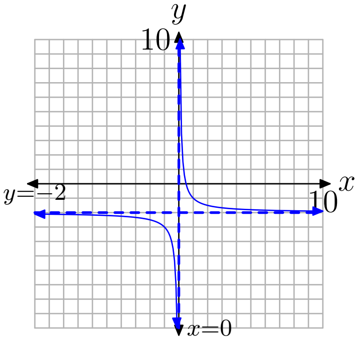
**3.** $D = \{x : x \neq 4\}$, $R = \{y : y \neq 0\}$



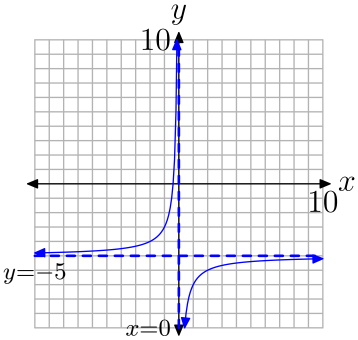
**5.** $D = \{x : x \neq 5\}$, $R = \{y : y \neq 0\}$



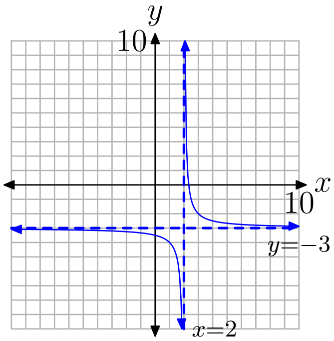
**7.** $D = \{x : x \neq 0\}$, $R = \{y : y \neq -2\}$



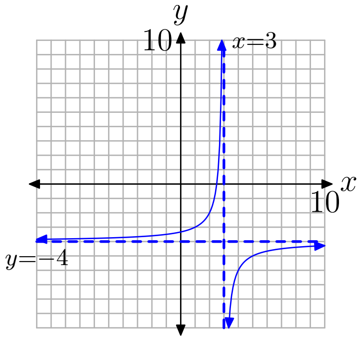
**9.** $D = \{x : x \neq 0\}$, $R = \{y : y \neq -5\}$



**11.** $D = \{x : x \neq 2\}$, $R = \{y : y \neq -3\}$



**13.** $D = \{x : x \neq 3\}$, $R = \{y : y \neq -4\}$



**15.** Vertical asymptote: $x = -1$

**17.** Vertical asymptote: $x = -2$

**19.** Vertical asymptote: $x = -5$

**21.** Vertical asymptote: $x = -8$

**23.** Horizontal asymptote: $y = 9$

**25.** Horizontal asymptote: $y = -1$

**27.** Horizontal asymptote: $y = -9$

**29.** Horizontal asymptote: $y = -4$

**31.** Domain $= \{x : x \neq -5\}$

**33.** Domain $= \{x : x \neq 5\}$

**35.** Domain $= \{x : x \neq -7\}$

**37.** Domain $= \{x : x \neq -2\}$

**39.** Range $= \{y : y \neq 8\}$

**41.** Range $= \{y : y \neq -5\}$

**43.** Range $= \{y : y \neq 5\}$

**45.** Range $= \{y : y \neq -2\}$

# 7.2 Reducing Rational Functions

The goal of this section is to learn how to reduce a rational expression to "lowest terms." Of course, that means that we will have to understand what is meant by the phrase "lowest terms." With that thought in mind, we begin with a discussion of the *greatest common divisor* of a pair of integers.

First, we define what we mean by "divisibility."

**Definition 1.** *Suppose that we have a pair of integers $a$ and $b$. We say that "$a$ is a divisor of $b$," or "$a$ divides $b$" if and only if there is another integer $k$ so that $b = ak$. Another way of saying the same thing is to say that $a$ divides $b$ if, upon dividing $b$ by $a$, the remainder is zero.*

Let's look at an example.

► **Example 2.** *What are the divisors of 12?*

Because $12 = 1 \times 12$, both 1 and 12 are divisors[6] of 12. Because $12 = 2 \times 6$, both 2 and 6 are divisors of 12. Finally, because $12 = 3 \times 4$, both 3 and 4 are divisors of 12. If we list them in ascending order, the divisors of 12 are

$$1,\ 2,\ 3,\ 4,\ 6,\ \text{and } 12.$$

Let's look at another example.

► **Example 3.** *What are the divisors of 18?*

Because $18 = 1 \times 18$, both 1 and 18 are divisors of 18. Similarly, $18 = 2 \times 9$ and $18 = 3 \times 6$, so in ascending order, the divisors of 18 are

$$1,\ 2,\ 3,\ 6,\ 9,\ \text{and } 18.$$

The *greatest common divisor* of two or more integers is the largest divisor the integers share in common. An example should make this clear.

► **Example 4.** *What is the greatest common divisor of 12 and 18?*

In **Example 2** and **Example 3**, we saw the following.

$$\begin{array}{l}
\text{Divisors of } 12 : \boxed{1},\ \boxed{2},\ \boxed{3},\ 4,\ \boxed{6},\ 12 \\
\text{Divisors of } 18 : \boxed{1},\ \boxed{2},\ \boxed{3},\ \boxed{6},\ 9,\ 18
\end{array}$$

---

[5] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

[6] The word "divisor" and the word "factor" are synonymous.

We've framed the divisors that 12 and 18 have in common. They are 1, 2, 3, and 6. The "greatest" of these "common" divisors is 6. Hence, we say that "the greatest common divisor of 12 and 18 is 6."

◆

**Definition 5.** *The greatest common divisor of two integers $a$ and $b$ is the largest divisor they have in common. We will use the notation*

$$GCD(a, b)$$

*to represent the greatest common divisor of $a$ and $b$.*

Thus, as we saw in **Example 4**, $\text{GCD}(12, 18) = 6$.

When the greatest common divisor of a pair of integers is one, we give that pair a special name.

**Definition 6.** *Let $a$ and $b$ be integers. If the greatest common divisor of $a$ and $b$ is one, that is, if $GCD(a, b) = 1$, then we say that $a$ and $b$ are* **relatively prime**.

For example:

- 9 and 12 are **not** relatively prime because $\text{GCD}(9, 12) = 3$.
- 10 and 15 are **not** relatively prime because $\text{GCD}(10, 15) = 5$.
- 8 and 21 **are** relatively prime because $\text{GCD}(8, 21) = 1$.

We can now define what is meant when we say that a rational number is reduced to lowest terms.

**Definition 7.** *A rational number in the form $p/q$, where $p$ and $q$ are integers, is said to be reduced to lowest terms if and only if $GCD(p, q) = 1$. That is, $p/q$ is reduced to lowest terms if the greatest common divisor of both numerator and denominator is 1.*

As we saw in **Example 4**, the greatest common divisor of 12 and 18 is 6. Therefore, the fraction 12/18 is **not** reduced to lowest terms. However, we can reduce 12/18 to lowest terms by dividing both numerator and denominator by their greatest common divisor. That is,

$$\frac{12}{18} = \frac{12 \div 6}{18 \div 6} = \frac{2}{3}.$$

Note that $\text{GCD}(2, 3) = 1$, so $2/3$ is reduced to lowest terms.

When it is difficult to ascertain the greatest common divisor, we'll find it more efficient to proceed as follows:

- Prime factor both numerator and denominator.
- Cancel common factors.

Thus, to reduce 12/18 to lowest terms, first express both numerator and denominator as a product of prime numbers, then cancel common primes.

$$\frac{12}{18}=\frac{2\cdot 2\cdot 3}{2\cdot 3\cdot 3}=\frac{\cancel{2}\cdot 2\cdot \cancel{3}}{\cancel{2}\cdot 3\cdot \cancel{3}}=\frac{2}{3} \tag{8}$$

When you cancel a 2, you're actually dividing both numerator and denominator by 2. When you cancel a 3, you're actually dividing both numerator and denominator by 3. Note that doing both (dividing by 2 and then dividing by 3) is equivalent to dividing both numerator and denominator by 6.

We will favor this latter technique, precisely because it is identical to the technique we will use to reduce rational functions to lowest terms. However, this "cancellation" technique has some pitfalls, so let's take a moment to discuss some common cancellation mistakes.

## *Cancellation*

You can spark some pretty heated debate amongst mathematics educators by innocently mentioning the word "cancellation." There seem to be two diametrically opposed camps, those who don't mind when their students use the technique of cancellation, and on the other side, those that refuse to even use the term "cancellation" in their classes.

Both sides of the argument have merit. As we showed in **equation (8)**, we can reduce 12/18 quite efficiently by simply canceling common factors. On the other hand, instructors from the second camp prefer to use the phrase "factor out a 1" instead of the phrase "cancel," encouraging their students to reduce 12/18 as follows.

$$\frac{12}{18}=\frac{2\cdot 2\cdot 3}{2\cdot 3\cdot 3}=\frac{2}{3}\cdot\frac{2\cdot 3}{2\cdot 3}=\frac{2}{3}\cdot 1=\frac{2}{3}$$

This is a perfectly valid technique and one that, quite honestly, avoids the quicksand of "cancellation mistakes." Instructors who grow weary of watching their students "cancel" when they shouldn't are quite likely to promote this latter technique.

However, if we can help our students avoid "cancellation mistakes," we prefer to allow our students to cancel common factors (as we did in **equation (8)**) when reducing fractions such as 12/18 to lowest terms. So, with these thoughts in mind, let's discuss some of the most common cancellation mistakes.

Let's begin with a most important piece of advice.

**How to Avoid Cancellation Mistakes**. You may only cancel factors, not addends. To avoid cancellation mistakes, factor **completely** before you begin to cancel.

**Warning 9.** *Many of the ensuing calculations are incorrect. They are examples of common mistakes that are made when performing cancellation. Make sure that you read carefully and avoid just "scanning" these calculations.*

As a first example, consider the rational expression

$$\frac{2+6}{2},$$

which clearly equals 8/2, or 4. However, if you cancel in this situation, as in

$$\frac{2+6}{2} = \frac{\cancel{2}+6}{\cancel{2}}, \tag{10}$$

you certainly do not get the same result. So, what happened?

Note that in the numerator of **equation (10)**, the 2 and the 6 are separated by a plus sign. Thus, they are not factors; they are addends! You are not allowed to cancel addends, only factors.

Suppose, for comparison, that the rational expression had been

$$\frac{2 \cdot 6}{2},$$

which clearly equals 12/2, or 6. In this case, the 2 and the 6 in the numerator are separated by a multiplication symbol, so they are factors and cancellation is allowed, as in

$$\frac{2 \cdot 6}{2} = \frac{\cancel{2} \cdot 6}{\cancel{2}} = 6. \tag{11}$$

Now, before you dismiss these examples as trivial, consider the following examples which are identical in structure. First, consider

$$\frac{x+(x+2)}{x} = \frac{\cancel{x}+(x+2)}{\cancel{x}} = x+2.$$

This cancellation is identical to that performed in **equation (10)** and is not allowed. In the numerator, note that $x$ and $(x+2)$ are separated by an addition symbol, so they are addends. You are not allowed to cancel addends!

Conversely, consider the following example.

$$\frac{x(x+2)}{x} = \frac{\cancel{x}(x+2)}{\cancel{x}} = x+2$$

In the numerator of this example, $x$ and $(x+2)$ are separated by implied multiplication. Hence, they are factors and cancellation is permissible.

Look again at **equation (10)**, where the correct answer should have been 8/2, or 4. We mistakenly found the answer to be 6, because we cancelled addends. A workaround would be to first factor the numerator of **equation (10)**, then cancel, as follows.

$$\frac{2+6}{2} = \frac{2(1+3)}{2} = \frac{\cancel{2}(1+3)}{\cancel{2}} = 1+3=4$$

Note that we cancelled **factors** in this approach, which is permissible, and got the correct answer 4.

**Warning 12.** *We are finished discussing common cancellation mistakes and you may not continue reading with confidence that all mathematics is correctly presented.*

## *Reducing Rational Expressions in $x$*

Now that we've discussed some fundamental ideas and techniques, let's apply what we've learned to rational expressions that are functions of an independent variable (usually $x$). Let's start with a simple example.

▶ **Example 13.** *Reduce the rational expression*

$$\frac{2x-6}{x^2-7x+12} \tag{14}$$

*to lowest terms. For what values of $x$ is your result valid?*

In the numerator, factor out a 2, as in $2x - 6 = 2(x - 3)$.

The denominator is a quadratic trinomial with $ac = (1)(12) = 12$. The integer pair $-3$ and $-4$ has product 12 and sum $-7$, so the denominator factors as shown.

$$\frac{2x-6}{x^2-7x+12} = \frac{2(x-3)}{(x-3)(x-4)}.$$

Now that both numerator and denominator are factored, we can cancel common factors.

$$\frac{2x-6}{x^2-7x+12} = \frac{2\cancel{(x-3)}}{\cancel{(x-3)}(x-4)} = \frac{2}{x-4}$$

Thus, we have shown that

$$\frac{2x-6}{x^2-7x+12} = \frac{2}{x-4}. \tag{15}$$

In **equation (15)**, we are stating that the expression on the left (the original expression) is *identical* to the expression on the right for all values of $x$.

Actually, there are two notable exceptions, the first of which is $x = 3$. If we substitute $x = 3$ into the left-hand side of **equation (15)**, we get

$$\frac{2x-6}{x^2-7x+12} = \frac{2(3)-6}{(3)^2-7(3)+12} = \frac{0}{0}$$

We cannot divide by zero, so the left-hand side of **equation (15)** is undefined if $x = 3$. Therefore, the result in **equation (15)** is not valid if $x = 3$.

Similarly, if we insert $x = 4$ in the left-hand side of **equation (15)**,

$$\frac{2x-6}{x^2-7x+12} = \frac{2(4)-6}{(4)^2-7(4)+12} = \frac{2}{0}.$$

Again, division by zero is undefined. The left-hand side of **equation (15)** is undefined if $x = 4$, so the result in **equation (15)** is not valid if $x = 4$. Note that the right-hand side of **equation (15)** is also undefined at $x = 4$.

However, the algebraic work we did above guarantees that the left-hand side of **equation (15)** will be identical to the right-hand side of **equation (15)** for all other values of $x$. For example, if we substitute $x = 5$ into the left-hand side of **equation (15)**,

$$\frac{2x-6}{x^2-7x+12} = \frac{2(5)-6}{(5)^2-7(5)+12} = \frac{4}{2} = 2.$$

On the other hand, if we substitute $x = 5$ into the right-hand side of **equation (15)**,

$$\frac{2}{x-4} = \frac{2}{5-4} = 2.$$

Hence, both sides of **equation (15)** are identical when $x = 5$. In a similar manner, we could check the validity of the identity in **equation (15)** for all other values of $x$.

You can use the graphing calculator to verify the identity in **equation (15)**. Load the left- and right-hand sides of **equation (15)** in `Y=` menu, as shown in **Figure 1**(a). Press `2nd TBLSET` and adjust settings as shown in **Figure 1**(b). Be sure that you highlight `AUTO` for both independent and dependent variables and press `ENTER` on each to make the selection permanent. In **Figure 1**(b), note that we've set `TblStart = 0` and $\Delta$`Tbl` $= 1$. Press `2nd TABLE` to produce the tabular results shown in **Figure 1**(c).

```
 Plot1  Plot2  Plot3
\Y1=(2*X-6)/(X^2
-7*X+12)
\Y2=2/(X-4)
\Y3=
\Y4=
\Y5=
\Y6=
```

(a)

```
TABLE SETUP
 TblStart=0
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

(b)

```
   X   |  Y1    |  Y2
   0   | ⁻.5    | ⁻.5
   1   | ⁻.6667 | ⁻.6667
   2   | ⁻1     | ⁻1
   3   | ERR:   | ⁻2
   4   | ERR:   | ERR:
   5   | 2      | 2
   6   | 1      | 1
X=0
```

(c)

**Figure 1.** Using the graphing calculator to check that the left- and right-hand sides of **equation (15)** are identical.

Remember that we placed the left- and right-hand sides of **equation (15)** in `Y1` and `Y2`, respectively.

- In the tabular results of **Figure 1**(c), note the `ERR` (error) message in `Y1` when $x = 3$ and $x = 4$. This agrees with our findings above, where the left-hand side of **equation (15)** was undefined because of the presence of zero in the denominator when $x = 3$ or $x = 4$.
- In the tabular results of **Figure 1**(c), note that the value of `Y1` and `Y2` agree for all other values of $x$.

◆

We are led to the following key result.

**Restrictions.** In general, when you reduce a rational expression to lowest terms, the expression obtained should be **identical** to the original expression for all values of the variables in each expression, save those values of the variables that make any denominator equal to zero. This applies to the denominator in the original expression, all intermediate expressions in your work, and the final result. We will refer to any values of the variable that make any denominator equal to zero as **restrictions**.

Let's look at another example.

▶ **Example 16.** *Reduce the expression*

$$\frac{2x^2+5x-12}{4x^3+16x^2-9x-36} \tag{17}$$

*to lowest terms. State all restrictions.*

The numerator is a quadratic trinomial with $ac = (2)(-12) = -24$. The integer pair $-3$ and 8 have product $-24$ and sum 5. Break the middle term of the polynomial in the numerator into a sum using this integer pair, then factor by grouping.

$$\begin{aligned}2x^2+5x-12 &= 2x^2-3x+8x-12\\ &= x(2x-3)+4(2x-3)\\ &= (x+4)(2x-3)\end{aligned}$$

Factor the denominator by grouping.

$$\begin{aligned}4x^3+16x^2-9x-36 &= 4x^2(x+4)-9(x+4)\\ &= (4x^2-9)(x+4)\\ &= (2x+3)(2x-3)(x+4)\end{aligned}$$

Note how the difference of two squares pattern was used to factor $4x^2-9 = (2x+3)(2x-3)$ in the last step.

Now that we've factored both numerator and denominator, we cancel common factors.

$$\begin{aligned}\frac{2x^2+5x-12}{4x^3+16x^2-9x-36} &= \frac{(x+4)(2x-3)}{(2x+3)(2x-3)(x+4)}\\ &= \frac{\cancel{(x+4)}\cancel{(2x-3)}}{(2x+3)\cancel{(2x-3)}\cancel{(x+4)}}\\ &= \frac{1}{2x+3}\end{aligned}$$

We must now determine the restrictions. This means that we must find those values of $x$ that make any denominator equal to zero.

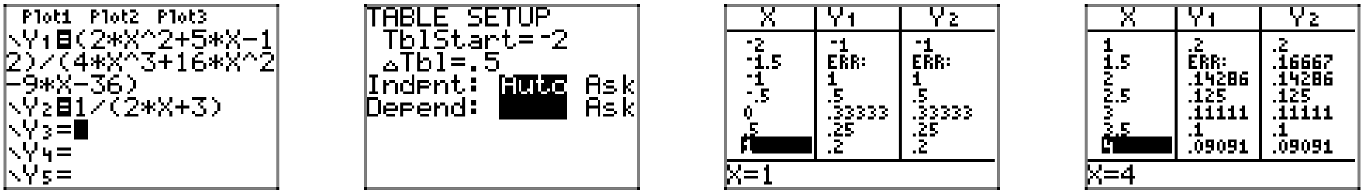
- In the body of our work, we have the denominator $(2x+3)(2x-3)(x+4)$. If we set this equal to zero, the zero product property implies that
  $$2x+3=0 \quad \text{or} \quad 2x-3=0 \quad \text{or} \quad x+4=0.$$
  Each of these linear factors can be solved independently.
  $$x=-3/2 \quad \text{or} \quad x=3/2 \quad \text{or} \quad x=-4$$
  Each of these $x$-values is a restriction.
- In the final rational expression, the denominator is $2x+3$. This expression equals zero when $x=-3/2$ and provides no new restrictions.
- Because the denominator of the original expression, namely $4x^3+16x^2-9x-36$, is identical to its factored form in the body our work, this denominator will produce no new restrictions.

  Thus, for all values of $x$,

$$\frac{2x^2+5x-12}{4x^3+16x^2-9x-36}=\frac{1}{2x+3}, \tag{18}$$

provided $x \neq -3/2$, $3/2$, or $-4$. These are the restrictions. The two expressions are identical for all other values of $x$.

Finally, let's check this result with our graphing calculator. Load each side of **equation (18)** into the `Y=` menu, as shown in **Figure 2**(a). We know that we have a restriction at $x=-3/2$, so let's set `TblStart` $=-2$ and `ΔTbl` $=0.5$, as shown in **Figure 2**(b). Be sure that you have `AUTO` set for both independent and dependent variables. Push the `TABLE` button to produce the tabular display shown in **Figure 2**(c).



```
 Plot1 Plot2 Plot3
\Y1■(2*X^2+5*X-1
2)/(4*X^3+16*X^2
-9*X-36)
\Y2■1/(2*X+3)
\Y3=■
\Y4=
\Y5=
```

(a)

```
TABLE SETUP
 TblStart=-2
 ΔTbl=.5
Indpnt: Auto Ask
Depend: Auto Ask
```

(b)

```
   X   |  Y1    |  Y2
  -2   |  -1    |  -1
  -1.5 |  ERR:  |  ERR:
  -1   |  1     |  1
  -.5  |  .5    |  .5
  0    |  .33333|  .33333
  .5   |  .25   |  .25
  1    |  .2    |  .2
X=1
```

(c)

```
   X   |  Y1    |  Y2
  1    |  .2    |  .2
  1.5  |  ERR:  |  .16667
  2    |  .14286|  .14286
  2.5  |  .125  |  .125
  3    |  .11111|  .11111
  3.5  |  .1    |  .1
  4    |  .09091|  .09091
X=4
```

(c)

**Figure 2.** Using the graphing calculator to check that the left- and right-hand sides of **equation (18)** are identical.

Remember that we placed the left- and right-hand sides of **equation (18)** in `Y1` and `Y2`, respectively.

- In **Figure 2**(c), note that the expressions `Y1` and `Y2` agree at all values of $x$ except $x=-1.5$. This is the restriction $-3/2$ we found above.
- Use the down arrow key to scroll down in the table shown in **Figure 2**(c) to produce the tabular view shown in **Figure 2**(d). Note that `Y1` and `Y2` agree for all values of $x$ except $x=1.5$. This is the restriction $3/2$ we found above.
- We leave it to our readers to uncover the restriction at $x=-4$ by using the up-arrow to scroll up in the table until you reach an $x$-value of $-4$. You should uncover

another `ERR` (error) message at this $x$-value because it is a restriction. You get the `ERR` message due to the fact that the denominator of the left-hand side of **equation (18)** is zero at $x = -4$.

◆

## *Sign Changes*

It is not uncommon that you will have to manipulate the signs in a fraction in order to obtain common factors that can be then cancelled. Consider, for example, the rational expression

$$\frac{3-x}{x-3}. \tag{19}$$

One possible approach is to factor $-1$ out of the numerator to obtain

$$\frac{3-x}{x-3} = \frac{-(x-3)}{x-3}.$$

You can now cancel common factors.[7]

$$\frac{3-x}{x-3} = \frac{-(x-3)}{x-3} = \frac{-\cancel{(x-3)}}{\cancel{x-3}} = -1$$

This result is valid for all values of $x$, provided $x \neq 3$.

Let's look at another example.

▶ **Example 20.** *Reduce the rational expression*

$$\frac{2x-2x^3}{3x^3+4x^2-3x-4} \tag{21}$$

*to lowest terms. State all restrictions.*

In the numerator, factor out $2x$, then complete the factorization using the difference of two squares pattern.

$$2x - 2x^3 = 2x(1-x^2) = 2x(1+x)(1-x)$$

The denominator can be factored by grouping.

$$\begin{aligned}3x^3 + 4x^2 - 3x - 4 &= x^2(3x+4) - 1(3x+4)\\ &= (x^2-1)(3x+4)\\ &= (x+1)(x-1)(3x+4)\end{aligned}$$

Note how the difference of two squares pattern was applied in the last step.

---

[7] When everything cancels, the resulting rational expression equals 1. For example, consider 6/6, which surely is equal to 1. If we factor and cancel common factors, everything cancels.

$$\frac{6}{6} = \frac{2\cdot 3}{2\cdot 3} = \frac{\cancel{2}\cdot\cancel{3}}{\cancel{2}\cdot\cancel{3}} = 1$$

At this point,

$$\frac{2x-2x^3}{3x^3+4x^2-3x-4}=\frac{2x(1+x)(1-x)}{(x+1)(x-1)(3x+4)}.$$

Because we have $1-x$ in the numerator and $x-1$ in the denominator, we will factor out a $-1$ from $1-x$, and because the order of factors does not affect their product, we will move the $-1$ out to the front of the numerator.

$$\frac{2x-2x^3}{3x^3+4x^2-3x-4}=\frac{2x(1+x)(-1)(x-1)}{(x+1)(x-1)(3x+4)}=\frac{-2x(1+x)(x-1)}{(x+1)(x-1)(3x+4)}$$

We can now cancel common factors.

$$\begin{aligned}\frac{2x-2x^3}{3x^3+4x^2-3x-4}&=\frac{-2x(1+x)(x-1)}{(x+1)(x-1)(3x+4)}\\&=\frac{-2x\cancel{(1+x)}\cancel{(x-1)}}{\cancel{(x+1)}\cancel{(x-1)}(3x+4)}\\&=\frac{-2x}{3x+4}\end{aligned}$$

Note that $x+1$ is identical to $1+x$ and cancels. Thus,

$$\frac{2x-2x^3}{3x^3+4x^2-3x-4}=\frac{-2x}{3x+4} \tag{22}$$

for all values of $x$, provided $x \neq -1$, 1, or $-4/3$. These are the restrictions, values of $x$ that make denominators equal to zero.

## *The Sign Change Rule for Fractions*

Let's look at an alternative approach to the last example. First, let's share the precept that every fraction has three signs, one on the numerator, one on the denominator, and a third on the fraction bar. Thus,

$$\frac{-2}{3} \quad \text{has understood signs} \quad +\frac{-2}{+3}.$$

Let's state the *sign change rule* for fractions.

**The Sign Change Rule for Fractions**. Every fraction has three signs, one on the numerator, one on the denominator, and one on the fraction bar. If you don't see an explicit sign, then a plus sign is understood. If you negate any two of these parts,

- numerator and denominator, or
- numerator and fraction bar, or
- fraction bar and denominator,

then the fraction remains unchanged.

For example, let's start with $-2/3$, then do two negations: numerator and fraction bar. Then,

$$+\frac{-2}{+3} = -\frac{+2}{+3}, \qquad \text{or with understood plus signs,} \qquad \frac{-2}{3} = -\frac{2}{3}.$$

This is a familiar result, as negative two divided by a positive three equals a negative two-thirds.

On another note, we might decide to negate numerator and denominator. Then $-2/3$ becomes

$$+\frac{-2}{+3} = \frac{+2}{-3}, \qquad \text{or with understood plus signs,} \qquad \frac{-2}{3} = \frac{2}{-3}.$$

Again, a familiar result. Certainly, negative two divided by positive three is the same as positive two divided by negative three. They both equal minus two-thirds.

So there you have it. Negate any two parts of a fraction and it remains unchanged. On the surface, this seems a trivial remark, but it can be put to good use when reducing rational expressions. Suppose, for example, that we take the original rational expression from **Example 20** and negate the numerator and fraction bar.

$$\frac{2x-2x^3}{3x^3+4x^2-3x-4} = -\frac{2x^3-2x}{3x^3+4x^2-3x-4}$$

Note how we've made two sign changes. We've negated the fraction bar, we've negated the numerator ($-(2x-2x^3) = 2x^3-2x$), and left the denominator alone. Therefore, the fraction is unchanged according to our sign change rule.

Now, factor and cancel common factors (we leave the steps for our readers — they're similar to those we took in **Example 20**).

$$\begin{aligned}\frac{2x-2x^3}{3x^3+4x^2-3x-4} &= -\frac{2x^3-2x}{3x^3+4x^2-3x-4}\\ &= -\frac{2x(x+1)(x-1)}{(x+1)(x-1)(3x+4)}\\ &= -\frac{2x\cancel{(x+1)}\cancel{(x-1)}}{\cancel{(x+1)}\cancel{(x-1)}(3x+4)}\\ &= -\frac{2x}{3x+4}\end{aligned}$$

But does this answer match the answer in **equation (22)**? It does, as can be seen by making two negations, fraction bar and numerator.

$$-\frac{2x}{3x+4} = \frac{-2x}{3x+4}$$

## *The Secant Line*

Consider the graph of the function $f$ that we've drawn in **Figure 3**. Note that we've chosen two points on the graph of $f$, namely $(a, f(a))$ and $(x, f(x))$, and we've drawn a line $L$ through them that mathematicians call the "secant line."

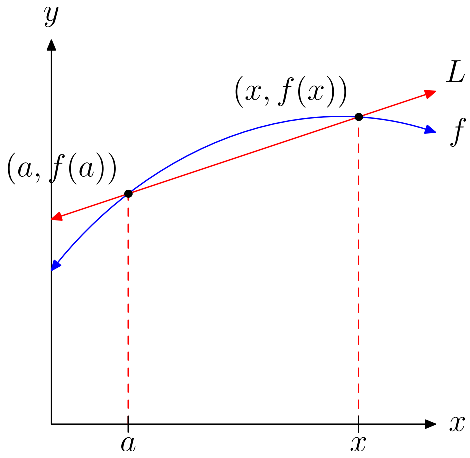


**Figure 3.** The secant line passes through $(a, f(a))$ and $(x, f(x))$.

The slope of the secant line $L$ is found by dividing the change in $y$ by the change in $x$.

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a} \tag{23}$$

This slope provides the average rate of change of the variable $y$ with respect to the variable $x$. Students in calculus use this "average rate of change" to develop the notion of "instantaneous rate of change." However, we'll leave that task for the calculus students and concentrate on the challenge of simplifying the expression **equation (23)** for the average rate of change.

▶ **Example 24.** *Given the function $f(x) = x^2$, simplify the expression for the average rate of change, namely*

$$\frac{f(x) - f(a)}{x - a}.$$

First, note that $f(x) = x^2$ and $f(a) = a^2$, so we can write

$$\frac{f(x) - f(a)}{x - a} = \frac{x^2 - a^2}{x - a}.$$

We can now use the difference of two squares pattern to factor the numerator and cancel common factors.

$$\frac{x^2 - a^2}{x - a} = \frac{(x + a)\cancel{(x - a)}}{\cancel{x - a}} = x + a$$

Thus,

$$\frac{f(x)-f(a)}{x-a}=x+a,$$

provided, of course, that $x \neq a$.

◆

Let's look at another example.

▶ **Example 25.** *Consider the function* $f(x)=x^2-3x-5$. *Simplify*

$$\frac{f(x)-f(2)}{x-2}.$$

First, $f(x)=x^2-3x-5$ and therefore $f(2)=(2)^2-3(2)-5=-7$, so we can write

$$\frac{f(x)-f(2)}{x-2}=\frac{(x^2-3x-5)-(-7)}{x-2}=\frac{x^2-3x+2}{x-2}.$$

We can now factor the numerator and cancel common factors.

$$\frac{x^2-3x+2}{x-2}=\frac{\cancel{(x-2)}(x-1)}{\cancel{x-2}}=x-1$$

Thus,

$$\frac{f(x)-f(2)}{x-2}=x-1,$$

provided, of course, that $x \neq 2$.

◆

## 7.2 Exercises

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In **Exercises 1-12**, reduce each rational number to lowest terms by applying the following steps:

i. Prime factor both numerator and denominator.
ii. Cancel common prime factors.
iii. Simplify the numerator and denominator of the result.

**1.** $\dfrac{147}{98}$

**2.** $\dfrac{3087}{245}$

**3.** $\dfrac{1715}{196}$

**4.** $\dfrac{225}{50}$

**5.** $\dfrac{1715}{441}$

**6.** $\dfrac{56}{24}$

**7.** $\dfrac{108}{189}$

**8.** $\dfrac{75}{500}$

**9.** $\dfrac{100}{28}$

**10.** $\dfrac{98}{147}$

**11.** $\dfrac{1125}{175}$

**12.** $\dfrac{3087}{8575}$

In **Exercises 13-18**, reduce the given expression to lowest terms. State all restrictions.

**13.** $\dfrac{x^2-10x+9}{5x-5}$

**14.** $\dfrac{x^2-9x+20}{x^2-x-20}$

**15.** $\dfrac{x^2-2x-35}{x^2-7x}$

**16.** $\dfrac{x^2-15x+54}{x^2+7x-8}$

**17.** $\dfrac{x^2+2x-63}{x^2+13x+42}$

**18.** $\dfrac{x^2+13x+42}{9x+63}$

In **Exercises 19-24**, negate any two parts of the fraction, then factor (if necessary) and cancel common factors to reduce the rational expression to lowest terms. State all restrictions.

**19.** $\dfrac{x+2}{-x-2}$

**20.** $\dfrac{4-x}{x-4}$

**21.** $\dfrac{2x-6}{3-x}$

**22.** $\dfrac{3x+12}{-x-4}$

**23.** $\dfrac{3x^2+6x}{-x-2}$

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**24.** $\dfrac{8x - 2x^2}{x - 4}$

---

In **Exercises 25-38**, reduce each of the given rational expressions to lowest terms. State all restrictions.

**25.** $\dfrac{x^2 - x - 20}{25 - x^2}$

**26.** $\dfrac{x - x^2}{x^2 - 3x + 2}$

**27.** $\dfrac{x^2 + 3x - 28}{x^2 + 5x - 36}$

**28.** $\dfrac{x^2 + 10x + 9}{x^2 + 15x + 54}$

**29.** $\dfrac{x^2 - x - 56}{8x - x^2}$

**30.** $\dfrac{x^2 - 7x + 10}{5x - x^2}$

**31.** $\dfrac{x^2 + 13x + 42}{x^2 - 2x - 63}$

**32.** $\dfrac{x^2 - 16}{x^2 - x - 12}$

**33.** $\dfrac{x^2 - 9x + 14}{49 - x^2}$

**34.** $\dfrac{x^2 + 7x + 12}{9 - x^2}$

**35.** $\dfrac{x^2 - 3x - 18}{x^2 - 6x + 5}$

**36.** $\dfrac{x^2 + 5x - 6}{x^2 - 1}$

**37.** $\dfrac{x^2 - 3x - 10}{-9x - 18}$

**38.** $\dfrac{x^2 - 6x + 8}{16 - x^2}$

---

In **Exercises 39-42**, reduce each rational function to lowest terms, and then perform each of the following tasks.

i. Load the original rational expression into `Y1` and the reduced rational expression (your answer) into `Y2` of your graphing calculator.

ii. In `TABLE SETUP`, set `TblStart` equal to zero, `ΔTbl` equal to 1, then make sure both independent and dependent variables are set to `Auto`. Select `TABLE` and scroll with the up- and down-arrows on your calculator until the smallest restriction is in view. Copy both columns of the table onto your homework paper, showing the agreement between `Y1` and `Y2` and what happens at all restrictions.

**39.** $\dfrac{x^2 - 8x + 7}{x^2 - 11x + 28}$

**40.** $\dfrac{x^2 - 5x}{x^2 - 9x}$

**41.** $\dfrac{8x - x^2}{x^2 - x - 56}$

**42.** $\dfrac{x^2 + 13x + 40}{-2x - 16}$

---

Given $f(x) = 2x + 5$, simplify each of the expressions in **Exercises 43-46**. Be sure to reduce your answer to lowest terms and state any restrictions.

**43.** $\dfrac{f(x) - f(3)}{x - 3}$

**44.** $\dfrac{f(x) - f(6)}{x - 6}$

**45.** $\dfrac{f(x) - f(a)}{x - a}$

**46.** $\dfrac{f(a + h) - f(a)}{h}$

Given $f(x) = x^2 + 2x$, simplify each of the expressions in **Exercises 47-50**. Be sure to reduce your answer to lowest terms and state any restrictions.

**47.** $\dfrac{f(x) - f(1)}{x - 1}$

**48.** $\dfrac{f(x) - f(a)}{x - a}$

**49.** $\dfrac{f(a + h) - f(a)}{h}$

**50.** $\dfrac{f(x + h) - f(x)}{h}$

**Drill for Skill.** In **Exercises 51-54**, evaluate the given function at the given expression and simplify your answer.

**51.** Suppose that $f$ is the function

$$f(x) = -\frac{x - 6}{8x + 7}.$$

Evaluate $f(-3x + 2)$ and simplify your answer.

**52.** Suppose that $f$ is the function

$$f(x) = -\frac{5x + 3}{7x + 6}.$$

Evaluate $f(-5x + 1)$ and simplify your answer.

**53.** Suppose that $f$ is the function

$$f(x) = -\frac{3x - 6}{4x + 6}.$$

Evaluate $f(-x-3)$ and simplify your answer.

**54.** Suppose that $f$ is the function

$$f(x) = \frac{4x - 1}{2x - 4}.$$

Evaluate $f(5x)$ and simplify your answer.

## 7.2 *Answers*

---

**1.** $\dfrac{3}{2}$

**3.** $\dfrac{35}{4}$

**5.** $\dfrac{35}{9}$

**7.** $\dfrac{4}{7}$

**9.** $\dfrac{25}{7}$

**11.** $\dfrac{45}{7}$

**13.** $\dfrac{x-9}{5}$, provided $x \neq 1$

**15.** $\dfrac{x+5}{x}$, provided $x \neq 0,\ 7$

**17.** $\dfrac{(x-7)(x+9)}{(x+7)(x+6)}$, provided $x \neq -7,\ -6$

**19.** $-1$, provided $x \neq -2$

**21.** $-2$, provided $x \neq 3$

**23.** $-3x$, provided $x \neq -2$

**25.** $-\dfrac{x+4}{x+5}$, provided $x \neq -5,\ 5$

**27.** $\dfrac{x+7}{x+9}$, provided $x \neq 4,\ -9$

**29.** $-\dfrac{x+7}{x}$, provided $x \neq 0,\ 8$

**31.** $\dfrac{x+6}{x-9}$, provided $x \neq -7,\ 9$

**33.** $-\dfrac{x-2}{x+7}$, provided $x \neq 7,\ -7$

**35.** $\dfrac{(x-6)(x+3)}{(x-1)(x-5)}$, provided $x \neq 1,\ 5$

**37.** $-\dfrac{x-5}{9}$, provided $x \neq -2$

**39.** $\dfrac{x-1}{x-4}$, provided $x \neq 7,\ 4$

| X | Y1 | Y2 |
|---|---|---|
| 3 | -2 | -2 |
| 4 | Err: | Err: |
| 5 | 4 | 4 |
| 6 | 2.5 | 2.5 |
| 7 | Err: | 2 |
| 8 | 1.75 | 1.75 |

**41.** $-\dfrac{x}{x+7}$, provided $x \neq -7, 8$

| X | Y1 | Y2 |
|---|---|---|
| -8 | -8 | -8 |
| -7 | Err: | Err: |
| -6 | 6 | 6 |
| -5 | 2.5 | 2.5 |
| -4 | 1.33333 | 1.33333 |
| -3 | 0.75 | 0.75 |
| -2 | 0.4 | 0.4 |
| -1 | 0.166667 | 0.166667 |
| 0 | -0 | -0 |
| 1 | -0.125 | -0.125 |
| 2 | -0.222222 | -0.222222 |
| 3 | -0.3 | -0.3 |
| 4 | -0.363636 | -0.363636 |
| 5 | -0.416667 | -0.416667 |
| 6 | -0.461538 | -0.461538 |
| 7 | -0.5 | -0.5 |
| 8 | Err: | -0.533333 |
| 9 | -0.5625 | -0.5625 |

**43.** 2, provided $x \neq 3$

**45.** 2, provided $x \neq a$

**47.** $x+3$, provided $x \neq 1$

**49.** $2a+h+2$, provided $h \neq 0$

**51.** $-\dfrac{3x+4}{24x-23}$

**53.** $-\dfrac{3x+15}{4x+6}$

# 7.3 Graphing Rational Functions

We've seen that the denominator of a rational function is never allowed to equal zero; division by zero is not defined. So, with rational functions, there are special values of the independent variable that are of particular importance. Now, it comes as no surprise that near values that make the denominator zero, rational functions exhibit special behavior, but here, we will also see that values that make the numerator zero sometimes create additional special behavior in rational functions.

We begin our discussion by focusing on the domain of a rational function.

## *The Domain of a Rational Function*

When presented with a rational function of the form

$$f(x) = \frac{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n}{b_0 + b_1x + b_2x^2 + \cdots + b_mx^m}, \tag{1}$$

the first thing we must do is identify the domain. Equivalently, we must identify the **restrictions**, values of the independent variable (usually $x$) that are **not** in the domain. To facilitate the search for restrictions, we should factor the denominator of the rational function (it won't hurt to factor the numerator at this time as well, as we will soon see). Once the domain is established and the restrictions are identified, here are the pertinent facts.

**Behavior of a Rational Function at Its Restrictions**. A rational function can only exhibit one of two behaviors at a restriction (a value of the independent variable that is not in the domain of the rational function).

1. The graph of the rational function will have a vertical asymptote at the restricted value.
2. The graph will exhibit a "hole" at the restricted value.

In the next two examples, we will examine each of these behaviors. In this first example, we see a restriction that leads to a vertical asymptote.

► **Example 2.** *Sketch the graph of*

$$f(x) = \frac{1}{x+2}.$$

The first step is to identify the domain. Note that $x = -2$ makes the denominator of $f(x) = 1/(x+2)$ equal to zero. Division by zero is undefined. Hence, $x = -2$ is **not** in the domain of $f$; that is, $x = -2$ is a restriction. Equivalently, the domain of $f$ is $\{x : x \neq -2\}$.

Now that we've identified the restriction, we can use the theory of Section 7.1 to shift the graph of $y = 1/x$ two units to the left to create the graph of $f(x) = 1/(x+2)$, as shown in **Figure 1**.

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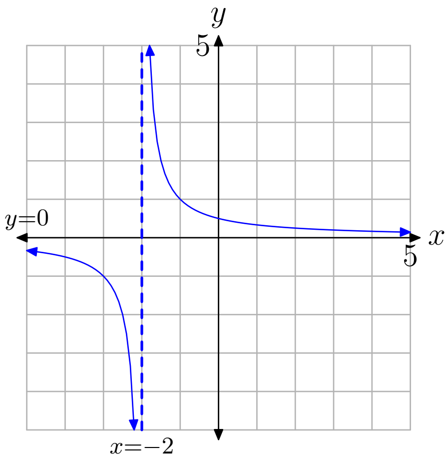


**Figure 1.** The function $f(x) = 1/(x+2)$ has a restriction at $x = -2$. The graph of $f$ has a vertical asymptote with equation $x = -2$.

The function $f(x) = 1/(x+2)$ has a restriction at $x = -2$ and the graph of $f$ exhibits a vertical asymptote having equation $x = -2$.

◆

It is important to note that although the restricted value $x = -2$ makes the denominator of $f(x) = 1/(x+2)$ equal to zero, it does **not** make the numerator equal to zero. We'll soon have more to say about this observation.

Let's look at an example of a rational function that exhibits a "hole" at one of its restricted values.

▶ **Example 3.** *Sketch the graph of*

$$f(x) = \frac{x-2}{x^2-4}.$$

We highlight the first step.

**Factor Numerators and Denominators**. When working with rational functions, the first thing you should always do is factor both numerator and denominator of the rational function.

Following this advice, we factor both numerator and denominator of $f(x) = (x-2)/(x^2-4)$.

$$f(x) = \frac{x-2}{(x-2)(x+2)}$$

It is easier to spot the restrictions when the denominator of a rational function is in factored form. Clearly, $x = -2$ and $x = 2$ will both make the denominator of

$f(x) = (x-2)/((x-2)(x+2))$ equal to zero. Hence, $x = -2$ and $x = 2$ are restrictions of the rational function $f$.

Now that the restrictions of the rational function $f$ are established, we proceed to the second step.

**Reduce to Lowest Terms.** After you establish the restrictions of the rational function, the second thing you should do is reduce the rational function to lowest terms.

Following this advice, we cancel common factors and reduce the rational function $f(x) = (x-2)/((x-2)(x+2))$ to lowest terms, obtaining a new function,

$$g(x) = \frac{1}{x+2}.$$

The functions $f(x) = (x-2)/((x-2)(x+2))$ and $g(x) = 1/(x+2)$ are **not** identical functions. They have different domains. The domain of $f$ is $D_f = \{x : x \neq -2, 2\}$, but the domain of $g$ is $D_g = \{x : x \neq -2\}$. Hence, the only difference between the two functions occurs at $x = 2$. The number 2 is in the domain of $g$, but not in the domain of $f$.

We know what the graph of the function $g(x) = 1/(x+2)$ looks like. We drew this graph in **Example 2** and we picture it anew in **Figure 2**.

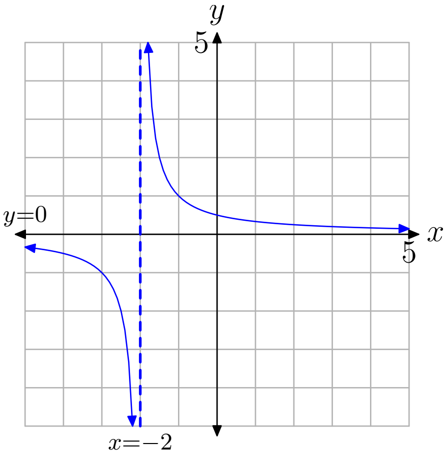


**Figure 2.** The graph of $g(x) = 1/(x+2)$ exhibits a vertical asymptote at its restriction $x = -2$.

The difficulty we now face is the fact that we've been asked to draw the graph of $f$, not the graph of $g$. However, we know that the functions $f$ and $g$ agree at all values of $x$ except $x = 2$. If we remove this value from the graph of $g$, then we will have the graph of $f$.

So, what point should we remove from the graph of $g$? We should remove the point that has an $x$-value equal to 2. Therefore, we evaluate the function $g(x) = 1/(x + 2)$ at $x = 2$ and find

$$g(2) = \frac{1}{2+2} = \frac{1}{4}.$$

Because $g(2) = 1/4$, we remove the point $(2, 1/4)$ from the graph of $g$ to produce the graph of $f$. The result is shown in **Figure 3**.

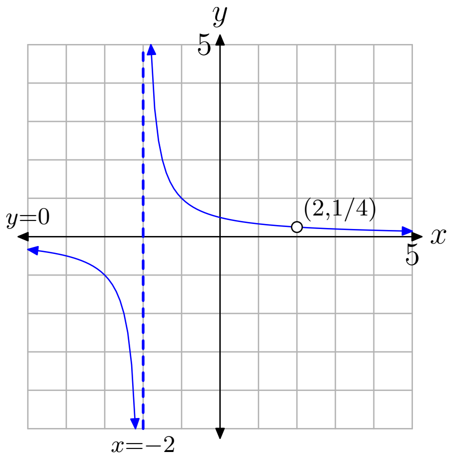


**Figure 3.** The graph of $f(x) = (x - 2)/((x - 2)(x + 2))$ exhibits a vertical asymptote at its restriction $x = -2$ and a hole at its second restriction $x = 2$.

◆

We pause to make an important observation. In **Example 3**, we started with the function

$$f(x) = \frac{x-2}{(x-2)(x+2)},$$

which had restrictions at $x = 2$ and $x = -2$. After reducing, the function

$$g(x) = \frac{1}{x+2}$$

no longer had a restriction at $x = 2$. The function $g$ had a single restriction at $x = -2$. The result, as seen in **Figure 3**, was a vertical asymptote at the remaining restriction, and a hole at the restriction that “went away” due to cancellation. This leads us to the following procedure.

**Asymptote or Hole?** To determine whether the graph of a rational function has a vertical asymptote or a hole at a restriction, proceed as follows:

1. Factor numerator and denominator of the original rational function $f$. Identify the restrictions of $f$.
2. Reduce the rational function to lowest terms, naming the new function $g$. Identify the restrictions of the function $g$.
3. Those restrictions of $f$ that remain restrictions of the function $g$ will introduce vertical asymptotes into the graph of $f$.
4. Those restrictions of $f$ that are no longer restrictions of the function $g$ will introduce "holes" into the graph of $f$. To determine the coordinates of the holes, substitute each restriction of $f$ that is not a restriction of $g$ into the function $g$ to determine the $y$-value of the hole.

We now turn our attention to the zeros of a rational function.

## *The Zeros of a Rational Function*

We've seen that division by zero is undefined. That is, if we have a fraction $N/D$, then $D$ (the denominator) must not equal zero. Thus, $5/0$, $-15/0$, and $0/0$ are all undefined. On the other hand, in the fraction $N/D$, if $N = 0$ and $D \neq 0$, then the fraction is equal to zero. For example, $0/5$, $0/(-15)$, and $0/\pi$ are all equal to zero.

Therefore, when working with an arbitrary rational function, such as

$$f(x) = \frac{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n}{b_0 + b_1x + b_2x^2 + \cdots + b_mx^m}, \tag{4}$$

whatever value of $x$ that will make the numerator zero without simultaneously making the denominator equal to zero will be a zero of the rational function $f$.

This discussion leads to the following procedure for identifying the zeros of a rational function.

**Finding Zeros of Rational Functions**. To determine the zeros of a rational function, proceed as follows.

1. Factor both numerator and denominator of the rational function $f$.
2. Identify the restrictions of the rational function $f$.
3. Identify the values of the independent variable (usually $x$) that make the numerator equal to zero.
4. The zeros of the rational function $f$ will be those values of $x$ that make the numerator zero but are not restrictions of the rational function $f$.

Let's look at an example.

▶ **Example 5.** *Find the zeros of the rational function defined by*

$$f(x)=\frac{x^2+3x+2}{x^2-2x-3}. \tag{6}$$

Factor numerator and denominator of the rational function $f$.

$$f(x)=\frac{(x+1)(x+2)}{(x+1)(x-3)}$$

The values $x=-1$ and $x=3$ make the denominator equal to zero and are restrictions.

Next, note that $x=-1$ and $x=-2$ both make the numerator equal to zero. However, $x=-1$ is also a restriction of the rational function $f$, so it will not be a zero of $f$. On the other hand, the value $x=-2$ is not a restriction and will be a zero of $f$.

Although we've correctly identified the zeros of $f$, it's instructive to check the values of $x$ that make the numerator of $f$ equal to zero. If we substitute $x=-1$ into original function defined by **equation (6)**, we find that

$$f(-1)=\frac{(-1)^2+3(-1)+2}{(-1)^2-2(-1)-3}=\frac{0}{0}$$

is undefined. Hence, $x=-1$ is **not** a zero of the rational function $f$. The difficulty in this case is that $x=-1$ also makes the denominator equal to zero.

On the other hand, when we substitute $x=-2$ in the function defined by **equation (6)**,

$$f(-2)=\frac{(-2)^2+3(-2)+2}{(-2)^2-2(-2)-3}=\frac{0}{5}=0.$$

In this case, $x=-2$ makes the numerator equal to zero without making the denominator equal to zero. Hence, $x=-2$ is a zero of the rational function $f$.

◆

It's important to note that you must work with the **original** rational function, and not its reduced form, when identifying the zeros of the rational function.

▶ **Example 7.** *Identify the zeros of the rational function*

$$f(x)=\frac{x^2-6x+9}{x^2-9}.$$

Factor both numerator and denominator.

$$f(x)=\frac{(x-3)^2}{(x+3)(x-3)}$$

Note that $x=-3$ and $x=3$ are restrictions. Further, the only value of $x$ that will make the numerator equal to zero is $x=3$. However, this is also a restriction. Hence, the function $f$ has no zeros.

The point to make here is what would happen if you work with the reduced form of the rational function in attempting to find its zeros. Cancelling like factors leads to a new function,

$$g(x) = \frac{x-3}{x+3}.$$

Note that $g$ has only one restriction, $x = -3$. Further, $x = 3$ makes the numerator of $g$ equal to zero and is not a restriction. Hence, $x = 3$ is a zero of the function $g$, **but it is not a zero of the function $f$.**

This example demonstrates that we must identify the zeros of the rational function before we cancel common factors.

◇

## *Drawing the Graph of a Rational Function*

In this section we will use the zeros and asymptotes of the rational function to help draw the graph of a rational function. We will also investigate the end-behavior of rational functions. Let's begin with an example.

▶ **Example 8.** *Sketch the graph of the rational function*

$$f(x) = \frac{x+2}{x-3}. \tag{9}$$

First, note that both numerator and denominator are already factored. The function has one restriction, $x = 3$. Next, note that $x = -2$ makes the numerator of **equation (9)** zero and is not a restriction. Hence, $x = -2$ is a zero of the function. Recall that a function is zero where its graph crosses the horizontal axis. Hence, the graph of $f$ will cross the $x$-axis at $(-2, 0)$, as shown in **Figure 4**.

Note that the rational function (**9**) is already reduced to lowest terms. Hence, the restriction at $x = 3$ will place a vertical asymptote at $x = 3$, which is also shown in **Figure 4**.

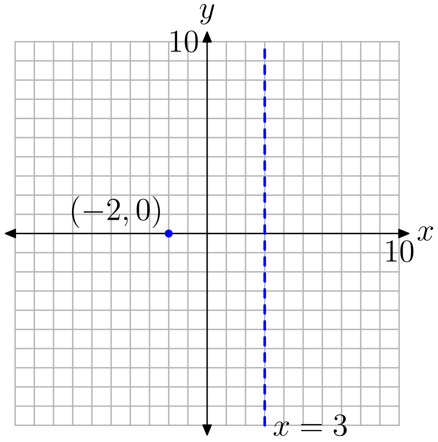
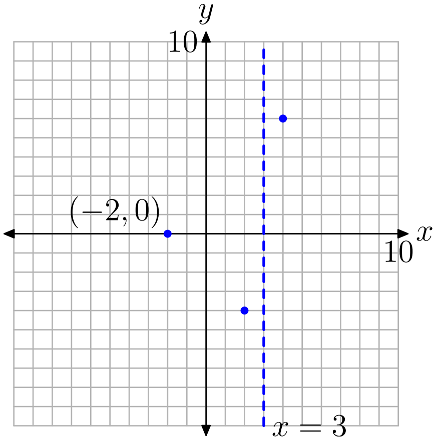


**Figure 4.** Plot and label the $x$-intercept and vertical asymptote.

At this point, we know two things:

1. The graph will cross the $x$-axis at $(-2, 0)$.
2. On each side of the vertical asymptote at $x = 3$, one of two things can happen. Either the graph will rise to positive infinity or the graph will fall to negative infinity.

To discover the behavior near the vertical asymptote, let's plot one point on each side of the vertical asymptote, as shown in **Figure 5**.



| $x$ | $y = (x+2)/(x-3)$ |
|---|---|
| 2 | $-4$ |
| 4 | 6 |

**Figure 5.** Additional points help determine the behavior near the vertical asymptote.

Consider the right side of the vertical asymptote and the plotted point $(4, 6)$ through which our graph must pass. As the graph approaches the vertical asymptote at $x = 3$, only one of two things can happen. Either the graph rises to positive infinity or the graph falls to negative infinity. However, in order for the latter to happen, the graph must first pass through the point $(4, 6)$, then *cross* the $x$-axis between $x = 3$ and $x = 4$ on its descent to minus infinity. But we already know that the only $x$-intercept is at the point $(2, 0)$, so this cannot happen. Hence, on the right, the graph must pass through the point $(4, 6)$, then rise to positive infinity, as shown in **Figure 6**.

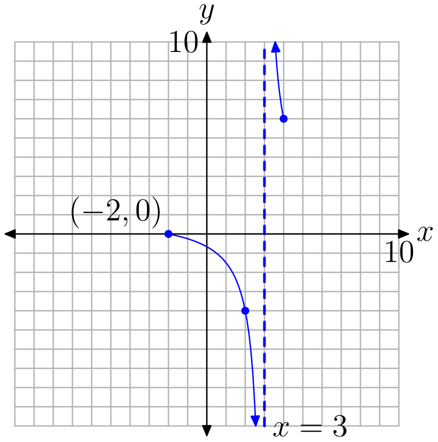


**Figure 6.** Behavior near the vertical asymptote.

A similar argument holds on the left of the vertical asymptote at $x = 3$. The graph cannot pass through the point $(2, -4)$ and rise to positive infinity as it approaches the vertical asymptote, because to do so would require that it cross the $x$-axis between $x = 2$ and $x = 3$. However, there is no $x$-intercept in this region available for this purpose. Hence, on the left, the graph must pass through the point $(2, -4)$ and fall to negative infinity as it approaches the vertical asymptote at $x = 3$. This behavior is shown in **Figure 6**.

Finally, what about the end-behavior of the rational function? What happens to the graph of the rational function as $x$ increases without bound? What happens when $x$ decreases without bound? One simple way to answer these questions is to use a table to investigate the behavior numerically. The graphing calculator facilitates this task.

First, enter your function as shown in **Figure 7**(a), then press `2nd TBLSET` to open the window shown in **Figure 7**(b). For what we are about to do, all of the settings in this window are irrelevant, save one. Make sure you use the arrow keys to highlight `ASK` for the `Indpnt` (independent) variable and press `ENTER` to select this option. Finally, select `2nd TABLE`, then enter the $x$-values 10, 100, 1 000, and 10 000, pressing `ENTER` after each one.

```
Plot1 Plot2 Plot3
\Y1=(X+2)/(X-3)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
TABLE SETUP
 TblStart=0
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

```
  X    | Y1
10     | 1.7143
100    | 1.0515
1000   | 1.005
10000  | 1.0005
X=
```

(a) (b) (c)

**Figure 7.** Using the table feature of the graphing calculator to investigate the end-behavior as $x$ approaches positive infinity.

Note the resulting $y$-values in the second column of the table (the `Y1` column) in **Figure 7**(c). As $x$ is increasing without bound, the $y$-values are greater than 1, yet appear to be approaching the number 1. Therefore, as our graph moves to the extreme right, it must approach the *horizontal asymptote* at $y = 1$, as shown in **Figure 9**.

A similar effort predicts the end-behavior as $x$ decreases without bound, as shown in the sequence of pictures in **Figure 8**. As $x$ decreases without bound, the $y$-values are less than 1, but again approach the number 1, as shown in **Figure 8**(c).

```
Plot1 Plot2 Plot3
\Y1=(X+2)/(X-3)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
TABLE SETUP
 TblStart=10
 ΔTbl=
Indpnt: Auto Ask
Depend: Auto Ask
```

```
 X      | Y1
 -10    | .61538
 -100   | .95146
 -1000  | .99501
 -10000 | .9995

X=
```

(a) (b) (c)

**Figure 8.** Using the table feature of the graphing calculator to investigate the end-behavior as $x$ approaches negative infinity.

The evidence in **Figure 8**(c) indicates that as our graph moves to the extreme left, it must approach the *horizontal asymptote* at $y = 1$, as shown in **Figure 9**.

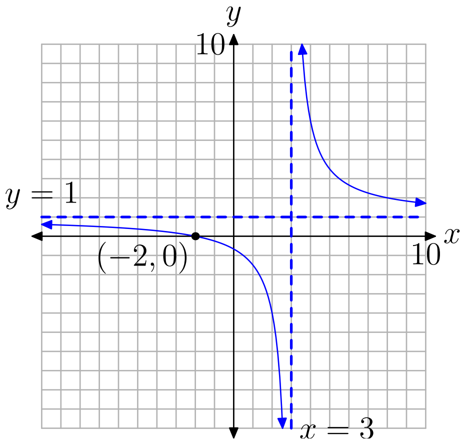
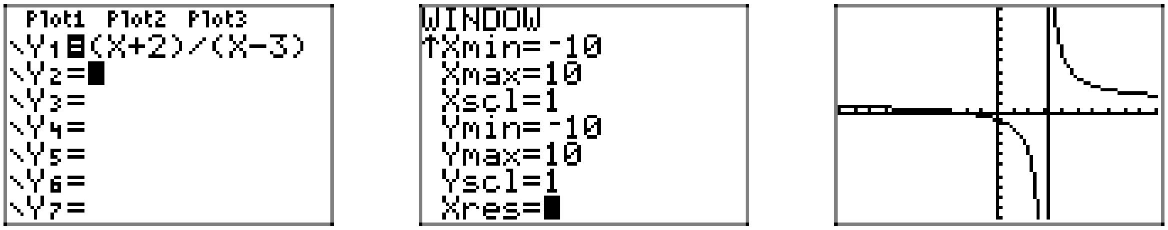


**Figure 9.** The graph approaches the horizontal asymptote $y = 1$ at the extreme right- and left-ends.

What kind of job will the graphing calculator do with the graph of this rational function? In **Figure 10**(a), we enter the function, adjust the window parameters as shown in **Figure 10**(b), then push the `GRAPH` button to produce the result in **Figure 10**(c).



```
Plot1 Plot2 Plot3
\Y1=(X+2)/(X-3)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

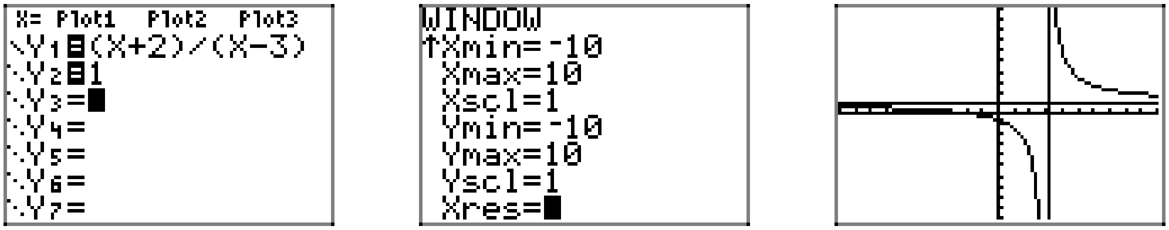
```
WINDOW
↑Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=
```

(a) (b) (c)

**Figure 10.** Drawing the graph of the rational function with the graphing calculator.

As was discussed in the first section, the graphing calculator manages the graphs of "continuous" functions extremely well, but has difficulty drawing graphs with discontinuities. In the case of the present rational function, the graph "jumps" from negative infinity to positive infinity across the vertical asymptote $x = 3$. The calculator knows only one thing: plot a point, then connect it to the previously plotted point with a line segment. Consequently, it does what it is told, and "connects" infinities when it shouldn't.

However, if we have prepared in advance, identifying zeros and vertical asymptotes, then we can interpret what we see on the screen in **Figure 10**(c), and use that information to produce the correct graph that is shown in **Figure 9**. We can even add the horizontal asymptote to our graph, as shown in the sequence in **Figure 11**.



```
Plot1  Plot2  Plot3
\Y1=(X+2)/(X-3)
\Y2=1
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
WINDOW
↑Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=
```

(a) (b) (c)

**Figure 11.** Adding a suspected horizontal asymptote.

◆

This is an appropriate point to pause and summarize the steps required to draw the graph of a rational function.

**Procedure for Graphing Rational Functions.** Consider the rational function

$$f(x)=\frac{a_0+a_1x+a_2x^2+\cdots+a_nx^n}{b_0+b_1x+b_2x^2+\cdots+b_mx^m}.$$

To draw the graph of this rational function, proceed as follows:

1. Factor the numerator and denominator of the rational function $f$.
2. Identify the domain of the rational function $f$ by listing each restriction, values of the independent variable (usually $x$) that make the denominator equal to zero.
3. Identify the values of the independent variable that make the numerator of $f$ equal to zero and are not restrictions. These are the zeros of $f$ and they provide the $x$-coordinates of the $x$-intercepts of the graph of the rational function. Plot these intercepts on a coordinate system and label them with their coordinates.
4. Cancel common factors to reduce the rational function to lowest terms.
   - The restrictions of $f$ that remain restrictions of this reduced form will place vertical asymptotes in the graph of $f$. Draw the vertical asymptotes on your coordinate system as dashed lines and label them with their equations.
   - The restrictions of $f$ that are not restrictions of the reduced form will place "holes" in the graph of $f$. We'll deal with the holes in step 8 of this procedure.
5. To determine the behavior near each vertical asymptote, calculate and plot one point on each side of each vertical asymptote.
6. To determine the end-behavior of the given rational function, use the table capability of your calculator to determine the limit of the function as $x$ approaches positive and/or negative infinity (as we did in the sequences shown in **Figure 7** and **Figure 8**). This determines the horizontal asymptote. Sketch the horizontal asymptote as a dashed line on your coordinate system and label it with its equation.
7. Draw the graph of the rational function.
8. If you determined that a restriction was a "hole," use the restriction and the reduced form of the rational function to determine the $y$-value of the "hole." Draw an open circle at this position to represent the "hole" and label the "hole" with its coordinates.
9. Finally, use your calculator to check the validity of your result.

Let's look at another example.

▶ **Example 10.** *Sketch the graph of the rational function*

$$f(x)=\frac{x-2}{x^2-3x-4}. \tag{11}$$

We will follow the outline presented in the Procedure for Graphing Rational Functions.

**Step 1**: First, factor both numerator and denominator.

$$f(x) = \frac{x-2}{(x+1)(x-4)} \tag{12}$$

**Step 2**: Thus, $f$ has two restrictions, $x = -1$ and $x = 4$. That is, the domain of $f$ is $D_f = \{s : x \neq -1, 4\}$.

**Step 3**: The numerator of **equation (12)** is zero at $x = 2$ and this value is not a restriction. Thus, 2 is a zero of $f$ and $(2, 0)$ is an $x$-intercept of the graph of $f$, as shown in **Figure 12**.

**Step 4**: Note that the rational function is already reduced to lowest terms (if it weren't, we'd reduce at this point). Note that the restrictions $x = -1$ and $x = 4$ are still restrictions of the reduced form. Hence, these are the locations and equations of the vertical asymptotes, which are also shown in **Figure 12**.

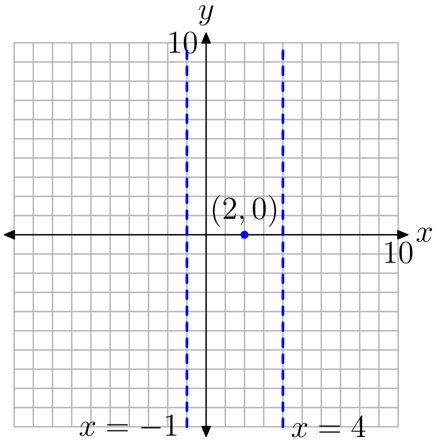
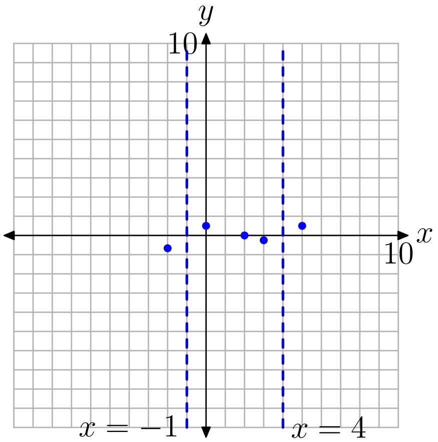


**Figure 12.** Plot the $x$-intercepts and draw the vertical asymptotes.

All of the restrictions of the original function remain restrictions of the reduced form. Therefore, there will be no "holes" in the graph of $f$.

**Step 5**: Plot points to the immediate right and left of each asymptote, as shown in **Figure 13**. These additional points completely determine the behavior of the graph near each vertical asymptote. For example, consider the point $(5, 1/2)$ to the immediate right of the vertical asymptote $x = 4$ in **Figure 13**. Because there is no $x$-intercept between $x = 4$ and $x = 5$, and the graph is already above the $x$-axis at the point $(5, 1/2)$, the graph is forced to increase to positive infinity as it approaches the vertical asymptote $x = 4$. Similar comments are in order for the behavior on each side of each vertical asymptote.

**Step 6**: Use the table utility on your calculator to determine the end-behavior of the rational function as $x$ decreases and/or increases without bound. To determine the end-behavior as $x$ goes to infinity (increases without bound), enter the equation in your calculator, as shown in **Figure 14**(a). Select `2nd TBLSET` and highlight `ASK` for the independent variable. Select `2nd TABLE`, then enter 10, 100, 1 000, and 10 000, as shown in **Figure 14**(c).



| $x$ | $y = \dfrac{x-2}{(x+1)(x-4)}$ |
|---|---|
| $-2$ | $-2/3$ |
| $0$ | $1/2$ |
| $3$ | $-1/4$ |
| $5$ | $1/2$ |

**Figure 13.** Additional points help determine the behavior near the vertical asymptote.

```
X= Plot1  Plot2  Plot3
\Y1=(X-2)/((X+1)
*(X-4))
\Y2=■
\Y3=
\Y4=
\Y5=
\Y6=
```

(a)

```
TABLE SETUP
 TblStart=10
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

(b)

```
  X     Y1
 10     .12121
 100    .01011
 1000   .001
 10000  1E-4

X=
```

(c)

**Figure 14.** Examining end-behavior as $x$ approaches positive infinity.

If you examine the $y$-values in **Figure 14**(c), you see that they are heading towards zero (`1e-4` means $1 \times 10^{-4}$, which equals 0.0001). This implies that the line $y = 0$ (the $x$-axis) is acting as a horizontal asymptote.

You can also determine the end-behavior as $x$ approaches negative infinity (decreases without bound), as shown in the sequence in **Figure 15**. The result in **Figure 15**(c) provides clear evidence that the $y$-values approach zero as $x$ goes to negative infinity. Again, this makes $y = 0$ a horizontal asymptote.

```
X= Plot1  Plot2  Plot3
\Y1■(X-2)/((X+1)
*(X-4))
\Y2=■
\Y3=
\Y4=
\Y5=
\Y6=
```

(a)

```
TABLE SETUP
 TblStart=10
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

(b)

```
  X      Y1
 -10     -.0952
 -100    -.0099
 -1000   -1E-3
 -10000  -1E-4

X=
```

(c)

**Figure 15.** Examining end-behavior as $x$ approaches negative infinity.

Add the horizontal asymptote $y = 0$ to the image in **Figure 13**.

**Step 7**: We can use all the information gathered to date to draw the image shown in **Figure 16**.

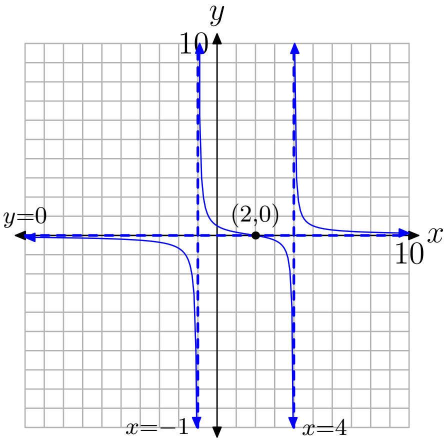
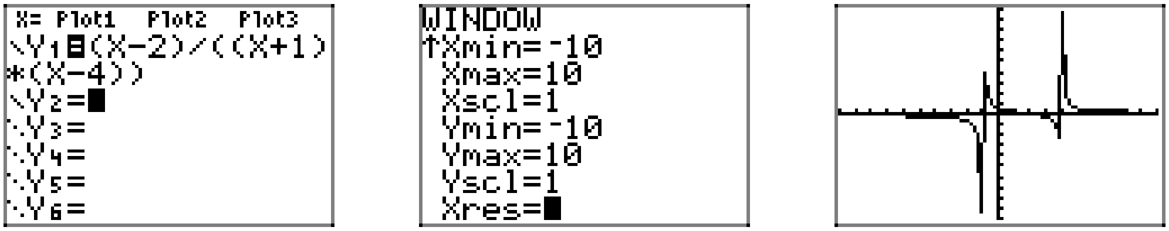


**Figure 16.** The completed graph runs up against vertical and horizontal asymptotes and crosses the $x$-axis at the zero of the function.

**Step 8**: As stated above, there are no “holes” in the graph of $f$.

**Step 9**: Use your graphing calculator to check the validity of your result. Note how the graphing calculator handles the graph of this rational function in the sequence in **Figure 17**. The image in **Figure 17**(c) is nowhere near the quality of the image we have in **Figure 16**, but there is enough there to intuit the actual graph if you prepare properly in advance (zeros, vertical asymptotes, end-behavior analysis, etc.).



```
 Plot1  Plot2  Plot3
\Y1=(X-2)/((X+1)
*(X-4))
\Y2=█
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
↑Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=█
```

(a) (b) (c)

**Figure 17.** The user of the graphing calculator must decipher the image in the calculator’s view screen.

◇

## 7.3 Exercises

For rational functions **Exercises 1-20**, follow the Procedure for Graphing Rational Functions in the narrative, performing each of the following tasks.

For rational functions **Exercises 1-20**, perform each of the following tasks.

i. Set up a coordinate system on graph paper. Label and scale each axis. Remember to draw all lines with a ruler.
ii. Perform each of the nine steps listed in the Procedure for Graphing Rational Functions in the narrative.

**1.** $f(x) = (x-3)/(x+2)$

**2.** $f(x) = (x+2)/(x-4)$

**3.** $f(x) = (5-x)/(x+1)$

**4.** $f(x) = (x+2)/(4-x)$

**5.** $f(x) = (2x-5)/(x+1)$

**6.** $f(x) = (2x+5/(3-x)$

**7.** $f(x) = (x+2)/(x^2-2x-3)$

**8.** $f(x) = (x-3)/(x^2-3x-4)$

**9.** $f(x) = (x+1)/(x^2+x-2)$

**10.** $f(x) = (x-1)/(x^2-x-2)$

**11.** $f(x) = (x^2-2x)/(x^2+x-2)$

**12.** $f(x) = (x^2-2x)/(x^2-2x-8)$

**13.** $f(x) = (2x^2-2x-4)/(x^2-x-12)$

**14.** $f(x) = (8x-2x^2)/(x^2-x-6)$

**15.** $f(x) = (x-3)/(x^2-5x+6)$

**16.** $f(x) = (2x-4)/(x^2-x-2)$

**17.** $f(x) = (2x^2-x-6)/(x^2-2x)$

**18.** $f(x) = (2x^2-x-6)/(x^2-2x)$

**19.** $f(x) = (4+2x-2x^2)/(x^2+4x+3)$

**20.** $f(x) = (3x^2-6x-9)/(1-x^2)$

In **Exercises 21-28**, find the coordinates of the $x$-intercept(s) of the graph of the given rational function.

**21.** $f(x) = \dfrac{81-x^2}{x^2+10x+9}$

**22.** $f(x) = \dfrac{x-x^2}{x^2+5x-6}$

**23.** $f(x) = \dfrac{x^2-x-12}{x^2+2x-3}$

**24.** $f(x) = \dfrac{x^2-81}{x^2-4x-45}$

**25.** $f(x) = \dfrac{6x-18}{x^2-7x+12}$

**26.** $f(x) = \dfrac{4x+36}{x^2+15x+54}$

**27.** $f(x) = \dfrac{x^2-9x+14}{x^2-2x}$

**28.** $f(x) = \dfrac{x^2-5x-36}{x^2-9x+20}$

---

[10] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

In **Exercises 29-36**, find the equations of all vertical asymptotes.

**29.** $f(x) = \dfrac{x^2 - 7x}{x^2 - 2x}$

**30.** $f(x) = \dfrac{x^2 + 4x - 45}{3x + 27}$

**31.** $f(x) = \dfrac{x^2 - 6x + 8}{x^2 - 16}$

**32.** $f(x) = \dfrac{x^2 - 11x + 18}{2x - x^2}$

**33.** $f(x) = \dfrac{x^2 + x - 12}{-4x + 12}$

**34.** $f(x) = \dfrac{x^2 - 3x - 54}{9x - x^2}$

**35.** $f(x) = \dfrac{16 - x^2}{x^2 + 7x + 12}$

**36.** $f(x) = \dfrac{x^2 - 11x + 30}{-8x + 48}$

---

In **Exercises 37-42**, use a graphing calculator to determine the behavior of the given rational function as $x$ approaches both positive and negative infinity by performing the following tasks:

i. Load the rational function into the `Y=` menu of your calculator.
ii. Use the `TABLE` feature of your calculator to determine the value of $f(x)$ for $x = 10$, 100, 1000, and 10000. Record these results on your homework in table form.
iii. Use the `TABLE` feature of your calculator to determine the value of $f(x)$ for $x = -10$, $-100$, $-1000$, and $-10000$. Record these results on your homework in table form.
iv. Use the results of your tabular exploration to determine the equation of the horizontal asymptote.

**37.** $f(x) = (2x + 3)/(x - 8)$

**38.** $f(x) = (4 - 3x)/(x + 2)$

**39.** $f(x) = (4 - x^2)/(x^2 + 4x + 3)$

**40.** $f(x) = (10 - 2x^2)/(x^2 - 4)$

**41.** $f(x) = (x^2 - 2x - 3)/(2x^2 - 3x - 2)$

**42.** $f(x) = (2x^2 - 3x - 5)/(x^2 - x - 6)$

---

In **Exercises 43-48**, use a purely analytical method to determine the domain of the given rational function. Describe the domain using set-builder notation.

**43.** $f(x) = \dfrac{x^2 - 5x - 6}{-9x - 9}$

**44.** $f(x) = \dfrac{x^2 + 4x + 3}{x^2 - 5x - 6}$

**45.** $f(x) = \dfrac{x^2 + 5x - 24}{x^2 - 3x}$

**46.** $f(x) = \dfrac{x^2 - 3x - 4}{x^2 - 5x - 6}$

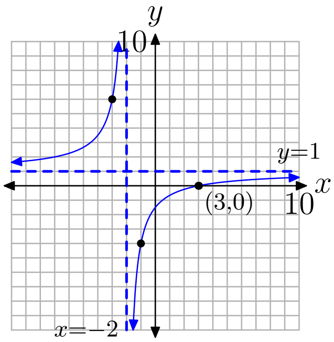
**47.** $f(x) = \dfrac{x^2 - 4x + 3}{x - x^2}$

**48.** $f(x) = \dfrac{x^2 - 4}{x^2 - 9x + 14}$

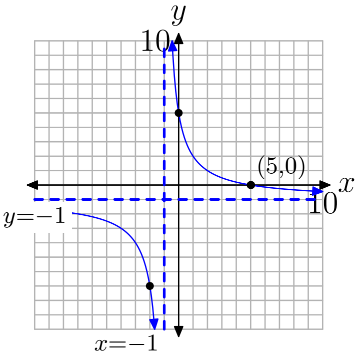
## 7.3 Answers

---

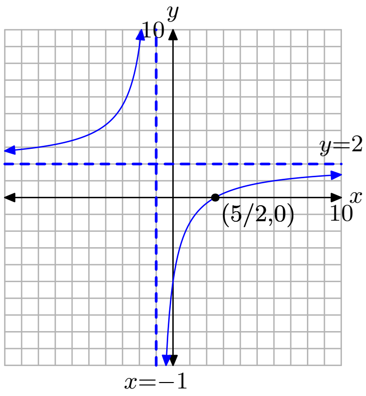
**1.**



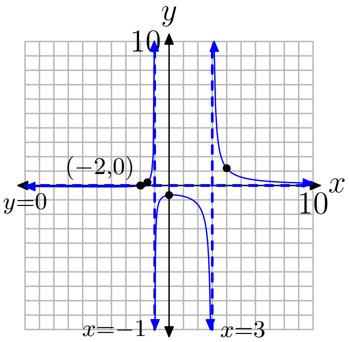
**3.**



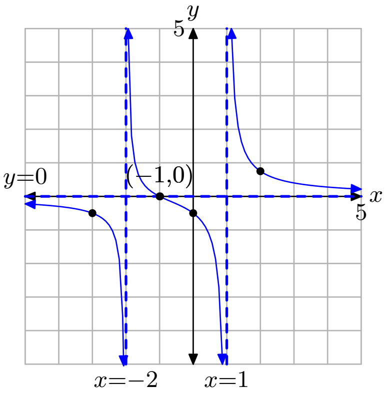
**5.**



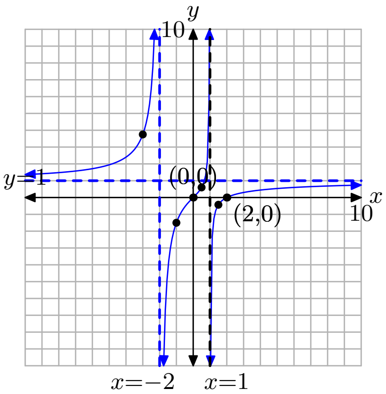
**7.**



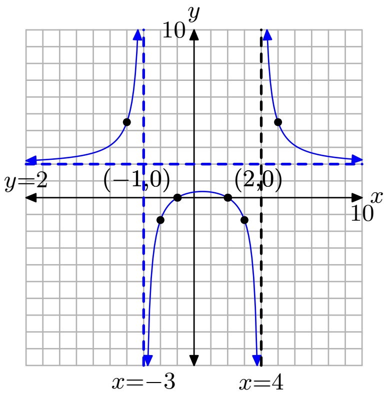
**9.**



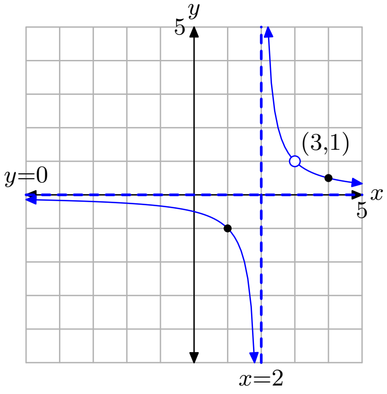
**11.**



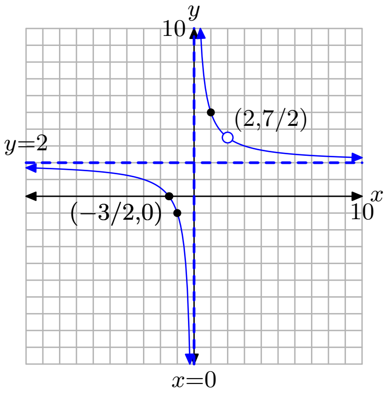
**13.**



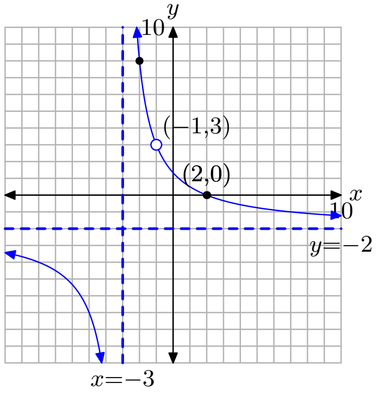
**15.**



**17.**



**19.**



**21.** $(9, 0)$

**23.** $(4, 0)$

**25.** no $x$-intercepts

**27.** $(7, 0)$

**29.** $x = 2$

**31.** $x = -4$

**33.** no vertical asymptotes

**35.** $x = -3$

**37.** Horizontal asymptote at $y = 2$.

| X | Y1 | X | Y1 |
|---|---|---|---|
| 10 | 11.5 | -10 | 0.944444 |
| 100 | 2.20652 | -100 | 1.82407 |
| 1000 | 2.01915 | -1000 | 1.98115 |
| 10000 | 2.0019 | -10000 | 1.998 |

**39.** Horizontal asymptote at $y = -1$.

| X | Y1 | X | Y1 |
|---|---|---|---|
| 10 | -0.671329 | -10 | -1.52381 |
| 100 | -0.960877 | -100 | -1.04092 |
| 1000 | -0.996009 | -1000 | -1.00401 |
| 10000 | -0.9996 | -10000 | -1 |

**41.** Horizontal asymptote at $y = 1/2$.

| X | Y1 | X | Y1 |
|---|---|---|---|
| 10 | 0.458333 | -10 | 0.513158 |
| 100 | 0.49736 | -100 | 0.502365 |
| 1000 | 0.499749 | -1000 | 0.500249 |
| 10000 | 0.49997 | -10000 | 0.50002 |

**43.** Domain $= \{x : x \neq -1\}$

**45.** Domain $= \{x : x \neq 3, 0\}$

**47.** Domain $= \{x : x \neq 0, 1\}$

# 7.4 Products and Quotients of Rational Functions

In this section we deal with products and quotients of rational expressions. Before we begin, we'll need to establish some fundamental definitions and technique. We begin with the definition of the product of two rational numbers.

**Definition 1.** *Let $a/b$ and $c/d$ be rational numbers. The product of these rational numbers is defined by*

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}, \qquad \text{or more compactly,} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}. \tag{2}$$

The definition simply states that you should multiply the numerators of each rational number to obtain the numerator of the product, and you also multiply the denominators of each rational number to obtain the denominator of the product. For example,

$$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}.$$

Of course, you should also check to make sure your final answer is reduced to lowest terms.

Let's look at an example.

▶ **Example 3.** *Simplify the product of rational numbers*

$$\frac{6}{231} \cdot \frac{35}{10}. \tag{4}$$

First, multiply numerators and denominators together as follows.

$$\frac{6}{231} \cdot \frac{35}{10} = \frac{6 \cdot 35}{231 \cdot 10} = \frac{210}{2310}.$$

However, the answer is not reduced to lowest terms. We can express the numerator as a product of primes.

$$210 = 21 \cdot 10 = 3 \cdot 7 \cdot 2 \cdot 5 = 2 \cdot 3 \cdot 5 \cdot 7$$

It's not necessary to arrange the factors in ascending order, but every little bit helps. The denominator can also be expressed as a product of primes.

$$2310 = 10 \cdot 231 = 2 \cdot 5 \cdot 7 \cdot 33 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$$

We can now cancel common factors.

$$\frac{210}{2310} = \frac{2 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11} = \frac{\cancel{2} \cdot \cancel{3} \cdot \cancel{5} \cdot \cancel{7}}{\cancel{2} \cdot \cancel{3} \cdot \cancel{5} \cdot \cancel{7} \cdot 11} = \frac{1}{11} \tag{5}$$

---

[11] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

However, this approach is not the most efficient way to proceed, as multiplying numerators and denominators allows the products to grow to larger numbers, as in 210/2310. It is then a little bit harder to prime factor the larger numbers.

A better approach is to factor the smaller numerators and denominators immediately, as follows.

$$\frac{6}{231}\cdot\frac{35}{10}=\frac{2\cdot 3}{3\cdot 7\cdot 11}\cdot\frac{5\cdot 7}{2\cdot 5}$$

We could now multiply numerators and denominators, then cancel common factors, which would match identically the last computation in **equation (5)**.

However, we can also employ the following cancellation rule.

**Cancellation Rule.** When working with the product of two or more rational expressions, factor all numerators and denominators, then cancel. The cancellation rule is simple: cancel a factor "on the top" for an identical factor "on the bottom." Speaking more technically, cancel any factor in any numerator for an identical factor in any denominator.

Thus, we can finish our computation by canceling common factors, canceling "something on the top for something on the bottom."

$$\frac{6}{231}\cdot\frac{35}{10}=\frac{2\cdot 3}{3\cdot 7\cdot 11}\cdot\frac{5\cdot 7}{2\cdot 5}=\frac{\cancel{2}\cdot\cancel{3}}{\cancel{3}\cdot\cancel{7}\cdot 11}\cdot\frac{\cancel{5}\cdot\cancel{7}}{\cancel{2}\cdot\cancel{5}}=\frac{1}{11}$$

Note that we canceled a 2, 3, 5, and a 7 "on the top" for a 2, 3, 5, and 7 "on the bottom."[12]

◆

Thus, we have two choices when multiplying rational expressions:

- Multiply numerators and denominators, factor, then cancel.
- Factor numerators and denominators, cancel, then multiply numerators and denominators.

It is the latter approach that we will use in this section. Let's look at another example.

---

[12] Students will sometimes use the phrase "cross-cancel" when working with the product of rational expressions. Unfortunately, this term implies that cancellation can occur only in a diagonal direction, which is far from the truth. We like to tell our students that there is no such term as "cross-cancel." There is only "cancel," and the rule is: cancel something on the top for something on the bottom, which is vernacular for "cancel a factor from any numerator and the identical factor from any denominator."

▶ **Example 6.** *Simplify the expression*

$$\frac{x^2-x-6}{x^2+2x-15}\cdot\frac{x^2-x-30}{x^2-2x-8} \tag{7}$$

*State restrictions.*

Use the *ac*-test to factor each numerator and denominator. Then cancel as shown.

$$\begin{aligned}\frac{x^2-x-6}{x^2+2x-15}\cdot\frac{x^2-x-30}{x^2-2x-8} &= \frac{(x+2)(x-3)}{(x-3)(x+5)}\cdot\frac{(x+5)(x-6)}{(x+2)(x-4)}\\ &= \frac{\cancel{(x+2)}\cancel{(x-3)}}{\cancel{(x-3)}\cancel{(x+5)}}\cdot\frac{\cancel{(x+5)}(x-6)}{\cancel{(x+2)}(x-4)}\\ &= \frac{x-6}{x-4}\end{aligned}$$

The first fraction's denominator has factors $x-3$ and $x+5$. Hence, $x=3$ or $x=-5$ will make this denominator zero. Therefore, the 3 and $-5$ are restrictions.

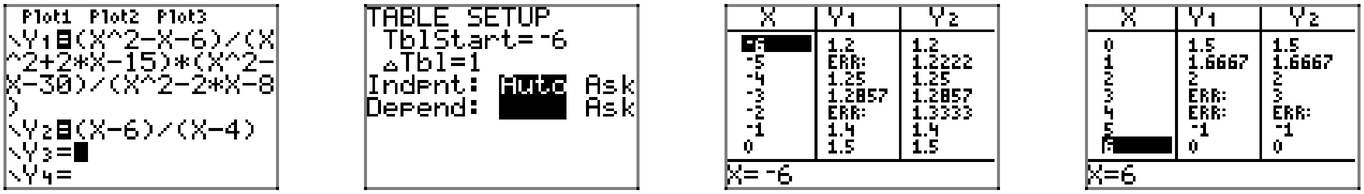
The second fraction's denominator has factors $x+2$ and $x-4$. Hence, $x=-2$ or $x=4$ will make this denominator zero. Therefore, $-2$ and 4 are restrictions.

Therefore, for all values of $x$, except the restrictions $-5$, $-2$, 3, and 4, the left side of

$$\frac{x^2-x-6}{x^2+2x-15}\cdot\frac{x^2-x-30}{x^2-2x-8} = \frac{x-6}{x-4} \tag{8}$$

is identical to its right side.

It's possible to use your graphing calculator to check your results. First, load the left- and right-hand sides of **equation (8)** into the calculator's into `Y1` and `Y2` in your graphing calculator's `Y=` menu, as shown in **Figure 1**(a). Press `2nd TBLSET` and set `TblStart` $=-6$ and $\Delta$`Tbl` $=1$, as shown in **Figure 1**(b). Make sure that `AUTO` is highlighted and selected with the `ENTER` key on both the independent and dependent variables. Press `2nd TABLE` to produce the tabular display in **Figure 1**(c).



```
Plot1 Plot2 Plot3
\Y1=(X^2-X-6)/(X
^2+2*X-15)*(X^2-
X-30)/(X^2-2*X-8
)
\Y2=(X-6)/(X-4)
\Y3=
\Y4=
```

(a)

```
TABLE SETUP
 TblStart=-6
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

(b)

```
X     Y1      Y2
-6    1.2     1.2
-5    ERR:    1.2222
-4    1.25    1.25
-3    1.2857  1.2857
-2    ERR:    1.3333
-1    1.4     1.4
0     1.5     1.5
X=-6
```

(c)

```
X     Y1      Y2
0     1.5     1.5
1     1.6667  1.6667
2     2       2
3     ERR:    3
4     ERR:    ERR:
5     -1      -1
6     0       0
X=6
```

(d)

**Figure 1.** Using the table features of the graphing calculator to check the result in **equation (8)**.

Remember that the left- and right-hand sides of **equation (8)** are loaded in `Y1` and `Y2`, respectively.

- In **Figure 1**(c), note the `ERR` (error) message at the restricted values of $x = -5$ and $x = -2$. However, other than at these two restrictions, the functions `Y1` and `Y2` agree at all other values of $x$ in **Figure 1**(c).
- Use the down arrow to scroll down in the table to produce the tabular results shown in **Figure 1**(d). Note the `ERR` (error) message at the restricted values of $x = 3$ and $x = 4$. However, other than at these two restrictions, the functions `Y1` and `Y2` agree at all other values of $x$ in **Figure 1**(d).
- If you scroll up or down in the table, you'll find that the functions `Y1` and `Y2` agree at all values of $x$ other than the restricted values $-5$, $-2$, 3, and 4.

◆

Let's look at another example.

▶ **Example 9.** *Simplify*

$$\frac{9-x^2}{x^2+3x} \cdot \frac{6x-2x^2}{x^2-6x+9} \tag{10}$$

*State any restrictions.*

The first numerator can be factored using the difference of two squares pattern.

$$9 - x^2 = (3+x)(3-x).$$

The second denominator is a perfect square trinomial and can be factored as the square of a binomial.

$$x^2 - 6x + 9 = (x-3)^2$$

You will want to remove the greatest common factor from the first denominator and second numerator.

$$x^2 + 3x = x(x+3) \qquad \text{and} \qquad 6x - 2x^2 = 2x(3-x)$$

Thus,

$$\frac{9-x^2}{x^2+3x} \cdot \frac{6x-2x^2}{x^2-6x+9}. = \frac{(3+x)(3-x)}{x(x+3)} \cdot \frac{2x(3-x)}{(x-3)^2}.$$

We'll need to execute a sign change or two to create common factors in the numerators and denominators. So, in both the first and second numerator, factor a $-1$ from the factor $3 - x$ to obtain $3 - x = -1(x - 3)$. Because the order of factors in a product doesn't matter, we'll slide the $-1$ to the front in each case.

$$\frac{9-x^2}{x^2+3x} \cdot \frac{6x-2x^2}{x^2-6x+9}. = \frac{-(3+x)(x-3)}{x(x+3)} \cdot \frac{-2x(x-3)}{(x-3)^2}.$$

We can now cancel common factors.

$$\begin{aligned}\frac{9-x^2}{x^2+3x}\cdot\frac{6x-2x^2}{x^2-6x+9} &= \frac{-(3+x)(x-3)}{x(x+3)}\cdot\frac{-2x(x-3)}{(x-3)^2}\\ &= \frac{-(\cancel{3+x})(\cancel{x-3})}{\cancel{x}(\cancel{x+3})}\cdot\frac{-2\cancel{x}(\cancel{x-3})}{\cancel{(x-3)^2}}\\ &= 2\end{aligned}$$

A few things to notice:

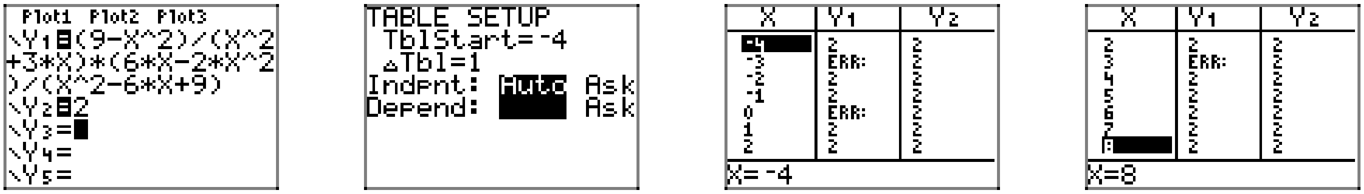
- The factors $3+x$ and $x+3$ are identical, so they may be cancelled, one on the top for one on the bottom.
- Two factors of $x-3$ on the top are cancelled for $(x-3)^2$ (which is equivalent to $(x-3)(x-3)$) on the bottom.
- An $x$ on top cancels an $x$ on the bottom.
- We're left with two minus signs (two $-1$'s) and a 2. So the solution is a positive 2.

Finally, the first denominator has factors $x$ and $x+3$, so $x=0$ and $x=-3$ are restrictions (they make this denominator equal to zero). The second denominator has two factors of $x-3$, so $x=3$ is an additional restriction.

Hence, for all values of $x$, except the restricted values $-3$, 0, and 3, the left-hand side of

$$\frac{9-x^2}{x^2+3x}\cdot\frac{6x-2x^2}{x^2-6x+9}=2 \tag{11}$$

is identical to the right-hand side. Again, this claim is easily tested on the graphing calculator which is evidenced in the sequence of screen captures in **Figure 2**.



```
 Plot1  Plot2  Plot3
\Y1■(9-X^2)/(X^2
+3*X)*(6*X-2*X^2
)/(X^2-6*X+9)
\Y2■2
\Y3=
\Y4=
\Y5=
```

(a)

```
TABLE SETUP
 TblStart=-4
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

(b)

```
   X   |  Y1  |  Y2
-4     | 2    | 2
-3     | ERR: | 2
-2     | 2    | 2
-1     | 2    | 2
0      | ERR: | 2
1      | 2    | 2
2      | 2    | 2
X=-4
```

(c)

```
   X   |  Y1  |  Y2
2      | 2    | 2
3      | ERR: | 2
4      | 2    | 2
5      | 2    | 2
6      | 2    | 2
7      | 2    | 2
8      | 2    | 2
X=8
```

(d)

**Figure 2.** Using the table features of the graphing calculator to check the result in **equation (11)**.

An alternate approach to the problem in **equation (10)** is to note differing orders in the numerators and denominators (descending, ascending powers of $x$) and *anticipate* the need for a sign change. That is, make the sign change before you factor.

For example, negate (multiply by $-1$) both numerator and fraction bar of the first fraction to obtain

$$\frac{9-x^2}{x^2+3x}=-\frac{x^2-9}{x^2+3x}$$

According to our sign change rule, negating any two parts of a fraction leaves the fraction unchanged.

If we perform a similar sign change on the second fraction (negate numerator and fraction bar), then we can factor and cancel common factors.

$$\begin{aligned}\frac{9-x^2}{x^2+3x}\cdot\frac{6x-2x^2}{x^2-6x+9} &= -\frac{x^2-9}{x^2+3x}\cdot-\frac{2x^2-6x}{x^2-6x+9}\\ &= -\frac{(x+3)(x-3)}{x(x+3)}\cdot-\frac{2x(x-3)}{(x-3)^2}\\ &= -\frac{\cancel{(x+3)}\cancel{(x-3)}}{\cancel{x}\cancel{(x+3)}}\cdot-\frac{2\cancel{x}\cancel{(x-3)}}{\cancel{(x-3)^2}}\\ &= 2\end{aligned}$$

## *Division of Rational Expressions*

A simple definition will change a problem involving division of two rational expressions into one involving multiplication of two rational expressions. Then there's nothing left to explain, for we already know how to multiply two rational expressions.

So, let's motivate our definition of division. Suppose we ask the question, how many halves are in a whole? The answer is easy, as two halves make a whole. Thus, when we divide 1 by 1/2, we should get 2. There are two halves in one whole.

Let's raise the stakes a bit and ask how many halves are in six? To make the problem more precise, imagine you've ordered 6 pizzas and you cut each in half. How many halves do you have? Again, this is easy when you think about the problem in this manner, the answer is 12. Thus,

$$6\div\frac{1}{2}$$

(how many halves are in six) is identical to

$$6\cdot 2,$$

which, of course, is 12. Hopefully, thanks to this opening motivation, the following definition will not seem too strange.

**Definition 12.** *To perform the division*

$$\frac{a}{b}\div\frac{c}{d},$$

*invert the second fraction and multiply, as in*

$$\frac{a}{b}\cdot\frac{d}{c}.$$

Thus, if we want to know how many halves are in 12, we change the division into multiplication ("invert and multiply").

$$12 \div \frac{1}{2} = 12 \cdot 2 = 24$$

This makes sense, as there are 24 “halves” in 12. Let’s look at a harder example.

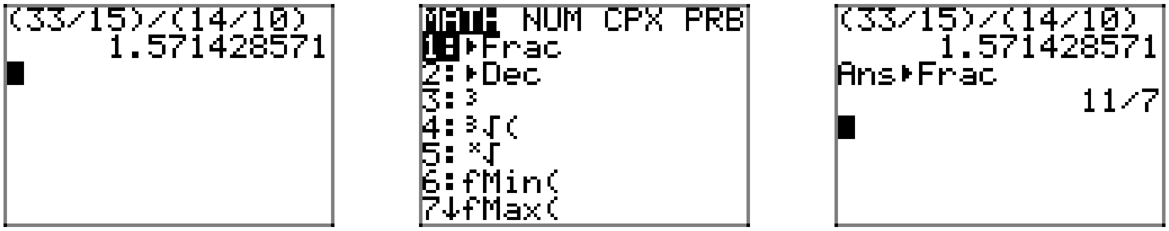
▶ **Example 13.** *Simplify*

$$\frac{33}{15} \div \frac{14}{10}. \tag{14}$$

Invert the second fraction and multiply. After that, all we need to do is factor numerators and denominators, then cancel common factors.

$$\frac{33}{15} \div \frac{14}{10} = \frac{33}{15} \cdot \frac{10}{14} = \frac{3 \cdot 11}{3 \cdot 5} \cdot \frac{2 \cdot 5}{2 \cdot 7} = \frac{\cancel{3} \cdot 11}{\cancel{3} \cdot \cancel{5}} \cdot \frac{\cancel{2} \cdot \cancel{5}}{\cancel{2} \cdot 7} = \frac{11}{7}$$

An interesting way to check this result on your calculator is shown in the sequence of screens in **Figure 3**.



```
(33/15)/(14/10)
      1.571428571
```

```
(33/15)/(14/10)
      1.571428571
Ans▸Frac
             11/7
```

(a) (b) (c)

**Figure 3.** Using the calculator to check division of fractions.

After entering the original problem in your calculator, press `ENTER`, then press the `MATH` button, then select `1:▶ Frac` from the menu and press `ENTER`. The result is shown in **Figure 3**(c), which agrees with our calculation above.

◆

Let’s look at another example.

▶ **Example 15.** *Simplify*

$$\frac{9 + 3x - 2x^2}{x^2 - 16} \div \frac{4x^3 - 9x}{2x^2 + 5x - 12}. \tag{16}$$

*State the restrictions.*

Note the order of the first numerator differs from the other numerators and denominators, so we “anticipate” the need for a sign change, negating the numerator and fraction bar of the first fraction. We also invert the second fraction and change the division to multiplication (“invert and multiply”).

$$-\frac{2x^2 - 3x - 9}{x^2 - 16} \cdot \frac{2x^2 + 5x - 12}{4x^3 - 9x} \tag{17}$$

The numerator in the first fraction in **equation (17)** is a quadratic trinomial, with $ac = (2)(-9) = -18$. The integer pair 3 and $-6$ has product $-18$ and sum $-3$. Hence,

$$\begin{aligned} 2x^2 - 3x - 9 &= 2x^2 + 3x - 6x - 9 \\ &= x(2x+3) - 3(2x+3) \\ &= (x-3)(2x+3). \end{aligned}$$

The denominator of the first fraction in **equation (17)** easily factors using the difference of two squares pattern.

$$x^2 - 16 = (x+4)(x-4)$$

The numerator of the second fraction in **equation (17)** is a quadratic trinomial, with $ac = (2)(-12) = -24$. The integer pair $-3$ and 8 have product $-24$ and sum 5. Hence,

$$\begin{aligned} 2x^2 + 5x - 12 &= 2x^2 - 3x + 8x - 12 \\ &= x(2x-3) + 4(2x-3) \\ &= (x+4)(2x-3). \end{aligned}$$

To factor the denominator of the last fraction in **equation (17)**, first pull the greatest common factor (in this case $x$), then complete the factorization using the difference of two squares pattern.

$$4x^3 - 9x = x(4x^2 - 9) = x(2x+3)(2x-3)$$

We can now replace each numerator and denominator in **equation (17)** with its factorization, then cancel common factors.

$$\begin{aligned} -\frac{2x^2 - 3x - 9}{x^2 - 16} \cdot \frac{2x^2 + 5x - 12}{4x^3 - 9x} &= -\frac{(x-3)(2x+3)}{(x+4)(x-4)} \cdot \frac{(x+4)(2x-3)}{x(2x+3)(2x-3)} \\ &= -\frac{(x-3)\cancel{(2x+3)}}{\cancel{(x+4)}(x-4)} \cdot \frac{\cancel{(x+4)}\cancel{(2x-3)}}{x\cancel{(2x+3)}\cancel{(2x-3)}} \\ &= -\frac{x-3}{x(x-4)} \end{aligned}$$

The last denominator has factors $x$ and $x - 4$, so $x = 0$ and $x = 4$ are restrictions. In the body of our work, the first fraction's denominator has factors $x + 4$ and $x - 4$. We've seen the factor $x - 4$ already, so only the factor $x + 4$ adds a new restriction, $x = -4$. Again, in the body of our work, the second fraction's denominator has factors $x$, $2x + 3$, and $2x - 3$, so we have added restrictions $x = 0$, $x = -3/2$, and $x = 3/2$.

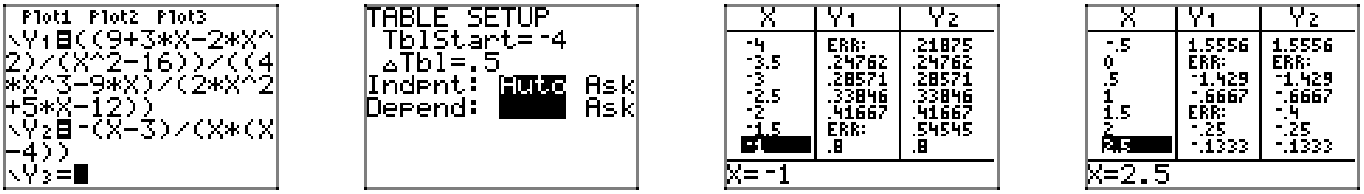
There's one bit of trickery here that can easily be overlooked. In the body of our work, the second fraction's numerator was originally a denominator before we inverted the fraction. So, we must consider what makes this numerator zero as well. Fortunately, the factors in this numerator are $x + 4$ and $2x - 3$ and we've already considered the restrictions produced by these factors.

Hence, for all values of $x$, except the restricted values $-4$, $-3/2$, 0, $3/2$, and 4, the left-hand side of

$$\frac{9 + 3x - 2x^2}{x^2 - 16} \div \frac{4x^3 - 9x}{2x^2 + 5x - 12} = -\frac{x-3}{x(x-4)} \tag{18}$$

is identical to the right-hand side.

Again, this claim is easily checked by using a graphing calculator, as is partially evidenced (you'll have to scroll downward to see the last restriction come into view) in the sequence of screen captures in **Figure 4**.



```
 Plot1  Plot2  Plot3
\Y1■((9+3*X-2*X^
2)/(X^2-16))/((4
*X^3-9*X)/(2*X^2
+5*X-12))
\Y2■-(X-3)/(X*(X
-4))
\Y3=■
```

(a)

```
TABLE SETUP
 TblStart=-4
 ΔTbl=.5
Indpnt: Auto Ask
Depend: Auto Ask
```

(b)

```
   X    | Y1     | Y2
 -4     | ERR:   | .21875
 -3.5   | .24762 | .24762
 -3     | .28571 | .28571
 -2.5   | .33846 | .33846
 -2     | .41667 | .41667
 -1.5   | ERR:   | .54545
 -1     | .8     | .8
X=-1
```

(c)

```
   X    | Y1     | Y2
 -.5    | 1.5556 | 1.5556
 0      | ERR:   | ERR:
 .5     | -1.429 | -1.429
 1      | -.6667 | -.6667
 1.5    | ERR:   | -.4
 2      | -.25   | -.25
 2.5    | -.1333 | -.1333
X=2.5
```

(d)

**Figure 4.** Using the table feature of the calculator to check the result in **equation (18)**.

◆

**Alternative Notation**. Note that the fractional expression $a/b$ means "$a$ divided by $b$," so we can use this equivalent notation for $a \div b$. For example, the expression

$$\frac{9+3x-2x^2}{x^2-16} \div \frac{4x^3-9x}{2x^2+5x-12} \tag{19}$$

is equivalent to the expression

$$\frac{\dfrac{9+3x-2x^2}{x^2-16}}{\dfrac{4x^3-9x}{2x^2+5x-12}}. \tag{20}$$

Let's look at an example of this notation in use.

▶ **Example 21.** *Given that*

$$f(x) = \frac{x}{x+3} \qquad \text{and} \qquad g(x) = \frac{x^2}{x+3},$$

*simplify both $f(x)g(x)$ and $f(x)/g(x)$.*

First, the multiplication. There is no possible cancellation, so we simply multiply numerators and denominators.

$$f(x)g(x) = \frac{x}{x+3} \cdot \frac{x^2}{x+3} = \frac{x^3}{(x+3)^2}.$$

This result is valid for all values of $x$ except $-3$.

On the other hand,

$$\frac{f(x)}{g(x)} = \frac{\dfrac{x}{x+3}}{\dfrac{x^2}{x+3}} = \frac{x}{x+3} \div \frac{x^2}{x+3}.$$

When we “invert and multiply,” then cancel, we obtain

$$\frac{f(x)}{g(x)} = \frac{x}{x+3} \cdot \frac{x+3}{x^2} = \frac{1}{x}.$$

This result is valid for all values of $x$ except $-3$ and $0$.

◆

## 7.4 Exercises

---

In **Exercises 1-10**, reduce the product to a single fraction in lowest terms.

**1.** $\dfrac{108}{14} \cdot \dfrac{6}{100}$

**2.** $\dfrac{75}{63} \cdot \dfrac{18}{45}$

**3.** $\dfrac{189}{56} \cdot \dfrac{12}{27}$

**4.** $\dfrac{45}{72} \cdot \dfrac{63}{64}$

**5.** $\dfrac{15}{36} \cdot \dfrac{28}{100}$

**6.** $\dfrac{189}{49} \cdot \dfrac{32}{25}$

**7.** $\dfrac{21}{100} \cdot \dfrac{125}{16}$

**8.** $\dfrac{21}{35} \cdot \dfrac{49}{45}$

**9.** $\dfrac{56}{20} \cdot \dfrac{98}{32}$

**10.** $\dfrac{27}{125} \cdot \dfrac{4}{12}$

---

In **Exercises 11-34**, multiply and simplify. State all restrictions.

**11.**

$$\frac{x+6}{x^2+16x+63} \cdot \frac{x^2+7x}{x+4}$$

**12.**

$$\frac{x^2+9x}{x^2-25} \cdot \frac{x^2-x-20}{-18-11x-x^2}$$

**13.**

$$\frac{x^2+7x+10}{x^2-1} \cdot \frac{-9+10x-x^2}{x^2+9x+20}$$

**14.**

$$\frac{x^2+5x}{x-4} \cdot \frac{x-2}{x^2+6x+5}$$

**15.**

$$\frac{x^2-5x}{x^2+2x-48} \cdot \frac{x^2+11x+24}{x^2-x}$$

**16.**

$$\frac{x^2-6x-27}{x^2+10x+24} \cdot \frac{x^2+13x+42}{x^2-11x+18}$$

**17.**

$$\frac{-x-x^2}{x^2-9x+8} \cdot \frac{x^2-4x+3}{x^2+4x+3}$$

**18.**

$$\frac{x^2-12x+35}{x^2+2x-15} \cdot \frac{45+4x-x^2}{x^2+x-30}$$

**19.**

$$\frac{x+2}{7-x} \cdot \frac{x^2+x-56}{x^2+7x+6}$$

**20.**

$$\frac{x^2-2x-15}{x^2+x} \cdot \frac{x^2+7x}{x^2+12x+27}$$

**21.**

$$\frac{x^2-9}{x^2-4x-45} \cdot \frac{x-6}{-3-x}$$

---

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**22.**

$$\frac{x^2-12x+27}{x-4}\cdot\frac{x-5}{x^2-18x+81}$$

**23.**

$$\frac{x+5}{x^2+12x+32}\cdot\frac{x^2-2x-24}{x+7}$$

**24.**

$$\frac{x^2-36}{x^2+11x+24}\cdot\frac{-8-x}{x+4}$$

**25.**

$$\frac{x-5}{x^2-8x+12}\cdot\frac{x^2-12x+36}{x-8}$$

**26.**

$$\frac{x^2-5x-36}{x-1}\cdot\frac{x-5}{x^2-81}$$

**27.**

$$\frac{x^2+2x-15}{x^2-10x+16}\cdot\frac{x^2-7x+10}{3x^2+13x-10}$$

**28.**

$$\frac{5x^2+14x-3}{x+9}\cdot\frac{x-7}{x^2+10x+21}$$

**29.**

$$\frac{x^2-4}{x^2+2x-63}\cdot\frac{x^2+6x-27}{x^2-6x-16}$$

**30.**

$$\frac{x^2+5x+6}{x^2-3x}\cdot\frac{x^2-5x}{x^2+9x+18}$$

**31.**

$$\frac{x-1}{x^2+2x-63}\cdot\frac{x^2-81}{x+4}$$

**32.**

$$\frac{x^2+9x}{x^2+7x+12}\cdot\frac{27+6x-x^2}{x^2-5x}$$

**33.**

$$\frac{5-x}{x+3}\cdot\frac{x^2+3x-18}{2x^2-7x-15}$$

**34.**

$$\frac{4x^2+21x+5}{18-7x-x^2}\cdot\frac{x^2+11x+18}{x^2-25}$$

---

In **Exercises 35-58**, divide and simplify. State all restrictions.

**35.**

$$\frac{\dfrac{x^2-14x+48}{x^2+10x+16}}{\dfrac{-24+11x-x^2}{x^2-x-72}}$$

**36.**

$$\frac{x-1}{x^2-14x+48}\div\frac{x+5}{x^2-3x-18}$$

**37.**

$$\frac{x^2-1}{x^2-7x+12}\div\frac{x^2+6x+5}{-24+10x-x^2}$$

**38.**

$$\frac{x^2-13x+42}{x^2-2x-63}\div\frac{x^2-x-42}{x^2+8x+7}$$

**39.**

$$\frac{x^2-25}{x+1}\div\frac{5x^2+23x-10}{x-3}$$

**40.**

$$\frac{\dfrac{x^2-3x}{x^2-7x+6}}{\dfrac{x^2-4x}{3x^2-11x-42}}$$

**41.**

$$\frac{\dfrac{x^2+10x+21}{x-4}}{\dfrac{x^2+3x}{x+8}}$$

**42.**

$$\frac{x^2+8x+15}{x^2-14x+45} \div \frac{x^2+11x+30}{-30+11x-x^2}$$

**43.**

$$\frac{\dfrac{x^2-6x-16}{x^2+x-42}}{\dfrac{x^2-64}{x^2+12x+35}}$$

**44.**

$$\frac{\dfrac{x^2+3x+2}{x^2-9x+18}}{\dfrac{x^2+7x+6}{x^2-6x}}$$

**45.**

$$\frac{\dfrac{x^2+12x+35}{x+4}}{\dfrac{x^2+10x+25}{x+9}}$$

**46.**

$$\frac{x^2-8x+7}{x^2+3x-18} \div \frac{x^2-7x}{x^2+6x-27}$$

**47.**

$$\frac{x^2+x-30}{x^2+5x-36} \div \frac{-6-x}{x+8}$$

**48.**

$$\frac{\dfrac{2x-x^2}{x^2-15x+54}}{\dfrac{x^2+x}{x^2-11x+30}}$$

**49.**

$$\frac{\dfrac{x^2-9x+8}{x^2-9}}{\dfrac{x^2-8x}{-15-8x-x^2}}$$

**50.**

$$\frac{x+5}{x^2+2x+1} \div \frac{x-2}{x^2+10x+9}$$

**51.**

$$\frac{\dfrac{x^2-4}{x+8}}{\dfrac{x^2-10x+16}{x+3}}$$

**52.**

$$\frac{27-6x-x^2}{x^2+9x+20} \div \frac{x^2-12x+27}{x^2+5x}$$

**53.**

$$\frac{\dfrac{x^2+5x+6}{x^2-36}}{\dfrac{x-7}{-6-x}}$$

**54.**

$$\frac{2-x}{x-5} \div \frac{x^2+3x-10}{x^2-14x+48}$$

**55.**

$$\frac{\dfrac{x+3}{x^2+4x-12}}{\dfrac{x-4}{x^2-36}}$$

**56.**

$$\frac{x+3}{x^2-x-2} \div \frac{x}{x^2-3x-4}$$

**57.**

$$\frac{x^2-11x+28}{x^2+5x+6} \div \frac{7x^2-30x+8}{x^2-x-6}$$

**58.**

$$\frac{\dfrac{x-7}{3-x}}{\dfrac{2x^2+3x-5}{x^2-12x+27}}$$

**59.** Let

$$f(x) = \frac{x^2-7x+10}{x^2+4x-21}$$

and

$$g(x) = \frac{5x-x^2}{x^2+15x+56}$$

Compute $f(x)/g(x)$ and simplify your answer.

**60.** Let

$$f(x) = \frac{x^2+15x+56}{x^2-x-20}$$

and

$$g(x) = \frac{-7-x}{x+1}$$

Compute $f(x)/g(x)$ and simplify your answer.

**61.** Let

$$f(x) = \frac{x^2+12x+35}{x^2+4x-32}$$

and

$$g(x) = \frac{x^2-2x-35}{x^2+8x}$$

Compute $f(x)/g(x)$ and simplify your answer.

**62.** Let

$$f(x) = \frac{x^2+4x+3}{x-1}$$

and

$$g(x) = \frac{x^2-4x-21}{x+5}$$

Compute $f(x)/g(x)$ and simplify your answer.

**63.** Let

$$f(x) = \frac{x^2+x-20}{x}$$

and

$$g(x) = \frac{x-1}{x^2-2x-35}$$

Compute $f(x)g(x)$ and simplify your answer.

**64.** Let

$$f(x) = \frac{x^2+10x+24}{x^2-13x+42}$$

and

$$g(x) = \frac{x^2-6x-7}{x^2+8x+12}$$

Compute $f(x)g(x)$ and simplify your answer.

**65.** Let

$$f(x) = \frac{x+5}{-6-x}$$

and

$$g(x) = \frac{x^2+8x+12}{x^2-49}$$

Compute $f(x)g(x)$ and simplify your answer.

**66.** Let

$$f(x) = \frac{8-7x-x^2}{x^2-8x-9}$$

and

$$g(x) = \frac{x^2-6x-7}{x^2-6x+5}$$

Compute $f(x)g(x)$ and simplify your answer.

## 7.4 Answers

---

**1.** $\frac{81}{175}$

**3.** $\frac{3}{2}$

**5.** $\frac{7}{60}$

**7.** $\frac{105}{64}$

**9.** $\frac{343}{40}$

**11.** Provided $x \neq -9, -7, -4$,
$$\frac{x(x+6)}{(x+9)(x+4)}$$

**13.** Provided $x \neq 1, -1, -4, -5$,
$$-\frac{(x+2)(x-9)}{(x+1)(x+4)}$$

**15.** Provided $x \neq -8, 6, 1, 0$,
$$\frac{(x-5)(x+3)}{(x-6)(x-1)}$$

**17.** Provided $x \neq 1, 8, -3, -1$,
$$-\frac{x(x-3)}{(x-8)(x+3)}$$

**19.** Provided $x \neq 7, -1, -6$,
$$-\frac{(x+2)(x+8)}{(x+1)(x+6)}$$

**21.** Provided $x \neq -3, -5, 9$,
$$-\frac{(x-3)(x-6)}{(x+5)(x-9)}$$

**23.** Provided $x \neq -8, -4, -7$,
$$\frac{(x+5)(x-6)}{(x+8)(x+7)}$$

**25.** Provided $x \neq 2, 6, 8$,
$$\frac{(x-5)(x-6)}{(x-2)(x-8)}$$

**27.** Provided $x \neq 2, 8, 2/3, -5$,
$$\frac{(x-3)(x-5)}{(3x-2)(x-8)}$$

**29.** Provided $x \neq -9, 7, 8, -2$,
$$\frac{(x-2)(x-3)}{(x-7)(x-8)}$$

**31.** Provided $x \neq 7, -9, -4$,
$$\frac{(x-1)(x-9)}{(x-7)(x+4)}$$

**33.** Provided $x \neq -3, -3/2, 5$,
$$-\frac{(x+6)(x-3)}{(2x+3)(x+3)}$$

**35.** Provided $x \neq -8, -2, 9, 3, 8$,
$$-\frac{(x-6)(x-9)}{(x+2)(x-3)}$$

**37.** Provided $x \neq 4, 3, 6, -5, -1$,
$$-\frac{(x-1)(x-6)}{(x-3)(x+5)}$$

**39.** Provided $x \neq -1, 2/5, -5, 3$,
$$\frac{(x-5)(x-3)}{(5x-2)(x+1)}$$

**41.** Provided $x \neq 4, 0, -3, -8$,

$$\frac{(x+7)(x+8)}{x(x-4)}$$

**43.** Provided $x \neq -7, 6, -5, -8, 8$,

$$\frac{(x+2)(x+5)}{(x-6)(x+8)}$$

**45.** Provided $x \neq -4, -5, -9$,

$$\frac{(x+7)(x+9)}{(x+4)(x+5)}$$

**47.** Provided $x \neq 4, -9, -8, -6$,

$$-\frac{(x-5)(x+8)}{(x-4)(x+9)}$$

**49.** Provided $x \neq -3, 3, -5, 0, 8$,

$$-\frac{(x-1)(x+5)}{x(x-3)}$$

**51.** Provided $x \neq -8, 8, 2, -3$,

$$\frac{(x+2)(x+3)}{(x+8)(x-8)}$$

**53.** Provided $x \neq 6, -6, 7$,

$$-\frac{(x+2)(x+3)}{(x-6)(x-7)}$$

**55.** Provided $x \neq 2, -6, 4, 6$,

$$\frac{(x+3)(x-6)}{(x-2)(x-4)}$$

**57.** Provided $x \neq -2, -3, 3, 2/7, 4$,

$$\frac{(x-7)(x-3)}{(7x-2)(x+3)}$$

**59.** Provided $x \neq -7, 3, -8, 0, 5$,

$$-\frac{(x-2)(x+8)}{x(x-3)}$$

**61.** Provided $x \neq -8, 4, 0, 7, -5$,

$$\frac{x(x+7)}{(x-4)(x-7)}$$

**63.** Provided $x \neq 0, 7, -5$,

$$\frac{(x-4)(x-1)}{x(x-7)}$$

**65.** Provided $x \neq -6, -7, 7$,

$$-\frac{(x+5)(x+2)}{(x+7)(x-7)}$$

# 7.5 Sums and Differences of Rational Functions

In this section we concentrate on finding sums and differences of rational expressions. However, before we begin, we need to review some fundamental ideas and technique.

First and foremost is the concept of the multiple of an integer. This is best explained with a simple example. The multiples of 8 is the set of integers $\{8k : k \text{ is an integer}\}$. In other words, if you multiply 8 by 0, $\pm 1$, $\pm 2$, $\pm 3$, $\pm 4$, etc., you produce what is known as the multiples of 8.

$$\text{Multiples of 8 are:}\quad 0,\ \pm 8,\ \pm 16,\ \pm 24,\ \pm 32,\ \text{etc.}$$

However, for our purposes, only the positive multiples are of interest. So we will say:

$$\text{Multiples of 8 are:}\quad 8, 16, \boxed{24}, 32, 40, \boxed{48}, 56, 64, \boxed{72}, \ldots$$

Similarly, we can list the positive multiples of 6.

$$\text{Multiples of 6 are:}\quad 6, 12, 18, \boxed{24}, 30, 36, 42, \boxed{48}, 54, 60, 66, \boxed{72}, \ldots$$

We've framed those numbers that are multiples of both 8 and 6. These are called the *common multiples* of 8 and 6.

$$\text{Common multiples of 8 and 6 are:}\quad 24, 48, 72, \ldots$$

The smallest of this list of common multiples of 8 and 6 is called the *least common multiple* of 8 and 6. We will use the following notation to represent the least common multiple of 8 and 6: $\text{LCM}(8, 6)$.

Hopefully, you will now feel comfortable with the following definition.

**Definition 1.** *Let $a$ and $b$ be integers. The* **least common multiple** *of $a$ and $b$, denoted $LCM(a, b)$, is the smallest positive multiple that $a$ and $b$ have in common.*

For larger numbers, listing multiples until you find one in common can be impractical and time consuming. Let's find the least common multiple of 8 and 6 a second time, only this time let's use a different technique.

First, write each number as a product of primes in exponential form.

$$8 = 2^3$$
$$6 = 2 \cdot 3$$

---

[14] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

Here's the rule.

**A Procedure to Find the LCM.** To find the LCM of two integers, proceed as follows.

1. Express the prime factorization of each integer in exponential format.
2. To find the least common multiple, write down every prime number that appears, then affix the largest exponent of that prime that appears.

In our example, the primes that occur are 2 and 3. The highest power of 2 that occurs is $2^3$. The highest power of 3 that occurs is $3^1$. Thus, the LCM(8, 6) is

$$\text{LCM}(8, 6) = 2^3 \cdot 3^1 = 24.$$

Note that this result is identical to the result found above by listing all common multiples and choosing the smallest.

◇

Let's try a harder example.

▶ **Example 2.** *Find the least common multiple of 24 and 36.*

Using the first technique, we list the multiples of each number, framing the multiples in common.

Multiples of 24: $24, 48, \boxed{72}, 96, 120, \boxed{144}, 168, \ldots$
Multiples of 36: $36, \boxed{72}, 108, \boxed{144}, 180 \ldots$

The multiples in common are 72, 144, etc., and the least common multiple is $\text{LCM}(24, 36) = 72$.

Now, let's use our second technique to find the least common multiple (LCM). First, express each number as a product of primes in exponential format.

$$\begin{aligned} 24 &= 2^3 \cdot 3 \\ 36 &= 2^2 \cdot 3^2 \end{aligned}$$

To find the least common multiple, write down every prime that occurs and affix the highest power of that prime that occurs. Thus, the highest power of 2 that occurs is $2^3$, and the highest power of 3 that occurs is $3^2$. Thus, the least common multiple is

$$\text{LCM}(24, 36) = 2^3 \cdot 3^2 = 8 \cdot 9 = 72.$$

◇

## *Addition and Subtraction Defined*

Imagine a pizza that has been cut into 12 equal slices. Then, each slice of pizza represents 1/12 of the entire pizza.

If Jimmy eats 3 slices, then he has consumed 3/12 of the entire pizza. If Margaret eats 2 slices, then she has consumed 2/12 of the entire pizza. It's clear that together they have consumed

$$\frac{3}{12}+\frac{2}{12}=\frac{5}{12}$$

of the pizza. It would seem that adding two fractions with a common denominator is as simple as eating pizza! Hopefully, the following definition will seem reasonable.

**Definition 3.** *To add two fractions with a common denominator, such as $a/c$ and $b/c$, add the numerators and divide by the common denominator. In symbols,*

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}.$$

Note how this definition agrees precisely with our pizza consumption discussed above. Here are some examples of adding fractions having common denominators.

$$\begin{aligned}\frac{5}{21}+\frac{3}{21}&=\frac{5+3}{21}\\&=\frac{8}{21}\end{aligned}\qquad\qquad\begin{aligned}\frac{2}{x+2}+\frac{x-3}{x+2}&=\frac{2+(x-3)}{x+2}\\&=\frac{2+x-3}{x+2}\\&=\frac{x-1}{x+2}\end{aligned}$$

Subtraction works in much the same way as does addition.

**Definition 4.** *To subtract two fractions with a common denominator, such as $a/c$ and $b/c$, subtract the numerators and divide by the common denominator. In symbols,*

$$\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}.$$

Here are some examples of subtracting fractions already having common denominators.

$$\begin{aligned}\frac{5}{21}-\frac{3}{21}&=\frac{5-3}{21}\\&=\frac{2}{21}\end{aligned}\qquad\qquad\begin{aligned}\frac{2}{x+2}-\frac{x-3}{x+2}&=\frac{2-(x-3)}{x+2}\\&=\frac{2-x+3}{x+2}\\&=\frac{5-x}{x+2}\end{aligned}$$

In the example on the right, note that it is extremely important to use grouping symbols when subtracting numerators. Note that the minus sign in front of the parenthetical expression changes the sign of each term inside the parentheses.

There are times when a sign change will provide a common denominator.

▶ **Example 5.** *Simplify*

$$\frac{x}{x-3} - \frac{2}{3-x}. \tag{6}$$

*State all restrictions.*

At first glance, it appears that we do not have a common denominator. On second glance, if we make a sign change on the second fraction, it might help. So, on the second fraction, let's negate the denominator and fraction bar to obtain

$$\frac{x}{x-3} - \frac{2}{3-x} = \frac{x}{x-3} + \frac{2}{x-3} = \frac{x+2}{x-3}.$$

The denominators $x-3$ or $3-x$ are zero when $x = 3$. Hence, 3 is a restricted value. For all other values of $x$, the left-hand side of

$$\frac{x}{x-3} - \frac{2}{3-x} = \frac{x+2}{x-3}. \tag{7}$$

is identical to the right-hand side.

This is easily tested using the table utility on the graphing calculator, as shown in the sequence of screenshots in **Figure 1**. First load the left- and right-hand sides of **equation (7)** into `Y1` and `Y2` in the `Y=` menu of your graphing calculator, as shown in **Figure 1**(a). Press `2nd TBLSET` and make the changes shown in **Figure 1**(b). Press `2nd TABLE` to produce the table shown in **Figure 1**(c). Note the `ERR` (error) message at the restriction $x = 3$, but note also the agreement of `Y1` and `Y2` for all other values of $x$.

```
Plot1  Plot2  Plot3
\Y1=X/(X-3)-2/(3
-X)
\Y2=(X+2)/(X-3)
\Y3=
\Y4=
\Y5=
\Y6=
```

(a)

```
TABLE SETUP
 TblStart=0
 ΔTbl=1
Indpnt: Auto Ask
Depend:      Ask
```

(b)

```
 X     Y1      Y2
 0     -.6667  -.6667
 1     -1.5    -1.5
 2     -4      -4
 3     ERR:    ERR:
 4     6       6
 5     3.5     3.5
 6     2.6667  2.6667
X=6
```

(c)

**Figure 1.** Using the table feature of the graphing calculator to check the result in **equation (7)**.

◇

## Equivalent Fractions

If you slice a pizza into four equal pieces, then consume two of the four slices, you've consumed half of the pizza. This motivates the fact that

$$\frac{1}{2}=\frac{2}{4}.$$

Indeed, if you slice the pizza into six equal pieces, then consume three slices, you've consumed half of the pizza, so it's fair to say that $3/6 = 1/2$. Indeed, all of the following fractions are equivalent:

$$\frac{1}{2}=\frac{2}{4}=\frac{3}{6}=\frac{4}{8}=\frac{5}{10}=\frac{6}{12}=\frac{7}{14}=\cdots$$

A more formal way to demonstrate that $1/2$ and $7/14$ are equal is to start with the fact that $1/2 = 1/2 \times 1$, then replace 1 with $7/7$ and multiply.

$$\frac{1}{2}=\frac{1}{2}\times 1=\frac{1}{2}\times\frac{7}{7}=\frac{7}{14}$$

Here's another example of this principle in action, only this time we replace 1 with $(x-2)/(x-2)$.

$$\frac{3}{x+2}=\frac{3}{x+2}\cdot 1=\frac{3}{x+2}\cdot\frac{x-2}{x-2}=\frac{3(x-2)}{(x+2)(x-2)}$$

In the next example we replace 1 with $(x(x-3))/(x(x-3))$.

$$\frac{2}{x-4}=\frac{2}{x-4}\cdot 1=\frac{2}{x-4}\cdot\frac{x(x-3)}{x(x-3)}=\frac{2x(x-3)}{x(x-4)(x-3)}$$

Now, let's apply the concept of equivalent fractions to add and subtract fractions with different denominators.

## Adding and Subtracting Fractions with Different Denominators

In this section we show our readers how to add and subtract fractions having different denominators. For example, suppose we are asked to add the following fractions.

$$\frac{5}{12}+\frac{5}{18} \qquad (8)$$

First, we must find a "common denominator." Fortunately, the machinery to find the "common denominator" is already in place. It turns out that the *least common denominator* for 12 and 18 is the *least common multiple* of 12 and 18.

$$\begin{aligned}18&=2\cdot 3^2\\12&=2^2\cdot 3\\\text{LCD}(12,18)&=2^2\cdot 3^2=36\end{aligned}$$

The next step is to create equivalent fractions using the LCD as the denominator. So, in the case of 5/12,

$$\frac{5}{12}=\frac{5}{12}\cdot 1=\frac{5}{12}\cdot\frac{3}{3}=\frac{15}{36}.$$

In the case of 5/18,

$$\frac{5}{18}=\frac{5}{18}\cdot 1=\frac{5}{18}\cdot\frac{2}{2}=\frac{10}{36}.$$

If we replace the fractions in **equation (8)** with their equivalent fractions, we can then add the numerators and divide by the common denominator, as in

$$\frac{5}{12}+\frac{5}{18}=\frac{15}{36}+\frac{10}{36}=\frac{15+10}{36}=\frac{25}{36}.$$

Let's examine a method of organizing the work that is more compact. Consider the following arrangement, where we've used color to highlight the form of 1 required to convert the fractions to equivalent fractions with a common denominator of 36.

$$\begin{aligned}\frac{5}{12}+\frac{5}{18}&=\frac{5}{12}\cdot\frac{3}{3}+\frac{5}{18}\cdot\frac{2}{2}\\&=\frac{15}{36}+\frac{10}{36}\\&=\frac{25}{36}\end{aligned}$$

◆

Let's look at a more complicated example.

▶ **Example 9.** *Simplify the expression*

$$\frac{x+3}{x+2}-\frac{x+2}{x+3}. \tag{10}$$

*State all restrictions.*

The denominators are already factored. If we take each factor that appears to the highest exponential power that appears, our least common denominator is $(x+2)(x+3)$. Our first task is to make equivalent fractions having this common denominator.

$$\begin{aligned}\frac{x+3}{x+2}-\frac{x+2}{x+3}&=\frac{x+3}{x+2}\cdot\frac{x+3}{x+3}-\frac{x+2}{x+3}\cdot\frac{x+2}{x+2}\\&=\frac{x^2+6x+9}{(x+2)(x+3)}-\frac{x^2+4x+4}{(x+2)(x+3)}\end{aligned}$$

Now, subtract the numerators and divide by the common denominator.

$$\begin{aligned}\frac{x+3}{x+2}-\frac{x+2}{x+3}&=\frac{(x^2+6x+9)-(x^2+4x+4)}{(x+2)(x+3)}\\&=\frac{x^2+6x+9-x^2-4x-4}{(x+2)(x+3)}\\&=\frac{2x+5}{(x+2)(x+3)}\end{aligned}$$

Note the use of parentheses when we subtracted the numerators. Note further how the minus sign negates each term in the parenthetical expression that follows the minus sign.

**Tip 11.** *Always use grouping symbols when subtracting the numerators of fractions.*

In the final answer, the factors $x + 2$ and $x + 3$ in the denominator are zero when $x = -2$ or $x = -3$. These are the restrictions. No other denominators, in the original problem or in the body of our work, provide additional restrictions.

Thus, for all values of $x$, except the restricted values $-2$ and $-3$, the left-hand side of

$$\frac{x+3}{x+2}-\frac{x+2}{x+3}=\frac{2x+5}{(x+2)(x+3)} \tag{12}$$

is identical to the right-hand side. This claim is easily tested on the graphing calculator which is evidenced in the sequence of screen captures in **Figure 2**. Note the `ERR` (error) message at each restricted value of $x$ in **Figure 2**(c), but also note the agreement of `Y1` and `Y2` for all other values of $x$.

```
Plot1 Plot2 Plot3
\Y1=(X+3)/(X+2)-
(X+2)/(X+3)
\Y2=(2*X+5)/((X+
2)*(X+3))
\Y3=
\Y4=
\Y5=
```

```
TABLE SETUP
 TblStart=-4
 ΔTbl=1
Indpnt: Auto Ask
Depend:      Ask
```

```
  X   | Y1     | Y2
------+--------+--------
 -4   | -1.5   | -1.5
 -3   | ERR:   | ERR:
 -2   | ERR:   | ERR:
 -1   | 1.5    | 1.5
 0    | .83333 | .83333
 1    | .58333 | .58333
 2    | .45    | .45
X=2
```

(a) (b) (c)

**Figure 2.** Using the table feature of the graphing calculator to check the result in **equation (12)**.

◆

Let's look at another example.

▶ **Example 13.** *Simplify the expression*

$$\frac{4}{x^2+6x+5}-\frac{2}{x^2+8x+15}.$$

*State all restrictions.*

First, factor each denominator.

$$\frac{4}{x^2+6x+5}-\frac{2}{x^2+8x+15}=\frac{4}{(x+1)(x+5)}-\frac{2}{(x+3)(x+5)}$$

The least common denominator, or least common multiple (LCM), requires that we write down each factor that occurs, then affix the highest power of that factor that occurs. Because all factors in the denominators are raised to an understood power of one, the LCD (least common denominator) or LCM is $(x+1)(x+5)(x+3)$.

Next, we make equivalent fractions having this common denominator.

$$\begin{aligned}\frac{4}{x^2+6x+5}-\frac{2}{x^2+8x+15}&=\frac{4}{(x+1)(x+5)}\cdot\frac{x+3}{x+3}-\frac{2}{(x+3)(x+5)}\cdot\frac{x+1}{x+1}\\&=\frac{4x+12}{(x+3)(x+5)(x+1)}-\frac{2x+2}{(x+3)(x+5)(x+1)}\end{aligned}$$

Subtract the numerators and divide by the common denominator. Be sure to use grouping symbols, particularly with the minus sign that is in play.

$$\begin{aligned}\frac{4}{x^2+6x+5}-\frac{2}{x^2+8x+15}&=\frac{(4x+12)-(2x+2)}{(x+3)(x+5)(x+1)}\\&=\frac{4x+12-2x-2}{(x+3)(x+5)(x+1)}\\&=\frac{2x+10}{(x+3)(x+5)(x+1)}\end{aligned}$$

Finally, we should always make sure that our answer is reduced to lowest terms. With that thought in mind, we factor the numerator in hopes that we can get a common factor to cancel.

$$\begin{aligned}\frac{4}{x^2+6x+5}-\frac{2}{x^2+8x+15}&=\frac{2(x+5)}{(x+3)(x+5)(x+1)}\\&=\frac{2\cancel{(x+5)}}{(x+3)\cancel{(x+5)}(x+1)}\\&=\frac{2}{(x+3)(x+1)}\end{aligned}$$

The denominators have factors of $x+3$, $x+5$ and $x+1$, so the restrictions are $x=-3$, $x=-5$, and $x=-1$, respectively. For all other values of $x$, the left-hand side of

$$\frac{4}{x^2+6x+5}-\frac{2}{x^2+8x+15}=\frac{2}{(x+3)(x+1)} \tag{14}$$

is identical to its right-hand side. Again, this is easily tested using the table feature of the graphing calculator, as shown in the screenshots in **Figure 3**. Again, note the `ERR` (error) messages at each restricted value of $x$, but also note that `Y1` and `Y2` agree for all other values of $x$.

```
Plot1  Plot2  Plot3
\Y1■4/(X^2+6*X+5
)-2/(X^2+8*X+15)

\Y2■2/((X+3)*(X+
1))
\Y3=■
\Y4=
```

```
TABLE SETUP
 TblStart=-6
  ΔTbl=1
Indpnt: Auto Ask
Depend:      Ask
```

```
  X   | Y1     | Y2
 -6   | .13333 | .13333
 -5   | ERR:   | .25
 -4   | .66667 | .66667
 -3   | ERR:   | ERR:
 -2   | -2     | -2
 -1   | ERR:   | ERR:
 0    | .66667 | .66667
X=0
```

(a) (b) (c)

**Figure 3.** Using the table feature of the graphing calculator to check the result in **equation (14)**.

Let's look at another example.

▶ **Example 15.** *Simplify the expression*

$$\frac{x-3}{x^2-1}+\frac{1}{x+1}-\frac{1}{1-x}.$$

*State all restrictions.*

First, factor all denominators.

$$\frac{x-3}{x^2-1}+\frac{1}{x+1}-\frac{1}{1-x}=\frac{x-3}{(x+1)(x-1)}+\frac{1}{x+1}-\frac{1}{1-x}$$

If we're not careful, we might be tempted to take one of each factor and use $(x+1)(x-1)(1-x)$ as a common denominator. However, let's first make two negations of the last of the three fractions on the right, negating the fraction bar and denominator to get

$$\frac{x-3}{x^2-1}+\frac{1}{x+1}-\frac{1}{1-x}=\frac{x-3}{(x+1)(x-1)}+\frac{1}{x+1}+\frac{1}{x-1}.$$

Now we can see that a common denominator of $(x+1)(x-1)$ will suffice. Let's make equivalent fractions with this common denominator.

$$\begin{aligned}\frac{x-3}{x^2-1}+\frac{1}{x+1}-\frac{1}{1-x}&=\frac{x-3}{(x+1)(x-1)}+\frac{1}{x+1}\cdot\frac{x-1}{x-1}+\frac{1}{x-1}\cdot\frac{x+1}{x+1}\\&=\frac{x-3}{(x+1)(x-1)}+\frac{x-1}{(x+1)(x-1)}+\frac{x+1}{(x+1)(x-1)}\end{aligned}$$

Add the numerators and divide by the common denominator. Even though grouping symbols are not as critical in this problem (because of the plus signs), we still think it good practice to use them.

$$\begin{aligned}\frac{x-3}{x^2-1}+\frac{1}{x+1}-\frac{1}{1-x}&=\frac{(x-3)+(x-1)+(x+1)}{(x+1)(x-1)}\\&=\frac{3x-3}{(x+1)(x-1)}\end{aligned}$$

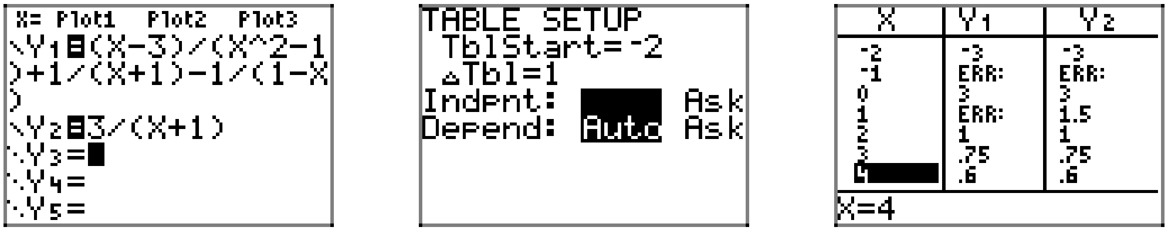
Finally, always make sure that your final answer is reduced to lowest terms. With that thought in mind, we factor the numerator in hopes that we can get a common factor to cancel.

$$\begin{aligned}\frac{x-3}{x^2-1}+\frac{1}{x+1}-\frac{1}{1-x}&=\frac{3(x-1)}{(x+1)(x-1)}\\&=\frac{3\cancel{(x-1)}}{(x+1)\cancel{(x-1)}}\\&=\frac{3}{x+1}\end{aligned}$$

The factors $x+1$ and $x-1$ in the denominator produce restrictions $x=-1$ and $x=1$, respectively. However, for all other values of $x$, the left-hand side of

$$\frac{x-3}{x^2-1}+\frac{1}{x+1}-\frac{1}{1-x}=\frac{3}{x+1} \tag{16}$$

is identical to the right-hand side. Again, this is easily checked on the graphing calculator as shown in the sequence of screenshots in **Figure 4**.



```
X= Plot1  Plot2  Plot3
\Y1=(X-3)/(X^2-1
)+1/(X+1)-1/(1-X
)
\Y2=3/(X+1)
\Y3=
\Y4=
\Y5=
```

```
TABLE SETUP
 TblStart=-2
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

```
  X   | Y1   | Y2
-2    | -3   | -3
-1    | ERR: | ERR:
0     | 3    | 3
1     | ERR: | 1.5
2     | 1    | 1
3     | .75  | .75
      | .6   | .6
X=4
```

(a) (b) (c)

**Figure 4.** Using the table feature of the graphing calculator to check the result in **equation (16)**.

Again, note the `ERR` (error) messages at each restriction, but also note that the values of `Y1` and `Y2` agree for all other values of $x$.

◆

Let's look at an example using function notation.

▶ **Example 17.** *If the function $f$ and $g$ are defined by the rules*

$$f(x) = \frac{x}{x+2} \qquad \text{and} \qquad g(x) = \frac{1}{x},$$

*simplify $f(x) - g(x)$.*

First,

$$f(x) - g(x) = \frac{x}{x+2} - \frac{1}{x}.$$

Note how tempting it would be to cancel. However, canceling would be an error in this situation, because subtraction requires a common denominator.

$$\begin{aligned} f(x) - g(x) &= \frac{x}{x+2} \cdot \frac{x}{x} - \frac{1}{x} \cdot \frac{x+2}{x+2} \\ &= \frac{x^2}{x(x+2)} - \frac{x+2}{x(x+2)} \end{aligned}$$

Subtract numerators and divide by the common denominator. This requires that we "distribute" the minus sign.

$$\begin{aligned} f(x) - g(x) &= \frac{x^2 - (x+2)}{x(x+2)} \\ &= \frac{x^2 - x - 2}{x(x+2)} \end{aligned}$$

This result is valid for all values of $x$ except 0 and $-2$. We leave it to our readers to verify that this result is reduced to lowest terms. You might want to check the result on your calculator as well.

◆

## 7.5 Exercises

In **Exercises 1-16**, add or subtract the rational expressions, as indicated, and simplify your answer. State all restrictions.

**1.** $\dfrac{7x^2 - 49x}{x-6} + \dfrac{42}{x-6}$

**2.** $\dfrac{2x^2 - 110}{x-7} - \dfrac{12}{7-x}$

**3.** $\dfrac{27x - 9x^2}{x+3} + \dfrac{162}{x+3}$

**4.** $\dfrac{2x^2 - 28}{x+2} - \dfrac{10x}{x+2}$

**5.** $\dfrac{4x^2 - 8}{x-4} + \dfrac{56}{4-x}$

**6.** $\dfrac{4x^2}{x-2} - \dfrac{36x - 56}{x-2}$

**7.** $\dfrac{9x^2}{x-1} + \dfrac{72x - 63}{1-x}$

**8.** $\dfrac{5x^2 + 30}{x-6} - \dfrac{35x}{x-6}$

**9.** $\dfrac{4x^2 - 60x}{x-7} + \dfrac{224}{x-7}$

**10.** $\dfrac{3x^2}{x-7} - \dfrac{63 - 30x}{7-x}$

**11.** $\dfrac{3x^2}{x-2} - \dfrac{48 - 30x}{2-x}$

**12.** $\dfrac{4x^2 - 164}{x-6} - \dfrac{20}{6-x}$

**13.** $\dfrac{9x^2}{x-2} - \dfrac{81x - 126}{x-2}$

**14.** $\dfrac{9x^2}{x-8} + \dfrac{144x - 576}{8-x}$

**15.** $\dfrac{3x^2 - 12}{x-3} + \dfrac{15}{3-x}$

**16.** $\dfrac{7x^2}{x-9} - \dfrac{112x - 441}{x-9}$

In **Exercises 17-34**, add or subtract the rational expressions, as indicated, and simplify your answer. State all restrictions.

**17.** $\dfrac{3x}{x^2 - 6x + 5} + \dfrac{15}{x^2 - 14x + 45}$

**18.** $\dfrac{7x}{x^2 - 4x} + \dfrac{28}{x^2 - 12x + 32}$

**19.** $\dfrac{9x}{x^2 + 4x - 12} - \dfrac{54}{x^2 + 20x + 84}$

**20.** $\dfrac{9x}{x^2 - 25} - \dfrac{45}{x^2 + 20x + 75}$

**21.** $\dfrac{5x}{x^2 - 21x + 98} - \dfrac{35}{7x - x^2}$

**22.** $\dfrac{7x}{7x - x^2} + \dfrac{147}{x^2 + 7x - 98}$

**23.** $\dfrac{-7x}{x^2 - 8x + 15} - \dfrac{35}{x^2 - 12x + 35}$

**24.** $\dfrac{-6x}{x^2 + 2x} + \dfrac{12}{x^2 + 6x + 8}$

**25.** $\dfrac{-9x}{x^2 - 12x + 32} - \dfrac{36}{x^2 - 4x}$

**26.** $\dfrac{5x}{x^2 - 12x + 32} - \dfrac{20}{4x - x^2}$

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**27.** $\dfrac{6x}{x^2-21x+98}-\dfrac{42}{7x-x^2}$

**28.** $\dfrac{-2x}{x^2-3x-10}+\dfrac{4}{x^2+11x+18}$

**29.** $\dfrac{-9x}{x^2-6x+8}-\dfrac{18}{x^2-2x}$

**30.** $\dfrac{6x}{5x-x^2}+\dfrac{90}{x^2+5x-50}$

**31.** $\dfrac{8x}{5x-x^2}+\dfrac{120}{x^2+5x-50}$

**32.** $\dfrac{-5x}{x^2+5x}+\dfrac{25}{x^2+15x+50}$

**33.** $\dfrac{-5x}{x^2+x-30}+\dfrac{30}{x^2+23x+102}$

**34.** $\dfrac{9x}{x^2+12x+32}-\dfrac{36}{x^2+4x}$

---

**35.** Let

$$f(x)=\frac{8x}{x^2+6x+8}$$

and

$$g(x)=\frac{16}{x^2+2x}$$

Compute $f(x)-g(x)$ and simplify your answer.

**36.** Let

$$f(x)=\frac{-7x}{x^2+8x+12}$$

and

$$g(x)=\frac{42}{x^2+16x+60}$$

Compute $f(x)+g(x)$ and simplify your answer.

**37.** Let

$$f(x)=\frac{11x}{x^2+12x+32}$$

and

$$g(x)=\frac{44}{-4x-x^2}$$

Compute $f(x)+g(x)$ and simplify your answer.

**38.** Let

$$f(x)=\frac{8x}{x^2-6x}$$

and

$$g(x)=\frac{48}{x^2-18x+72}$$

Compute $f(x)+g(x)$ and simplify your answer.

**39.** Let

$$f(x)=\frac{4x}{-x-x^2}$$

and

$$g(x)=\frac{4}{x^2+3x+2}$$

Compute $f(x)+g(x)$ and simplify your answer.

**40.** Let

$$f(x)=\frac{5x}{x^2-x-12}$$

and

$$g(x)=\frac{15}{x^2+13x+30}$$

Compute $f(x)-g(x)$ and simplify your answer.

## 7.5 *Answers*

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**1.** $7(x-1)$, provided $x \neq 6$.

**3.** $-9(x-6)$, provided $x \neq -3$.

**5.** $4(x+4)$, provided $x \neq 4$.

**7.** $9(x-7)$, provided $x \neq 1$.

**9.** $4(x-8)$, provided $x \neq 7$.

**11.** $3(x-8)$, provided $x \neq 2$.

**13.** $9(x-7)$, provided $x \neq 2$.

**15.** $3(x+3)$, provided $x \neq 3$.

**17.** Provided $x \neq 5, 1, 9$,
$$\frac{3(x+1)}{(x-1)(x-9)}$$

**19.** Provided $x \neq -6, 2, -14$,
$$\frac{9(x+2)}{(x-2)(x+14)}$$

**21.** Provided $x \neq 7, 14, 0$,
$$\frac{5(x+14)}{x(x-14)}$$

**23.** Provided $x \neq 5, 3, 7$,
$$\frac{-7(x+3)}{(x-3)(x-7)}$$

**25.** Provided $x \neq 4, 8, 0$,
$$\frac{-9(x+8)}{x(x-8)}$$

**27.** Provided $x \neq 7, 14, 0$,
$$\frac{6(x+14)}{x(x-14)}$$

**29.** Provided $x \neq 2, 4, 0$,
$$\frac{-9(x+4)}{x(x-4)}$$

**31.** Provided $x \neq 5, 0, -10$,
$$\frac{-8}{x+10}$$

**33.** Provided $x \neq -6, 5, -17$,
$$\frac{-5(x+5)}{(x-5)(x+17)}$$

**35.** Provided $x \neq -2, -4, 0$,
$$\frac{8(x-4)}{x(x+4)}$$

**37.** Provided $x \neq -4, -8, 0$,
$$\frac{11(x-8)}{x(x+8)}$$

**39.** Provided $x \neq -1, 0, -2$,
$$\frac{-4}{x+2}$$

# 7.6 Complex Fractions

In this section we learn how to simplify what are called *complex fractions*, an example of which follows.

$$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}+\frac{2}{3}} \tag{1}$$

Note that both the numerator and denominator are fraction problems in their own right, lending credence to why we refer to such a structure as a "complex fraction."

There are two very different techniques we can use to simplify the complex fraction (**1**). The first technique is a "natural" choice.

**Simplifying Complex Fractions — First Technique.** To simplify a complex fraction, proceed as follows:

1. Simplify the numerator.
2. Simplify the denominator.
3. Simplify the division problem that remains.

Let's follow this outline to simplify the complex fraction (**1**). First, add the fractions in the numerator as follows.

$$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6} \tag{2}$$

Secondly, add the fractions in the denominator as follows.

$$\frac{1}{4}+\frac{2}{3}=\frac{3}{12}+\frac{8}{12}=\frac{11}{12} \tag{3}$$

Substitute the results from (**2**) and (**3**) into the numerator and denominator of (**1**), respectively.

$$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}+\frac{2}{3}}=\frac{\frac{5}{6}}{\frac{11}{12}} \tag{4}$$

The right-hand side of (**4**) is equivalent to

$$\frac{5}{6}\div\frac{11}{12}.$$

This is a division problem, so invert and multiply, factor, then cancel common factors.

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$$\begin{aligned}\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}+\frac{2}{3}} &= \frac{5}{6}\cdot\frac{12}{11}\\ &= \frac{5}{2\cdot 3}\cdot\frac{2\cdot 2\cdot 3}{11}\\ &= \frac{5}{\cancel{2}\cdot\cancel{3}}\cdot\frac{\cancel{2}\cdot 2\cdot\cancel{3}}{11}\\ &= \frac{10}{11}\end{aligned}$$

Here is an arrangement of the work, from start to finish, presented without comment. This is a good template to emulate when doing your homework.

$$\begin{aligned}\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}+\frac{2}{3}} &= \frac{\frac{3}{6}+\frac{2}{6}}{\frac{3}{12}+\frac{8}{12}}\\ &= \frac{\frac{5}{6}}{\frac{11}{12}}\\ &= \frac{5}{6}\cdot\frac{12}{11}\\ &= \frac{5}{2\cdot 3}\cdot\frac{2\cdot 2\cdot 3}{11}\\ &= \frac{5}{\cancel{2}\cdot\cancel{3}}\cdot\frac{\cancel{2}\cdot 2\cdot\cancel{3}}{11}\\ &= \frac{10}{11}\end{aligned}$$

Now, let's look at a second approach to the problem. We saw that simplifying the numerator in (**2**) required a common denominator of 6. Simplifying the denominator in (**3**) required a common denominator of 12. So, let's choose another common denominator, this one a common denominator for both numerator and denominator, namely, 12. Now, multiply top and bottom (numerator and denominator) of the complex fraction (**1**) by 12, as follows.

$$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}+\frac{2}{3}} = \frac{\left(\frac{1}{2}+\frac{1}{3}\right)12}{\left(\frac{1}{4}+\frac{2}{3}\right)12} \tag{5}$$

Distribute the 12 in both numerator and denominator and simplify.

$$\frac{\left(\frac{1}{2}+\frac{1}{3}\right)12}{\left(\frac{1}{4}+\frac{2}{3}\right)12}=\frac{\left(\frac{1}{2}\right)12+\left(\frac{1}{3}\right)12}{\left(\frac{1}{4}\right)12+\left(\frac{2}{3}\right)12}=\frac{6+4}{3+8}=\frac{10}{11}$$

Let's summarize this second technique.

**Simplifying Complex Fractions — Second Technique.** To simplify a complex fraction, proceed as follows:

1. Find a common denominator for both numerator and denominator.
2. Clear fractions from the numerator and denomaintor by multiplying each by the common denominator found in the first step.

Note that for this particular problem, the second method is much more efficient. It saves both space and time and is more aesthetically pleasing. It is the technique that we will favor in the rest of this section.

Let's look at another example.

► **Example 6.** *Use both the First and Second Techniques to simplify the expression*

$$\frac{\frac{1}{x}-1}{1-\frac{1}{x^2}}. \tag{7}$$

*State all restrictions.*

Let's use the first technique, simplifying numerator and denominator separately before dividing. First, make equivalent fractions with a common denominator for the subtraction problem in the numerator of (7) and simplify. Do the same for the denominator.

$$\frac{\frac{1}{x}-1}{1-\frac{1}{x^2}}=\frac{\frac{1}{x}-\frac{x}{x}}{\frac{x^2}{x^2}-\frac{1}{x^2}}=\frac{\frac{1-x}{x}}{\frac{x^2-1}{x^2}}$$

Next, invert and multiply, then factor.

$$\frac{\frac{1}{x}-1}{1-\frac{1}{x^2}}=\frac{1-x}{x}\cdot\frac{x^2}{x^2-1}=\frac{1-x}{x}\cdot\frac{x^2}{(x+1)(x-1)}$$

Let's invoke the sign change rule and negate two parts of the fraction $(1-x)/x$, numerator and fraction bar, then cancel the common factors.

$$\frac{\dfrac{1}{x}-1}{1-\dfrac{1}{x^2}} = -\frac{x-1}{x}\cdot\frac{x^2}{(x+1)(x-1)} = -\frac{\cancel{x-1}}{\cancel{x}}\cdot\frac{x\cancel{x}}{(x+1)\cancel{(x-1)}}$$

Hence,

$$\frac{\dfrac{1}{x}-1}{1-\dfrac{1}{x^2}} = -\frac{x}{x+1}.$$

Now, let's try the problem a second time, multiplying numerator and denominator by $x^2$ to clear fractions from both the numerator and denominator.

$$\frac{\dfrac{1}{x}-1}{1-\dfrac{1}{x^2}} = \frac{\left(\dfrac{1}{x}-1\right)x^2}{\left(1-\dfrac{1}{x^2}\right)x^2} = \frac{\left(\dfrac{1}{x}\right)x^2-(1)x^2}{(1)x^2-\left(\dfrac{1}{x^2}\right)x^2} = \frac{x-x^2}{x^2-1}$$

The order in the numerator of the last fraction intimates that a sign change would be helpful. Negate the numerator and fraction bar, factor, then cancel common factors.

$$\frac{\dfrac{1}{x}-1}{1-\dfrac{1}{x^2}} = -\frac{x^2-x}{x^2-1} = -\frac{x(x-1)}{(x+1)(x-1)} = -\frac{x\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} = -\frac{x}{x+1}$$

This is precisely the same answer found with the first technique. To list the restrictions, we must make sure that no values of $x$ make any denominator equal to zero, at the beginning of the problem, in the body of our work, or in the final answer.

In the original problem, if $x=0$, then both $1/x$ and $1/x^2$ are undefined, so $x=0$ is a restriction. In the body of our work, the factors $x+1$ and $x-1$ found in various denominators make $x=-1$ and $x=1$ restrictions. No other denominators supply restrictions that have not already been listed. Hence, for all $x$ other than $-1$, 0, and 1, the left-hand side of

$$\frac{\dfrac{1}{x}-1}{1-\dfrac{1}{x^2}} = -\frac{x}{x+1} \tag{8}$$

is identical to the right-hand side. Again, the calculator's table utility provides ample evidence of this fact in the screenshots shown in **Figure 1**.

Note the `ERR` (error) messages at each of the restricted values of $x$, but also note the perfect agreement of `Y1` and `Y2` at all other values of $x$.

◆

Let's look at another example, an important example involving function notation.

```
Plot1 Plot2 Plot3
\Y1■(1/X-1)/(1-1
/X^2)
\Y2■-X/(X+1)
\Y3=■
\Y4=
\Y5=
\Y6=
```

(a)

```
TABLE SETUP
 TblStart=-2
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

(b)

```
  X   | Y1    | Y2
 -2   | -2    | -2
 -1   | ERR:  | ERR:
 0    | ERR:  | 0
 1    | ERR:  | -.5
 2    | -.6667| -.6667
 3    | -.75  | -.75
 4    | -.8   | -.8
X=4
```

(c)

**Figure 1.** Using the table feature of the graphing calculator to check the identity in (8).

▶ **Example 9.** *Given that*

$$f(x)=\frac{1}{x},$$

*simplify the expression*

$$\frac{f(x)-f(2)}{x-2}.$$

*List all restrictions.*

Remember, $f(2)$ means substitute 2 for $x$. Because $f(x)=1/x$, we know that $f(2)=1/2$, so

$$\frac{f(x)-f(2)}{x-2}=\frac{\dfrac{1}{x}-\dfrac{1}{2}}{x-2}.$$

To clear the fractions from the numerator, we'd use a common denominator of $2x$. There are no fractions in the denominator that need clearing, so the common denominator for numerator and denominator is $2x$. Multiply numerator and denominator by $2x$.

$$\frac{f(x)-f(2)}{x-2}=\frac{\left(\dfrac{1}{x}-\dfrac{1}{2}\right)2x}{(x-2)2x}=\frac{\left(\dfrac{1}{x}\right)2x-\left(\dfrac{1}{2}\right)2x}{(x-2)2x}=\frac{2-x}{2x(x-2)}$$

Negate the numerator and fraction bar, then cancel common factors.

$$\frac{f(x)-f(2)}{x-2}=-\frac{x-2}{2x(x-2)}=-\frac{\cancel{x-2}}{2x\cancel{(x-2)}}=-\frac{1}{2x}$$

In the original problem, we have a denominator of $x-2$, so $x=2$ is a restriction. If the body of our work, there is a fraction $1/x$, which is undefined when $x=0$, so $x=0$ is also a restriction. The remaining denominators provide no other restrictions. Hence, for all values of $x$ except 0 and 2, the left-hand side of

$$\frac{f(x)-f(2)}{x-2}=-\frac{1}{2x}$$

is identical to the right-hand side.

◆

Let's look at another example involving function notation.

▶ **Example 10.** *Given*

$$f(x) = \frac{1}{x^2},$$

*simplify the expression*

$$\frac{f(x+h) - f(x)}{h}. \tag{11}$$

*List all restrictions.*

The function notation $f(x+h)$ is asking us to replace each instance of $x$ in the formula $1/x^2$ with $x+h$. Thus, $f(x+h) = 1/(x+h)^2$.

Here is another way to think of this substitution. Suppose that we remove the $x$ from

$$f(x) = \frac{1}{x^2},$$

so that it reads

$$f(\quad) = \frac{1}{(\quad)^2}. \tag{12}$$

Now, if you want to compute $f(2)$, simply insert a 2 in the blank area between parentheses. In our case, we want to compute $f(x+h)$, so we insert an $x+h$ in the blank space between parentheses in (**12**) to get

$$f(x+h) = \frac{1}{(x+h)^2}.$$

With these preliminary remarks in mind, let's return to the problem. First, we interpret the function notation as in our preliminary remarks and write

$$\frac{f(x+h) - f(x)}{h} = \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}.$$

The common denominator for the numerator is found by listing each factor to the highest power that it occurs. Hence, the common denominator is $x^2(x+h)^2$. The denominator has no fractions to be cleared, so it suffices to multiply both numerator and denominator by $x^2(x+h)^2$.

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{\left(\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}\right) x^2(x+h)^2}{hx^2(x+h)^2} \\
&= \frac{\left(\dfrac{1}{(x+h)^2}\right) x^2(x+h)^2 - \left(\dfrac{1}{x^2}\right) x^2(x+h)^2}{hx^2(x+h)^2} \\
&= \frac{x^2 - (x+h)^2}{hx^2(x+h)^2}
\end{aligned}$$

We will now expand the numerator. Don't forget to use parentheses and distribute that minus sign.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}\\ &= \frac{x^2-x^2-2xh-h^2}{hx^2(x+h)^2}\\ &= \frac{-2xh-h^2}{hx^2(x+h)^2}\end{aligned}$$

Finally, factor a $-h$ out of the numerator in hopes of finding a common factor to cancel.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{-h(2x+h)}{hx^2(x+h)^2}\\ &= \frac{-\cancel{h}(2x+h)}{\cancel{h}x^2(x+h)^2}\\ &= \frac{-(2x+h)}{x^2(x+h)^2}\end{aligned}$$

We must now discuss the restrictions. In the original question (**11**), the $h$ in the denominator must not equal zero. Hence, $h = 0$ is a restriction. In the final simplified form, the factor of $x^2$ in the denominator is undefined if $x = 0$. Hence, $x = 0$ is a restriction. Finally, the factor of $(x+h)^2$ in the final denominator is undefined if $x+h = 0$, so $x = -h$ is a restriction. The remaining denominators provide no additional restrictions. Hence, provided $h \neq 0$, $x \neq 0$, and $x \neq -h$, for all other combinations of $x$ and $h$, the left-hand side of

$$\frac{f(x+h)-f(x)}{h} = \frac{-(2x+h)}{x^2(x+h)^2}$$

is identical to the right-hand side.

Let's look at one final example using function notation.

▶ **Example 13.** *If*

$$f(x) = \frac{x}{x+1} \tag{14}$$

*simplify* $f(f(x))$.

We first evaluate $f$ at $x$, then evaluate $f$ at the result of the first computation. Thus, we work the inner function first to obtain

$$f(f(x)) = f\left(\frac{x}{x+1}\right).$$

The notation $f(x/(x+1))$ is asking us to replace each occurrence of $x$ in the formula $x/(x+1)$ with the expression $x/(x+1)$. Confusing? Here is an easy way to think of this substitution. Suppose that we remove $x$ from

$$f(x)=\frac{x}{x+1},$$

replacing each occurrence of $x$ with empty parentheses, which will produce the template

$$f(\ )=\frac{(\ )}{(\ )+1}. \tag{15}$$

Now, if asked to compute $f(3)$, simply insert 3 into the blank areas between parentheses. In this case, we want to compute $f(x/(x+1))$, so we insert $x/(x+1)$ in the blank space between each set of parentheses in (**15**) to obtain

$$f\left(\frac{x}{x+1}\right)=\frac{\dfrac{x}{x+1}}{\dfrac{x}{x+1}+1}.$$

We now have a complex fraction. The common denominator for both top and bottom of this complex fraction is $x+1$. Thus, we multiply both numerator and denominator of our complex fraction by $x+1$ and use the distributive property as follows.

$$\frac{\dfrac{x}{x+1}}{\dfrac{x}{x+1}+1}=\frac{\left(\dfrac{x}{x+1}\right)(x+1)}{\left(\dfrac{x}{x+1}+1\right)(x+1)}=\frac{\left(\dfrac{x}{x+1}\right)(x+1)}{\left(\dfrac{x}{x+1}\right)(x+1)+(1)(x+1)}$$

Cancel and simplify.

$$\frac{\left(\dfrac{x}{x+1}\right)(x+1)}{\left(\dfrac{x}{x+1}\right)(x+1)+(1)(x+1)}=\frac{x}{x+(x+1)}=\frac{x}{2x+1}$$

In the final denominator, the value $x=-1/2$ makes the denominator $2x+1$ equal to zero. Hence, $x=-1/2$ is a restriction. In the body of our work, several fractions have denominators of $x+1$ and are therefore undefined at $x=-1$. Thus, $x=-1$ is a restriction. No other denominators add additional restrictions.

Hence, for all values of $x$, except $x=-1/2$ and $x=-1$, the left-hand side of

$$f(f(x))=\frac{x}{2x+1}$$

is identical to the right-hand side.

◆

## 7.6 Exercises

In **Exercises 1-6**, evaluate the function at the given rational number. Then use the first or second technique for simplifying complex fractions explained in the narrative to simplify your answer.

**1.** Given

$$f(x)=\frac{x+1}{2-x},$$

evaluate and simplify $f(1/2)$.

**2.** Given

$$f(x)=\frac{2-x}{x+5},$$

evaluate and simplify $f(3/2)$.

**3.** Given

$$f(x)=\frac{2x+3}{4-x},$$

evaluate and simplify $f(1/3)$.

**4.** Given

$$f(x)=\frac{3-2x}{x+5},$$

evaluate and simplify $f(2/5)$.

**5.** Given

$$f(x)=\frac{5-2x}{x+4},$$

evaluate and simplify $f(3/5)$.

**6.** Given

$$f(x)=\frac{2x-9}{11-x},$$

evaluate and simplify $f(4/3)$.

In **Exercises 7-46**, simplify the given complex rational expression. State all restrictions.

**7.**

$$\frac{5+\dfrac{6}{x}}{\dfrac{25}{x}-\dfrac{36}{x^3}}$$

**8.**

$$\frac{7+\dfrac{9}{x}}{\dfrac{49}{x}-\dfrac{81}{x^3}}$$

**9.**

$$\frac{\dfrac{7}{x-2}-\dfrac{5}{x-7}}{\dfrac{8}{x-7}+\dfrac{3}{x+8}}$$

**10.**

$$\frac{\dfrac{9}{x+4}-\dfrac{7}{x-9}}{\dfrac{9}{x-9}+\dfrac{5}{x-4}}$$

**11.**

$$\frac{3+\dfrac{7}{x}}{\dfrac{9}{x^2}-\dfrac{49}{x^4}}$$

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**12.**

$$\frac{2-\frac{5}{x}}{\frac{4}{x^2}-\frac{25}{x^4}}$$

**13.**

$$\frac{\frac{9}{x+4}+\frac{7}{x+9}}{\frac{9}{x+9}+\frac{2}{x-8}}$$

**14.**

$$\frac{\frac{4}{x-6}+\frac{9}{x-9}}{\frac{9}{x-6}+\frac{8}{x-9}}$$

**15.**

$$\frac{\frac{5}{x-7}-\frac{4}{x-4}}{\frac{10}{x-4}-\frac{5}{x+2}}$$

**16.**

$$\frac{\frac{3}{x+6}+\frac{7}{x+9}}{\frac{9}{x+6}-\frac{4}{x+9}}$$

**17.**

$$\frac{\frac{6}{x-3}+\frac{5}{x-8}}{\frac{9}{x-3}+\frac{7}{x-8}}$$

**18.**

$$\frac{\frac{7}{x-7}-\frac{4}{x-2}}{\frac{7}{x-7}-\frac{6}{x-2}}$$

**19.**

$$\frac{\frac{4}{x-2}+\frac{7}{x-7}}{\frac{5}{x-2}+\frac{2}{x-7}}$$

**20.**

$$\frac{\frac{9}{x+2}-\frac{7}{x+5}}{\frac{4}{x+2}+\frac{3}{x+5}}$$

**21.**

$$\frac{5+\frac{4}{x}}{\frac{25}{x}-\frac{16}{x^3}}$$

**22.**

$$\frac{\frac{6}{x+5}+\frac{5}{x+4}}{\frac{8}{x+5}-\frac{3}{x+4}}$$

**23.**

$$\frac{\frac{9}{x-5}+\frac{8}{x+4}}{\frac{5}{x-5}-\frac{4}{x+4}}$$

**24.**

$$\frac{\frac{4}{x-6}+\frac{4}{x-9}}{\frac{6}{x-6}+\frac{6}{x-9}}$$

**25.**

$$\frac{\frac{6}{x+8}+\frac{5}{x-2}}{\frac{5}{x-2}-\frac{2}{x+2}}$$

**26.**

$$\frac{\frac{7}{x+9}+\frac{9}{x-2}}{\frac{4}{x-2}+\frac{7}{x+1}}$$

**27.**

$$\frac{\frac{7}{x+7}-\frac{5}{x+4}}{\frac{8}{x+7}-\frac{3}{x+4}}$$

**28.**

$$\frac{25-\frac{16}{x^2}}{5+\frac{4}{x}}$$

**29.**

$$\frac{\frac{64}{x}-\frac{25}{x^3}}{8-\frac{5}{x}}$$

**30.**

$$\frac{\frac{4}{x+2}+\frac{5}{x-6}}{\frac{7}{x-6}-\frac{5}{x+7}}$$

**31.**

$$\frac{\frac{2}{x-6}-\frac{4}{x+9}}{\frac{3}{x-6}-\frac{6}{x+9}}$$

**32.**

$$\frac{\frac{3}{x+6}-\frac{4}{x+4}}{\frac{6}{x+6}-\frac{8}{x+4}}$$

**33.**

$$\frac{\frac{9}{x^2}-\frac{64}{x^4}}{3-\frac{8}{x}}$$

**34.**

$$\frac{\frac{9}{x^2}-\frac{25}{x^4}}{3-\frac{5}{x}}$$

**35.**

$$\frac{\frac{4}{x-4}-\frac{8}{x-7}}{\frac{4}{x-7}+\frac{2}{x+2}}$$

**36.**

$$\frac{2-\frac{7}{x}}{4-\frac{49}{x^2}}$$

**37.**

$$\frac{\frac{3}{x^2+8x-9}+\frac{3}{x^2-81}}{\frac{9}{x^2-81}+\frac{9}{x^2-8x-9}}$$

**38.**

$$\frac{\frac{7}{x^2-5x-14}+\frac{2}{x^2-7x-18}}{\frac{5}{x^2-7x-18}+\frac{8}{x^2-6x-27}}$$

**39.**

$$\frac{\frac{2}{x^2+8x+7}+\frac{5}{x^2+13x+42}}{\frac{7}{x^2+13x+42}+\frac{6}{x^2+3x-18}}$$

**40.**

$$\frac{\dfrac{3}{x^2+5x-14}+\dfrac{3}{x^2-7x-98}}{\dfrac{3}{x^2-7x-98}+\dfrac{3}{x^2-15x+14}}$$

**41.**

$$\frac{\dfrac{6}{x^2+11x+24}-\dfrac{6}{x^2+13x+40}}{\dfrac{9}{x^2+13x+40}-\dfrac{9}{x^2-3x-40}}$$

**42.**

$$\frac{\dfrac{7}{x^2+13x+30}+\dfrac{7}{x^2+19x+90}}{\dfrac{9}{x^2+19x+90}+\dfrac{9}{x^2+7x-18}}$$

**43.**

$$\frac{\dfrac{7}{x^2-6x+5}+\dfrac{7}{x^2+2x-35}}{\dfrac{8}{x^2+2x-35}+\dfrac{8}{x^2+8x+7}}$$

**44.**

$$\frac{\dfrac{2}{x^2-4x-12}-\dfrac{2}{x^2-x-30}}{\dfrac{2}{x^2-x-30}-\dfrac{2}{x^2-4x-45}}$$

**45.**

$$\frac{\dfrac{4}{x^2+6x-7}-\dfrac{4}{x^2+2x-3}}{\dfrac{4}{x^2+2x-3}-\dfrac{4}{x^2+5x+6}}$$

**46.**

$$\frac{\dfrac{9}{x^2+3x-4}+\dfrac{8}{x^2-7x+6}}{\dfrac{4}{x^2-7x+6}+\dfrac{9}{x^2-10x+24}}$$

**47.** Given $f(x)=2/x$, simplify

$$\frac{f(x)-f(3)}{x-3}.$$

State all restrictions.

**48.** Given $f(x)=5/x$, simplify

$$\frac{f(x)-f(2)}{x-2}.$$

State all restrictions.

**49.** Given $f(x)=3/x^2$, simplify

$$\frac{f(x)-f(1)}{x-1}.$$

State all restrictions.

**50.** Given $f(x)=5/x^2$, simplify

$$\frac{f(x)-f(2)}{x-2}.$$

State all restrictions.

**51.** Given $f(x)=7/x$, simplify

$$\frac{f(x+h)-f(x)}{h}.$$

State all restrictions.

**52.** Given $f(x)=4/x$, simplify

$$\frac{f(x+h)-f(x)}{h}.$$

State all restrictions.

**53.** Given

$$f(x)=\frac{x+1}{3-x},$$

find and simplify $f(1/x)$. State all restrictions.

**54.** Given

$$f(x) = \frac{2 - x}{3x + 4},$$

find and simplify $f(2/x)$. State all restrictions.

**55.** Given

$$f(x) = \frac{x + 1}{2 - 5x},$$

find and simplify $f(5/x)$. State all restrictions.

**56.** Given

$$f(x) = \frac{2x - 3}{4 + x},$$

find and simplify $f(1/x)$. State all restrictions.

**57.** Given

$$f(x) = \frac{x}{x + 2},$$

find and simplify $f(f(x))$. State all restrictions.

**58.** Given

$$f(x) = \frac{2x}{x + 5},$$

find and simplify $f(f(x))$. State all restrictions.

## 7.6 *Answers*

---

**1.** 1

**3.** 1

**5.** 19/23

**7.** Provided $x \neq 0$, $-6/5$, or $6/5$,

$$\frac{x^2}{5x-6}.$$

**9.** Provided $x \neq 2$, $7$, $-8$, of $-43/11$,

$$\frac{(2x-39)(x+8)}{(11x+43)(x-2)}.$$

**11.** Provided $x \neq 0$, $-7/3$, or $7/3$,

$$\frac{x^3}{3x-7}.$$

**13.** Provided $x \neq -4$, $-9$, $8$, or $54/11$,

$$\frac{(16x+109)(x-8)}{(11x-54)(x+4)}.$$

**15.** Provided $x \neq 7$, $4$, $-2$, or $-8$,

$$\frac{x+2}{5(x-7)}.$$

**17.** Provided $x \neq 3$, $8$, or $93/16$,

$$\frac{11x-63}{16x-93}.$$

**19.** Provided $x \neq 2$, $7$, or $39/7$,

$$\frac{11x-42}{7x-39}.$$

**21.** Provided $x \neq 0$, $-4/5$, or $4/5$,

$$\frac{x^2}{5x-4}.$$

**23.** Provided $x \neq 5$, $-4$, or $-40$,

$$\frac{17x-4}{x+40}.$$

**25.** Provided $x \neq -8$, $2$, $-2$, or $-14/3$,

$$\frac{(11x+28)(x+2)}{(3x+14)(x+8)}.$$

**27.** Provided $x \neq -7$, $-4$, or $-11/5$,

$$\frac{2x-7}{5x+11}.$$

**29.** Provided $x \neq 0$ or $5/8$,

$$\frac{8x+5}{x^2}.$$

**31.** Provided $x \neq 6$, $-9$, or $21$,

$$\frac{2}{3}.$$

**33.** Provided $x \neq 0$ or $8/3$,

$$\frac{3x+8}{x^3}.$$

**35.** Provided $x \neq 4$, $7$, $-2$, or $1$,

$$\frac{-2(x+2)}{3(x-4)}.$$

**37.** Provided $x \neq 1, -9, 9, -1, -5$,

$$\frac{(x-5)(x+1)}{3(x+5)(x-1)}$$

**39.** Provided $x \neq -1, -7, -6, 3, -21/13$,

$$\frac{(7x+17)(x-3)}{(13x+21)(x+1)}$$

**41.** Provided $x \neq -3, -8, -5, 8$,

$$\frac{-1(x-8)}{12(x+3)}$$

**43.** Provided $x \neq 1, 5, -7, -1, 2$,

$$\frac{7(x+3)(x+1)}{8(x-2)(x-1)}$$

**45.** Provided $x \neq -7, 1, -3, -2$,

$$\frac{-4(x+2)}{3(x+7)}$$

**47.** Provided $x \neq 0$, 3,

$$-\frac{2}{3x}$$

**49.** Provided $x \neq 0$, 1,

$$-\frac{3(x+1)}{x^2}$$

**51.** Provided $x \neq 0, -h$, and $h \neq 0$,

$$-\frac{7}{h(x+h)}$$

**53.** Provided $x \neq 0$, 1/3,

$$\frac{x+1}{3x-1}$$

**55.** Provided $x \neq 0$, 25/2,

$$\frac{x+5}{2x-25}$$

**57.** Provided $x \neq -2$, $-4/3$,

$$\frac{x}{3x+4}$$

## 7.7 Solving Rational Equations

When simplifying complex fractions in the previous section, we saw that multiplying both numerator and denominator by the appropriate expression could "clear" all fractions from the numerator and denominator, greatly simplifying the rational expression.

In this section, a similar technique is used.

**Clear the Fractions from a Rational Equation.** If your equation has rational expressions, multiply both sides of the equation by the least common denominator to clear the equation of rational expressions.

Let's look at an example.

▶ **Example 1.** *Solve the following equation for $x$.*

$$\frac{x}{2}-\frac{2}{3}=\frac{3}{4} \qquad (2)$$

To clear this equation of fractions, we will multiply both sides by the common denominator for 2, 3, and 4, which is 12. Distribute 12 in the second step.

$$\begin{aligned} 12\left(\frac{x}{2}-\frac{2}{3}\right) &= \left(\frac{3}{4}\right)12 \\ 12\left(\frac{x}{2}\right)-12\left(\frac{2}{3}\right) &= \left(\frac{3}{4}\right)12 \end{aligned}$$

Multiply.

$$6x-8=9$$

We've succeeded in clearing the rational expressions from the equation by multiplying through by the common denominator. We now have a simple linear equation which can be solved by first adding 8 to both sides of the equation, followed by dividing both sides of the equation by 6.

$$\begin{aligned} 6x &= 17 \\ x &= \frac{17}{6} \end{aligned}$$

We'll leave it to our readers to check this solution.

◆

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[18] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

Let's try another example.

▶ **Example 3.** *Solve the following equation for $x$.*

$$6 = \frac{5}{x} + \frac{6}{x^2} \tag{4}$$

In this equation, the denominators are 1, $x$, and $x^2$, and the common denominator for both sides of the equation is $x^2$. Consequently, we begin the solution by first multiplying both sides of the equation by $x^2$.

$$\begin{aligned} x^2\,(6) &= \left(\frac{5}{x} + \frac{6}{x^2}\right) x^2 \\ x^2\,(6) &= \left(\frac{5}{x}\right) x^2 + \left(\frac{6}{x^2}\right) x^2 \end{aligned}$$

Simplify.

$$6x^2 = 5x + 6$$

Note that multiplying both sides of the original equation by the least common denominator clears the equation of all rational expressions. This last equation is nonlinear,[19] so make one side of the equation equal to zero by subtracting $5x$ and 6 from both sides of the equation.

$$6x^2 - 5x - 6 = 0$$

To factor the left-hand side of this equation, note that it is a quadratic trinomial with $ac = (6)(-6) = -36$. The integer pair 4 and $-9$ have product $-36$ and sum $-5$. Split the middle term using this pair and factor by grouping.

$$\begin{aligned} 6x^2 + 4x - 9x - 6 &= 0 \\ 2x(3x + 2) - 3(3x + 2) &= 0 \\ (2x - 3)(3x + 2) &= 0 \end{aligned}$$

The zero product property forces either

$$2x - 3 = 0 \qquad \text{or} \qquad 3x + 2 = 0.$$

Each of these linear equations is easily solved.

$$x = \frac{3}{2} \qquad \text{or} \qquad x = -\frac{2}{3}$$

Of course, we should always check our solutions. Substituting $x = 3/2$ into the right-hand side of the original **equation (4)**,

---

[19] Whenever an equation in $x$ has a power of $x$ other than 1, the equation is *nonlinear* (the graphs involved are not all lines). As we've seen in previous chapters, the approach to solving a quadratic (second degree) equation should be to make one side of the equation equal to zero, then factor or use the quadratic formula to find the solutions.

$$\frac{5}{x}+\frac{6}{x^2}=\frac{5}{3/2}+\frac{6}{(3/2)^2}=\frac{5}{3/2}+\frac{6}{9/4}.$$

In the final expression, multiply top and bottom of the first fraction by 2, top and bottom of the second fraction by 4.

$$\frac{5}{3/2}\cdot\frac{2}{2}+\frac{6}{9/4}\cdot\frac{4}{4}=\frac{10}{3}+\frac{24}{9}$$

Make equivalent fractions with a common denominator of 9 and add.

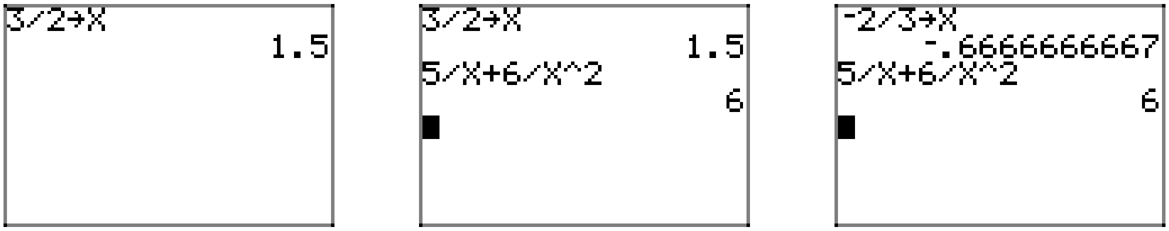
$$\frac{10}{3}\cdot\frac{3}{3}+\frac{24}{9}=\frac{30}{9}+\frac{24}{9}=\frac{54}{9}=6$$

Note that this result is identical to the left-hand side of the original **equation (4)**. Thus, $x=3/2$ checks.

This example clearly demonstrates that the check can be as difficult and as time consuming as the computation used to originally solve the equation. For this reason, we tend to get lazy and not check our answers as we should. There is help, however, as the graphing calculator can help us check the solutions of equations.

First, enter the solution 3/2 in your calculator screen, push the `STO▶` button, then push the `X` button, and execute the resulting command on the screen by pushing the `ENTER` key. The result is shown in **Figure 1**(a).

Next, enter the expression `5/X+6/X^2` and execute the resulting command on the screen by pushing the `ENTER` key. The result is shown in **Figure 1**(b). Note that the result is 6, the same as computed by hand above, and it matches the left-hand side of the original **equation (4)**. We've also used the calculator to check the second solution $x=-2/3$. This is shown in **Figure 4**(c).



```
3/2→X
                1.5
```

(a)

```
3/2→X
                1.5
5/X+6/X^2
                  6
```

(b)

```
⁻2/3→X
       ⁻.6666666667
5/X+6/X^2
                  6
```

(c)

**Figure 1.** Using the graphing calculator to check the solutions of **equation (4)**.

◆

Let's look at another example.

▶ **Example 5.** *Solve the following equation for $x$.*

$$\frac{2}{x^2}=1-\frac{2}{x} \qquad (6)$$

First, multiply both sides of **equation (6)** by the common denominator $x^2$.

$$\begin{aligned} x^2\left(\frac{2}{x^2}\right) &= \left(1-\frac{2}{x}\right)x^2 \\ 2 &= x^2-2x \end{aligned}$$

Make one side zero.

$$0 = x^2 - 2x - 2$$

The right-hand side is a quadratic trinomial with $ac = (1)(-2) = -2$. There are no integer pairs with product $-2$ that sum to $-2$, so this quadratic trinomial does not factor. Fortunately, the equation is quadratic (second degree), so we can use the quadratic formula with $a = 1$, $b = -2$, and $c = -2$.

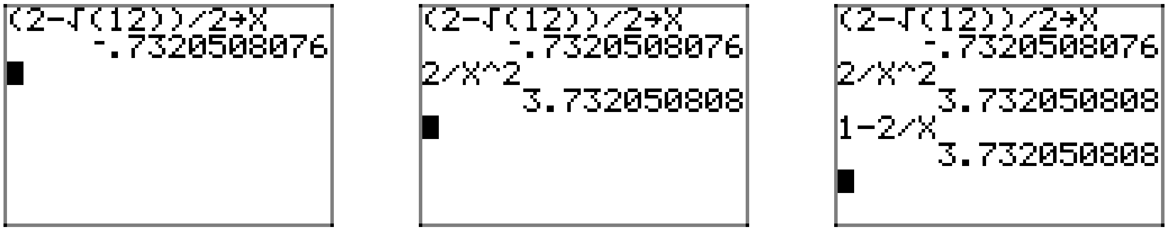
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2-4(1)(-2)}}{2(1)} = \frac{2 \pm \sqrt{12}}{2}$$

This gives us two solutions, $x = (2-\sqrt{12})/2$ and $x = (2+\sqrt{12})/2$. Let's check the solution $x = (2-\sqrt{12})/2$. First, enter this result in your calculator, press the `STO▶` button, press `X`, then press the `ENTER` key to execute the command and store the solution in the variable `X`. This command is shown in **Figure 2**(a).

Enter the left-hand side of the original **equation (6)** as `2/x^2` and press the `ENTER` key to execute this command. This is shown in **Figure 2**(b).

Enter the right-hand side of the original **equation (6)** as `1-2/X` and press the `ENTER` key to execute this command. This is shown in **Figure 2**(c). Note that the left- and right-hand sides of **equation (6)** are both shown to equal `3.732050808` at $x = (2-\sqrt{12})/2$ (at `X = -0.7320508076`), as shown in **Figure 2**(c). This shows that $x = (2-\sqrt{12})/2$ is a solution of **equation (6)**.

We leave it to our readers to check the second solution, $x = (2+\sqrt{12})/2$.



```
(2-√(12))/2→X
    -.7320508076
```

```
(2-√(12))/2→X
    -.7320508076
2/X^2
     3.732050808
```

```
(2-√(12))/2→X
    -.7320508076
2/X^2
     3.732050808
1-2/X
     3.732050808
```

(a) (b) (c)

**Figure 2.** Using the graphing calculator to check the solutions of **equation (6)**.

◆

Let's look at another example, this one involving function notation.

▶ **Example 7.** *Consider the function defined by*

$$f(x) = \frac{1}{x} + \frac{1}{x-4}. \tag{8}$$

*Solve the equation $f(x) = 2$ for $x$ using both graphical and analytical techniques, then compare solutions. Perform each of the following tasks.*

a. *Sketch the graph of $f$ on graph paper. Label the zeros of $f$ with their coordinates and the asymptotes of $f$ with their equations.*
b. *Add the graph of $y = 2$ to your plot and estimate the coordinates of where the graph of $f$ intersects the graph of $y = 2$.*
c. *Use the* `intersect` *utility on your calculator to find better approximations of the points where the graphs of $f$ and $y = 2$ intersect.*
d. *Solve the equation $f(x) = 2$ algebraically and compare your solutions to those found in part (c).*

For the graph in part (a), we need to find the zeros of $f$ and the equations of any vertical or horizontal asymptotes.

To find the zero of the function $f$, we find a common denominator and add the two rational expressions in **equation (8)**.

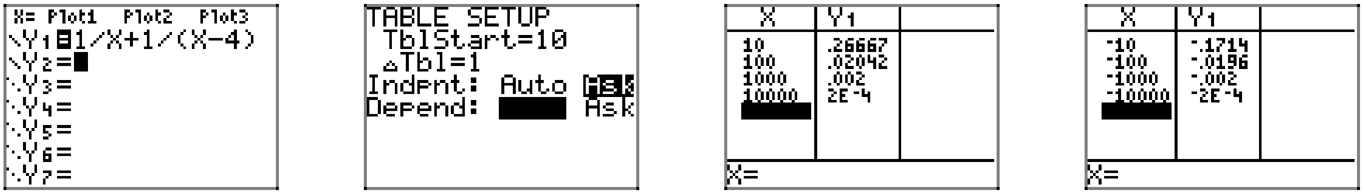
$$f(x) = \frac{1}{x} + \frac{1}{x-4} = \frac{x-4}{x(x-4)} + \frac{x}{x(x-4)} = \frac{2x-4}{x(x-4)} \tag{9}$$

Note that the numerator of this result equal zero (but not the denominator) when $x = 2$. This is the zero of $f$. Thus, the graph of $f$ has $x$-intercept at $(2, 0)$, as shown in **Figure 4**.

Note that the rational function in **equation (9)** is reduced to lowest terms. The denominators of $x$ and $x + 4$ in **equation (9)** are zero when $x = 0$ and $x = 4$. These are our vertical asymptotes, as shown in **Figure 4**.

To find the horizontal asymptotes, we need to examine what happens to the function values as $x$ increases (or decreases) without bound. Enter the function in the `Y=` menu with `1/X+1/(X-4)`, as shown in **Figure 3**(a). Press `2nd TBLSET`, then highlight `ASK` for the independent variable and press `ENTER` to make this selection permanent, as shown in **Figure 3**(b).

Press `2nd TABLE`, then enter 10, 100, 1,000, and 10,000, as shown in **Figure 3**(c). Note how the values of `Y1` approach zero. In **Figure 3**(d), as $x$ decreases without bound, the end-behavior is the same. This is an indication of a horizontal asymptote at $y = 0$, as shown in **Figure 4**.



```
Plot1  Plot2  Plot3
\Y1=1/X+1/(X-4)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

(a)

```
TABLE SETUP
 TblStart=10
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

(b)

```
  X    | Y1
 10    | .26667
 100   | .02042
 1000  | .002
 10000 | 2E-4
X=
```

(c)

```
  X     | Y1
 ⁻10    | ⁻.1714
 ⁻100   | ⁻.0196
 ⁻1000  | ⁻.002
 ⁻10000 | ⁻2E-4
X=
```

(d)

**Figure 3.** Examining the end-behavior of $f$ with the graphing calculator.

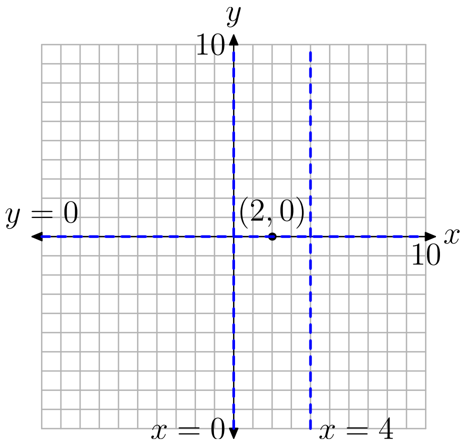


**Figure 4.** Placing the horizontal and vertical asymptotes and the $x$-intercept of the graph of the function $f$.

At this point, we already have our function $f$ loaded in `Y1`, so we can press the `ZOOM` button and select `6:ZStandard` to produce the graph shown in **Figure 5**. As expected, the graphing calculator does not do a very good job with the rational function $f$, particularly near the discontinuities at the vertical asymptotes. However, there is enough information in **Figure 5**, couple with our advanced work summarized in **Figure 4**, to draw a very nice graph of the rational function on our graph paper, as shown in **Figure 6**(a). *Note: We haven't labeled asymptotes with equations, nor zeros with coordinates, in* ***Figure 6***(a)*, as we thought the picture might be a little crowded. However, you should label each of these parts on your graph paper, as we did in* ***Figure 4****.*

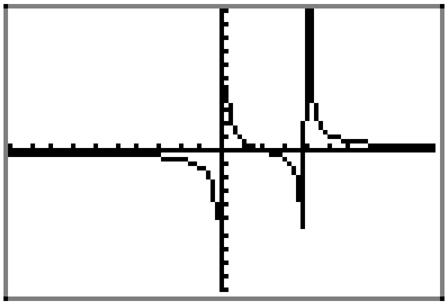
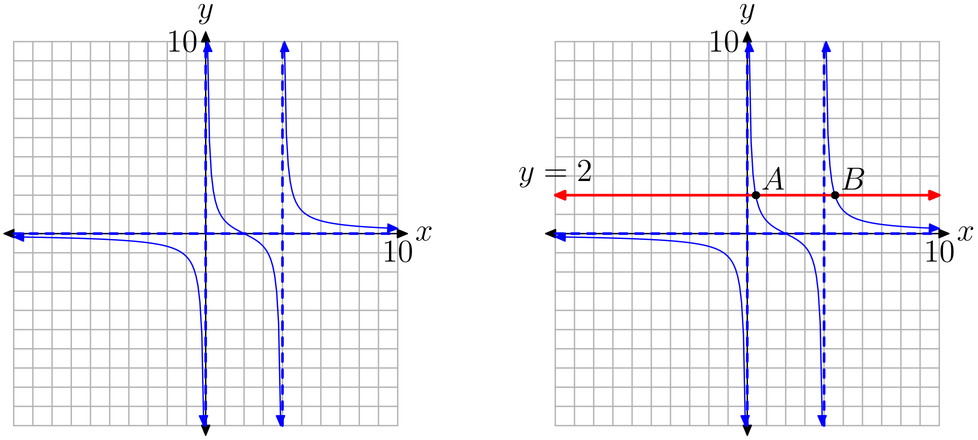


**Figure 5.** The graph of $f$ as drawn on the calculator.

Let's now address part (b) by adding the horizontal line $y = 2$ to the graph, as shown in **Figure 6**(b). Note that the graph of $y = 2$ intersects the graph of the rational function $f$ at two points $A$ and $B$. The $x$-values of points $A$ and $B$ are the solutions to our equation $f(x) = 2$.

We can get a crude estimate of the $x$-coordinates of points $A$ and $B$ right off our graph paper. The $x$-value of point $A$ is approximately $x \approx 0.3$, while the $x$-value of point $B$ appears to be approximately $x \approx 4.6$.



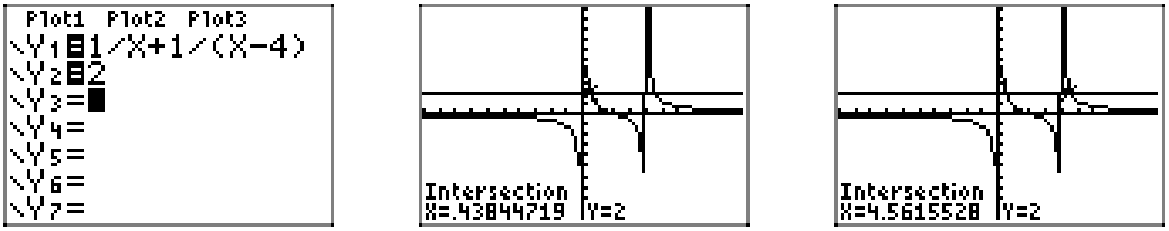
(a) (b)

**Figure 6.** Solving $f(x) = 2$ graphically.

Next, let's address the task required in part (c). We have very reasonable estimates of the solutions of $f(x) = 2$ based on the data presented in **Figure 6**(b). Let's use the graphing calculator to improve upon these estimates.

First, load the equation `Y2=2` into the `Y=` menu, as shown in **Figure 7**(a). We need to find where the graph of `Y1` intersects the graph of `Y2`, so we press `2nd CALC` and select `5:intersect` from the menu. In the usual manner, select "First curve," "Second curve," and move the cursor close to the point you wish to estimate. This is your "Guess." Perform similar tasks for the second point of intersection.

Our results are shown in **Figures 7**(b) and **Figures 7**(c). The estimate in **Figure 7**(b) has $x \approx 0.43844719$, while that in **Figure 7**(c) has $x \approx 4.5615528$. Note that these are more accurate than the approximations of $x \approx 0.3$ and $x \approx 4.6$ captured from our hand drawn image in **Figure 6**(b).



```
Plot1 Plot2 Plot3
\Y1=1/X+1/(X-4)
\Y2=2
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
Intersection
X=.43844719  Y=2
```

```
Intersection
X=4.5615528  Y=2
```

(a) (b) (c)

**Figure 7.** Solving $f(x) = 2$ graphically.

Finally, let's address the request for an algebraic solution of $f(x) = 2$ in part (d). First, replace $f(x)$ with $1/x + 1/(x - 4)$ to obtain

$$\begin{aligned} f(x) &= 2 \\ \frac{1}{x} + \frac{1}{x-4} &= 2. \end{aligned}$$

Multiply both sides of this equation by the common denominator $x(x - 4)$.

$$\begin{aligned} x(x-4)\left[\frac{1}{x}+\frac{1}{x-4}\right] &= [2]\, x(x-4) \\ x(x-4)\left[\frac{1}{x}\right]+x(x-4)\left[\frac{1}{x-4}\right] &= [2]\, x(x-4) \end{aligned}$$

Cancel.

$$\begin{aligned} \cancel{x}(x-4)\left[\frac{1}{\cancel{x}}\right]+x\cancel{(x-4)}\left[\frac{1}{\cancel{x-4}}\right] &= [2]\, x(x-4) \\ (x-4)+x &= 2x(x-4) \end{aligned}$$

Simplify each side.

$$2x-4=2x^2-8x$$

This last equation is nonlinear, so we make one side zero by subtracting $2x$ and adding 4 to both sides of the equation.

$$\begin{aligned} 0 &= 2x^2-8x-2x+4 \\ 0 &= 2x^2-10x+4 \end{aligned}$$

Note that each coefficient on the right-hand side of this last equation is divisible by 2. Let's divide both sides of the equation by 2, distributing the division through each term on the right-hand side of the equation.

$$0=x^2-5x+2$$

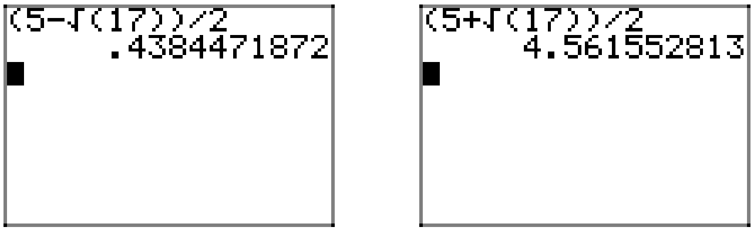
The trinomial on the right is a quadratic with $ac=(1)(2)=2$. There are no integer pairs having product 2 and sum $-5$, so this trinomial doesn't factor. We will use the quadratic formula instead, with $a=1$, $b=-5$ and $c=2$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(2)}}{2(1)}=\frac{5\pm\sqrt{17}}{2}$$

It remains to compare these with the graphical solutions found in part (c). So, enter the solution `(5-√(17))/(2)` in your calculator screen, as shown in **Figure 8**(a). Enter `(5+√(17))/(2)`, as shown in **Figure 8**(b). Thus,

$$\frac{5-\sqrt{17}}{2}\approx 0.4384471872 \qquad \text{and} \qquad \frac{5+\sqrt{17}}{2}\approx 4.561552813.$$

Note the close agreement with the approximations found in part (c).



```
(5-√(17))/2
      .4384471872
```

```
(5+√(17))/2
      4.561552813
```

(a) (b)

**Figure 8.** Approximating the exact solutions.

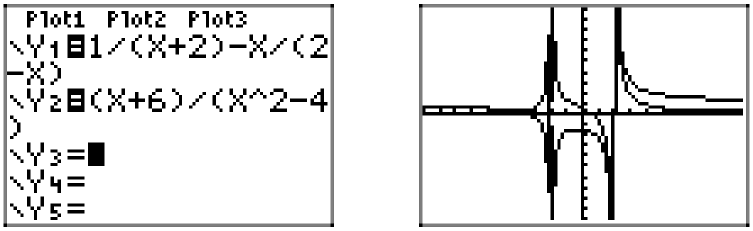
◆

Let's look at another example.

▶ **Example 10.** *Solve the following equation for $x$, both graphically and analytically.*

$$\frac{1}{x+2}-\frac{x}{2-x}=\frac{x+6}{x^2-4} \tag{11}$$

We start the graphical solution in the usual manner, loading the left- and right-hand sides of **equation (11)** into `Y1` and `Y2`, as shown in **Figure 9**(a). Note that in the resulting plot, shown in **Figure 9**(b), it is very difficult to interpret where the graph of the left-hand side intersects the graph of the right-hand side of **equation (11)**.



```
Plot1 Plot2 Plot3
\Y1■1/(X+2)-X/(2
-X)
\Y2■(X+6)/(X^2-4
)
\Y3=■
\Y4=
\Y5=
```

(a) (b)

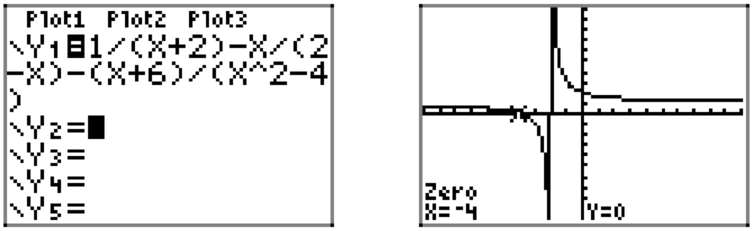
**Figure 9.** Sketch the left- and right-hand sides of **equation (11)**.

In this situation, a better strategy is to make one side of **equation (11)** equal to zero.

$$\frac{1}{x+2}-\frac{x}{2-x}-\frac{x+6}{x^2-4}=0 \tag{12}$$

Our approach will now change. We'll plot the left-hand side of **equation (12)**, then find where the left-hand side is equal to zero; that is, we'll find where the graph of the left-hand side of **equation (12)** intercepts the $x$-axis.

With this thought in mind, load the left-hand side of **equation (12)** into `Y1`, as shown in **Figure 10**(a). Note that the graph in **Figure 10**(b) appears to have only one vertical asymptote at $x=-2$ (some cancellation must remove the factor of $x-2$ from the denominator when you combine the terms of the left-hand side of **equation (12)**[20]). Further, when you use the `zero` utility in the `CALC` menu of the graphing calculator, there appears to be a zero at $x=-4$, as shown in **Figure 10**(b).



```
Plot1 Plot2 Plot3
\Y1■1/(X+2)-X/(2
-X)-(X+6)/(X^2-4
)
\Y2=■
\Y3=
\Y4=
\Y5=
```

(a) (b)

**Figure 10.** Finding the zero of the left-hand side of **equation (12)**.

---

[20] Closer analysis might reveal a "hole" in the graph, but we push on because our check at the end of the problem will reveal a false solution.

Therefore, **equation (12)** seems to have only one solution, namely $x = 4$.

Next, let's seek an analytical solution of **equation (11)**. We'll need to factor the denominators in order to discover a common denominator.

$$\frac{1}{x+2} - \frac{x}{2-x} = \frac{x+6}{(x+2)(x-2)}$$

It's tempting to use a denominator of $(x+2)(2-x)(x-2)$. However, the denominator of the second term on the left-hand side of this last equation, $2-x$, is in a different order than the factors in the other denominators, $x-2$ and $x+2$, so let's perform a sign change on this term and reverse the order. We will negate the fraction bar and negate the denominator. That's two sign changes, so the term remains unchanged when we write

$$\frac{1}{x+2} + \frac{x}{x-2} = \frac{x+6}{(x+2)(x-2)}.$$

Now we see that a common denominator of $(x+2)(x-2)$ will suffice. Let's multiply both sides of the last equation by $(x+2)(x-2)$.

$$\begin{aligned}(x+2)(x-2)\left[\frac{1}{x+2} + \frac{x}{x-2}\right] &= \left[\frac{x+6}{(x+2)(x-2)}\right](x+2)(x-2)\\ (x+2)(x-2)\left[\frac{1}{x+2}\right] + (x+2)(x-2)\left[\frac{x}{x-2}\right] &= \left[\frac{x+6}{(x+2)(x-2)}\right](x+2)(x-2)\end{aligned}$$

Cancel.

$$\begin{aligned}\cancel{(x+2)}(x-2)\left[\frac{1}{\cancel{x+2}}\right] + (x+2)\cancel{(x-2)}\left[\frac{x}{\cancel{x-2}}\right] &= \left[\frac{x+6}{\cancel{(x+2)}\cancel{(x-2)}}\right]\cancel{(x+2)}\cancel{(x-2)}\\ (x-2) + x(x+2) &= x+6\end{aligned}$$

Simplify.

$$\begin{aligned}x - 2 + x^2 + 2x &= x + 6\\ x^2 + 3x - 2 &= x + 6\end{aligned}$$

This last equation is nonlinear because of the presence of a power of $x$ larger than 1 (note the $x^2$ term). Therefore, the strategy is to make one side of the equation equal to zero. We will subtract $x$ and subtract 6 from both sides of the equation.

$$\begin{aligned}x^2 + 3x - 2 - x - 6 &= 0\\ x^2 + 2x - 8 &= 0\end{aligned}$$

The left-hand side is a quadratic trinomial with $ac = (1)(-8) = -8$. The integer pair 4 and $-2$ have product $-8$ and sum 2. Thus,

$$(x+4)(x-2) = 0.$$

Using the zero product property, either

$$x + 4 = 0 \qquad \text{or} \qquad x - 2 = 0,$$

so

$$x = -4 \qquad \text{or} \qquad x = 2.$$

The fact that we have found two answers using an analytical method is troubling. After all, the graph in **Figure 10**(b) indicates only one solution, namely $x = -4$. It is comforting that one of our analytical solutions is also $x = -4$, but it is still disconcerting that our analytical approach reveals a second "answer," namely $x = 2$.

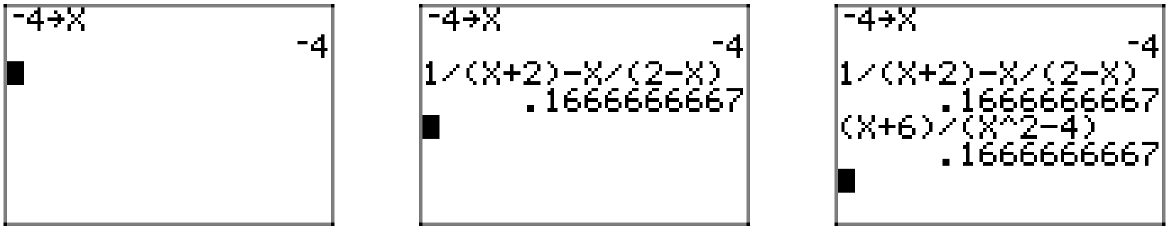
However, notice that we haven't paid any attention to the restrictions caused by denominators up to this point. Indeed, careful consideration of **equation (11)** reveals factors of $x+2$ and $x-2$ in the denominators. Hence, $x = -2$ and $x = 2$ are restrictions.

Note that one of our answers, namely $x = 2$, is a restricted value. It will make some of the denominators in **equation (11)** equal to zero, so it cannot be a solution. Thus, the only viable solution is $x = -4$. One can certainly check this solution by hand, but let's use the graphing calculator to assist us in the check.

First, enter `-4`, press the `STO▶` button, press `X`, then press `ENTER` to execute the resulting command and store `-4` in the variable `X`. The result is shown in **Figure 11**(a).

Next, we calculate the value of the left-hand side of **equation (11)** at this value of `X`. Enter the left-hand side of **equation (11)** as `1/(X+2)-X/(2-X)`, then press the `ENTER` key to execute the statement and produce the result shown in **Figure 11**(b).

Finally, enter the right-hand side of **equation (11)** as `(X+6)/(x^2-4)` and press the `ENTER` key to execute the statement. The result is shown in **Figure 11**(c). Note that both sides of the equation equal `.1666666667` at `X=-4`. Thus, the solution $x = -4$ checks.



```
-4→X
                -4
```

(a)

```
-4→X
                -4
1/(X+2)-X/(2-X)
       .1666666667
```

(b)

```
-4→X
                -4
1/(X+2)-X/(2-X)
       .1666666667
(X+6)/(X^2-4)
       .1666666667
```

(c)

**Figure 11.** Using the graphing calculator to check the solution $x = -4$ of **equation (11)**.

◇

## 7.7 Exercises

For each of the rational functions given in **Exercises 1-6**, perform each of the following tasks.

i. Set up a coordinate system on graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Plot the zero of the rational function on your coordinate system and label it with its coordinates. Plot the vertical and horizontal asymptotes on your coordinate system and label them with their equations. Use this information (and your graphing calculator) to draw the graph of $f$.
iii. Plot the horizontal line $y = k$ on your coordinate system and label this line with its equation.
iv. Use your calculator's `intersect` utility to help determine the solution of $f(x) = k$. Label this point on your graph with its coordinates.
v. Solve the equation $f(x) = k$ algebraically, placing the work for this solution on your graph paper next to your coordinate system containing the graphical solution. Do the answers agree?

**1.** $f(x) = \dfrac{x-1}{x+2}; \quad k = 3$

**2.** $f(x) = \dfrac{x+1}{x-2}; \quad k = -3$

**3.** $f(x) = \dfrac{x+1}{3-x}; \quad k = 2$

**4.** $f(x) = \dfrac{x+3}{2-x}; \quad k = 2$

**5.** $f(x) = \dfrac{2x+3}{x-1}; \quad k = -3$

**6.** $f(x) = \dfrac{5-2x}{x-1}; \quad k = 3$

In **Exercises 7-14**, use a strictly algebraic technique to solve the equation $f(x) = k$ for the given function and value of $k$. You are encouraged to check your result with your calculator.

**7.** $f(x) = \dfrac{16x-9}{2x-1}; \quad k = 8$

**8.** $f(x) = \dfrac{10x-3}{7x+7}; \quad k = 1$

**9.** $f(x) = \dfrac{5x+8}{4x+1}; \quad k = -11$

**10.** $f(x) = -\dfrac{6x-11}{7x-2}; \quad k = -6$

**11.** $f(x) = -\dfrac{35x}{7x+12}; \quad k = -5$

**12.** $f(x) = -\dfrac{66x-5}{6x-10}; \quad k = -11$

**13.** $f(x) = \dfrac{8x+2}{x-11}; \quad k = 11$

**14.** $f(x) = \dfrac{36x-7}{3x-4}; \quad k = 12$

In **Exercises 15-20**, use a strictly algebraic technique to solve the given equation. You are encouraged to check your result with your calculator.

**15.** $\dfrac{x}{7} + \dfrac{8}{9} = -\dfrac{8}{7}$

**16.** $\dfrac{x}{3} + \dfrac{9}{2} = -\dfrac{3}{8}$

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**17.** $-\dfrac{57}{x} = 27 - \dfrac{40}{x^2}$

**18.** $-\dfrac{117}{x} = 54 + \dfrac{54}{x^2}$

**19.** $\dfrac{7}{x} = 4 - \dfrac{3}{x^2}$

**20.** $\dfrac{3}{x^2} = 5 - \dfrac{3}{x}$

For each of the rational functions given in **Exercises 21-26**, perform each of the following tasks.

i. Set up a coordinate system on graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Plot the zero of the rational function on your coordinate system and label it with its coordinates. You may use your calculator's `zero` utility to find this, if you wish.
iii. Plot the vertical and horizontal asymptotes on your coordinate system and label them with their equations. Use the asymptote and zero information (and your graphing calculator) to draw the graph of $f$.
iv. Plot the horizontal line $y = k$ on your coordinate system and label this line with its equation.
v. Use your calculator's `intersect` utility to help determine the solution of $f(x) = k$. Label this point on your graph with its coordinates.
vi. Solve the equation $f(x) = k$ algebraically, placing the work for this soluton on your graph paper next to your coordinate system containing the graphical solution. Do the answers agree?

**21.** $f(x) = \dfrac{1}{x} + \dfrac{1}{x+5}, \quad k = 9/14$

**22.** $f(x) = \dfrac{1}{x} + \dfrac{1}{x-2}, \quad k = 8/15$

**23.** $f(x) = \dfrac{1}{x-1} - \dfrac{1}{x+1}, \quad k = 1/4$

**24.** $f(x) = \dfrac{1}{x-1} - \dfrac{1}{x+2}, \quad k = 1/6$

**25.** $f(x) = \dfrac{1}{x-2} + \dfrac{1}{x+2}, \quad k = 4$

**26.** $f(x) = \dfrac{1}{x-3} + \dfrac{1}{x+2}, \quad k = 5$

In **Exercises 27-34**, use a strictly algebraic technique to solve the given equation. You are encouraged to check your result with your calculator.

**27.** $\dfrac{2}{x+1} + \dfrac{4}{x+2} = -3$

**28.** $\dfrac{2}{x-5} - \dfrac{7}{x-7} = 9$

**29.** $\dfrac{3}{x+9} - \dfrac{2}{x+7} = -3$

**30.** $\dfrac{3}{x+9} - \dfrac{6}{x+7} = 9$

**31.** $\dfrac{2}{x+9} + \dfrac{2}{x+6} = -1$

**32.** $\dfrac{5}{x-6} - \dfrac{8}{x-7} = -1$

**33.** $\dfrac{3}{x+3} + \dfrac{6}{x+2} = -2$

**34.** $\dfrac{2}{x-4} - \dfrac{2}{x-1} = 1$

For each of the equations in **Exercises 35-40**, perform each of the following tasks.

i. Follow the lead of Example 10 in the text. Make one side of the equation equal to zero. Load the nonzero side into your calculator and draw its graph.
ii. Determine the vertical asymptotes of by analyzing the equation and the resulting graph on your calculator. Use the `TABLE` feature of your calculator to determine any horizontal asymptote behavior.
iii. Use the `zero` finding utility in the `CALC` menu to determine the zero of the nonzero side of the resulting equation.
iv. Set up a coordinate system on graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.* Draw the graph of the nonzero side of the equation. Draw the vertical and horizontal asymptotes and label them with their equations. Plot the $x$-intercept and label it with its coordinates.
v. Use an algebraic technique to determine the solution of the equation and compare it with the solution found by the graphical analysis above.

**35.** $\dfrac{x}{x+1}+\dfrac{8}{x^2-2x-3}=\dfrac{2}{x-3}$

**36.** $\dfrac{x}{x+4}-\dfrac{2}{x+1}=\dfrac{12}{x^2+5x+4}$

**37.** $\dfrac{x}{x+1}-\dfrac{4}{2x+1}=\dfrac{2x-1}{2x^2+3x+2}$

**38.** $\dfrac{2x}{x-4}-\dfrac{1}{x+1}=\dfrac{4x+24}{x^2-3x-4}$

**39.** $\dfrac{x}{x-2}+\dfrac{3}{x+2}=\dfrac{8}{4-x^2}$

**40.** $\dfrac{x}{x-1}-\dfrac{4}{x+1}=\dfrac{x-6}{1-x^2}$

---

In **Exercises 41-68**, use a strictly algebraic technique to solve the given equation. You are encouraged to check your result with your calculator.

**41.** $\dfrac{x}{3x-9}-\dfrac{9}{x}=\dfrac{1}{x-3}$

**42.** $\dfrac{5x}{x+2}+\dfrac{5}{x-5}=\dfrac{x+6}{x^2-3x-10}$

**43.** $\dfrac{3x}{x+2}-\dfrac{7}{x}=-\dfrac{1}{2x+4}$

**44.** $\dfrac{4x}{x+6}-\dfrac{4}{x+4}=\dfrac{x-4}{x^2+10x+24}$

**45.** $\dfrac{x}{x-5}+\dfrac{9}{4-x}=\dfrac{x+5}{x^2-9x+20}$

**46.** $\dfrac{6x}{x-5}-\dfrac{2}{x-3}=\dfrac{x-8}{x^2-8x+15}$

**47.** $\dfrac{2x}{x-4}+\dfrac{5}{2-x}=\dfrac{x+8}{x^2-6x+8}$

**48.** $\dfrac{x}{x-7}-\dfrac{8}{5-x}=\dfrac{x+7}{x^2-12x+35}$

**49.** $-\dfrac{x}{2x+2}-\dfrac{6}{x}=-\dfrac{2}{x+1}$

**50.** $\dfrac{7x}{x+3}-\dfrac{4}{2-x}=\dfrac{x+8}{x^2+x-6}$

**51.** $\dfrac{2x}{x+5}-\dfrac{2}{6-x}=\dfrac{x-2}{x^2-x-30}$

**52.** $\dfrac{4x}{x+1}+\dfrac{6}{x+3}=\dfrac{x-9}{x^2+4x+3}$

**53.** $\dfrac{x}{x+7}-\dfrac{2}{x+5}=\dfrac{x+1}{x^2+12x+35}$

**54.** $\dfrac{5x}{6x+4}+\dfrac{6}{x}=\dfrac{1}{3x+2}$

**55.** $\dfrac{2x}{3x+9}-\dfrac{4}{x}=-\dfrac{2}{x+3}$

**56.** $\dfrac{7x}{x+1}-\dfrac{4}{x+2}=\dfrac{x+6}{x^2+3x+2}$

**57.** $\dfrac{x}{2x-8}+\dfrac{8}{x}=\dfrac{2}{x-4}$

**58.** $\dfrac{3x}{x-6}+\dfrac{6}{x-6}=\dfrac{x+2}{x^2-12x+36}$

**59.** $\dfrac{x}{x+2}+\dfrac{2}{x}=-\dfrac{5}{2x+4}$

**60.** $\dfrac{4x}{x-2}+\dfrac{2}{2-x}=\dfrac{x+4}{x^2-4x+4}$

**61.** $-\dfrac{2x}{3x-9}-\dfrac{3}{x}=-\dfrac{2}{x-3}$

**62.** $\dfrac{2x}{x+1}-\dfrac{2}{x}=\dfrac{1}{2x+2}$

**63.** $\dfrac{x}{x+1}+\dfrac{5}{x}=\dfrac{1}{4x+4}$

**64.** $\dfrac{2x}{x-4}-\dfrac{8}{x-7}=\dfrac{x+2}{x^2-11x+28}$

**65.** $-\dfrac{9x}{8x-2}+\dfrac{2}{x}=-\dfrac{2}{4x-1}$

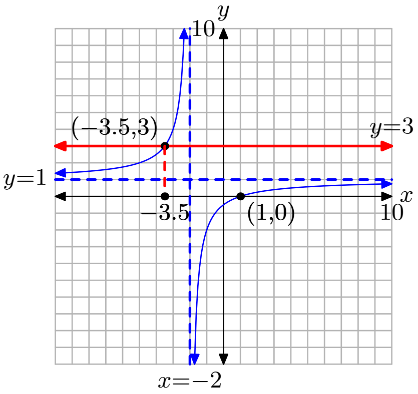
**66.** $\dfrac{2x}{x-3}-\dfrac{4}{4-x}=\dfrac{x-9}{x^2-7x+12}$

**67.** $\dfrac{4x}{x+6}-\dfrac{5}{7-x}=\dfrac{x-5}{x^2-x-42}$

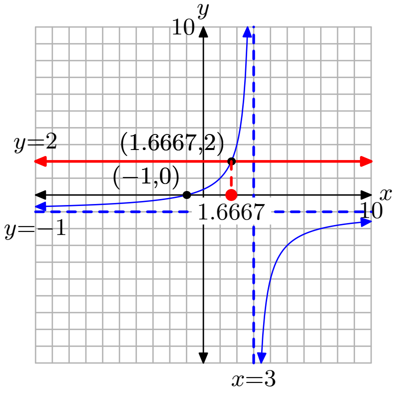
**68.** $\dfrac{x}{x-1}-\dfrac{4}{x}=\dfrac{1}{5x-5}$

## 7.7 *Answers*

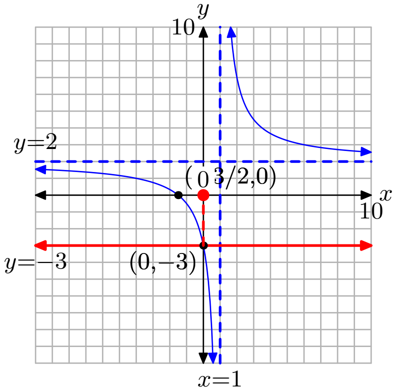
**1.** $x = -7/2$



**3.** $x = 5/3$



**5.** $x = 0$



**7.** none

**9.** $-\dfrac{19}{49}$

**11.** none

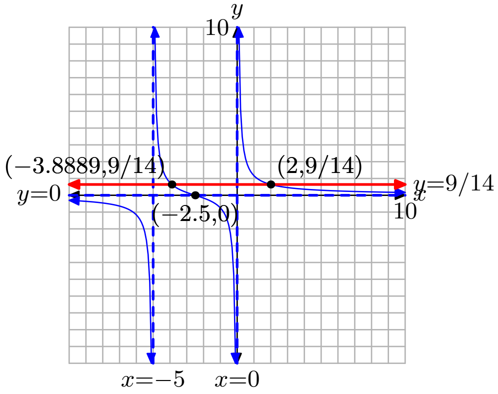
**13.** 41

**15.** $-\dfrac{128}{9}$

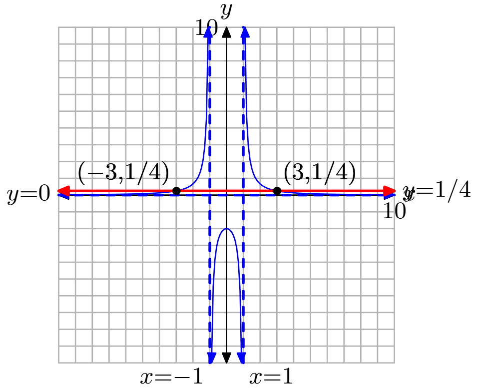
**17.** $-\dfrac{8}{3}, \dfrac{5}{9}$

**19.** $\dfrac{7+\sqrt{97}}{8}, \dfrac{7-\sqrt{97}}{8}$

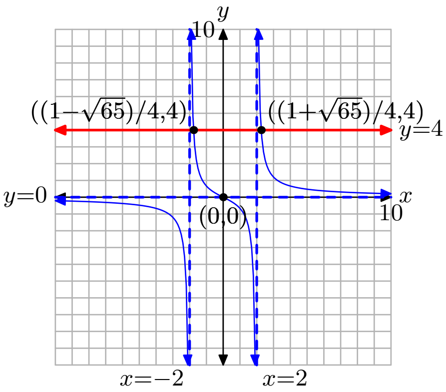
**21.** $x = -35/9$ or $x = 2$



**23.** $x = -3$ or $x = 3$



**25.** $x = \dfrac{1+\sqrt{65}}{4}, \dfrac{1-\sqrt{65}}{4}$



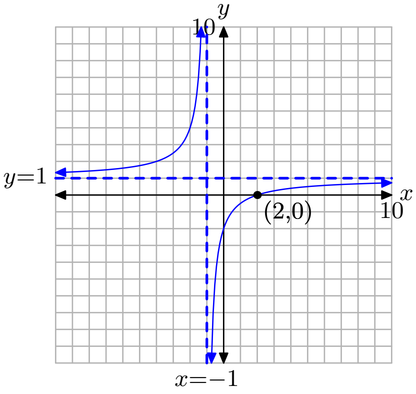
**27.** $\dfrac{-15+\sqrt{57}}{6}, \dfrac{-15-\sqrt{57}}{6}$

**29.** $\dfrac{-49+\sqrt{97}}{6}, \dfrac{-49-\sqrt{97}}{6}$

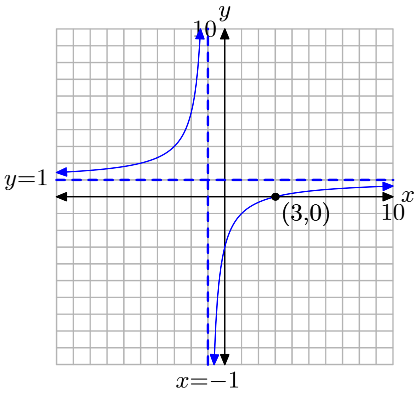
**31.** $-7, -12$

**33.** $\dfrac{-19+\sqrt{73}}{4}, \dfrac{-19-\sqrt{73}}{4}$

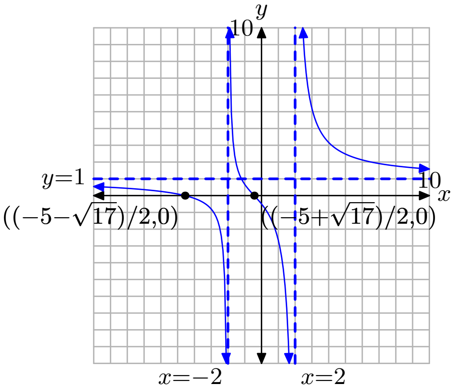
**35.** $x = 2$



**37.** $x = 3$



**39.** $x = \dfrac{-5+\sqrt{17}}{2}, \dfrac{-5-\sqrt{17}}{2}$



**41.** 27

**43.** $\dfrac{7}{2}, -\dfrac{4}{3}$

**45.** 10

**47.** 3

**49.** $-6, -2$

**51.** $4, \frac{3}{2}$

**53.** 3

**55.** 6

**57.** $-16$

**59.** $\dfrac{-9+\sqrt{17}}{4}, \dfrac{-9-\sqrt{17}}{4}$

**61.** $-\dfrac{9}{2}$

**63.** $\dfrac{-19+\sqrt{41}}{8}, \dfrac{-19-\sqrt{41}}{8}$

**65.** $\dfrac{2}{9}, 2$

**67.** $\frac{7}{2}, \frac{5}{2}$

# 7.8 Applications of Rational Functions

In this section, we will investigate the use of rational functions in several applications.

## *Number Problems*

We start by recalling the definition of the *reciprocal* of a number.

**Definition 1.** *For any nonzero real number $a$, the* **reciprocal** *of $a$ is the number $1/a$. Note that the product of a number and its reciprocal is always equal to the number 1. That is,*

$$a \cdot \frac{1}{a} = 1.$$

For example, the reciprocal of the number 3 is 1/3. Note that we simply "invert" the number 3 to obtain its reciprocal 1/3. Further, note that the product of 3 and its reciprocal 1/3 is

$$3 \cdot \frac{1}{3} = 1.$$

As a second example, to find the reciprocal of $-3/5$, we could make the calculation

$$\frac{1}{-\frac{3}{5}} = 1 \div \left(-\frac{3}{5}\right) = 1 \cdot \left(-\frac{5}{3}\right) = -\frac{5}{3},$$

but it's probably faster to simply "invert" $-3/5$ to obtain its reciprocal $-5/3$. Again, note that the product of $-3/5$ and its reciprocal $-5/3$ is

$$\left(-\frac{3}{5}\right) \cdot \left(-\frac{5}{3}\right) = 1.$$

Let's look at some applications that involve the reciprocals of numbers.

► **Example 2.** *The sum of a number and its reciprocal is* 29/10. *Find the number(s).*

Let $x$ represent a nonzero number. The reciprocal of $x$ is $1/x$. Hence, the sum of $x$ and its reciprocal is represented by the rational expression $x + 1/x$. Set this equal to 29/10.

$$x + \frac{1}{x} = \frac{29}{10}$$

To clear fractions from this equation, multiply both sides by the common denominator $10x$.

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$$\begin{aligned} 10x\left(x+\frac{1}{x}\right) &= \left(\frac{29}{10}\right)10x \\ 10x^2+10 &= 29x \end{aligned}$$

This equation is nonlinear (it has a power of $x$ larger than 1), so make one side equal to zero by subtracting $29x$ from both sides of the equation.

$$10x^2 - 29x + 10 = 0$$

Let's try to use the *ac*-test to factor. Note that $ac = (10)(10) = 100$. The integer pair $\{-4, -25\}$ has product 100 and sum $-29$. Break up the middle term of the quadratic trinomial using this pair, then factor by grouping.

$$\begin{aligned} 10x^2 - 4x - 25x + 10 &= 0 \\ 2x(5x-2) - 5(5x-2) &= 0 \\ (2x-5)(5x-2) &= 0 \end{aligned}$$

Using the zero product property, either

$$2x - 5 = 0 \quad \text{or} \quad 5x - 2 = 0.$$

Each of these linear equations is easily solved.

$$x = \frac{5}{2} \quad \text{or} \quad x = \frac{2}{5}$$

Hence, we have two solutions for $x$. However, they both lead to the same number-reciprocal pair. That is, if $x = 5/2$, then its reciprocal is $2/5$. On the other hand, if $x = 2/5$, then its reciprocal is $5/2$.

Let's check our solution by taking the sum of the solution and its reciprocal. Note that

$$\frac{5}{2} + \frac{2}{5} = \frac{25}{10} + \frac{4}{10} = \frac{29}{10},$$

as required by the problem statement.

◆

Let's look at another application of the reciprocal concept.

▶ **Example 3.** *There are two numbers. The second number is 1 larger than twice the first number. The sum of the reciprocals of the two numbers is $7/10$. Find the two numbers.*

Let $x$ represent the first number. If the second number is 1 larger than twice the first number, then the second number can be represented by the expression $2x + 1$.

Thus, our two numbers are $x$ and $2x+1$. Their reciprocals, respectively, are $1/x$ and $1/(2x+1)$. Therefore, the sum of their reciprocals can be represented by the rational expression $1/x + 1/(2x+1)$. Set this equal to $7/10$.

$$\frac{1}{x} + \frac{1}{2x+1} = \frac{7}{10}$$

Multiply both sides of this equation by the common denominator $10x(2x+1)$.

$$10x(2x+1)\left[\frac{1}{x} + \frac{1}{2x+1}\right] = \left[\frac{7}{10}\right] 10x(2x+1)$$
$$10(2x+1) + 10x = 7x(2x+1)$$

Expand and simplify each side of this result.

$$\begin{aligned} 20x + 10 + 10x &= 14x^2 + 7x \\ 30x + 10 &= 14x^2 + 7x \end{aligned}$$

Again, this equation is nonlinear. We will move everything to the right-hand side of this equation. Subtract $30x$ and 10 from both sides of the equation to obtain

$$\begin{aligned} 0 &= 14x^2 + 7x - 30x - 10 \\ 0 &= 14x^2 - 23x - 10. \end{aligned}$$

Note that the right-hand side of this equation is quadratic with $ac = (14)(-10) = -140$. The integer pair $\{5, -28\}$ has product $-140$ and sum $-23$. Break up the middle term using this pair and factor by grouping.

$$\begin{aligned} 0 &= 14x^2 + 5x - 28x - 10 \\ 0 &= x(14x+5) - 2(14x+5) \\ 0 &= (x-2)(14x+5) \end{aligned}$$

Using the zero product property, either

$$x - 2 = 0 \qquad \text{or} \qquad 14x + 5 = 0.$$

These linear equations are easily solved for $x$, providing

$$x = 2 \qquad \text{or} \qquad x = -\frac{5}{14}.$$

We still need to answer the question, which was to find two numbers such that the sum of their reciprocals is $7/10$. Recall that the second number was 1 more than twice the first number and the fact that we let $x$ represent the first number.

Consequently, if the first number is $x = 2$, then the second number is $2x + 1$, or $2(2) + 1$. That is, the second number is 5. Let's check to see if the pair $\{2, 5\}$ is a solution by computing the sum of the reciprocals of 2 and 5.

$$\frac{1}{2} + \frac{1}{5} = \frac{5}{10} + \frac{2}{10} = \frac{7}{10}$$

Thus, the pair $\{2, 5\}$ is a solution.

However, we found a second value for the first number, namely $x = -5/14$. If this is the first number, then the second number is

$$2\left(-\frac{5}{14}\right) + 1 = -\frac{5}{7} + \frac{7}{7} = \frac{2}{7}.$$

Thus, we have a second pair $\{-5/14, 2/7\}$, but what is the sum of the reciprocals of these two numbers? The reciprocals are $-14/5$ and $7/2$, and their sum is

$$-\frac{14}{5} + \frac{7}{2} = -\frac{28}{10} + \frac{35}{10} = \frac{7}{10},$$

as required by the problem statement. Hence, the pair $\{-14/5, 7/2\}$ is also a solution.

◇

## Distance, Speed, and Time Problems

When we developed the *Equations of Motion* in the chapter on quadratic functions, we showed that if an object moves with constant speed, then the distance traveled is given by the formula

$$d = vt, \tag{4}$$

where $d$ represents the distance traveled, $v$ represents the speed, and $t$ represents the time of travel.

For example, if a car travels down a highway at a constant speed of 50 miles per hour (50 mi/h) for 4 hours (4 h), then it will travel

$$\begin{aligned} d &= vt \\ d &= 50\,\frac{\text{mi}}{\text{h}} \times 4\,\text{h} \\ d &= 200\,\text{mi}. \end{aligned}$$

Let's put this relation to use in some applications.

▶ **Example 5.** *A boat travels at a constant speed of 3 miles per hour in still water. In a river with unknown current, it takes the boat twice as long to travel 60 miles upstream (against the current) than it takes for the 60 mile return trip (with the current). What is the speed of the current in the river?*

The speed of the boat in still water is 3 miles per hour. When the boat travels upstream, the current is against the direction the boat is traveling and works to reduce the actual speed of the boat. When the boat travels downstream, then the actual speed of the boat is its speed in still water increased by the speed of the current. If we let $c$ represent the speed of the current in the river, then the boat's speed upstream (against the current) is $3 - c$, while the boat's speed downstream (with the current) is $3 + c$. Let's summarize what we know in a distance-speed-time table (see **Table 1**).

| | $d$ (mi) | $v$ (mi/h) | $t$ (h) |
|---|---|---|---|
| Upstream | 60 | $3-c$ | ? |
| Downstream | 60 | $3+c$ | ? |

**Table 1.** A distance, speed, and time table.

Here is a useful piece of advice regarding distance, speed, and time tables.

**Distance, Speed, and Time Tables.** Because distance, speed, and time are related by the equation $d = vt$, whenever you have two boxes in a row of the table completed, the third box in that row can be calculated by means of the formula $d = vt$.

Note that each row of **Table 1** has two entries entered. The third entry in each row is time. Solve the equation $d = vt$ for $t$ to obtain

$$t = \frac{d}{v}.$$

The relation $t = d/v$ can be used to compute the time entry in each row of **Table 1**.

For example, in the first row, $d = 60$ miles and $v = 3 - c$ miles per hour. Therefore, the time of travel is

$$t = \frac{d}{v} = \frac{60}{3-c}.$$

Note how we've filled in this entry in **Table 2**. In similar fashion, the time to travel downstream is calculated with

$$t = \frac{d}{v} = \frac{60}{3+c}.$$

We've also added this entry to the time column in **Table 2**.

| | $d$ (mi) | $v$ (mi/h) | $t$ (h) |
|---|---|---|---|
| Upstream | 60 | $3-c$ | $\frac{60}{3-c}$ |
| Downstream | 60 | $3+c$ | $\frac{60}{3+c}$ |

**Table 2.** Calculating the time column entries.

To set up an equation, we need to use the fact that the time to travel upstream is twice the time to travel downstream. This leads to the result

$$\frac{60}{3-c} = 2\left(\frac{60}{3+c}\right),$$

or equivalently,

$$\frac{60}{3-c} = \frac{120}{3+c}.$$

Multiply both sides by the common denominator, in this case, $(3-c)(3+c)$.

$$\begin{aligned} (3-c)(3+c)\left[\frac{60}{3-c}\right] &= \left[\frac{120}{3+c}\right](3-c)(3+c) \\ 60(3+c) &= 120(3-c) \end{aligned}$$

Expand each side of this equation.

$$180 + 60c = 360 - 120c$$

This equation is *linear* (no power of $c$ other than 1). Hence, we want to isolate all terms containing $c$ on one side of the equation. We add $120c$ to both sides of the equation, then subtract 180 from both sides of the equation.

$$60c + 120c = 360 - 180$$

From here, it is simple to solve for $c$.

$$\begin{aligned} 180c &= 180 \\ c &= 1. \end{aligned}$$

Hence, the speed of the current is 1 mile per hour.

It is important to check that the solution satisfies the constraints of the problem statement.

- If the speed of the boat in still water is 3 miles per hour and the speed of the current is 1 mile per hour, then the speed of the boat upstream (against the current) will be 2 miles per hour. It will take 30 hours to travel 60 miles at this rate.
- The speed of the boat as it goes downstream (with the current) will be 4 miles per hour. It will take 15 hours to travel 60 miles at this rate.

Note that the time to travel upstream (30 hours) is twice the time to travel downstream (15 hours), so our solution is correct.

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Let's look at another example.

▶ **Example 6.** *A speedboat can travel 32 miles per hour in still water. It travels 150 miles upstream against the current then returns to the starting location. The total time of the trip is 10 hours. What is the speed of the current?*

Let $c$ represent the speed of the current. Going upstream, the boat struggles against the current, so its net speed is $32-c$ miles per hour. On the return trip, the boat benefits from the current, so its net speed on the return trip is $32+c$ miles per hour. The trip each way is 150 miles. We've entered this data in **Table 3**.

| | $d$ (mi) | $v$ (mi/h) | $t$ (h) |
|---|---|---|---|
| Upstream | 150 | $32 - c$ | ? |
| Downstream | 150 | $32 + c$ | ? |

**Table 3.** Entering the given data in a distance, speed, and time table.

Solving $d = vt$ for the time $t$,

$$t = \frac{d}{v}.$$

In the first row of **Table 3**, we have $d = 150$ miles and $v = 32 - c$ miles per hour. Hence, the time it takes the boat to go upstream is given by

$$t = \frac{d}{v} = \frac{150}{32 - c}.$$

Similarly, upon examining the data in the second row of **Table 3**, the time it takes the boat to return downstream to its starting location is

$$t = \frac{d}{v} = \frac{150}{32 + c}.$$

These results are entered in **Table 4**.

| | $d$ (mi) | $v$ (mi/h) | $t$ (h) |
|---|---|---|---|
| Upstream | 150 | $32 - c$ | $150/(32 - c)$ |
| Downstream | 150 | $32 + c$ | $150/(32 + c)$ |

**Table 4.** Calculating the time to go upstream and return.

Because the total time to go upstream and return is 10 hours, we can write

$$\frac{150}{32 - c} + \frac{150}{32 + c} = 10.$$

Multiply both sides by the common denominator $(32 - c)(32 + c)$.

$$\begin{aligned} (32 - c)(32 + c)\left(\frac{150}{32 - c} + \frac{150}{32 + c}\right) &= 10(32 - c)(32 + c) \\ 150(32 + c) + 150(32 - c) &= 10(1024 - c^2) \end{aligned}$$

We can make the numbers a bit smaller by noting that both sides of the last equation are divisible by 10.

$$15(32 + c) + 15(32 - c) = 1024 - c^2$$

Expand, simplify, make one side zero, then factor.

$$\begin{aligned}480 + 15c + 480 - 15c &= 1024 - c^2\\ 960 &= 1024 - c^2\\ 0 &= 64 - c^2\\ 0 &= (8 + c)(8 - c)\end{aligned}$$

Using the zero product property, either

$$8 + c = 0 \qquad \text{or} \qquad 8 - c = 0,$$

providing two solutions for the current,

$$c = -8 \qquad \text{or} \qquad c = 8.$$

Discarding the negative answer (speed is a positive quantity in this case), the speed of the current is 8 miles per hour.

Does our answer make sense?

- Because the speed of the current is 8 miles per hour, the boat travels 150 miles upstream at a net speed of 24 miles per hour. This will take 150/24 or 6.25 hours.
- The boat travels downstream 150 miles at a net speed of 40 miles per hour. This will take 150/40 or 3.75 hours.

Note that the total time to go upstream and return is $6.25 + 3.75$, or 10 hours.

◇

Let's look at another class of problems.

## Work Problems

A nice application of rational functions involves the amount of work a person (or team of persons) can do in a certain amount of time. We can handle these applications involving work in a manner similar to the method we used to solve distance, speed, and time problems. Here is the guiding principle.

**Work, Rate, and Time.** The amount of work done is equal to the product of the rate at which work is being done and the amount of time required to do the work. That is,

$$\text{Work} = \text{Rate} \times \text{Time}.$$

For example, suppose that Emilia can mow lawns at a rate of 3 lawns per hour. After 6 hours,

$$\text{Work} = 3\,\frac{\text{lawns}}{\text{hr}} \times 6\,\text{hr} = 18\,\text{lawns}.$$

A second important concept is the fact that rates add. For example, if Emilia can mow lawns at a rate of 3 lawns per hour and Michele can mow the same lawns at a rate of 2 lawns per hour, then together they can mow the lawns at a combined rate of 5 lawns per hour.

Let's look at an example.

► **Example 7.** *Bill can finish a report in 2 hours. Maria can finish the same report in 4 hours. How long will it take them to finish the report if they work together?*

A common misconception is that the times add in this case. That is, it takes Bill 2 hours to complete the report and it takes Maria 4 hours to complete the same report, so if Bill and Maria work together it will take 6 hours to complete the report. A little thought reveals that this result is nonsense. Clearly, if they work together, it will take them less time than it takes Bill to complete the report alone; that is, the combined time will surely be less than 2 hours.

However, as we saw above, the rates at which they are working will add. To take advantage of this fact, we set up what we know in a Work, Rate, and Time table (see **Table 5**).

- It takes Bill 2 hours to complete 1 report. This is reflected in the entries in the first row of **Table 5**.
- It takes Maria 4 hours to complete 1 report. This is reflected in the entries in the second row of **Table 5**.
- Let $t$ represent the time it takes them to complete 1 report if they work together. This is reflected in the entries in the last row of **Table 5**.

| | $w$ (reports) | $r$ (reports/h) | $t$ (h) |
|---|---|---|---|
| Bill | 1 | ? | 2 |
| Maria | 1 | ? | 4 |
| Together | 1 | ? | $t$ |

**Table 5.** A work, rate, and time table.

We have advice similar to that given for distance, speed, and time tables.

**Work, Rate, and Time Tables.** Because work, rate, and time are related by the equation

$$\text{Work} = \text{Rate} \times \text{Time},$$

whenever you have two boxes in a row completed, the third box in that row can be calculated by means of the relation Work = Rate × Time.

In the case of **Table 5**, we can calculate the rate at which Bill is working by solving the equation Work = Rate × Time for the Rate, then substitute Bill's data from row one of **Table 5**.

$$\text{Rate} = \frac{\text{Work}}{\text{Time}} = \frac{1\,\text{report}}{2\,\text{h}}.$$

Thus, Bill is working at a rate of 1/2 report per hour. Note how we've entered this result in the first row of **Table 6**. Similarly, Maria is working at a rate of 1/4 report per hour, which we've also entered in **Table 6**.

We've let $t$ represent the time it takes them to write 1 report if they are working together (see **Table 5**), so the following calculation gives us the combined rate.

$$\text{Rate} = \frac{\text{Work}}{\text{Time}} = \frac{1\,\text{report}}{t\,\text{h}}.$$

That is, together they work at a rate of $1/t$ reports per hour. This result is also recorded in **Table 6**.

| | $w$ (reports) | $r$ (reports/h) | $t$ (h) |
|---|---|---|---|
| Bill | 1 | 1/2 | 2 |
| Maria | 1 | 1/4 | 4 |
| Together | 1 | $1/t$ | $t$ |

**Table 6.** Calculating the Rate entries.

In our discussion above, we pointed out the fact that rates add. Thus, the equation we seek lies in the Rate column of **Table 6**. Bill is working at a rate of 1/2 report per hour and Maria is working at a rate of 1/4 report per hour. Therefore, their combined rate is $1/2+1/4$ reports per hour. However, the last row of **Table 6** indicates that the combined rate is also $1/t$ reports per hour. Thus,

$$\frac{1}{2}+\frac{1}{4}=\frac{1}{t}.$$

Multiply both sides of this equation by the common denominator $4t$.

$$\begin{aligned}(4t)\left[\frac{1}{2}+\frac{1}{4}\right] &= \left[\frac{1}{t}\right](4t)\\ 2t+t &= 4,\end{aligned}$$

This equation is linear (no power of $t$ other than 1) and is easily solved.

$$\begin{aligned}3t &= 4\\ t &= 4/3\end{aligned}$$

Thus, it will take 4/3 of an hour to complete 1 report if Bill and Maria work together.

Again, it is very important that we check this result.

- We know that Bill does 1/2 reports per hour. In 4/3 of an hour, Bill will complete

  $$\text{Work} = \frac{1}{2}\,\frac{\text{reports}}{\text{h}} \times \frac{4}{3}\,\text{h} = \frac{2}{3}\,\text{reports}.$$

  That is, Bill will complete 2/3 of a report.
- We know that Maria does 1/4 reports per hour. In 4/3 of an hour, Maria will complete

$$\text{Work} = \frac{1}{4}\frac{\text{reports}}{\text{h}} \times \frac{4}{3}\,\text{h} = \frac{1}{3}\,\text{reports}.$$

That is, Maria will complete 1/3 of a report.

Clearly, working together, Bill and Maria will complete $2/3 + 1/3$ reports, that is, one full report.

◇

Let's look at another example.

▶ **Example 8.** *It takes Liya 7 more hours to paint a kitchen than it takes Hank to complete the same job. Together, they can complete the same job in 12 hours. How long does it take Hank to complete the job if he works alone?*

Let $H$ represent the time it take Hank to complete the job of painting the kitchen when he works alone. Because it takes Liya 7 more hours than it takes Hank, let $H+7$ represent the time it takes Liya to paint the kitchen when she works alone. This leads to the entries in **Table 7**.

| | $w$ (kitchens) | $r$ (kitchens/h) | $t$ (h) |
|---|---|---|---|
| Hank | 1 | ? | $H$ |
| Liya | 1 | ? | $H+7$ |
| Together | 1 | ? | 12 |

**Table 7.** Entering the given data for Hank and Liya.

We can calculate the rate at which Hank is working alone by solving the equation Work = Rate × Time for the Rate, then substituting Hank's data from row one of **Table 7**.

$$\text{Rate} = \frac{\text{Work}}{\text{Time}} = \frac{1\,\text{kitchen}}{H\,\text{hour}}$$

Thus, Hank is working at a rate of $1/H$ kitchens per hour. Similarly, Liya is working at a rate of $1/(H+7)$ kitchens per hour. Because it takes them 12 hours to complete the task when working together, their combined rate is 1/12 kitchens per hour. Each of these rates is entered in **Table 8**.

| | $w$ (kitchens) | $r$ (kitchens/h) | $t$ (h) |
|---|---|---|---|
| Hank | 1 | $1/H$ | $H$ |
| Liya | 1 | $1/(H+7)$ | $H+7$ |
| Together | 1 | $1/12$ | 12 |

**Table 8.** Calculating the rates.

Because the rates add, we can write

$$\frac{1}{H} + \frac{1}{H+7} = \frac{1}{12}.$$

Multiply both sides of this equation by the common denominator $12H(H+7)$.

$$12H(H+7)\left(\frac{1}{H}+\frac{1}{H+7}\right)=\left(\frac{1}{12}\right)12H(H+7)$$
$$12(H+7)+12H=H(H+7)$$

Expand and simplify.

$$\begin{aligned}12H+84+12H&=H^2+7H\\ 24H+84&=H^2+7H\end{aligned}$$

This last equation is nonlinear, so make one side zero by subtracting $24H$ and 84 from both sides of the equation.

$$\begin{aligned}0&=H^2+7H-24H-84\\ 0&=H^2-17H-84\end{aligned}$$

Note that $ac=(1)(-84)=-84$. The integer pair $\{4,-21\}$ has product $-84$ and sums to $-17$. Hence,

$$0=(H+4)(H-21).$$

Using the zero product property, either

$$H+4=0 \qquad \text{or} \qquad H-21=0,$$

leading to the solutions

$$H=-4 \qquad \text{or} \qquad H=21.$$

We eliminate the solution $H=-4$ from consideration (it doesn't take Hank negative time to paint the kitchen), so we conclude that it takes Hank 21 hours to paint the kitchen.

Does our solution make sense?

- It takes Hank 21 hours to complete the kitchen, so he is finishing 1/21 of the kitchen per hour.
- It takes Liya 7 hours longer than Hank to complete the kitchen, namely 28 hours, so she is finishing 1/28 of the kitchen per hour.

Together, they are working at a combined rate of

$$\frac{1}{21}+\frac{1}{28}=\frac{4}{84}+\frac{3}{84}=\frac{7}{84}=\frac{1}{12},$$

or 1/12 of a kitchen per hour. This agrees with the combined rate in **Table 8**.

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## 7.8 Exercises

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**1.** The sum of the reciprocals of two consecutive odd integers is $-\frac{16}{63}$. Find the two numbers.

**2.** The sum of the reciprocals of two consecutive odd integers is $\frac{28}{195}$. Find the two numbers.

**3.** The sum of the reciprocals of two consecutive integers is $-\frac{19}{90}$. Find the two numbers.

**4.** The sum of a number and its reciprocal is $\frac{41}{20}$. Find the number(s).

**5.** The sum of the reciprocals of two consecutive even integers is $\frac{5}{12}$. Find the two numbers.

**6.** The sum of the reciprocals of two consecutive integers is $\frac{19}{90}$. Find the two numbers.

**7.** The sum of a number and twice its reciprocal is $\frac{9}{2}$. Find the number(s).

**8.** The sum of a number and its reciprocal is $\frac{5}{2}$. Find the number(s).

**9.** The sum of the reciprocals of two consecutive even integers is $\frac{11}{60}$. Find the two numbers.

**10.** The sum of a number and twice its reciprocal is $\frac{17}{6}$. Find the number(s).

**11.** The sum of the reciprocals of two numbers is $15/8$, and the second number is 2 larger than the first. Find the two numbers.

**12.** The sum of the reciprocals of two numbers is $16/15$, and the second number is 1 larger than the first. Find the two numbers.

---

**13.** Moira can paddle her kayak at a speed of 2 mph in still water. She paddles 3 miles upstream against the current and then returns to the starting location. The total time of the trip is 9 hours. What is the speed (in mph) of the current? Round your answer to the nearest hundredth.

**14.** Boris is kayaking in a river with a 6 mph current. Suppose that he can kayak 4 miles upstream in the same amount of time as it takes him to kayak 9 miles downstream. Find the speed (mph) of Boris's kayak in still water.

**15.** Jacob can paddle his kayak at a speed of 6 mph in still water. He paddles 5 miles upstream against the current and then returns to the starting location. The total time of the trip is 5 hours. What is the speed (in mph) of the current? Round your answer to the nearest hundredth.

**16.** Boris can paddle his kayak at a speed of 6 mph in still water. If he can paddle 5 miles upstream in the same amount of time as it takes his to paddle 9 miles downstream, what is the speed of the current?

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[23] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

**17.** Jacob is canoeing in a river with a 5 mph current. Suppose that he can canoe 4 miles upstream in the same amount of time as it takes him to canoe 8 miles downstream. Find the speed (mph) of Jacob's canoe in still water.

**18.** The speed of a freight train is 16 mph slower than the speed of a passenger train. The passenger train travels 518 miles in the same time that the freight train travels 406 miles. Find the speed of the freight train.

**19.** The speed of a freight train is 20 mph slower than the speed of a passenger train. The passenger train travels 440 miles in the same time that the freight train travels 280 miles. Find the speed of the freight train.

**20.** Emily can paddle her canoe at a speed of 2 mph in still water. She paddles 5 miles upstream against the current and then returns to the starting location. The total time of the trip is 6 hours. What is the speed (in mph) of the current? Round your answer to the nearest hundredth.

**21.** Jacob is canoeing in a river with a 2 mph current. Suppose that he can canoe 2 miles upstream in the same amount of time as it takes him to canoe 5 miles downstream. Find the speed (mph) of Jacob's canoe in still water.

**22.** Moira can paddle her kayak at a speed of 2 mph in still water. If she can paddle 4 miles upstream in the same amount of time as it takes her to paddle 8 miles downstream, what is the speed of the current?

**23.** Boris can paddle his kayak at a speed of 6 mph in still water. If he can paddle 5 miles upstream in the same amount of time as it takes his to paddle 10 miles downstream, what is the speed of the current?

**24.** The speed of a freight train is 19 mph slower than the speed of a passenger train. The passenger train travels 544 miles in the same time that the freight train travels 392 miles. Find the speed of the freight train.

---

**25.** It takes Jean 15 hours longer to complete an inventory report than it takes Sanjay. If they work together, it takes them 10 hours. How many hours would it take Sanjay if he worked alone?

**26.** Jean can paint a room in 5 hours. It takes Amelie 10 hours to paint the same room. How many hours will it take if they work together?

**27.** It takes Amelie 18 hours longer to complete an inventory report than it takes Jean. If they work together, it takes them 12 hours. How many hours would it take Jean if she worked alone?

**28.** Sanjay can paint a room in 5 hours. It takes Amelie 9 hours to paint the same room. How many hours will it take if they work together?

**29.** It takes Ricardo 12 hours longer to complete an inventory report than it takes Sanjay. If they work together, it takes them 8 hours. How many hours would it take Sanjay if he worked alone?

**30.** It takes Ricardo 8 hours longer to complete an inventory report than it takes Amelie. If they work together, it takes them 3 hours. How many hours would it take Amelie if she worked alone?

**31.** Jean can paint a room in 4 hours. It takes Sanjay 7 hours to paint the same room. How many hours will it take if they work together?

**32.** Amelie can paint a room in 5 hours. It takes Sanjay 9 hours to paint the same room. How many hours will it take if they work together?

## 7.8 Answers

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**1.** $-9$, $-7$

**3.** $-10$, $-9$

**5.** 4, 6

**7.** $\frac{1}{2}$, 4

**9.** 10, 12

**11.** $\{2/3, 8/3\}$ and $\{-8/5, 2/5\}$

**13.** 1.63 mph

**15.** 4.90 mph

**17.** 15 mph

**19.** 35 mph

**21.** $\frac{14}{3}$ mph

**23.** 2 mph

**25.** 15 hours

**27.** 18 hours

**29.** 12 hours

**31.** $\frac{28}{11}$ hours

## 7.9 Index

# 8 Exponential and Logarithmic Functions

In this chapter, we will investigate two more families of functions: exponential functions and logarithmic functions. These are two of the most important functions in mathematics, and both types of functions are used extensively in the study of real-world phenomena. In particular, a good understanding of the concepts of exponential growth and decay is necessary for students in both the natural and social sciences.

Our main focus will be on the nature of exponential functions, and their use in describing and solving problems involving compound interest, population growth, and radioactive decay. Our work with logarithmic functions will be a more limited introduction, mostly concentrating on their relationship with exponential functions and their use in solving exponential equations.

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# 8.1 Exponents and Roots

Before defining the next family of functions, the *exponential functions*, we will need to discuss exponent notation in detail. As we shall see, exponents can be used to describe not only powers (such as $5^2$ and $2^3$), but also roots (such as square roots and cube roots). Along the way, we'll define higher roots and develop a few of their properties. More detailed work with roots will then be taken up in the next chapter.

## Integer Exponents

Recall that use of a positive integer exponent is simply a shorthand for repeated multiplication. For example,

$$5^2 = 5 \cdot 5 \tag{1}$$

and

$$2^3 = 2 \cdot 2 \cdot 2. \tag{2}$$

In general, $b^n$ stands for the quanitity $b$ multiplied by itself $n$ times. With this definition, the following *Laws of Exponents* hold.

**Laws of Exponents**

1. $b^r b^s = b^{r+s}$
2. $\dfrac{b^r}{b^s} = b^{r-s}$
3. $(b^r)^s = b^{rs}$

The Laws of Exponents are illustrated by the following examples.

▶ **Example 3.**

a) $2^3 2^2 = (2 \cdot 2 \cdot 2)(2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 2^{3+2}$

b) $\dfrac{2^4}{2^2} = \dfrac{2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2} = \dfrac{2 \cdot 2 \cdot \cancel{2} \cdot \cancel{2}}{\cancel{2} \cdot \cancel{2}} = 2 \cdot 2 = 2^2 = 2^{4-2}$

c) $(2^3)^2 = (2^3)(2^3) = (2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 2^{3 \cdot 2}$

◆

Note that the second law only makes sense for $r > s$, since otherwise the exponent $r - s$ would be negative or 0. But actually, it turns out that we can create definitions for negative exponents and the 0 exponent, and consequently remove this restriction.

---

[1] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

Negative exponents, as well as the 0 exponent, are simply defined in such a way that the Laws of Exponents will work for *all* integer exponents.

- For the 0 exponent, the first law implies that $b^0 b^1 = b^{0+1}$, and therefore $b^0 b = b$. If $b \neq 0$, we can divide both sides by $b$ to obtain $b^0 = 1$ (there is one exception: $0^0$ is not defined).
- For negative exponents, the second law implies that

  $$b^{-n} = b^{0-n} = \frac{b^0}{b^n} = \frac{1}{b^n},$$

  provided that $b \neq 0$. For example, $2^{-3} = 1/2^3 = 1/8$, and $2^{-4} = 1/2^4 = 1/16$.

Therefore, negative exponents and the 0 exponent are defined as follows:

**Definition 4.**

$$b^{-n} = \frac{1}{b^n} \qquad \text{and} \qquad b^0 = 1$$

*provided that* $b \neq 0$.

▶ **Example 5.** *Compute the exact values of* $4^{-3}$, $6^0$, *and* $\left(\frac{1}{5}\right)^{-2}$.

a) $4^{-3} = \dfrac{1}{4^3} = \dfrac{1}{64}$

b) $6^0 = 1$

c) $\left(\dfrac{1}{5}\right)^{-2} = \dfrac{1}{\left(\frac{1}{5}\right)^2} = \dfrac{1}{\frac{1}{25}} = 25$

◇

We now have $b^n$ defined for all integers $n$, in such a way that the Laws of Exponents hold. It may be surprising to learn that we can likewise define expressions using rational exponents, such as $2^{1/3}$, in a consistent manner. Before doing so, however, we'll need to take a detour and define *roots*.

## Roots

**Square Roots**: Let's begin by defining the square root of a real number. We've used the square root in many sections in this text, so it should be a familiar concept. Nevertheless, in this section we'll look at square roots in more detail.

**Definition 6.** *Given a real number* $a$*, a "square root of* $a$*" is a number* $x$ *such that* $x^2 = a$.

For example, 3 is a square root of 9 since $3^2 = 9$. Likewise, $-4$ is a square root of 16 since $(-4)^2 = 16$. In a sense, taking a square root is the "opposite" of squaring, so the definition of square root must be intimately connected with the graph of $y = x^2$, the squaring function. We investigate square roots in more detail by looking for solutions of the equation

$$x^2 = a. \tag{7}$$

There are three cases, each depending on the value and sign of $a$. In each case, the graph of the left-hand side of $x^2 = a$ is the parabola shown in **Figures 1**(a), (b), and (c).

- Case I: $a < 0$

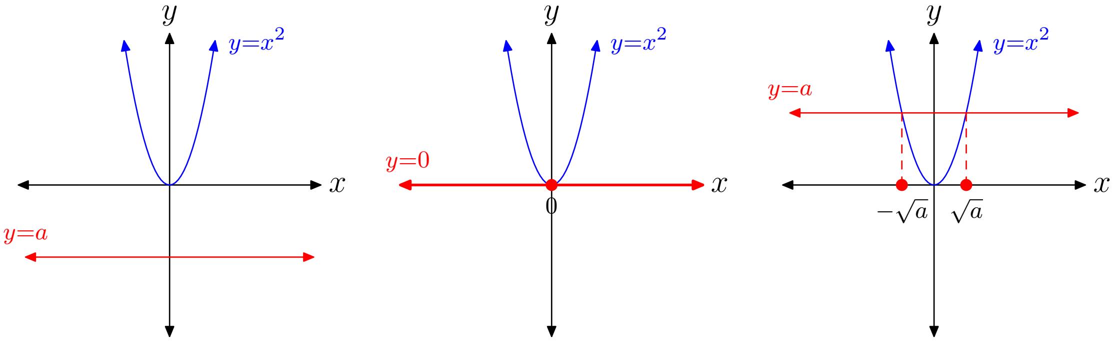
  The graph of the right-hand side of $x^2 = a$ is a horizontal line located $a$ units *below* the $x$-axis. Hence, the graphs of $y = x^2$ and $y = a$ do not intersect and the equation $x^2 = a$ has no real solutions. This case is shown in **Figure 1**(a). It follows that a negative number has no square root.

- Case II: $a = 0$

  The graph of the right-hand side of $x^2 = 0$ is a horizontal line that coincides with the $x$-axis. The graph of $y = x^2$ intersects the graph of $y = 0$ at one point, at the vertex of the parabola. Thus, the only solution of $x^2 = 0$ is $x = 0$, as seen in **Figure 1**(b). The solution is the square root of 0, and is denoted $\sqrt{0}$, so it follows that $\sqrt{0} = 0$.

- Case III: $a > 0$

  The graph of the right-hand side of $x^2 = a$ is a horizontal line located $a$ units *above* the $x$-axis. The graphs of $y = x^2$ and $y = a$ have two points of intersection, and therefore the equation $x^2 = a$ has two real solutions, as shown in **Figure 1**(c). The solutions of $x^2 = a$ are $x = \pm\sqrt{a}$. Note that we have two notations, one that calls for the positive solution and a second that calls for the negative solution.



(a) No real solutions. (b) One real solution. (c) Two real solutions.

**Figure 1.** The solutions of $x^2 = a$ depend upon the sign and value of $a$.

Let's look at some examples.

▶ **Example 8.** *What are the solutions of* $x^2 = -5$*?*

The graph of the left-hand side of $x^2 = -5$ is the parabola depicted in **Figure 1**(a). The graph of the right-hand side of $x^2 = -5$ is a horizontal line located 5 units below the $x$-axis. Thus, the graphs do not intersect and the equation $x^2 = -5$ has no real solutions.

You can also reason as follows. We're asked to find a solution of $x^2 = -5$, so you must find a number whose square equals $-5$. However, whenever you square a real number, the result is always nonnegative (zero or positive). It is not possible to square a real number and get $-5$.

Note that this also means that it is not possible to take the square root of a negative number. That is, $\sqrt{-5}$ is not a real number.

◆

▶ **Example 9.** *What are the solutions of* $x^2 = 0$*?*

There is only one solution, namely $x = 0$. Note that this means that $\sqrt{0} = 0$.

◆

▶ **Example 10.** *What are the solutions of* $x^2 = 25$*?*

The graph of the left-hand side of $x^2 = 25$ is the parabola depicted in **Figure 1**(c). The graph of the right-hand side of $x^2 = 25$ is a horizontal line located 25 units above the $x$-axis. The graphs will intersect in two points, so the equation $x^2 = 25$ has two real solutions.

The solutions of $x^2 = 25$ are called *square roots* of 25 and are written $x = \pm\sqrt{25}$. In this case, we can simplify further and write $x = \pm 5$.

It is extremely important to note the symmetry in **Figure 1**(c) and note that we have two real solutions, one negative and one positive. Thus, we need two notations, one for the positive square root of 25 and one for the negative square root 25.

Note that $(5)^2 = 25$, so $x = 5$ is the positive solution of $x^2 = 25$. For the positive solution, we use the notation

$$\sqrt{25} = 5.$$

This is pronounced "the positive square root of 25 is 5."

On the other hand, note that $(-5)^2 = 25$, so $x = -5$ is the negative solution of $x^2 = 25$. For the negative solution, we use the notation

$$-\sqrt{25} = -5.$$

This is pronounced "the negative square root of 25 is $-5$."

◆

This discussion leads to the following detailed summary.

**Summary: Square Roots**

The solutions of $x^2 = a$ are called "square roots of $a$."

- Case I: $a < 0$. The equation $x^2 = a$ has no real solutions.
- Case II: $a = 0$. The equation $x^2 = a$ has one real solution, namely $x = 0$. Thus, $\sqrt{0} = 0$.
- Case III: $a > 0$. The equation $x^2 = a$ has two real solutions, $x = \pm\sqrt{a}$. The notation $\sqrt{a}$ calls for the positive square root of $a$, that is, the positive solution of $x^2 = a$. The notation $-\sqrt{a}$ calls for the negative square root of $a$, that is, the negative solution of $x^2 = a$.

**Cube Roots**: Let's move on to the definition of cube roots.

**Definition 11.** *Given a real number $a$, a "cube root of $a$" is a number $x$ such that $x^3 = a$.*

For example, 2 is a cube root of 8 since $2^3 = 8$. Likewise, $-4$ is a cube root of $-64$ since $(-4)^3 = -64$. Thus, taking the cube root is the "opposite" of cubing, so the definition of cube root must be closely connected to the graph of $y = x^3$, the cubing function. Therefore, we look for solutions of

$$x^3 = a. \tag{12}$$

Because of the shape of the graph of $y = x^3$, there is only one case to consider. The graph of the left-hand side of $x^3 = a$ is shown in **Figure 2**. The graph of the right-hand side of $x^3 = a$ is a horizontal line, located $a$ units above, on, or below the $x$-axis, depending on the sign and value of $a$. Regardless of the location of the horizontal line $y = a$, there will only be one point of intersection, as shown in **Figure 2**.

A detailed summary of cube roots follows.

**Summary: Cube Roots**

The solutions of $x^3 = a$ are called the "cube roots of $a$." Whether $a$ is negative, zero, or positive makes no difference. There is exactly one real solution, namely $x = \sqrt[3]{a}$.

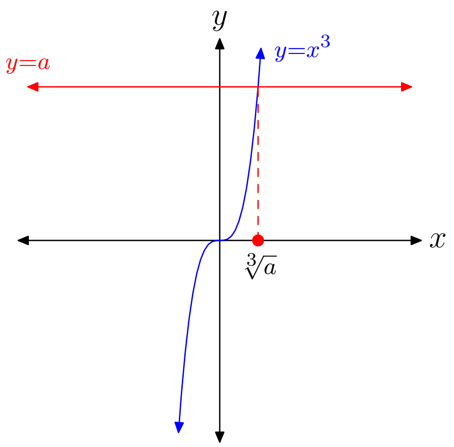


**Figure 2.** The graph of $y = x^3$ intersect the graph of $y = a$ in exactly one place.

Let's look at some examples.

► **Example 13.** *What are the solutions of* $x^3 = 8$*?*

The graph of the left-hand side of $x^3 = 8$ is the cubic polynomial shown in **Figure 2**. The graph of the right-hand side of $x^3 = 8$ is a horizontal line located 8 units above the $x$-axis. The graphs have one point of intersection, so the equation $x^3 = 8$ has exactly one real solution.[2]

The solutions of $x^3 = 8$ are called "cube roots of 8." As shown from the graph, there is exactly one real solution of $x^3 = 8$, namely $x = \sqrt[3]{8}$. Now since $(2)^3 = 8$, it follows that $x = 2$ is a real solution of $x^3 = 8$. Consequently, the cube root of 8 is 2, and we write

$$\sqrt[3]{8} = 2.$$

Note that in the case of cube root, there is no need for the two notations we saw in the square root case (one for the positive square root, one for the negative square root). This is because there is only one real cube root. Thus, the notation $\sqrt[3]{8}$ is pronounced "the cube root of 8."

◆

► **Example 14.** *What are the solutions of* $x^3 = 0$*?*

There is only one solution of $x^3 = 0$, namely $x = 0$. This means that $\sqrt[3]{0} = 0$.

◆

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[2] There are also two other solutions, but they are both complex numbers, not real numbers. This textbook does not discuss complex numbers, but you may learn about them in more advanced courses.

▶ **Example 15.** *What are the solutions of $x^3 = -8$?*

The graph of the left-hand side of $x^3 = -8$ is the cubic polynomial shown in **Figure 2**. The graph of the right-hand side of $x^3 = -8$ is a horizontal line located 8 units below the $x$-axis. The graphs have only one point of intersection, so the equation $x^3 = -8$ has exactly one real solution, denoted $x = \sqrt[3]{-8}$. Now since $(-2)^3 = -8$, it follows that $x = -2$ is a real solution of $x^3 = -8$. Consequently, the cube root of $-8$ is $-2$, and we write

$$\sqrt[3]{-8} = -2.$$

Again, because there is only one real solution of $x^3 = -8$, the notation $\sqrt[3]{-8}$ is pronounced "the cube root of $-8$." Note that, unlike the square root of a negative number, the cube root of a negative number is allowed.

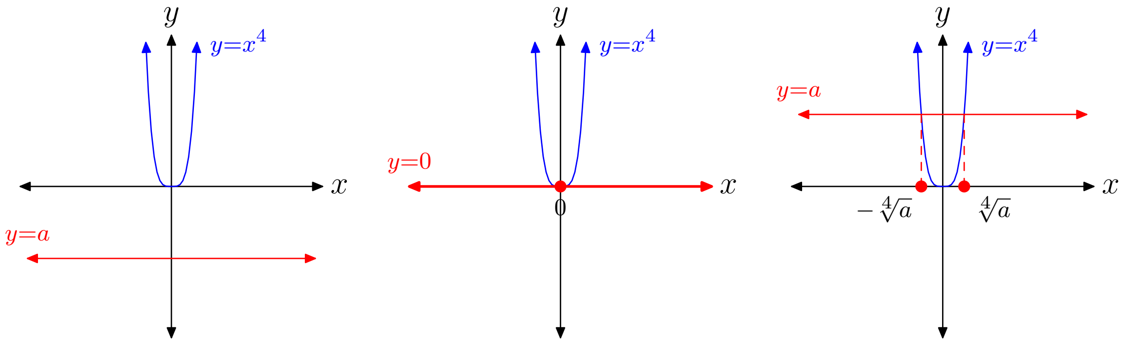
◆

**Higher Roots**: The previous discussions generalize easily to higher roots, such as fourth roots, fifth roots, sixth roots, etc.

**Definition 16.** *Given a real number $a$ and a positive integer $n$, an "$n$th root of $a$" is a number $x$ such that $x^n = a$.*

For example, 2 is a 6th root of 64 since $2^6 = 64$, and $-3$ is a fifth root of $-243$ since $(-3)^5 = -243$.

The case of *even* roots (i.e., when $n$ is even) closely parallels the case of square roots. That's because when the exponent $n$ is even, the graph of $y = x^n$ closely resembles that of $y = x^2$. For example, observe the case for fourth roots shown in **Figures 3**(a), (b), and (c).



(a) No real solutions. (b) One real solution. (c) Two real solutions.

**Figure 3.** The solutions of $x^4 = a$ depend upon the sign and value of $a$.

The discussion for *even* $n$th roots closely parallels that presented in the introduction of square roots, so without further ado, we go straight to the summary.

**Summary: Even $n$th Roots**

If $n$ is a positive even integer, then the solutions of $x^n = a$ are called "$n$th roots of $a$."

- Case I: $a < 0$. The equation $x^n = a$ has no real solutions.
- Case II: $a = 0$. The equation $x^n = a$ has exactly one real solution, namely $x = 0$. Thus, $\sqrt[n]{0} = 0$.
- Case III: $a > 0$. The equation $x^n = a$ has two real solutions, $x = \pm\sqrt[n]{a}$. The notation $\sqrt[n]{a}$ calls for the positive $n$th root of $a$, that is, the positive solution of $x^n = a$. The notation $-\sqrt[n]{a}$ calls for the negative $n$th root of $a$, that is, the negative solution of $x^n = a$.

Likewise, the case of *odd* roots (i.e., when $n$ is odd) closely parallels the case of cube roots. That's because when the exponent $n$ is odd, the graph of $y = x^n$ closely resembles that of $y = x^3$. For example, observe the case for fifth roots shown in **Figure 4**.

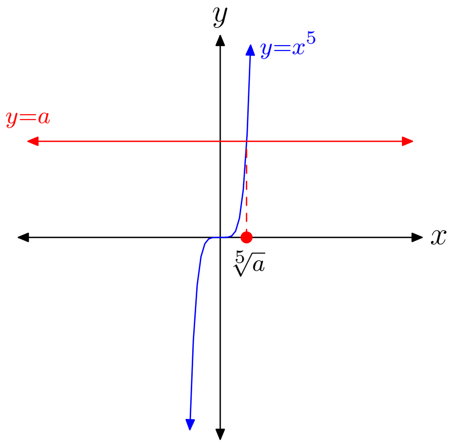


**Figure 4.** The graph of $y = x^5$ intersects the graph of $y = a$ in exactly one place.

The discussion of *odd* $n$th roots closely parallels the introduction of cube roots which we discussed earlier. So, without further ado, we proceed straight to the summary.

**Summary: Odd $n$th Roots**

If $n$ is a positive odd integer, then the solutions of $x^n = a$ are called the "$n$th roots of a." Whether $a$ is negative, zero, or positive makes no difference. There is exactly one real solution of $x^n = a$, denoted $x = \sqrt[n]{a}$.

**Remark 17.** The symbols $\sqrt{\phantom{x}}$ and $\sqrt[n]{\phantom{x}}$ for square root and $n$th root, respectively, are also called *radicals*.

We'll close this section with a few more examples.

▶ **Example 18.** *What are the solutions of* $x^4 = 16$*?*

The graph of the left-hand side of $x^4 = 16$ is the quartic polynomial shown in **Figure 3**(c). The graph of the right-hand side of $x^4 = 16$ is a horizontal line, located 16 units above the $x$-axis. The graphs will intersect in two points, so the equation $x^4 = 16$ has two real solutions.

The solutions of $x^4 = 16$ are called *fourth roots* of 16 and are written $x = \pm\sqrt[4]{16}$. It is extremely important to note the symmetry in **Figure 3**(c) and note that we have two real solutions of $x^4 = 16$, one of which is negative and the other positive. Hence, we need two notations, one for the positive fourth root of 16 and one for the negative fourth root of 16.

Note that $2^4 = 16$, so $x = 2$ is the positive real solution of $x^4 = 16$. For this positive solution, we use the notation

$$\sqrt[4]{16} = 2.$$

This is pronounced "the positive fourth root of 16 is 2."

On the other hand, note that $(-2)^4 = 16$, so $x = -2$ is the negative real solution of $x^4 = 16$. For this negative solution, we use the notation

$$-\sqrt[4]{16} = -2. \tag{19}$$

This is pronounced "the negative fourth root of 16 is −2."

◇

▶ **Example 20.** *What are the solutions of* $x^5 = -32$*?*

The graph of the left-hand side of $x^5 = -32$ is the quintic polynomial pictured in **Figure 4**. The graph of the right-hand side of $x^5 = -32$ is a horizontal line, located 32 units below the $x$-axis. The graphs have one point of intersection, so the equation $x^5 = -32$ has exactly one real solution.

The solutions of $x^5 = -32$ are called "fifth roots of −32." As shown from the graph, there is exactly one real solution of $x^5 = -32$, namely $x = \sqrt[5]{-32}$. Now since $(-2)^5 = -32$, it follows that $x = -2$ is a solution of $x^5 = -32$. Consequently, the fifth root of −32 is −2, and we write

$$\sqrt[5]{-32} = -2.$$

Because there is only one real solution, the notation $\sqrt[5]{-32}$ is pronounced "the fifth root of −32." Again, unlike the square root or fourth root of a negative number, the fifth root of a negative number is allowed.

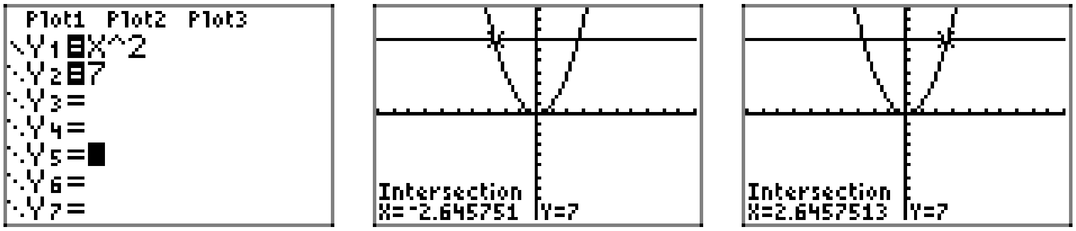
◇

Not all roots simplify to rational numbers. If that were the case, it would not even be necessary to implement radical notation. Consider the following example.

▶ **Example 21.** *Find all real solutions of the equation $x^2 = 7$, both graphically and algebraically, and compare your results.*

We could easily sketch rough graphs of $y = x^2$ and $y = 7$ by hand, but let's seek a higher level of accuracy by asking the graphing calculator to handle this task.

- Load the equation $y = x^2$ and $y = 7$ into `Y1` and `Y2` in the calculator's `Y=` menu, respectively. This is shown in **Figure 5**(a).
- Use the `intersect` utility on the graphing calculator to find the coordinates of the points of intersection. The $x$-coordinates of these points, shown in **Figure 5**(b) and (c), are the solutions to the equation $x^2 = 7$.



```
Plot1  Plot2  Plot3
\Y1=X^2
\Y2=7
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
Intersection
X=⁻2.645751  Y=7
```

```
Intersection
X=2.6457513  Y=7
```

(a) (b) (c)

**Figure 5.** The solutions of $x^2 = 7$ are $x \approx -2.645751$ or $x \approx 2.6457513$.

**Guidelines for Reporting Graphing Calculator Solutions**. Recall the standard method for reporting graphing calculator results on your homework:

- Copy the image from your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax, then label each graph with its equation, as shown in **Figure 6**.
- Drop dashed vertical lines from each point of intersection to the $x$-axis. Shade and label your solutions on the $x$-axis.

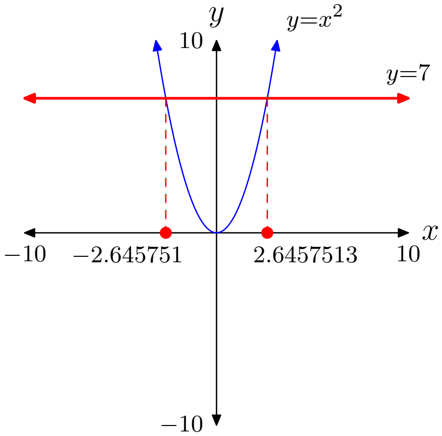


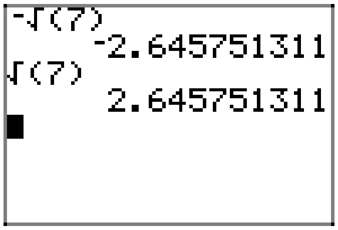
**Figure 6.** The solutions of $x^2 = 7$ are $x \approx -2.645751$ or $x \approx 2.6457513$.

Hence, the **approximate** solutions are $x \approx -2.645751$ or $x \approx 2.6457513$.

On the other hand, to find analytic solutions of $x^2 = 7$, we simply take plus or minus the square root of 7.

$$\begin{aligned} x^2 &= 7 \\ x &= \pm\sqrt{7} \end{aligned}$$

To compare these **exact** solutions with the approximate solutions found by using the graphing calculator, use a calculator to compute $\pm\sqrt{7}$, as shown in **Figure 7**.



```
-√(7)
      ⁻2.645751311
√(7)
       2.645751311
█
```

**Figure 7.** Approximating $\pm\sqrt{7}$.

Note that these approximations of $-\sqrt{7}$ and $\sqrt{7}$ agree quite nicely with the solutions found using the graphing calculator's `intersect` utility and reported in **Figure 6**.

Both $-\sqrt{7}$ and $\sqrt{7}$ are examples of *irrational* numbers, that is, numbers that cannot be expressed in the form $p/q$, where $p$ and $q$ are integers.

## *Rational Exponents*

As with the definition of negative and zero exponents, discussed earlier in this section, it turns out that rational exponents can be defined in such a way that the Laws of Exponents will still apply (and in fact, there's only one way to do it).

The third law gives us a hint on how to define rational exponents. For example, suppose that we want to define $2^{1/3}$. Then by the third law,

$$\left(2^{\frac{1}{3}}\right)^3 = 2^{\frac{1}{3}\cdot 3} = 2^1 = 2,$$

so, by taking cube roots of both sides, we must define $2^{1/3}$ by the formula[3]

$$2^{\frac{1}{3}} = \sqrt[3]{2}.$$

The same argument shows that if $n$ is any odd positive integer, then $2^{1/n}$ must be defined by the formula

$$2^{\frac{1}{n}} = \sqrt[n]{2}.$$

However, for an even integer $n$, there appears to be a choice. Suppose that we want to define $2^{1/2}$. Then

---

[3] Recall that the equation $x^3 = a$ has a unique solution $x = \sqrt[3]{a}$.

$$\left(2^{\frac{1}{2}}\right)^2 = 2^{\frac{1}{2}\cdot 2} = 2^1 = 2,$$

so

$$2^{\frac{1}{2}} = \pm\sqrt{2}.$$

However, the negative choice for the exponent 1/2 leads to problems, because then certain expressions are not defined. For example, it would follow from the third law that

$$(2^{\frac{1}{2}})^{\frac{1}{2}} = -\sqrt{-\sqrt{2}}.$$

But $-\sqrt{2}$ is negative, so $\sqrt{-\sqrt{2}}$ is not defined. Therefore, it only makes sense to use the positive choice. Thus, for all $n$, even and odd, $2^{1/n}$ is defined by the formula

$$2^{\frac{1}{n}} = \sqrt[n]{2}.$$

In a similar manner, for a general positive rational $\frac{m}{n}$, the third law implies that

$$2^{\frac{m}{n}} = (2^m)^{\frac{1}{n}} = \sqrt[n]{2^m}.$$

But also,

$$2^{\frac{m}{n}} = (2^{\frac{1}{n}})^m = (\sqrt[n]{2})^m.$$

Thus,

$$2^{\frac{m}{n}} = \sqrt[n]{2^m} = (\sqrt[n]{2})^m.$$

Finally, negative rational exponents are defined in the usual manner for negative exponents:

$$2^{-\frac{m}{n}} = \frac{1}{2^{\frac{m}{n}}}$$

More generally, here is the final general definition. With this definition, the Laws of Exponents hold for all rational exponents.

**Definition 22.** *For a positive rational exponent* $\frac{m}{n}$, *and* $b > 0$,

$$b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m. \tag{23}$$

*For a negative rational exponent* $-\frac{m}{n}$,

$$b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}}. \tag{24}$$

**Remark 25.** For $b < 0$, the same definitions make sense only when $n$ is odd. For example $(-2)^{\frac{1}{4}}$ is not defined.

▶ **Example 26.** *Compute the exact values of* (a) $4^{\frac{5}{2}}$, (b) $64^{\frac{2}{3}}$, *and* (c) $81^{-\frac{3}{4}}$.

a) $4^{\frac{5}{2}} = \left(4^{\frac{1}{2}}\right)^5 = (\sqrt{4})^5 = 2^5 = 32$

b) $64^{\frac{2}{3}} = \left(64^{\frac{1}{3}}\right)^2 = (\sqrt[3]{64})^2 = 4^2 = 16$

c) $81^{-\frac{3}{4}} = \dfrac{1}{81^{\frac{3}{4}}} = \dfrac{1}{\left(81^{\frac{1}{4}}\right)^3} = \dfrac{1}{(\sqrt[4]{81})^3} = \dfrac{1}{3^3} = \dfrac{1}{27}$

▶ **Example 27.** *Simplify the following expressions, and write them in the form* $x^r$*:*

$$\text{a) } x^{\frac{2}{3}}x^{\frac{1}{4}}, \qquad \text{b) } \frac{x^{\frac{2}{3}}}{x^{\frac{1}{4}}}, \qquad \text{c) } \left(x^{-\frac{2}{3}}\right)^{\frac{1}{4}}$$

a) $x^{\frac{2}{3}}x^{\frac{1}{4}} = x^{\frac{2}{3}+\frac{1}{4}} = x^{\frac{8}{12}+\frac{3}{12}} = x^{\frac{11}{12}}$

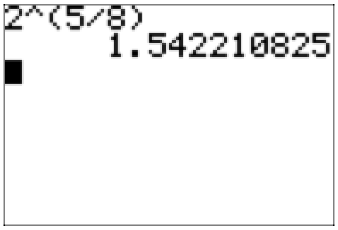
b) $\dfrac{x^{\frac{2}{3}}}{x^{\frac{1}{4}}} = x^{\frac{2}{3}-\frac{1}{4}} = x^{\frac{8}{12}-\frac{3}{12}} = x^{\frac{5}{12}}$

c) $\left(x^{-\frac{2}{3}}\right)^{\frac{1}{4}} = x^{-\frac{2}{3}\cdot\frac{1}{4}} = x^{-\frac{2}{12}} = x^{-\frac{1}{6}}$

▶ **Example 28.** *Use rational exponents to simplify* $\sqrt[5]{\sqrt{x}}$, *and write it as a single radical.*

$$\sqrt[5]{\sqrt{x}} = (\sqrt{x})^{\frac{1}{5}} = \left(x^{\frac{1}{2}}\right)^{\frac{1}{5}} = x^{\frac{1}{2}\cdot\frac{1}{5}} = x^{\frac{1}{10}} = \sqrt[10]{x}$$

▶ **Example 29.** *Use a calculator to approximate* $2^{5/8}$.



```
2^(5/8)
         1.542210825
█
```

**Figure 8.** $2^{5/8} \approx 1.542210825$

## *Irrational Exponents*

What about irrational exponents? Is there a way to define numbers like $2^{\sqrt{2}}$ and $3^{\pi}$? It turns out that the answer is yes. While a rigorous definition of $b^s$ when $s$ is irrational is beyond the scope of this book, it's not hard to see how one could proceed to find a value for such a number. For example, if we want to compute the value of $2^{\sqrt{2}}$, we can start with rational approximations for $\sqrt{2}$. Since $\sqrt{2} = 1.41421356237310\ldots$, the successive powers

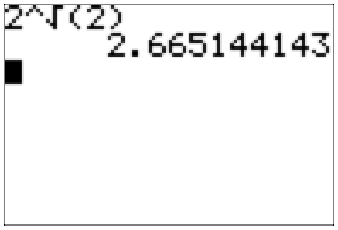
$$2^1,\ 2^{1.4},\ 2^{1.41},\ 2^{1.414},\ 2^{1.4142},\ 2^{1.41421},\ 2^{1.414213},$$
$$2^{1.4142135},\ 2^{1.41421356},\ 2^{1.414213562},\ 2^{1.4142135623},\ldots$$

should be closer and closer approximations to the desired value of $2^{\sqrt{2}}$.

In fact, using more advanced mathematical theory (ultimately based on the actual construction of the real number system), it can be shown that these powers approach a single real number, and we define $2^{\sqrt{2}}$ to be that number. Using your calculator, you can observe this convergence and obtain an approximation by computing the powers above.

| $t$ | $f(t) = 2^t$ |
|---|---|
| 1 | 2 |
| 1.4 | 2.639015822 |
| 1.41 | 2.657371628 |
| 1.414 | 2.664749650 |
| 1.4142 | 2.665119089 |
| 1.41421 | 2.665137562 |
| 1.414213 | 2.665143104 |
| 1.4142135 | 2.665144027 |
| 1.41421356 | 2.665144138 |
| 1.414213562 | 2.665144142 |
| 1.4142135623 | 2.665144143 |

(a) Approximations of $2^{\sqrt{2}}$



```
2^√(2)
         2.665144143
```

(b) $2^{\sqrt{2}} \approx 2.665144143$

**Figure 9.**

The last value in the table in **Figure 9**(a) is a correct approximation of $2^{\sqrt{2}}$ to 10 digits of accuracy. Your calculator will obtain this same approximation when you ask it to compute $2^{\sqrt{2}}$ directly (see **Figure 9**(b)).

In a similar manner, $b^s$ can be defined for any irrational exponent $s$ and any $b > 0$. Combined with the earlier work in this section, it follows that $b^s$ is defined for every real exponent $s$.

## 8.1 Exercises

---

In **Exercises 1-12**, compute the exact value.

**1.** $3^{-5}$

**2.** $4^2$

**3.** $(3/2)^3$

**4.** $(2/3)^1$

**5.** $6^{-2}$

**6.** $4^{-3}$

**7.** $(2/3)^{-3}$

**8.** $(1/3)^{-3}$

**9.** $7^1$

**10.** $(3/2)^{-4}$

**11.** $(5/6)^3$

**12.** $3^2$

---

In **Exercises 13-24**, perform each of the following tasks for the given equation.

i. Load the left- and right-hand sides of the given equation into `Y1` and `Y2`, respectively. Adjust the `WINDOW` parameters until all points of intersection (if any) are visible in your viewing window. Use the `intersect` utility in the `CALC` menu to determine the coordinates of any points of intersection.

ii. Make a copy of the image in your viewing window on your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label each graph with its equation. Drop dashed vertical lines from each point of intersection to the $x$-axis, then shade and label each solution of the given equation on the $x$-axis. *Remember to draw all lines with a ruler.*

iii. Solve each problem algebraically. Use a calculator to approximate any radicals and compare these solutions with those found in parts (i) and (ii).

**13.** $x^2 = 5$

**14.** $x^2 = 7$

**15.** $x^2 = -7$

**16.** $x^2 = -3$

**17.** $x^3 = -6$

**18.** $x^3 = -4$

**19.** $x^4 = 4$

**20.** $x^4 = -7$

**21.** $x^5 = 8$

**22.** $x^5 = 4$

**23.** $x^6 = -5$

**24.** $x^6 = 9$

---

In **Exercises 25-40**, simplify the given radical expression.

**25.** $\sqrt{49}$

**26.** $\sqrt{121}$

---

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**27.** $\sqrt{-36}$

**28.** $\sqrt{-100}$

**29.** $\sqrt[3]{27}$

**30.** $\sqrt[3]{-1}$

**31.** $\sqrt[3]{-125}$

**32.** $\sqrt[3]{64}$

**33.** $\sqrt[4]{-16}$

**34.** $\sqrt[4]{81}$

**35.** $\sqrt[4]{16}$

**36.** $\sqrt[4]{-625}$

**37.** $\sqrt[5]{-32}$

**38.** $\sqrt[5]{243}$

**39.** $\sqrt[5]{1024}$

**40.** $\sqrt[5]{-3125}$

---

**41.** Compare and contrast $\sqrt{(-2)^2}$ and $(\sqrt{-2})^2$.

**42.** Compare and contrast $\sqrt[4]{(-3)^4}$ and $(\sqrt[4]{-3})^4$.

**43.** Compare and contrast $\sqrt[3]{(-5)^3}$ and $(\sqrt[3]{-5})^3$.

**44.** Compare and contrast $\sqrt[5]{(-2)^5}$ and $(\sqrt[5]{-2})^5$.

---

In **Exercises 45-56**, compute the exact value.

**45.** $25^{-\frac{3}{2}}$

**46.** $16^{-\frac{5}{4}}$

**47.** $8^{\frac{4}{3}}$

**48.** $625^{-\frac{3}{4}}$

**49.** $16^{\frac{3}{2}}$

**50.** $64^{\frac{2}{3}}$

**51.** $27^{\frac{2}{3}}$

**52.** $625^{\frac{3}{4}}$

**53.** $256^{\frac{5}{4}}$

**54.** $4^{-\frac{3}{2}}$

**55.** $256^{-\frac{3}{4}}$

**56.** $81^{-\frac{5}{4}}$

---

In **Exercises 57-64**, simplify the product, and write your answer in the form $x^r$.

**57.** $x^{\frac{5}{4}}x^{\frac{5}{4}}$

**58.** $x^{\frac{5}{3}}x^{-\frac{5}{4}}$

**59.** $x^{-\frac{1}{3}}x^{\frac{5}{2}}$

**60.** $x^{-\frac{3}{5}}x^{\frac{3}{2}}$

**61.** $x^{\frac{4}{5}}x^{-\frac{4}{3}}$

**62.** $x^{-\frac{5}{4}}x^{\frac{1}{2}}$

**63.** $x^{-\frac{2}{5}}x^{-\frac{3}{2}}$

**64.** $x^{-\frac{5}{4}}x^{\frac{5}{2}}$

---

In **Exercises 65-72**, simplify the quotient, and write your answer in the form $x^r$.

**65.** $\dfrac{x^{-\frac{5}{4}}}{x^{\frac{1}{5}}}$

**66.** $\dfrac{x^{-\frac{2}{3}}}{x^{\frac{1}{4}}}$

**67.** $\dfrac{x^{-\frac{1}{2}}}{x^{-\frac{3}{5}}}$

**68.** $\dfrac{x^{-\frac{5}{2}}}{x^{\frac{2}{5}}}$

**69.** $\dfrac{x^{\frac{3}{5}}}{x^{-\frac{1}{4}}}$

**70.** $\dfrac{x^{\frac{1}{3}}}{x^{-\frac{1}{2}}}$

**71.** $\dfrac{x^{-\frac{5}{4}}}{x^{\frac{2}{3}}}$

**72.** $\dfrac{x^{\frac{1}{3}}}{x^{\frac{1}{2}}}$

---

In **Exercises 73-80**, simplify the expression, and write your answer in the form $x^r$.

**73.** $\left(x^{\frac{1}{2}}\right)^{\frac{4}{3}}$

**74.** $\left(x^{-\frac{1}{2}}\right)^{-\frac{1}{2}}$

**75.** $\left(x^{-\frac{5}{4}}\right)^{\frac{1}{2}}$

**76.** $\left(x^{-\frac{1}{5}}\right)^{-\frac{3}{2}}$

**77.** $\left(x^{-\frac{1}{2}}\right)^{\frac{3}{2}}$

**78.** $\left(x^{-\frac{1}{3}}\right)^{-\frac{1}{2}}$

**79.** $\left(x^{\frac{1}{5}}\right)^{-\frac{1}{2}}$

**80.** $\left(x^{\frac{2}{5}}\right)^{-\frac{1}{5}}$

## 8.1 Answers

---

**1.** $\dfrac{1}{243}$

**3.** $\dfrac{27}{8}$

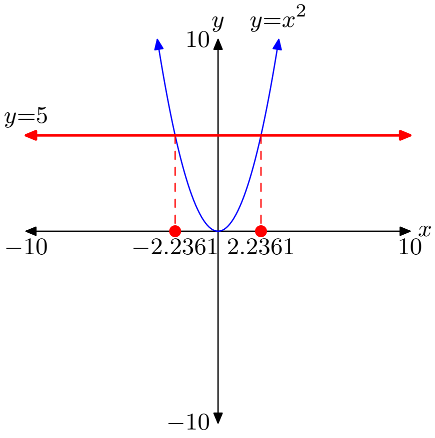
**5.** $\dfrac{1}{36}$

**7.** $\dfrac{27}{8}$

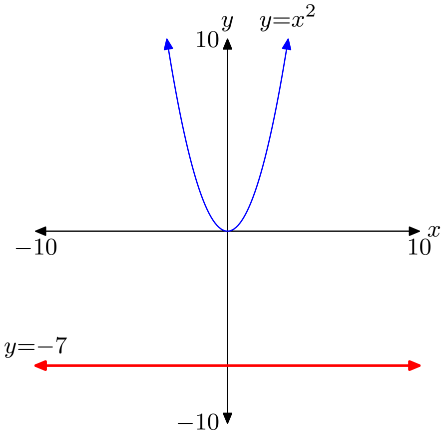
**9.** 7

**11.** $\dfrac{125}{216}$

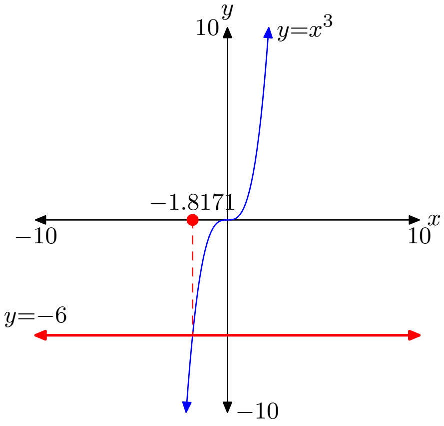
**13.** Solutions: $x = \pm\sqrt{5}$



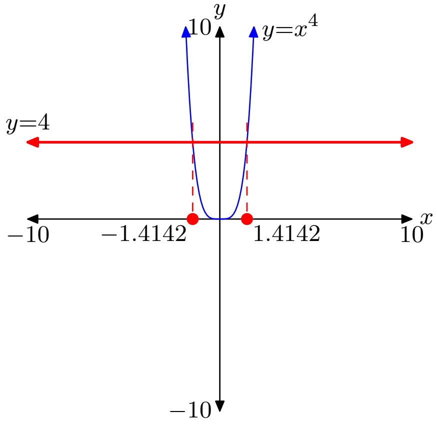
**15.** No real solutions.



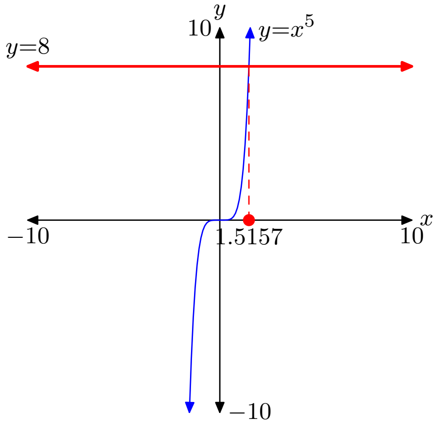
**17.** $x = \sqrt[3]{-6}$



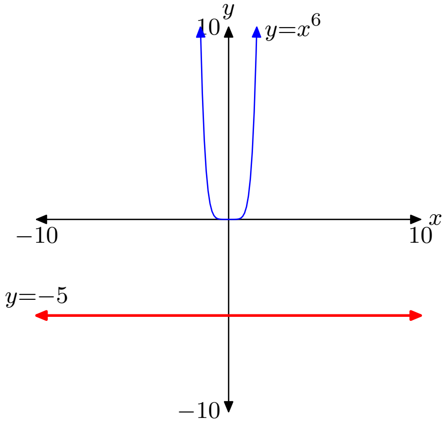
**19.** Solutions: $x = \pm\sqrt[4]{4}$



**21.** $x = \sqrt[5]{8}$



**23.** No real solutions.



**25.** 7

**27.** Not a real number.

**29.** 3

**31.** $-5$

**33.** Not a real number.

**35.** 2

**37.** $-2$

**39.** 4

**41.** $\sqrt{(-2)^2} = 2$, while $(\sqrt{-2})^2$ is not a real number.

**43.** Both equal $-5$.

**45.** $\dfrac{1}{125}$

**47.** 16

**49.** 64

**51.** 9

**53.** 1024

**55.** $\dfrac{1}{64}$

**57.** $x^{\frac{5}{2}}$

**59.** $x^{\frac{13}{6}}$

**61.** $x^{-\frac{8}{15}}$

**63.** $x^{-\frac{19}{10}}$

**65.** $x^{-\frac{29}{20}}$

**67.** $x^{\frac{1}{10}}$

**69.** $x^{\frac{17}{20}}$

**71.** $x^{-\frac{23}{12}}$

**73.** $x^{\frac{2}{3}}$

**75.** $x^{-\frac{5}{8}}$

**77.** $x^{-\frac{3}{4}}$

**79.** $x^{-\frac{1}{10}}$

# 8.2 Exponential Functions

Let's suppose that the current population of the city of Pleasantville is 10 000 and that the population is growing at a rate of 2% per year. In order to analyze the population growth over a period of years, we'll try to develop a formula for the population as a function of time, and then graph the result.

First, note that at the end of one year, the population increase is 2% of 10 000, or 200 people. We would now have 10 200 people in Pleasantville. At the end of the second year, take another 2% of 10 200, which is an increase of 204 people, for a total of 10 404. Because the increase each year is not constant, the graph of population versus time cannot be a line. Hence, our eventual population function will not be linear.

To develop our population formula, we start by letting the function $P(t)$ represent the population of Pleasantville at time $t$, where we measure $t$ in years. We will start time at $t = 0$ when the initial population of Pleasantville is 10 000. In other words, $P(0) = 10\,000$. The key to understanding this example is the fact that the population increases by 2% each year. We are making an assumption here that this overall growth accounts for births, deaths, and people coming into and leaving Pleasantville. That is, at the end of the first year, the population of Pleasantville will be 102% of the initial population. Thus,

$$P(1) = 1.02P(0) = 1.02(10\,000). \tag{1}$$

We could multiply out the right side of this equation, but it will actually be more useful to leave it in its current form.

Now each year the population increases by 2%. Therefore, at the end of the second year, the population will be 102% of the population at the end of the first year. In other words,

$$P(2) = 1.02P(1). \tag{2}$$

If we replace $P(1)$ in **equation (2)** with the result found in **equation (1)**, then

$$P(2) = (1.02)(1.02)(10\,000) = (1.02)^2(10\,000). \tag{3}$$

Let's iterate one more year. At the end of the third year, the population will be 102% of the population at the end of the second year, so

$$P(3) = 1.02P(2). \tag{4}$$

However, if we replace $P(2)$ in **equation (4)** with the result found in **equation (3)**, we obtain

$$P(3) = (1.02)(1.02)^2(10\,000) = (1.02)^3(10\,000). \tag{5}$$

The pattern should now be clear. The population at the end of $t$ years is given by the function

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$$P(t) = (1.02)^t(10\,000).$$

It is traditional in mathematics and science to place the initial population in front in this formula, writing instead

$$P(t) = 10\,000(1.02)^t. \tag{6}$$

Our function $P(t)$ is defined by **equation (6)** for all positive integers $\{1, 2, 3, \ldots\}$, and $P(0) = 10\,000$, the initial population. **Figure 1** shows a plot of our function. Although points are plotted only at integer values of $t$ from 0 to 40, that's enough to show the trend of the population over time. The population starts at 10 000, increases over time, and the yearly increase (the difference in population from one year to the next) also gets larger as time passes.

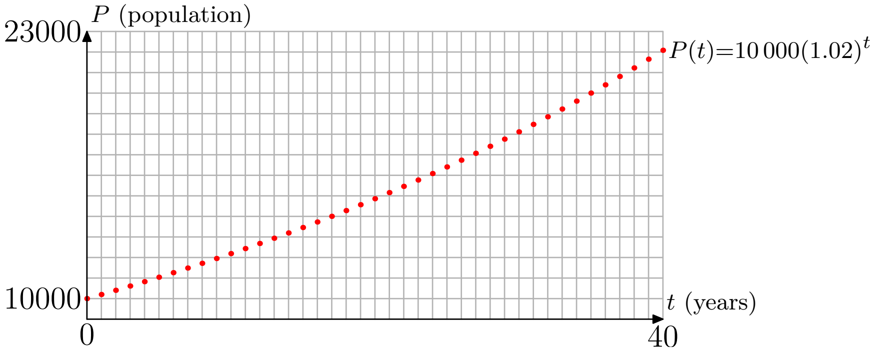


**Figure 1.** Graph of population $P(t)$ of Pleasantville for $t = 0, 1, 2, 3, \ldots$

▶ **Example 7.** *We can now use the function $P(t)$ to predict the population in later years. Assuming that the growth rate of 2% continues, what will the population of Pleasantville be after 40 years? What will it be after 100 years?*

Substitute $t = 40$ and $t = 100$ into **equation (6)**. The population in 40 years will be

$$P(40) = 10\,000(1.02)^{40} \approx 22\,080,$$

and the population in 100 years will be

$$P(100) = 10\,000(1.02)^{100} \approx 72\,446.$$

◆

What would be different if we had started with a population of 12 000? By tracing over our previous steps, it should be easy to see that the new formula would be

$$P(t) = 12\,000(1.02)^t.$$

Similarly, if the growth rate had been 3% per year instead of 2%, then we would have ended up with the formula

$$P(t) = 10\,000(1.03)^t.$$

Thus, by letting $P_0$ represent the initial population, and $r$ represent the growth rate (in decimal form), we can generalize the formula to

$$P(t) = P_0(1 + r)^t. \tag{8}$$

Note that our formula for the function $P(t)$ is different from the previous functions that we've studied so far, in that the input variable $t$ is part of the exponent in the formula. Thus, this is a new type of function.

Now let's contrast the situation in Pleasantville with the population dynamics of Ghosttown. Ghosttown also starts with a population of 10 000, but several factories have closed, so some people are leaving for better opportunities. In this case, the population of Ghosttown is *decreasing* at a rate of 2% per year. We'll again develop a formula for the population as a function of time, and then graph the result.

First, note that at the end of one year, the population decrease is 2% of 10 000, or 200 people. We would now have 9 800 people left in Ghosttown. At the end of the second year, take another 2% of 9 800, which is a decrease of 196 people, for a total of 9 604. As before, because the decrease each year is not constant, the graph of population versus time cannot be a line, so our eventual population function will not be linear.

Now let the function $P(t)$ represent the population of Ghosttown at time $t$, where we measure $t$ in years. The initial population of Ghosttown at $t = 0$ is 10 000, so $P(0) = 10\,000$. Since the population decreases by 2% each year, at the end of the first year the population of Ghosttown will be 98% of the initial population. Thus,

$$P(1) = 0.98P(0) = 0.98(10\,000). \tag{9}$$

Each year the population deccreases by 2%. Therefore, at the end of the second year, the population will be 98% of the population at the end of the first year. In other words,

$$P(2) = 0.98P(1). \tag{10}$$

If we replace $P(1)$ in **equation (10)** with the result found in **equation (9)**, then

$$P(2) = (0.98)(0.98)(10\,000) = (0.98)^2(10\,000). \tag{11}$$

Let's iterate one more year. At the end of the third year, the population will be 98% of the population at the end of the second year, so

$$P(3) = 0.98P(2). \tag{12}$$

However, if we replace $P(2)$ in **equation (12)** with the result found in **equation (11)**, we obtain

$$P(3) = (0.98)(0.98)^2(10\,000) = (0.98)^3(10\,000). \tag{13}$$

The pattern should now be clear. The population at the end of $t$ years is given by the function

$$P(t) = (0.98)^t(10\,000),$$

or equivalently,

$$P(t) = 10\,000(0.98)^t. \tag{14}$$

Our function $P(t)$ is defined by **equation (14)** for all positive integers $\{1, 2, 3, \ldots\}$, and $P(0) = 10\,000$, the initial population. **Figure 2** shows a plot of our function. Although points are plotted only at integer values of $t$ from 0 to 40, that's enough to show the trend of the population over time. The population starts at 10 000, decreases over time, and the yearly decrease (the difference in population from one year to the next) also gets smaller as time passes.

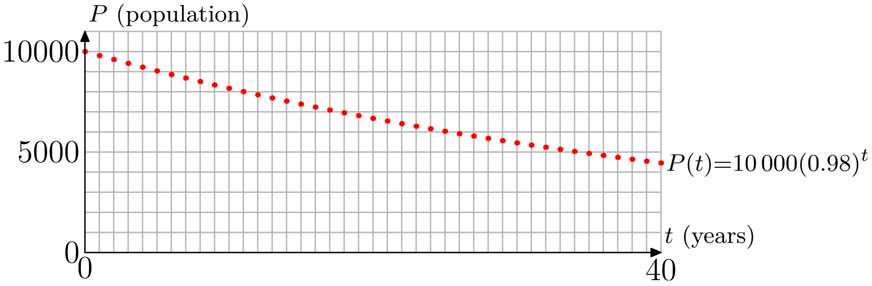


**Figure 2.** Graph of population $P(t)$ of Ghosttown for $t = 0, 1, 2, 3, \ldots$

▶ **Example 15.** *Assuming that the rate of decrease continues at 2%, predict the population of Ghosttown after 40 years and after 100 years.*

Substitute $t = 40$ and $t = 100$ into **equation (14)**. The population in 40 years will be

$$P(40) = 10\,000(0.98)^{40} \approx 4457,$$

and the population in 100 years will be

$$P(100) = 10\,000(0.98)^{100} \approx 1326.$$

◆

Note that if we had instead started with a population of 9 000, for example, then the new formula would be

$$P(t) = 9\,000(0.98)^t.$$

Similarly, if the rate of decrease had been 5% per year instead of 2%, then we would have ended up with the formula

$$P(t) = 10\,000(0.95)^t.$$

Thus, by letting $P_0$ represent the initial population, and $r$ represent the growth rate (in decimal form), we can generalize the formula to

$$P(t) = P_0(1 - r)^t. \tag{16}$$

## Definition

As noted before, our functions $P(t)$ in our Pleasantville and Ghosttown examples are a new type of function, because the input variable $t$ is part of the exponent in the formula.

**Definition 17.** *An* exponential function *is a function of the form*

$$f(t) = b^t,$$

*where $b > 0$ and $b \neq 1$. $b$ is called the* **base** *of the exponential function.*

*More generally, a function of the form*

$$f(t) = Ab^t,$$

*where $b > 0$, $b \neq 1$, and $A \neq 0$, is also referred to as an exponential function. In this case, the value of the function when $t = 0$ is $f(0) = A$, so $A$ is the* **initial amount**.

In applications, you will almost always encounter exponential functions in the more general form $Ab^t$. In fact, note that in the previous population examples, the function $P(t)$ has this form $P(t) = Ab^t$, with $A = P_0$, $b = 1 + r$ in Pleasantville, and $b = 1 - r$ in Ghosttown. In particular, $A = P_0$ is the initial population.

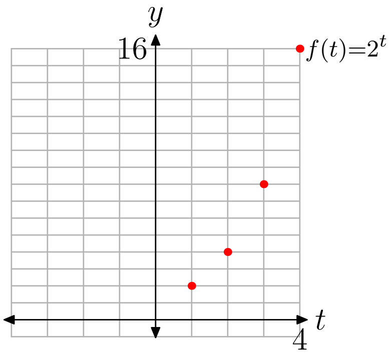
Since exponential functions are often used to model processes that vary with time, we usually use the input variable $t$ (although of course any variable can be used). Also, you may be curious why the definition says $b \neq 1$, since $1^t$ just equals 1. We'll explain this curiosity at the end of this section.

## Graphs of Exponential Functions

We'll develop the properties for the basic exponential function $b^t$ first, and then note the minor changes for the more general form $Ab^t$. For a working example, let's use base $b = 2$, and let's compute some values of $f(t) = 2^t$ and plot the result (see **Figure 3**).

| $t$ | $f(t) = 2^t$ |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |

(a)



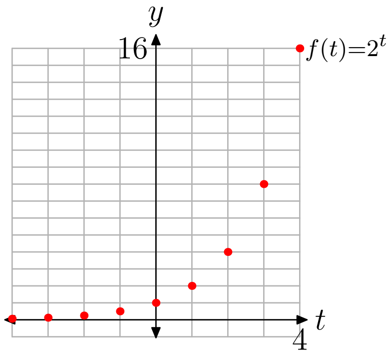
(b)

**Figure 3.** Plotting points $(t, f(t))$ defined by the function $f(t) = 2^t$, with $t = 1, 2, 3, 4, \ldots$.

Recall from the previous section that $2^t$ is also defined for negative exponents $t$ and the 0 exponent. Thus, the exponential function $f(t) = 2^t$ is defined for all integers. **Figure 4** shows a new table and plot with points added at 0 and negative integer values.

| $t$ | $f(t) = 2^t$ |
|---|---|
| $-4$ | $1/16$ |
| $-3$ | $1/8$ |
| $-2$ | $1/4$ |
| $-1$ | $1/2$ |
| 0 | 0 |
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |

(a)



(b)

**Figure 4.** Plotting points $(t, f(t))$ defined by the function $f(t) = 2^t$, with $t = \dots, -3, -2, -1, 0, 1, 2, 3 \dots$.

However, the previous section showed that $2^t$ is also defined for rational and irrational exponents. Therefore, the domain of the exponential function $f(t) = 2^t$ is the set of all real numbers. When we add in the values of the function at all rational and irrational values of $t$, we obtain a final continuous curve as shown in **Figure 5**.

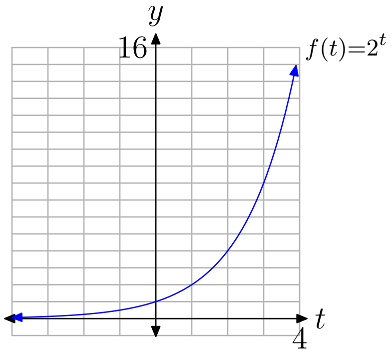


**Figure 5.**

Note several properties of the graph in **Figure 5**:

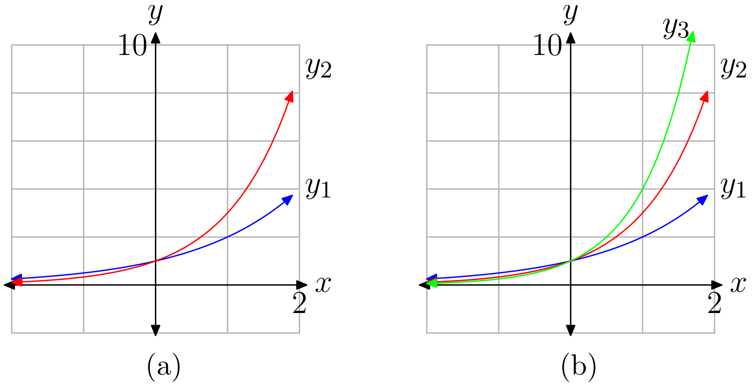
a) Moving from left to right, the curve rises, which means that the function increases as $t$ increases. In fact, the function increases rapidly for positive $t$.

b) The graph lies above the $t$-axis, so the values of the function are always positive. Therefore, the range of the function is $(0, \infty)$.

c) The graph has a horizontal asymptote $y = 0$ (the $t$-axis) on the left side. This means that the function almost "dies out" (the values get closer and closer to 0) as $t$ approaches $-\infty$.

What about the graphs of other exponential functions with different bases? We'll use the calculator to explore several of these.

First, use your calculator to compare $y_1(x) = 2^x$ and $y_2(x) = 3^x$. As can be seen in **Figure 6**(a), the graph of $3^x$ rises faster that $2^x$ for $x > 0$, and dies out faster for $x < 0$.

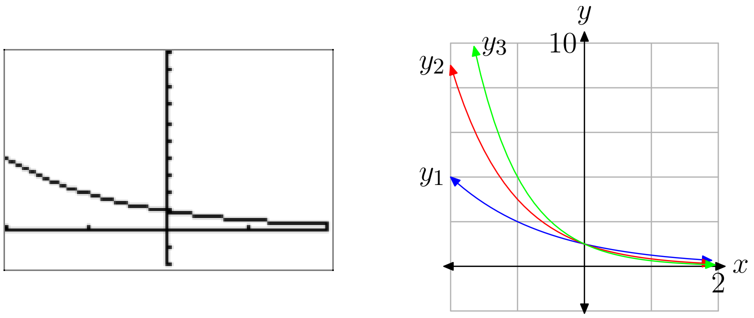


(a) (b)

**Figure 6.** Comparing functions $y_1(x) = 2^x$, $y_2(x) = 3^x$, and $y_3(x) = 4^x$

Next, add in $y_3(x) = 4^x$. The result is shown in **Figure 6**(b). Again, increasing the size of the base to $b = 4$ results in a function which rises even faster on the right and likewise dies out faster on the left. If you continue to increase the size of the base $b$, you'll see that this trend continues. That's not terribly surprising because, if we compute the value of these functions at a fixed positive $x$, for example at $x = 2$, then the values increase: $2^2 < 3^2 < 4^2 < \ldots$. Similarly, at $x = -2$, the values decrease: $2^{-2} > 3^{-2} > 4^{-2} > \ldots$.

All of the functions in our experiments so far share the properties listed in (a)–(c) above: the function increases, the range is $(0, \infty)$, and the graph has a horizontal asymptote $y = 0$ on the left side. Now let's try smaller values of the base $b$. First use the calculator to plot the graph of $y_1(x) = (1/2)^x$ (see **Figure 7**(a)).



(a) Graph of $y_1(x) = (1/2)^x$

(b) Comparing functions $y_1(x) = (1/2)^x$, $y_2(x) = (1/3)^x$, and $y_3(x) = (1/4)^x$

**Figure 7.**

This graph is much different. It rises rapidly to the left, and almost dies out on the right. Compare this with $y_2(x) = (1/3)^x$ and $y_3(x) = (1/4)^x$ (see **Figure 7**(b)). As the base gets smaller, the graph rises faster on the left, and dies out faster on the right.

Using reflection properties, it's easy to understand the appearance of these last three graphs. Note that

$$\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}, \tag{18}$$

so it follows that the graph of $\left(\frac{1}{2}\right)^x$ is just a reflection in the $y$-axis of the graph of $2^x$ (see **Figure 8**).

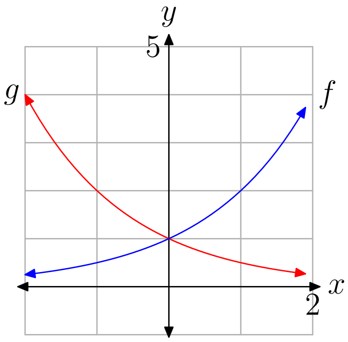


**Figure 8.** Comparing functions $f(x) = 2^x$ and $g(x) = (1/2)^x = 2^{-x}$

Thus, we seem to have two different types of graphs, and therefore two types of exponential functions: one type is increasing, and the other decreasing. Our experiments above, along with a little more experimentation, should convince you that $b^x$ is increasing for $b > 1$, and decreasing for $0 < b < 1$. The first type of functions are called *exponential growth* functions, and the second type are *exponential decay* functions.

**Properties of Exponential Growth Functions:** $f(x) = b^x$ with $b > 1$

- The domain is the set of all real numbers.
- Moving from left to right, the graph rises, which means that the function increases as $x$ increases. The function increases rapidly for positive $x$.
- The graph lies above the $x$-axis, so the values of the function are always positive. Therefore, the range is $(0, \infty)$.
- The graph has a horizontal asymptote $y = 0$ (the $x$-axis) on the left side. This means that the function almost "dies out" (the values get closer and closer to 0) as $x$ approaches $-\infty$.

The second property above deserves some additional explanation. Looking at **Figure 6**(b), it appears that $y_2$ and $y_3$ increase rapidly as $x$ increases, but $y_1$ appears to increase slowly. However, this is due to the fact that the graph of $y_1(x) = 2^x$

is only shown on the interval $[-2, 2]$. In **Figure 5**, the same function is graphed on the interval $[-4, 4]$, and it certainly appears to increase rapidly in that graph. The point here is that exponential growth functions *eventually* increase rapidly as $x$ increases. If you graph the function on a large enough interval, the function will eventually become very steep on the right side of the graph. This is an important property of the exponential growth functions, and will be explored further in the exercises.

**Properties of Exponential Decay Functions:** $f(x) = b^x$ with $0 < b < 1$

- The domain is the set of all real numbers.
- Moving from left to right, the graph falls, which means that the function decreases as $x$ increases. The function decreases rapidly for negative $x$.
- The graph lies above the $x$-axis, so the values of the function are always positive. Therefore, the range is $(0, \infty)$.
- The graph has a horizontal asymptote $y = 0$ (the $x$-axis) on the right side. This means that the function almost "dies out" (the values get closer and closer to 0) as $x$ approaches $\infty$.

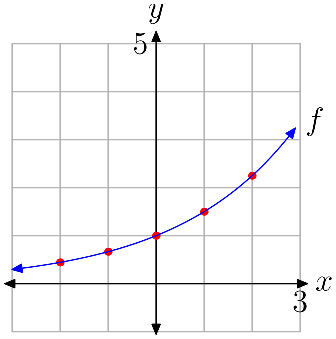
Why do we refrain from using the base $b = 1$? After all, $1^x$ is certainly defined: it has the value 1 for all $x$. But that means that $f(x) = 1^x$ is just a constant linear function – its graph is a horizontal line. Therefore, this function doesn't share the same properties as the other exponential functions, and we've already classified it as a linear function. Thus, $1^x$ is not considered to be an exponential function.

► **Example 19.** *Plot the graph of the function $f(x) = (1.5)^x$. Identify the range of the function and the horizontal asymptote.*

Since the base 1.5 is larger than 1, this is an exponential growth function. Therefore, its graph will have a shape similar to the graphs in **Figure 6**. The graph rises, there will be a horizontal asymptote $y = 0$ on the left side, and the range of the function is $(0, \infty)$. The graph can then be plotted by hand by using this knowledge along with approximate values at $x = -2, -1, 0, 1, 2$. See **Figure 9**.

| $x$ | $f(x) = (1.5)^x$ |
|---|---|
| $-2$ | 0.44 |
| $-1$ | 0.67 |
| 0 | 1 |
| 1 | 1.5 |
| 2 | 2.25 |

(a)



(b)

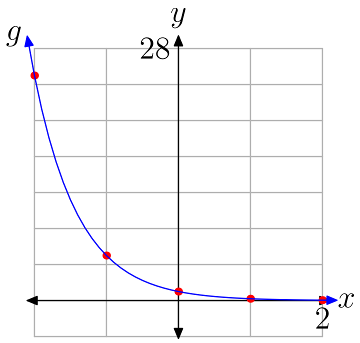
**Figure 9.** Graph of $f(x) = (1.5)^x$

▶ **Example 20.** *Plot the graph of the function $g(x) = (0.2)^x$. Identify the range of the function and the horizontal asymptote.*

Since the base 0.2 is smaller than 1, this is an exponential decay function. Therefore, its graph will have a shape similar to the graphs in **Figure 7**. The graph falls, there will be a horizontal asymptote $y = 0$ on the right side, and the range of the function is $(0, \infty)$. The graph can then be plotted by hand by using this knowledge along with approximate values at $x = -2, -1, 0, 1, 2$. See **Figure 10**.

| $x$ | $g(x) = (0.2)^x$ |
|---|---|
| $-2$ | 25 |
| $-1$ | 5 |
| 0 | 1 |
| 1 | 0.2 |
| 2 | 0.04 |

(a)



(b)

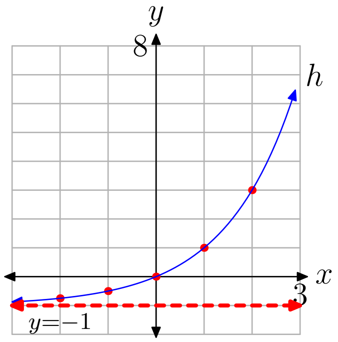
**Figure 10.** Graph of $g(x) = (0.2)^x$

▶ **Example 21.** *Plot the graph of the function $h(x) = 2^x - 1$. Identify the range of the function and the horizontal asymptote.*

The graph of $h$ can be obtained from the graph of $f(x) = 2^x$ (see **Figure 5**) by a vertical shift down 1 unit. Therefore, the horizontal asymptote $y = 0$ of the graph of $f$ will also be shifted down 1 unit, so the graph of $h$ has a horizontal asymptote $y = -1$. Similarly, the range of $f$ will be shifted down to $(-1, \infty) = \text{Range}(h)$. The graph can then be plotted by hand by using this knowledge along with approximate values at $x = -2, -1, 0, 1, 2$. See **Figure 11**.

| $x$ | $h(x) = 2^x - 1$ |
|---|---|
| $-2$ | $-0.75$ |
| $-1$ | $-0.5$ |
| 0 | 0 |
| 1 | 1 |
| 2 | 3 |

(a)



(b)

**Figure 11.** Graph of $h(x) = 2^x - 1$

In later sections of this chapter, we will also see more general exponential functions of the form $f(x) = Ab^x$ (in fact, the Pleasantville and Ghosttown functions at the beginning of this section are of this form). If $A$ is positive, then the graphs of these functions can be obtained from the basic exponential graphs by vertical scaling, so the graphs will have the same general shape as either the exponential growth curves (if $b > 1$) or the exponential decay curves (if $0 < b < 1$) we plotted earlier.

## 8.2 Exercises

---

**1.** The current population of Fortuna is 10,000 hearty souls. It is known that the population is growing at a rate of 4% per year. Assuming this rate remains constant, perform each of the following tasks.

a. Set up an equation that models the population $P(t)$ as a function of time $t$.
b. Use the model in the previous part to predict the population 40 years from now.
c. Use your calculator to sketch the graph of the population over the next 40 years.

**2.** The population of the town of Imagination currently numbers 12,000 people. It is known that the population is growing at a rate of 6% per year. Assuming this rate remains constant, perform each of the following tasks.

a. Set up an equation that models the population $P(t)$ as a function of time $t$.
b. Use the model in the previous part to predict the population 30 years from now.
c. Use your calculator to sketch the graph of the population over the next 30 years.

**3.** The population of the town of Despairia currently numbers 15,000 individuals. It is known that the population is decaying at a rate of 5% per year. Assuming this rate remains constant, perform each of the following tasks.

a. Set up an equation that models the population $P(t)$ as a function of time $t$.
b. Use the model in the previous part to predict the population 50 years from now.
c. Use your calculator to sketch the graph of the population over the next 50 years.

**4.** The population of the town of Hopeless currently numbers 25,000 individuals. It is known that the population is decaying at a rate of 6% per year. Assuming this rate remains constant, perform each of the following tasks.

a. Set up an equation that models the population $P(t)$ as a function of time $t$.
b. Use the model in the previous part to predict the population 40 years from now.
c. Use your calculator to sketch the graph of the population over the next 40 years.

---

In **Exercises 5-12**, perform each of the following tasks for the given function.

a. Find the $y$-intercept of the graph of the function. Also, use your calculator to find two points on the graph to the right of the $y$-axis, and two points to the left.
b. Using your five points from (a) as a guide, set up a coordinate system on graph paper. Choose and label appropriate scales for each axis. Plot the five points, and any additional points you feel are necessary to dis-

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[6] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

cern the shape of the graph.

c. Draw the horizontal asymptote with a dashed line, and label it with its equation.

d. Sketch the graph of the function.

e. Use interval notation to describe both the domain and range of the function.

**5.** $f(x) = (2.5)^x$

**6.** $f(x) = (0.1)^x$

**7.** $f(x) = (0.75)^x$

**8.** $f(x) = (1.1)^x$

**9.** $f(x) = 3^x + 1$

**10.** $f(x) = 4^x - 5$

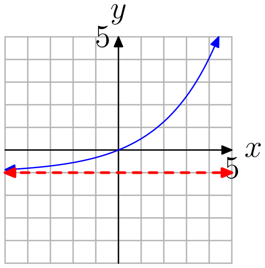
**11.** $f(x) = 2^x - 3$

**12.** $f(x) = 5^x + 2$

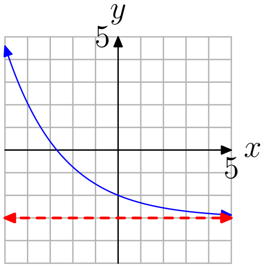
---

In **Exercises 13-20**, the graph of an exponential function of the form $f(x) = b^x + c$ is shown. The dashed red line is a horizontal asymptote. Determine the range of the function. Express your answer in interval notation.

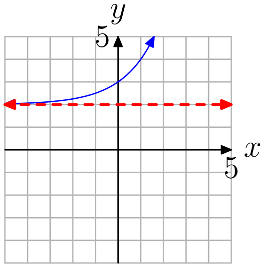
**13.**



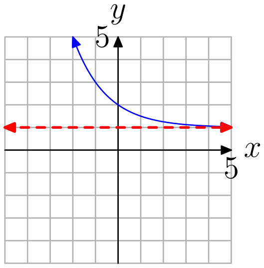
**14.**



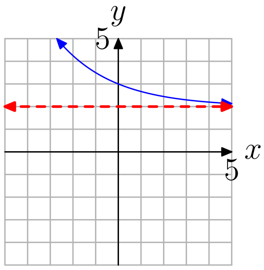
**15.**



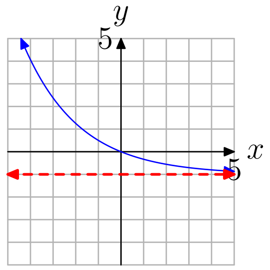
**16.**



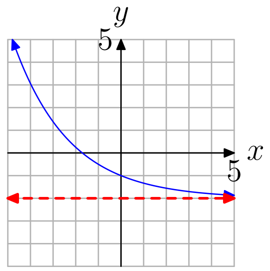
**17.**



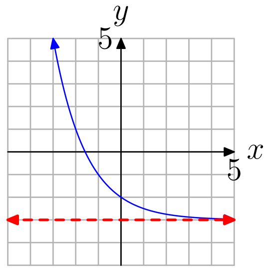
**18.**



**19.**



**20.**



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In **Exercises 21-32**, compute $f(p)$ at the given value $p$.

**21.** $f(x) = (1/3)^x$; $p = -4$

**22.** $f(x) = (3/4)^x$; $p = 1$

**23.** $f(x) = 5^x$; $p = 5$

**24.** $f(x) = (1/3)^x$; $p = 4$

**25.** $f(x) = 4^x$; $p = -4$

**26.** $f(x) = 5^x$; $p = -3$

**27.** $f(x) = (5/2)^x$; $p = -3$

**28.** $f(x) = 9^x$; $p = 3$

**29.** $f(x) = 5^x$; $p = -4$

**30.** $f(x) = 9^x$; $p = 0$

**31.** $f(x) = (6/5)^x$; $p = -4$

**32.** $f(x) = (3/5)^x$; $p = 0$

---

In **Exercises 33-40**, use your calculator to evaluate the function at the given value $p$. Round your answer to the nearest hundredth.

**33.** $f(x) = 10^x$; $p = -0.7$.

**34.** $f(x) = 10^x$; $p = -1.60$.

**35.** $f(x) = (2/5)^x$; $p = 3.67$.

**36.** $f(x) = 2^x$; $p = -3/4$.

**37.** $f(x) = 10^x$; $p = 2.07$.

**38.** $f(x) = 7^x$; $p = 4/3$.

**39.** $f(x) = 10^x$; $p = -1/5$.

**40.** $f(x) = (4/3)^x$; $p = 1.15$.

---

**41.** This exercise explores the property that exponential growth functions *eventually* increase rapidly as $x$ increases. Let $f(x) = 1.05^x$. Use your graphing calculator to graph $f$ on the intervals

(a) $[0, 10]$ and (b) $[0, 100]$.

For (a), use Ymin = 0 and Ymax = 10. For (b), use Ymin = 0 and Ymax = 100. Make accurate copies of the images in your viewing window on your homework paper. What do you observe when you compare the two graphs?

## 8.2 Answers

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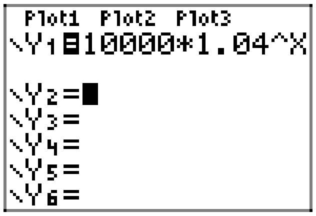
**1.**

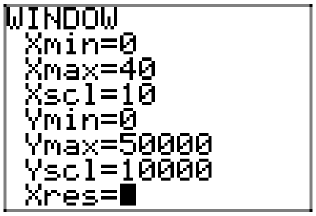
**a)** $P(t) = 10\,000(1.04)^t$

**b)** $P(40) \approx 48\,101$

**c)**



```
 Plot1  Plot2  Plot3
\Y1■10000*1.04^X
\Y2=█
\Y3=
\Y4=
\Y5=
\Y6=
```



```
WINDOW
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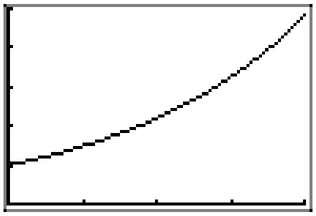
 Xmax=40
 Xscl=10
 Ymin=0
 Ymax=50000
 Yscl=10000
 Xres=█
```

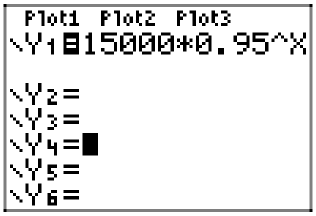


**3.**

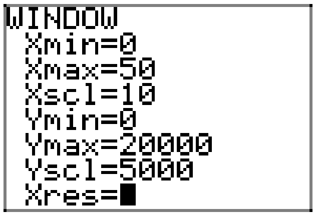
**a)** $P(t) = 15\,000(0.95)^t$

**b)** $P(50) \approx 1\,154$

**c)**



```
 Plot1  Plot2  Plot3
\Y1■15000*0.95^X
\Y2=
\Y3=
\Y4=█
\Y5=
\Y6=
```



```
WINDOW
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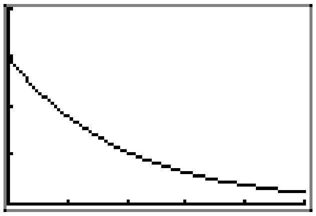
 Xmax=50
 Xscl=10
 Ymin=0
 Ymax=20000
 Yscl=5000
 Xres=█
```



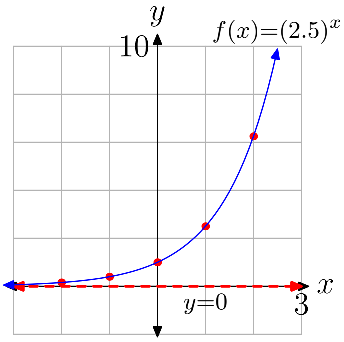
**5.**

**a)** The $y$-intercept is $(0, 1)$. Evaluate the function at $x = 1, 2, -1, -2$ to obtain the points $(1, 2.5)$, $(2, 6.25)$, $(-1, 0.4)$, $(-2, 0.16)$ (other answers are possible).

**b)** See the graph in part (d).

**c)** The horizontal asymptote is $y = 0$. See the graph in part (d).

**d)**



**e)** Domain $= (-\infty, \infty)$, Range $= (0, \infty)$

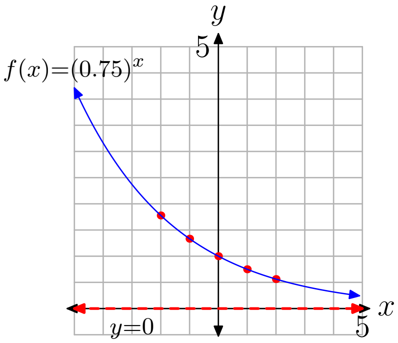
**7.**

**a)** The $y$-intercept is $(0,1)$. Evaluate the function at $x = 1, 2, -1, -2$ to obtain the points $(1, 0.75)$, $(2, 0.56)$, $(-1, 1.34)$, $(-2, 1.78)$ (other answers are possible).

**b)** See the graph in part (d).

**c)** The horizontal asymptote is $y = 0$. See the graph in part (d).

**d)**



**e)** Domain $= (-\infty, \infty)$, Range $= (0, \infty)$

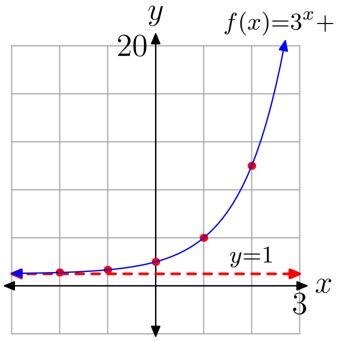
**9.**

**a)** The $y$-intercept is $(0,2)$. Evaluate the function at $x = 1, 2, -1, -2$ to obtain the points $(1, 4)$, $(2, 10)$, $(-1, 1.34)$, $(-2, 1.11)$ (other answers are possible).

**b)** See the graph in part (d).

**c)** The horizontal asymptote is $y = 1$. See the graph in part (d).

**d)**



**e)** Domain $= (-\infty, \infty)$, Range $= (1, \infty)$

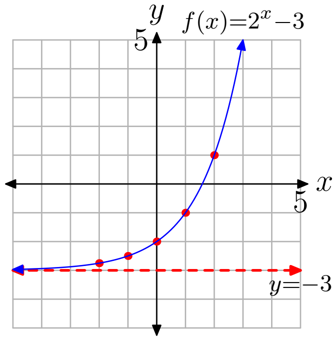
**11.**

**a)** The $y$-intercept is $(0,-2)$. Evaluate the function at $x = 1, 2, -1, -2$ to obtain the points $(1, -1)$, $(2, 1)$, $(-1, -2.5)$, $(-2, -2.75)$ (other answers are possible).

**b)** See the graph in part (d).

**c)** The horizontal asymptote is $y = -3$. See the graph in part (d).

**d)**



**e)** Domain $= (-\infty, \infty)$, Range $= (-3, \infty)$

**13.** $(-1, \infty)$

**15.** $(2, \infty)$

**17.** $(2, \infty)$

**19.** $(-2, \infty)$

**21.** 81

**23.** 3125

**25.** $\dfrac{1}{256}$

**27.** $\dfrac{8}{125}$

**29.** $\dfrac{1}{625}$

**31.** $\dfrac{625}{1296}$

**33.** 0.20

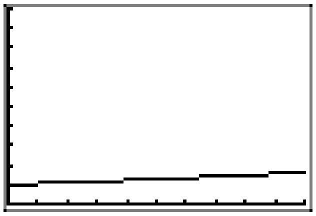
**35.** 0.03

**37.** 117.49

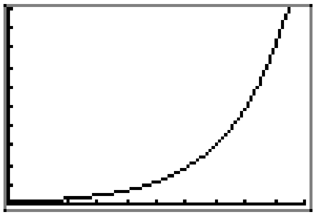
**39.** 0.63

**41.**

**a)** The graph on the interval $[0, 10]$ increases very slowly. In fact, the graph looks almost linear.



**b)** The graph on the interval $[0, 100]$ increases slowly at first, but then increases very rapidly on the second half of the interval.



# 8.3 Applications of Exponential Functions

In the preceding section, we examined a population growth problem in which the population grew at a fixed percentage each year. In that case, we found that the population can be described by an exponential function. A similar analysis will show that any process in which a quantity grows by a fixed percentage each year (or each day, hour, etc.) can be modeled by an exponential function. Compound interest is a good example of such a process.

## *Discrete Compound Interest*

If you put money in a savings account, then the bank will pay you interest (a percentage of your account balance) at the end of each time period, typically one month or one day. For example, if the time period is one month, this process is called *monthly compounding*. The term compounding refers to the fact that interest is added to your account each month and then in subsequent months you earn interest on the interest. If the time period is one day, it's called *daily compounding*.

Let's look at monthly compounding in more detail. Suppose that you deposit \$100 in your account, and the bank pays interest at an annual rate of 5%. Let the function $P(t)$ represent the amount of money that you have in your account at time $t$, where we measure $t$ in years. We will start time at $t = 0$ when the initial amount, called the *principal*, is \$100. In other words, $P(0) = 100$.

In the discussion that follows, we will compute the account balance at the end of each month. Since one month is 1/12 of a year, $P({}^{1}\!/_{12})$ represents the balance at the end of the first month, $P({}^{2}\!/_{12})$ represents the balance at the end of the second month, etc.

At the end of the first month, interest is added to the account balance. Since the annual interest rate 5%, the monthly interest rate is 5%/12, or .05/12 in decimal form. Although we could approximate .05/12 by a decimal, it will be more useful, as well as more accurate, to leave it in this form. Therefore, at the end of the first month, the interest earned will be 100(.05/12), so the total amount will be

$$P({}^{1}\!/_{12}) = 100 + 100\left(\frac{.05}{12}\right) = 100\left(1 + \frac{.05}{12}\right). \tag{1}$$

Now at the end of the second month, you will have the amount that you started that month with, namely $P({}^{1}\!/_{12})$, plus another month's worth of interest on that amount. Therefore, the total amount will be

$$P({}^{2}\!/_{12}) = P({}^{1}\!/_{12}) + P({}^{1}\!/_{12})\left(\frac{.05}{12}\right) = P({}^{1}\!/_{12})\left(1 + \frac{.05}{12}\right). \tag{2}$$

If we replace $P({}^{1}\!/_{12})$ in **equation (2)** with the result found in **equation (1)**, then

---

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$$P(2/12) = 100\left(1+\frac{.05}{12}\right)\left(1+\frac{.05}{12}\right) = 100\left(1+\frac{.05}{12}\right)^2. \tag{3}$$

Let's iterate one more month. At the end of the third month, you will have the amount that you started that month with, namely $P(2/12)$, plus another month's worth of interest on that amount. Therefore, the total amount will be

$$P(3/12) = P(2/12) + P(2/12)\left(\frac{.05}{12}\right) = P(2/12)\left(1+\frac{.05}{12}\right). \tag{4}$$

However, if we replace $P(2/12)$ in **equation (4)** with the result found in **equation (3)**, then

$$P(3/12) = 100\left(1+\frac{.05}{12}\right)^2\left(1+\frac{.05}{12}\right) = 100\left(1+\frac{.05}{12}\right)^3. \tag{5}$$

The pattern should now be clear. The amount of money you will have in the account at the end of $m$ months is given by the function

$$P(m/12) = 100\left(1+\frac{.05}{12}\right)^m.$$

We can rewrite this formula in terms of years $t$ by replacing $m/12$ by $t$. Then $m = 12t$, so the formula becomes

$$P(t) = 100\left(1+\frac{.05}{12}\right)^{12t}. \tag{6}$$

What would be different if you had started with a principal of 200? By tracing over our previous steps, it should be easy to see that the new formula would be

$$P(t) = 200\left(1+\frac{.05}{12}\right)^{12t}.$$

Similarly, if the interest rate had been 4% per year instead of 5%, then we would have ended up with the formula

$$P(t) = 100\left(1+\frac{.04}{12}\right)^{12t}.$$

Thus, if we let $P_0$ represent the principal, and $r$ represent the annual interest rate (in decimal form), then we can generalize the formula to

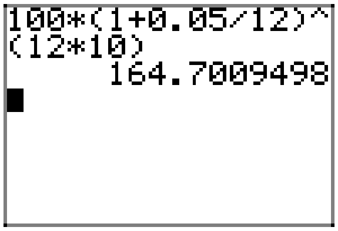
$$P(t) = P_0\left(1+\frac{r}{12}\right)^{12t}. \tag{7}$$

▶ **Example 8.** *If the principal is \$100, the annual interest rate is 5%, and interest is compounded monthly, how much money will you have after ten years?*

In formula (**7**), let $P_0 = 100$, $r = .05$, and $t = 10$:

$$P(10) = 100\left(1+\frac{.05}{12}\right)^{12\cdot 10}$$

We can use our graphing calculator to approximate this solution, as shown in **Figure 1**.



```
100*(1+0.05/12)^
(12*10)
      164.7009498
```

**Figure 1.** Computing the amount after compounding monthly for 10 years.

Thus, you would have \$164.70 after ten years.

▶ **Example 9.** *If the principal is \$10 000, the annual interest rate is 5%, and interest is compounded monthly, how much money will you have after forty years?*

In formula (7), let $P_0 = 10\,000$, $r = .05$, and $t = 40$:

$$P(40) = 10\,000\left(1+\frac{.05}{12}\right)^{12\cdot 40} \approx 73\,584.17$$

Thus, you would have \$73,584.17 after forty years.

These examples illustrate the "miracle of compound interest." In the last example, your account is more than seven times as large as the original, and your total "profit" (the amount of interest you've received) is \$63 584.17. Compare this to the amount you would have received if you had withdrawn the interest each month (i.e., no compounding). In that case, your "profit" would only be \$20 000:

$$\text{years}\cdot\frac{\text{months}}{\text{year}}\cdot\frac{\text{interest}}{\text{month}} = 40\cdot 12\cdot\left[(10\,000)\left(\frac{.05}{12}\right)\right] = 20\,000$$

The large difference can be attributed to the shape of the graph of the function $P(t)$. Recall from the preceding section that this is an exponential growth function, so as $t$ gets large, the graph will eventually rise steeply. Thus, if you can leave your money in the bank long enough, it will eventually grow dramatically.

What about daily compounding? Let's again analyze the situation in which the principal is \$100 and the annual interest rate is 5%. In this case, the time period over which interest is paid is one day, or 1/365 of a year, and the daily interest rate is 5%/365, or .05/365 in decimal form. Since we are measuring time in years, $P(1/365)$ represents the balance at the end of the first day, $P(2/365)$ represents the balance at the end of the second day, etc. We'll follow the same steps as in the earlier analysis for monthly compounding.

At the end of the first day, you will have

$$P(1/365) = 100 + 100\left(\frac{.05}{365}\right) = 100\left(1+\frac{.05}{365}\right). \tag{10}$$

At the end of the second day, you will have the amount that you started that day with, namely $P(1/365)$, plus another day's worth of interest on that amount. Therefore, the total amount will be

$$P(2/365) = P(1/365) + P(1/365)\left(\frac{.05}{365}\right) = P(1/365)\left(1 + \frac{.05}{365}\right). \tag{11}$$

If we replace $P(1/365)$ in **equation (11)** with the result found in **equation (10)**, then

$$P(2/365) = 100\left(1 + \frac{.05}{365}\right)\left(1 + \frac{.05}{365}\right) = 100\left(1 + \frac{.05}{365}\right)^2. \tag{12}$$

At the end of the third day, you will have the amount that you started that day with, namely $P(2/365)$, plus another day's worth of interest on that amount. Therefore, the total amount will be

$$P(3/365) = P(2/365) + P(2/365)\left(\frac{.05}{365}\right) = P(2/365)\left(1 + \frac{.05}{365}\right). \tag{13}$$

Again, replacing $P(2/365)$ in **equation (13)** with the result found in **equation (12)** yields

$$P(3/365) = 100\left(1 + \frac{.05}{365}\right)^2\left(1 + \frac{.05}{365}\right) = 100\left(1 + \frac{.05}{365}\right)^3. \tag{14}$$

Continuing this pattern shows that the amount of money you will have in the account at the end of $d$ days is given by the function

$$P(d/365) = 100\left(1 + \frac{.05}{365}\right)^d.$$

We can rewrite this formula in terms of years $t$ by replacing $d/365$ by $t$. Then $d = 365t$, so the formula becomes

$$P(t) = 100\left(1 + \frac{.05}{365}\right)^{365t}. \tag{15}$$

More generally, if you had started with a principal of $P_0$ and an annual interest rate of $r$ (in decimal form), then the formula would be

$$P(t) = P_0\left(1 + \frac{r}{365}\right)^{365t}. \tag{16}$$

Comparing formulas (**7**) and (**16**) for monthly and daily compounding, it should be apparent that the only difference is that the number 12 is used in the monthly compounding formula and the number 365 is used in the daily compounding formula. Looking at the respective analyses shows that this number arises from the portion of the year that interest is paid (1/12 in the case of monthly compounding, and 1/365 in the case of daily compounding). Thus, in each case, this number (12 or 365) also equals the number of times that interest is compounded per year. It follows that if interest is compounded quarterly (every three months, or 4 times per year), the formula would be

$$P(t) = P_0\left(1 + \frac{r}{4}\right)^{4t}.$$

Similarly, if interest is compounded hourly (8760 times per year), the formula would be

$$P(t) = P_0 \left(1 + \frac{r}{8760}\right)^{8760t}.$$

Summarizing, we have one final generalization:

**Discrete Compound Interest**

If $P_0$ is the principal, $r$ is the annual interest rate, and $n$ is the number of times that interest is compounded per year, then the balance at time $t$ years is

$$P(t) = P_0 \left(1 + \frac{r}{n}\right)^{nt}. \tag{17}$$

▶ **Example 18.** *If the principal is $100, the annual interest rate is 5%, and interest is compounded daily, what will be the balance after ten years?*

In formula (**17**), let $P_0 = 100$, $r = .05$, $n = 365$, and $t = 10$:

$$P(10) = 100 \left(1 + \frac{.05}{365}\right)^{365 \cdot 10} \approx 164.87$$

Thus, you would have $164.87 after ten years.

◆

▶ **Example 19.** *If the principal is $10 000, the annual interest rate is 5%, and interest is compounded daily, what will be the balance after forty years?*

In formula (**17**), let $P_0 = 10\,000$, $r = .05$, $n = 365$, and $t = 40$:

$$P(40) = 10\,000 \left(1 + \frac{.05}{365}\right)^{365 \cdot 40} \approx 73\,880.44$$

Thus, you would have $73 880.44 after forty years.

◆

As you can see from comparing Examples **8** and **18**, and Examples **9** and **19**, the difference between monthly and daily compounding is generally small. However, the difference can be substantial for large principals and/or large time periods.

▶ **Example 20.** *If the principal is $500, the annual interest rate is 8%, and interest is compounded quarterly, what will be the balance after 42 months?*

42 months is 3.5 years, so let $P_0 = 500$, $r = .08$, $n = 4$, and $t = 3.5$ in formula (**17**):

$$P(5) = 500 \left(1 + \frac{.08}{4}\right)^{4 \cdot 3.5} \approx 659.74$$

Thus, you would have $659.74 after 42 months.

◆

## *Continuous Compound Interest and the Number e*

Using formula (**17**), it is a simple matter to calculate the total amount for any type of compounding. Although most banks compound interest either daily or monthly, it could be done every hour, or every minute, or every second, etc. What happens to the total amount as the time period shortens? Equivalently, what happens as $n$ increases in formula (**17**)? **Table 1** shows the amount after one year with a principal of $P_0 = 100$, $r = .05$, and various values of $n$:

| compounding | $n$ | $P(1)$ |
|---|---|---|
| monthly | 12 | 105.11619 |
| daily | 365 | 105.12675 |
| hourly | 8760 | 105.12709 |
| every minute | 525600 | 105.12711 |
| every second | 31536000 | 105.12711 |

**Table 1.** Comparison of discrete compounding with $P_0 = 100$, $r = .05$, and $t = 1$ year.

Even if we carry out our computations to eight digits, it appears that the amounts in the right hand column of **Table 1** are stabilizing. In fact, using calculus, it can be shown that these amounts do indeed get closer and closer to a particular number, and we can calculate that number.

Starting with formula (**17**), we will let $n$ approach $\infty$. In other words, we will let $n$ get larger and larger without bound, as we started to do in **Table 1**. The first step is to use the Laws of Exponents to write

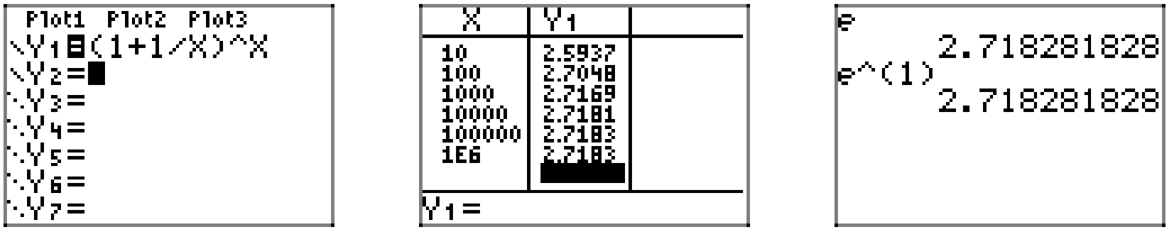
$$P_0\left(1+\frac{r}{n}\right)^{nt} = P_0\left[\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt}.$$

In the next step, replace $n/r$ by $m$. Since $n/r = m$, it follows that $r/n = 1/m$, and we have

$$P_0\left[\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt} = P_0\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}.$$

Now let $n$ approach $\infty$. Since $m = n/r$ and $r$ is fixed, it follows that $m$ also approaches $\infty$. We can use the `TABLE` feature of the graphing calculator to investigate the convergence of the expression in brackets as $m$ approaches infinity.

- Load $(1{+}1/m)^m$ into the `Y=` menu of the graphing calculator, as shown in **Figure 2**(a). Of course, you must use $x$ instead of $m$ and enter `(1+1/X)^X`.
- Use `TBLSET` and set `Indepnt` to `Ask`, select `TABLE`, then enter the numbers 10, 100, 1 000, 10 000, 100 000, and 1 000 000 to produce the result shown in **Figure 2**(b). Note that `(1+1/X)^X` appears to converge to the number 2.7183. If you move the cursor over the last result in the `Y1` column, you can see more precision, 2.71828046932.



```
Plot1 Plot2 Plot3
\Y1=(1+1/X)^X
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
  X     | Y1
10      | 2.5937
100     | 2.7048
1000    | 2.7169
10000   | 2.7181
100000  | 2.7183
1E6     | 2.7183
Y1=
```

```
e
          2.718281828
e^(1)
          2.718281828
```

(a) (b) (c)

**Figure 2.** Illustration of the convergence of $(1 + 1/m)^m$ to $e$ as $m$ increases to infinity

Note that the numbers in the second column in **Figure 2**(b) appear to stabilize. Indeed, it can be shown by using calculus that the expression in brackets above gets closer and closer to a single number, which is called $e$. To represent this convergence, we write

$$\left(1+\frac{1}{m}\right)^m \to e. \tag{21}$$

$e$ is an irrational number, approximately 2.7183, as shown by the computations in **Figure 2**(b). It follows that

$$P_0\left[\left(1+\frac{1}{m}\right)^m\right]^{rt} \to P_0e^{rt}.$$

Because we took the discrete compound interest formula (**17**) and let the number of times compounded per year ($n$) approach $\infty$, this process is known as *continuous compounding*.

**Continuous Compound Interest**

If $P_0$ is the principal, $r$ is the annual interest rate, and interest is compounded continuously, then the balance at time $t$ years is

$$P(t) = P_0e^{rt}. \tag{22}$$

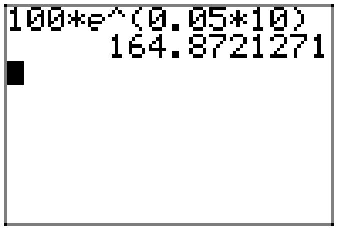
Before working the next examples, find the buttons on your calculator for the number $e$ and for the exponential function $e^x$. Typing either `e` or `e^(1)` (using the $e^x$ button) will yield an approximation to the number $e$, as shown in **Figure 2**(c). Compare this approximation with the one you obtained earlier in **Figure 2**(b).

▶ **Example 23.** *If the principal is \$100, the annual interest rate is 5%, and interest is compounded continuously, what will be the balance after ten years?*

In formula (**22**), let $P_0 = 100$, $r = 0.05$, and $t = 10$:

$$P(10) = 100e^{(0.05)(10)}$$

Use your calculator to approximate this result, as shown in **Figure 3**.



```
100*e^(0.05*10)
      164.8721271
```

**Figure 3.** Computing the amount after compounding continuously for 10 years.

Thus, you would have \$164.87 after ten years.

▶ **Example 24.** *If the principal is \$10,000, the annual interest rate is 5%, and interest is compounded continuously, what will be the balance after forty years?*

In formula (**22**), let $P_0 = 10\,000$, $r = 0.05$, and $t = 40$:

$$P(10) = 10\,000 e^{(0.05)(40)} \approx 73\,890.56$$

Thus, you would have \$73 890.56 after forty years.

Notice that the continuous compounding formula (**22**) is much simpler than the discrete compounding formula (**17**). Unless the principal is very large or the time period is very long, the preceding examples show that continuous compounding is also a close approximation to daily compounding. In **Example 23**, the amount \$164.87 is the same (rounded to the nearest cent) as the amount for daily compounding found in **Example 18**. With a larger principal and longer time period, the amount \$73 890.56 in **Example 24** using continuous compounding is still only about \$10 more than the amount \$73 880.44 for daily compounding found in **Example 19**.

**Remarks 25.**

1. The number $e$ may strike you as a mere curiosity. If so, that would be a big misconception. The number $e$ is actually one of the most important numbers in mathematics (it's probably the second most famous number, following $\pi$), and it arises naturally as the limit described in (**21**) above. Using notation from calculus, we write

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e \approx 2.71828. \tag{26}$$

   Although in our discussion above this limit arose in a man-made process, compound interest, it shows up in a similar manner in studies of many natural phenomena. We'll look at some of these applications later in this chapter.
2. Likewise, the exponential function $e^x$ is one of the most important functions used in mathematics, statistics, and many fields of science. For a variety of reasons, the base $e$ turns out to be the most natural base to use for an exponential function. Consequently, the function $f(x) = e^x$ is known as the *natural exponential function*.

## *Future Value and Present Value*

In this section we have derived two formulas, one for discrete compound interest, and the other for continuous compound interest. However, in the examples presented so far, we've only used these formulas to calculate *future value*: given a principal $P_0$ and interest rate $r$, how much will you have in your account in $t$ years?

Another type of question we can solve is known as a *present value* problem: how much money would you have to invest at interest $r$ in order to have $Q$ dollars in $t$ years? Here are a couple of examples:

▶ **Example 27.** *How much money would you have to invest at 4% interest compounded daily in order to have \$8000 dollars in* 6 *years?*

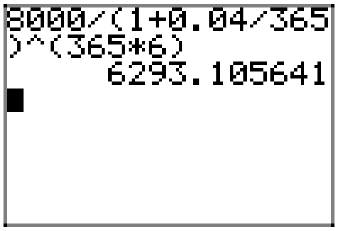
In this case, the principal $P_0$ is unknown, and we substitute $r = 0.04$, $n = 365$, and $t = 6$, into the discrete compounding formula (**17**). Since $P(6) = 8000$, we have the equation

$$8000 = P(6) = P_0\left(1 + \frac{0.04}{365}\right)^{(365)(6)}.$$

This equation can be solved by division:

$$\frac{8000}{\left(1 + \frac{0.04}{365}\right)^{(365)(6)}} = P_0$$

**Figure 4** shows a calculator approximation for this result.



```
8000/(1+0.04/365
)^(365*6)
      6293.105641
```

**Figure 4.** The present value of \$8000, compounded daily for six years.

Thus, the present value is approximately $P_0 \approx \$6293.11$. If this amount is invested now at 4% compounded daily, then its future value in 6 years will be \$8000.

▶ **Example 28.** *How much money would you have to invest at 7% interest compounded continuously in order to have \$5000 dollars in* 4 *years?*

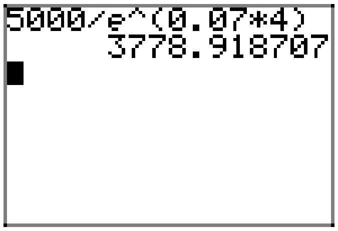
As in the last example, the principal $P_0$ is unknown, and this time $r = 0.07$ and $t = 4$ in the continuous compounding formula (**22**). Then $P(4) = 5000$ yields the equation

$$5000 = P(4) = P_0 e^{(0.07)(4)}.$$

As in the last example, this equation can also be solved by division:

$$\frac{5000}{e^{(0.07)(4)}} = P_0$$

A calculator approximation for this result is shown in **Figure 5**.



```
5000/e^(0.07*4)
      3778.918707
```

**Figure 5.** The present value of \$5000, compounded continuously for four years.

Thus, the present value is approximately $P_0 \approx \$3778.92$. If this amount is invested now at 7% compounded continuously, then its future value in 4 years will be \$5000.

◇

## Additional Questions

In terms of practical applications, there are also other types of questions that would be interesting to consider. Here are two examples:

1. If you deposit \$1000 in an account paying 6% compounded continuously, how long will it take for you to have \$1500 in your account?
2. If you deposit \$1000 in an account paying 5% compounded monthly, how long will it take for your money to double?

Let's look at the first question (the second is similar). In this case, $P_0 = 1000$ and $r = 0.06$. Inserting these values into the continuous compounding formula (**22**), we obtain

$$P(t) = 1000e^{0.06t}.$$

Now we want the future value $P(t)$ of the account at some time $t$ to equal \$1500. Therefore, we must solve the equation

$$1500 = 1000e^{0.06t}.$$

However, now we have a problem, because the variable $t$ is located in the exponent of the expression on the right side of the equation. Although we could approximate a solution graphically, we currently have no algebraic method for solving an equation such as this, where the variable is in the exponent (these types of equations are called *exponential equations*). Over the course of the next few sections, we will define another type of function, the logarithm function, which will in turn provide us with a method for solving exponential equations. Then we will return to these questions, and also discuss additional applications.

## 8.3 Exercises

**1.** Suppose that you invest \$15,000 at 7% interest compounded monthly. How much money will be in your account in 4 years? Round your answer to the nearest cent.

**2.** Suppose that you invest \$14,000 at 3% interest compounded monthly. How much money will be in your account in 7 years? Round your answer to the nearest cent.

**3.** Suppose that you invest \$14,000 at 4% interest compounded daily. How much money will be in your account in 6 years? Round your answer to the nearest cent.

**4.** Suppose that you invest \$15,000 at 8% interest compounded monthly. How much money will be in your account in 8 years? Round your answer to the nearest cent.

**5.** Suppose that you invest \$4,000 at 3% interest compounded monthly. How much money will be in your account in 7 years? Round your answer to the nearest cent.

**6.** Suppose that you invest \$3,000 at 5% interest compounded monthly. How much money will be in your account in 4 years? Round your answer to the nearest cent.

**7.** Suppose that you invest \$1,000 at 3% interest compounded monthly. How much money will be in your account in 4 years? Round your answer to the nearest cent.

**8.** Suppose that you invest \$19,000 at 2% interest compounded daily. How much money will be in your account in 9 years? Round your answer to the nearest cent.

**9.** Suppose that you can invest money at 4% interest compounded monthly. How much should you invest in order to have \$20,000 in 2 years? Round your answer to the nearest cent.

**10.** Suppose that you can invest money at 6% interest compounded daily. How much should you invest in order to have \$1,000 in 2 years? Round your answer to the nearest cent.

**11.** Suppose that you can invest money at 3% interest compounded daily. How much should you invest in order to have \$20,000 in 3 years? Round your answer to the nearest cent.

**12.** Suppose that you can invest money at 3% interest compounded monthly. How much should you invest in order to have \$10,000 in 7 years? Round your answer to the nearest cent.

**13.** Suppose that you can invest money at 9% interest compounded daily. How much should you invest in order to have \$4,000 in 9 years? Round your answer to the nearest cent.

**14.** Suppose that you can invest money at 8% interest compounded daily. How much should you invest in order to have \$18,000 in 6 years? Round your answer to the nearest cent.

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**15.** Suppose that you can invest money at 8% interest compounded daily. How much should you invest in order to have $17,000 in 6 years? Round your answer to the nearest cent.

**16.** Suppose that you can invest money at 9% interest compounded daily. How much should you invest in order to have $5,000 in 7 years? Round your answer to the nearest cent.

---

In **Exercises 17-24**, evaluate the function at the given value $p$. Round your answer to the nearest hundredth.

**17.** $f(x) = e^x$; $p = 1.57$.

**18.** $f(x) = e^x$; $p = 2.61$.

**19.** $f(x) = e^x$; $p = 3.07$.

**20.** $f(x) = e^x$; $p = -4.33$.

**21.** $f(x) = e^x$; $p = 1.42$.

**22.** $f(x) = e^x$; $p = -0.8$.

**23.** $f(x) = e^x$; $p = 4.75$.

**24.** $f(x) = e^x$; $p = 3.60$.

---

**25.** Suppose that you invest $3,000 at 4% interest compounded continuously. How much money will be in your account in 9 years? Round your answer to the nearest cent.

**26.** Suppose that you invest $8,000 at 8% interest compounded continuously. How much money will be in your account in 7 years? Round your answer to the nearest cent.

**27.** Suppose that you invest $1,000 at 2% interest compounded continuously. How much money will be in your account in 3 years? Round your answer to the nearest cent.

**28.** Suppose that you invest $3,000 at 8% interest compounded continuously. How much money will be in your account in 4 years? Round your answer to the nearest cent.

**29.** Suppose that you invest $15,000 at 2% interest compounded continuously. How much money will be in your account in 4 years? Round your answer to the nearest cent.

**30.** Suppose that you invest $8,000 at 2% interest compounded continuously. How much money will be in your account in 6 years? Round your answer to the nearest cent.

**31.** Suppose that you invest $13,000 at 9% interest compounded continuously. How much money will be in your account in 8 years? Round your answer to the nearest cent.

**32.** Suppose that you invest $16,000 at 4% interest compounded continuously. How much money will be in your account in 6 years? Round your answer to the nearest cent.

---

**33.** Suppose that you can invest money at 6% interest compounded continuously. How much should you invest in order to have $17,000 in 9 years? Round your answer to the nearest cent.

**34.** Suppose that you can invest money at 8% interest compounded continuously. How much should you invest in order to have $5,000 in 6 years? Round your answer to the nearest cent.

**35.** Suppose that you can invest money at 8% interest compounded continuously. How much should you invest in order to have \$10,000 in 6 years? Round your answer to the nearest cent.

**36.** Suppose that you can invest money at 6% interest compounded continuously. How much should you invest in order to have \$17,000 in 13 years? Round your answer to the nearest cent.

**37.** Suppose that you can invest money at 2% interest compounded continuously. How much should you invest in order to have \$13,000 in 8 years? Round your answer to the nearest cent.

**38.** Suppose that you can invest money at 9% interest compounded continuously. How much should you invest in order to have \$10,000 in 15 years? Round your answer to the nearest cent.

**39.** Suppose that you can invest money at 7% interest compounded continuously. How much should you invest in order to have \$18,000 in 10 years? Round your answer to the nearest cent.

**40.** Suppose that you can invest money at 9% interest compounded continuously. How much should you invest in order to have \$14,000 in 12 years? Round your answer to the nearest cent.

## 8.3 Answers

---

**1.** \$19830.81

**3.** \$17797.25

**5.** \$4933.42

**7.** \$1127.33

**9.** \$18464.78

**11.** \$18278.69

**13.** \$1779.61

**15.** \$10519.87

**17.** 4.81

**19.** 21.54

**21.** 4.14

**23.** 115.58

**25.** \$4299.99

**27.** \$1061.84

**29.** \$16249.31

**31.** \$26707.63

**33.** \$9906.72

**35.** \$6187.83

**37.** \$11077.87

**39.** \$8938.54

# 8.4 Inverse Functions

As we saw in the last section, in order to solve application problems involving exponential functions, we will need to be able to solve exponential equations such as

$$1500 = 1000e^{0.06t} \qquad \text{or} \qquad 300 = 2^x.$$

However, we currently don't have any mathematical tools at our disposal to solve for a variable that appears as an exponent, as in these equations. In this section, we will develop the concept of an inverse function, which will in turn be used to define the tool that we need, the logarithm, in Section 8.5.

## One-to-One Functions

**Definition 1.** *A given function $f$ is said to be* one-to-one *if for each value $y$ in the range of $f$, there is just one value $x$ in the domain of $f$ such that $y = f(x)$. In other words, $f$ is one-to-one if each output $y$ of $f$ corresponds to precisely one input $x$.*

It's easiest to understand this definition by looking at mapping diagrams and graphs of some example functions.

▶ **Example 2.** *Consider the two functions $h$ and $k$ defined according to the mapping diagrams in* **Figure 1**. *In* **Figure 1***(a), there are two values in the domain that are both mapped onto 3 in the range. Hence, the function $h$ is not one-to-one. On the other hand, in* **Figure 1***(b), for each output in the range of $k$, there is only one input in the domain that gets mapped onto it. Therefore, $k$ is a one-to-one function.*

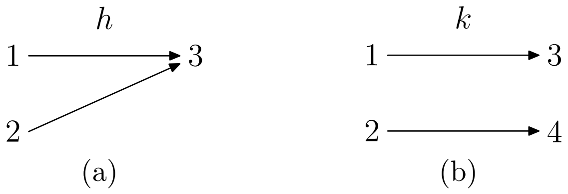

**Figure 1.** Mapping diagrams help to determine if a function is one-to-one.

◇

▶ **Example 3.** *The graph of a function is shown in* **Figure 2***(a). For this function $f$, the $y$-value 4 is the output corresponding to two input values, $x = -1$ and $x = 3$ (see the corresponding mapping diagram in* **Figure 2***(b)). Therefore, $f$ is* not *one-to-one.*

Graphically, this is apparent by drawing horizontal segments from the point $(0, 4)$ on the $y$-axis over to the corresponding points on the graph, and then drawing vertical segments to the $x$-axis. These segments meet the $x$-axis at $-1$ and 3.

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[9] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

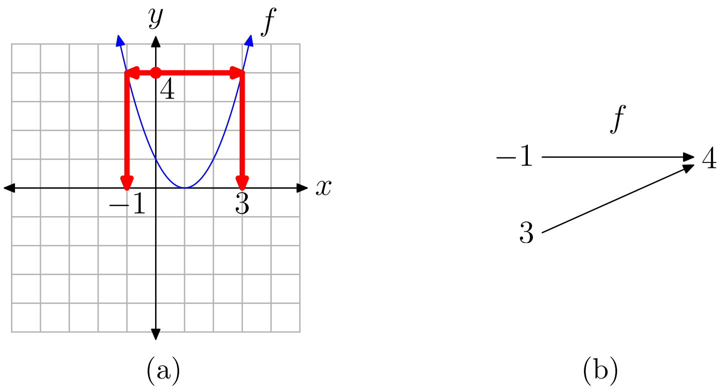


**Figure 2.** A function which is not one-to-one.

◆

▶ **Example 4.** *In* **Figure 3**, *each $y$-value in the range of $f$ corresponds to just one input value $x$. Therefore, this function* is *one-to-one.*

Graphically, this can be seen by mentally drawing a horizontal segment from each point on the $y$-axis over to the corresponding point on the graph, and then drawing a vertical segment to the $x$-axis. Several examples are shown in **Figure 3**. It's apparent that this procedure will always result in just one corresponding point on the $x$-axis, because each $y$-value only corresponds to one point on the graph. In fact, it's easiest to just note that since each horizontal line only intersects the graph once, then there can be only one corresponding input to each output.

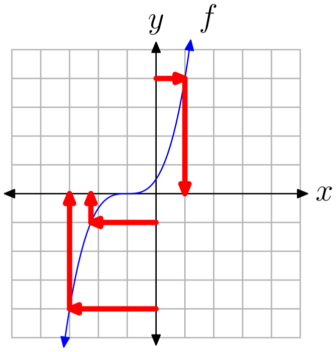


**Figure 3.** A one-to-one function

◆

The graphical process described in the previous example, known as the *horizontal line test*, provides a simple visual means of determining whether a function is one-to-one.

**Horizontal Line Test**

If each horizontal line intersects the graph of $f$ at most once, then $f$ is one-to-one. On the other hand, if some horizontal line intersects the graph of $f$ more than once, then $f$ is *not* one-to-one.

**Remark 5.** It follows from the horizontal line test that if $f$ is a strictly increasing function, then $f$ is one-to-one. Likewise, every strictly decreasing function is also one-to-one.

## *Inverse Functions*

If $f$ is one-to-one, then we can define an associated function $g$, called the *inverse function* of $f$. We will give a formal definition below, but the basic idea is that the inverse function $g$ simply sends the outputs of $f$ back to their corresponding inputs. In other words, the mapping diagram for $g$ is obtained by reversing the arrows in the mapping diagram for $f$.

► **Example 6.** *The function $f$ in* **Figure 4***(a) maps 1 to 5 and 2 to $-3$. Therefore, the inverse function $g$ in* **Figure 4***(b) maps the outputs of $f$ back to their corresponding inputs: 5 to 1 and $-3$ to 2. Note that reversing the arrows on the mapping diagram for $f$ yields the mapping diagram for $g$.*

(a) (b)

**Figure 4.** Reversing the arrows on the mapping diagram for $f$ yields the mapping diagram for $g$.

◆

Since the inverse function $g$ sends the outputs of $f$ back to their corresponding inputs, it follows that the inputs of $g$ are the outputs of $f$, and vice versa. Thus, the functions $g$ and $f$ are related by simply interchanging their inputs and outputs.

The original function must be one-to-one in order to have an inverse. For example, consider the function $h$ in **Example 2**. $h$ is not one-to-one. If we reverse the arrows in the mapping diagram for $h$ (see **Figure 1**(a)), then the resulting relation will not be a function, because 3 would map to both 1 and 2.

Before giving the formal definition of an inverse function, it's helpful to review the description of a function given in Section 2.1. While functions are often defined by means of a formula, remember that in general a function is just a *rule* that dictates how to associate a unique output value to each input value.

**Definition 7.** *Suppose that $f$ is a given one-to-one function. The inverse function $g$ is defined as follows: for each $y$ in the range of $f$, define $g(y)$ to be the unique value $x$ such that $y = f(x)$.*

To understand this definition, it's helpful to look at a diagram:

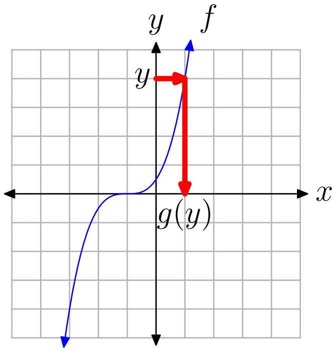


**Figure 5.**

The input for $g$ is any $y$-value in the range of $f$. Thus, the input in the above diagram is a value on the $y$-axis. The output of $g$ is the corresponding value on the $x$-axis which satisfies the condition $y = f(x)$. Note in particular that the $x$-value is unique because $f$ is one-to-one.

The relationship between the original function $f$ and its inverse function $g$ can be described by:[10]

**Property 8.** If $g$ is the inverse function of $f$, then

$$x = g(y) \quad \Longleftrightarrow \quad y = f(x).$$

In fact, this is really the defining relationship for the inverse function. An easy way to understand this relationship (and the entire concept of an inverse function) is to realize that it states that *inputs and outputs are interchanged*. The inputs of $g$ are the outputs of $f$, and vice versa. It follows that the Domain and Range of $f$ and $g$ are interchanged:

**Property 9.** If $g$ is the inverse function of $f$, then

$$\text{Domain}(g) = \text{Range}(f) \qquad \text{and} \qquad \text{Range}(g) = \text{Domain}(f).$$

---

[10] The $\Longleftrightarrow$ symbol means that these two statements are equivalent: if one is true, then so is the other.

The defining relationship in **Property 8** is also equivalent to the following two identities, so these provide an alternative characterization of inverse functions:

**Property 10.** If $g$ is the inverse function of $f$, then

$$g(f(x)) = x \text{ for every } x \text{ in Domain}(f)$$

and

$$f(g(y)) = y \text{ for every } y \text{ in Domain}(g).$$

Note that the first statement in **Property 10** says that $g$ maps the output $f(x)$ back to the input $x$. The second statement says the same with the roles of $f$ and $g$ reversed. Therefore, $f$ and $g$ must be inverses.

**Property 10** can also be interpreted to say that the functions $g$ and $f$ "undo" each other. If we first apply $f$ to an input $x$, and then apply $g$, we get $x$ back again. Likewise, if we apply $g$ to an input $y$, and then apply $f$, we get $y$ back again. So whatever action $f$ performs, $g$ reverses it, and vice versa.

▶ **Example 11.** *Suppose $f(x) = x^3$. Thus, $f$ is the "cubing" function. What operation will reverse the cubing process? Taking a cube root. Thus, the inverse of $f$ should be the function $g(y) = \sqrt[3]{y}$.*

Let's verify **Property 10**:

$$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

and

$$f(g(y)) = f(\sqrt[3]{y}) = (\sqrt[3]{y})^3 = y.$$

◇

▶ **Example 12.** *Suppose $f(x) = 4x - 1$. $f$ acts on an input $x$ by first multiplying by 4, and then subtracting 1. The inverse function must reverse the process: first add 1, and then divide by 4. Thus, the inverse function should be $g(y) = (y + 1)/4$.*

Again, let's verify **Property 10**:

$$g(f(x)) = g(4x - 1) = \frac{(4x - 1) + 1}{4} = \frac{4x}{4} = x$$

and

$$f(g(y)) = f\left(\frac{y+1}{4}\right) = 4\left(\frac{y+1}{4}\right) - 1 = (y + 1) - 1 = y.$$

◇

**Remarks 13.**

1. The computation $g(f(x))$, in which the output of one function is used as the input of another, is called the *composition of $g$ with $f$*. Thus, inverse functions "undo" each other in the sense of composition. Composition of functions is an important concept in many areas of mathematics, so more practice with composition of functions is provided in the exercises.
2. If $g$ is the inverse function of $f$, then $f$ is also the inverse of $g$. This follows from either **Property 8** or **Property 10**. (Note that the labels $x$ and $y$ for the variables are unimportant. The key idea is that two functions are inverses if their inputs and outputs are interchanged.)

**Notation:** In order to indicate that two functions $f$ and $g$ are inverses, we usually use the notation $f^{-1}$ for $g$. The symbol $f^{-1}$ is read "$f$ inverse". In addition, to avoid confusion with the typical roles of $x$ and $y$, it's often useful to use different labels for the variables. Rewriting **Property 8** with the $f^{-1}$ notation, and using new labels for the variables, we have the defining relationship:

**Property 14.**

$$v = f^{-1}(u) \qquad \Longleftrightarrow \qquad u = f(v)$$

Likewise, rewriting **Property 10**, we have the composition relationships:

**Property 15.**

$$f^{-1}(f(z)) = z \text{ for every } z \text{ in Domain}(f)$$

$$\text{and}$$

$$f(f^{-1}(z)) = z \text{ for every } z \text{ in Domain}(f^{-1})$$

However, the new notation comes with an important warning:

**Warning 16.**

$$f^{-1} \textit{ does } \textbf{not} \textit{ mean } \frac{1}{f}$$

*The $-1$ exponent is just notation in this context. When applied to a function, it stands for the inverse of the function, not the reciprocal of the function.*

## The Graph of an Inverse Function

How are the graphs of $f$ and $f^{-1}$ related? Suppose that the point $(a, b)$ is on the graph of $f$. That means that $b = f(a)$. Since inputs and outputs are interchanged for the inverse function, it follows that $a = f^{-1}(b)$, so $(b, a)$ is on the graph of $f^{-1}$. Now $(a, b)$

and $(b, a)$ are just reflections of each other across the line $y = x$ (see the discussion below for a detailed explanation), so it follows that the same is true of the graphs of $f$ and $f^{-1}$ if we graph both functions on the same coordinate system (i.e., as functions of $x$).

For example, consider the functions from **Example 11**. The functions $f(x) = x^3$ and $f^{-1}(x) = \sqrt[3]{x}$ are graphed in **Figure 6** along with the line $y = x$. Several reflected pairs of points are also shown on the graph.

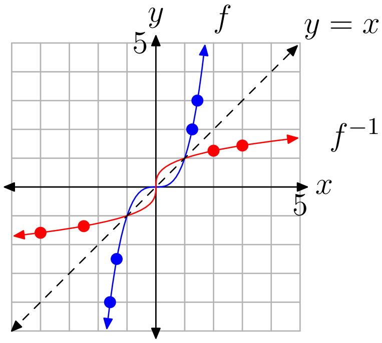


**Figure 6.** Graphs of $f(x) = x^3$ and $f^{-1}(x) = \sqrt[3]{x}$ are reflections across the line $y = x$.

To see why the points $(a, b)$ and $(b, a)$ are just reflections of each other across the line $y = x$, consider the segment $S$ between these two points (see **Figure 7**). It will be enough to show: (1) that $S$ is perpendicular to the line $y = x$, and (2) that the intersection point $P$ of the segment $S$ and the line $y = x$ is equidistant from each of $(a, b)$ and $(b, a)$.

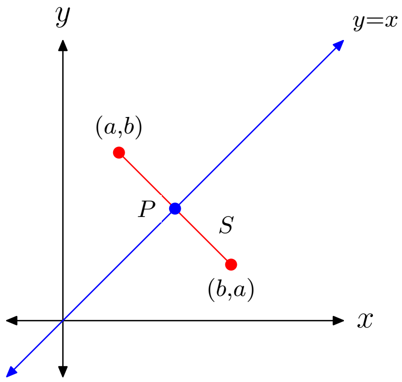


**Figure 7.** Switching the abscissa and ordinate reflects the point across the line $y = x$.

1. The slope of $S$ is

$$\frac{a-b}{b-a} = -1,$$

and the slope of the line $y = x$ is 1, so they are perpendicular.

2. The line containing $S$ has equation $y-b = -(x-a)$, or equivalently, $y = -x+(a+b)$. To find the intersection of $S$ and the line $y = x$, set $x = -x + (a + b)$ and solve for $x$ to get

$$x = \frac{a+b}{2}.$$

Since $y = x$, it follows that the intersection point is

$$P = \left(\frac{a+b}{2}, \frac{a+b}{2}\right).$$

Finally, we can use the distance formula presented in section 9.6 to compute the distance from $P$ to $(a, b)$ and the distance from $P$ to $(b, a)$. In both cases, the computed distance turns out to be

$$\frac{|a-b|}{\sqrt{2}}.$$

## Computing the Formula of an Inverse Function

How does one find the formula of an inverse function? In **Example 11**, it was easy to see that the inverse of the "cubing" function must be the cube root function. But how was the formula for the inverse in **Example 12** obtained?

Actually, there is a simple procedure for finding the formula for the inverse function (provided that such a formula exists; remember that not all functions can be described by a simple formula, so the procedure will not work for such functions). The following procedure works because the inputs and outputs (the $x$ and $y$ variables) are switched in step 3.

### Computing the Formula of an Inverse Function

1. Check the graph of the original function $f(x)$ to see if it passes the horizontal line test. If so, then $f$ is one-to-one and you can proceed.
2. Write the formula in $xy$-equation form, as $y = f(x)$.
3. Interchange the $x$ and $y$ variables.
4. Solve the new equation for $y$, if possible. The result will be the formula for $f^{-1}(x)$.

▶ **Example 17.** *Let's start by finding the inverse of the function* $f(x) = 4x - 1$ *from* **Example 12**.

**Step 1:** A check of the graph shows that $f$ is one-to-one (see **Figure 8**).

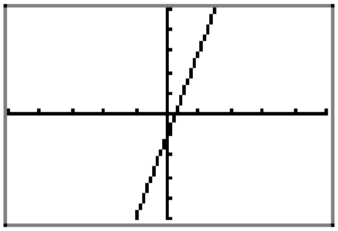


**Figure 8.** The graph of $f(x) = 4x-1$ passes the horizontal line test.

**Step 2:** Write the formula in $xy$-equation form: $y = 4x - 1$

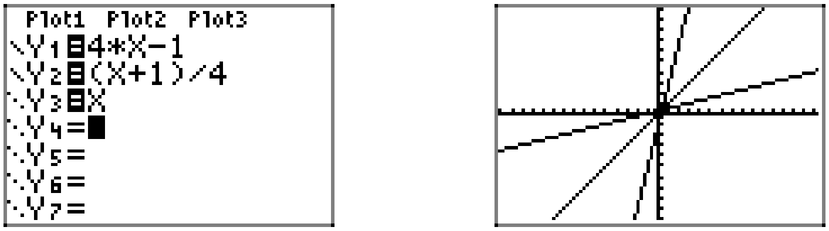
**Step 3:** Interchange $x$ and $y$: $x = 4y - 1$

**Step 4:** Solve for $y$:

$$\begin{aligned} x &= 4y - 1 \\ \implies x + 1 &= 4y \\ \implies \frac{x+1}{4} &= y \end{aligned}$$

Thus, $f^{-1}(x) = \dfrac{x+1}{4}$.

**Figure 9** demonstrates that the graph of $f^{-1}(x) = (x + 1)/4$ is a reflection of the graph of $f(x) = 4x - 1$ across the line $y = x$. In this figure, the `ZSquare` command in the `ZOOM` menu has been used to better illustrate the reflection (the `ZSquare` command equalizes the scales on both axes).



```
 Plot1  Plot2  Plot3
\Y1=4*X-1
\Y2=(X+1)/4
\Y3=X
\Y4=
\Y5=
\Y6=
\Y7=
```

(a) (b)

**Figure 9.** Symmetry across the line $y = x$

◆

▶ **Example 18.** *This time we'll find the inverse of* $f(x) = 2x^5 + 3$.

**Step 1:** A check of the graph shows that $f$ is one-to-one (this is left for the reader to verify).

**Step 2:** Write the formula in $xy$-equation form: $y = 2x^5 + 3$

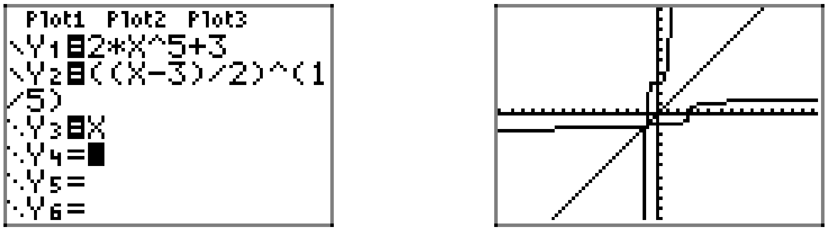
**Step 3:** Interchange $x$ and $y$: $x = 2y^5 + 3$

**Step 4:** Solve for $y$:

$$\begin{aligned} x &= 2y^5 + 3 \\ \Longrightarrow\quad x - 3 &= 2y^5 \\ \Longrightarrow\quad \frac{x-3}{2} &= y^5 \\ \Longrightarrow\quad \sqrt[5]{\frac{x-3}{2}} &= y \end{aligned}$$

Thus, $f^{-1}(x) = \sqrt[5]{\dfrac{x-3}{2}}$.

Again, note that the graph of $f^{-1}(x) = \sqrt[5]{(x-3)/2}$ is a reflection of the graph of $f(x) = 2x^5 + 3$ across the line $y = x$ (see **Figure 10**).



```
Plot1 Plot2 Plot3
\Y1■2*X^5+3
\Y2■((X-3)/2)^(1
/5)
·.Y3■X
·.Y4=■
·.Y5=
·.Y6=
```

(a) (b)

**Figure 10.** Symmetry across the line $y = x$

◆

▶ **Example 19.** *Find the inverse of* $f(x) = 5/(7 + x)$.

**Step 1:** A check of the graph shows that $f$ is one-to-one (this is left for the reader to verify).

**Step 2:** Write the formula in $xy$-equation form: $y = \dfrac{5}{7+x}$

**Step 3:** Interchange $x$ and $y$: $x = \dfrac{5}{7+y}$

**Step 4:** Solve for $y$:

$$\begin{aligned} x &= \frac{5}{7+y} \\ \Longrightarrow\ x(7+y) &= 5 \\ \Longrightarrow\ 7+y &= \frac{5}{x} \\ \Longrightarrow\ y &= \frac{5}{x} - 7 = \frac{5-7x}{x} \end{aligned}$$

Thus, $f^{-1}(x) = \dfrac{5-7x}{x}$.

◆

▶ **Example 20.** *This example is a bit more complicated: find the inverse of the function* $f(x) = (5x+2)/(x-3)$.

**Step 1:** A check of the graph shows that $f$ is one-to-one (this is left for the reader to verify).

**Step 2:** Write the formula in $xy$-equation form: $y = \dfrac{5x+2}{x-3}$

**Step 3:** Interchange $x$ and $y$: $x = \dfrac{5y+2}{y-3}$

**Step 4:** Solve for $y$:

$$\begin{aligned} x &= \frac{5y+2}{y-3} \\ \Longrightarrow\ x(y-3) &= 5y+2 \\ \Longrightarrow\ xy - 3x &= 5y+2 \end{aligned}$$

This equation is linear in $y$. Isolate the terms containing the variable $y$ on one side of the equation, factor, then divide by the coefficient of $y$.

$$\begin{aligned} xy - 3x &= 5y+2 \\ \Longrightarrow\ xy - 5y &= 3x+2 \\ \Longrightarrow\ y(x-5) &= 3x+2 \\ \Longrightarrow\ y &= \frac{3x+2}{x-5} \end{aligned}$$

Thus, $f^{-1}(x) = \dfrac{3x+2}{x-5}$.

◆

▶ **Example 21.** *According to the horizontal line test, the function* $h(x) = x^2$ *is certainly not one-to-one. However, if we only consider the right half or left half of the function (i.e., restrict the domain to either the interval* $[0,\infty)$ *or* $(-\infty,0]$*), then* *the function* would *be one-to-one, and therefore would have an inverse (***Figure 11***(a) shows the left half). For example, suppose $f$ is the function*

$$f(x) = x^2, \quad x \le 0.$$

In this case, the procedure still works, provided that we carry along the domain condition in all of the steps, as follows:

**Step 1:** The graph in **Figure 11**(a) passes the horizontal line test, so $f$ is one-to-one.

**Step 2:** Write the formula in $xy$-equation form:

$$y = x^2, \quad x \le 0$$

**Step 3:** Interchange $x$ and $y$:

$$x = y^2, \quad y \le 0$$

Note how $x$ and $y$ must also be interchanged in the domain condition.

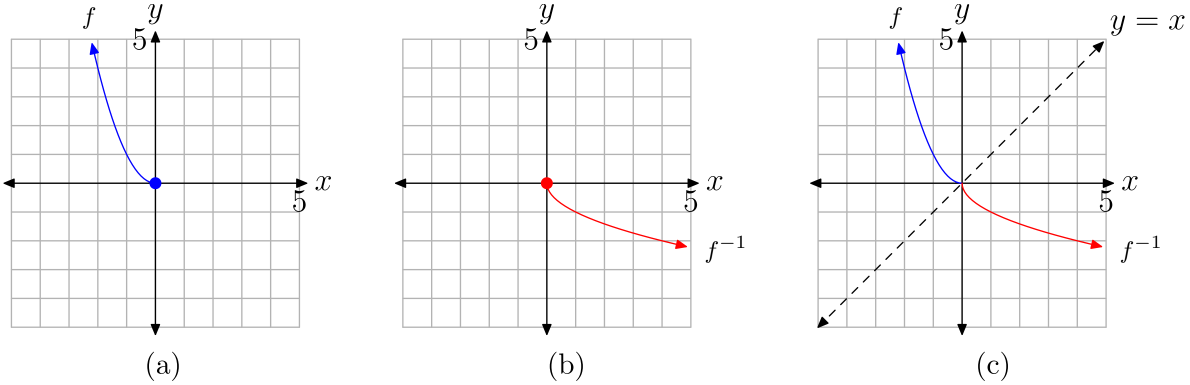
**Step 4:** Solve for $y$:

$$y = \pm\sqrt{x}, \quad y \le 0$$

Now there are two choices for $y$, one positive and one negative, but the condition $y \le 0$ tells us that the negative choice is the correct one. Thus, the last statement is equivalent to

$$y = -\sqrt{x}.$$

Thus, $f^{-1}(x) = -\sqrt{x}$. The graph of $f^{-1}$ is shown in **Figure 11**(b), and the graphs of both $f$ and $f^{-1}$ are shown in **Figure 11**(c) as reflections across the line $y = x$.



(a) (b) (c)

**Figure 11.**

◆

## 8.4 Exercises

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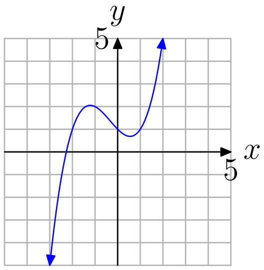
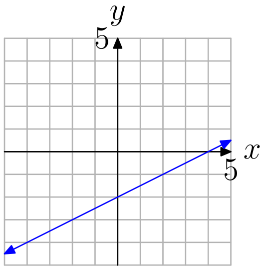
In **Exercises 1-12**, use the graph to determine whether the function is one-to-one.

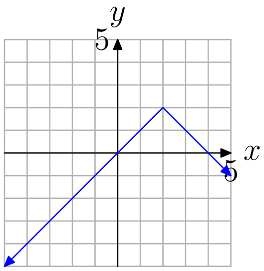
**1.**



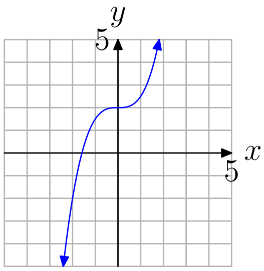
**2.**



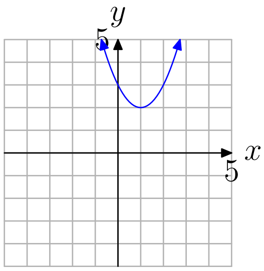
**3.**



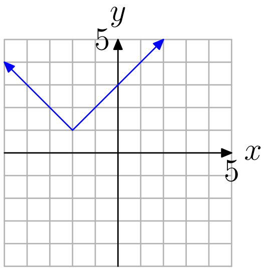
**4.**



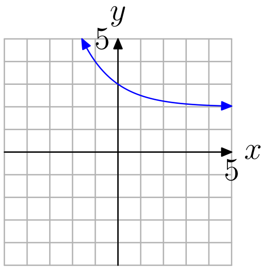
**5.**



**6.**



**7.**



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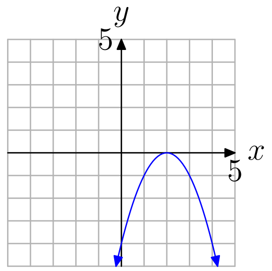
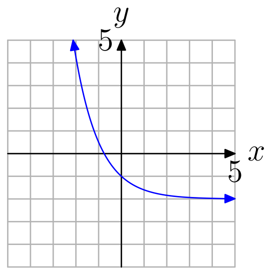
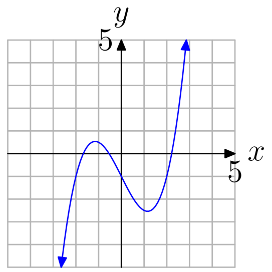
**8.**



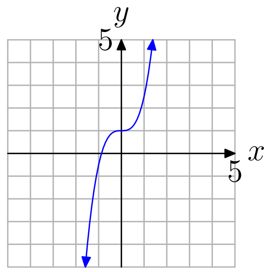
**9.**



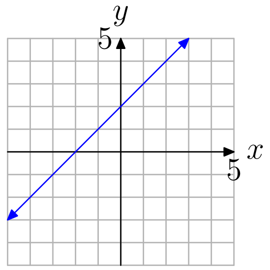
**10.**



**11.**



**12.**



In **Exercises 13-28**, evaluate the composition $g(f(x))$ and simplify your answer.

**13.** $g(x) = \dfrac{9}{x}$, $f(x) = -2x^2 + 5x - 2$

**14.** $f(x) = -\dfrac{5}{x}$, $g(x) = -4x^2 + x - 1$

**15.** $g(x) = 2\sqrt{x}$, $f(x) = -x - 3$

**16.** $f(x) = 3x^2 - 3x - 5$, $g(x) = \dfrac{6}{x}$

**17.** $g(x) = 3\sqrt{x}$, $f(x) = 4x + 1$

**18.** $f(x) = -3x - 5$, $g(x) = -x - 2$

**19.** $g(x) = -5x^2 + 3x - 4$, $f(x) = \dfrac{5}{x}$

**20.** $g(x) = 3x + 3$, $f(x) = 4x^2 - 2x - 2$

**21.** $g(x) = 6\sqrt{x}$, $f(x) = -4x + 4$

**22.** $g(x) = 5x - 3$, $f(x) = -2x - 4$

**23.** $g(x) = 3\sqrt{x}$, $f(x) = -2x + 1$

**24.** $g(x) = \dfrac{3}{x}$, $f(x) = -5x^2 - 5x - 4$

**25.** $f(x) = \dfrac{5}{x}$, $g(x) = -x + 1$

**26.** $f(x) = 4x^2 + 3x - 4$, $g(x) = \dfrac{2}{x}$

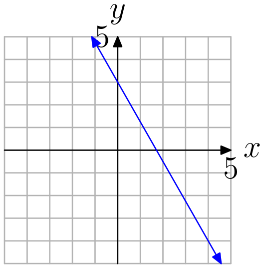
**27.** $g(x) = -5x + 1,\ f(x) = -3x - 2$

**28.** $g(x) = 3x^2 + 4x - 3,\ f(x) = \dfrac{8}{x}$

---

In **Exercises 29-36**, first copy the given graph of the one-to-one function $f(x)$ onto your graph paper. Then on the same coordinate system, sketch the graph of the inverse function $f^{-1}(x)$.

**29.**



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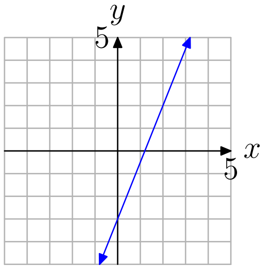

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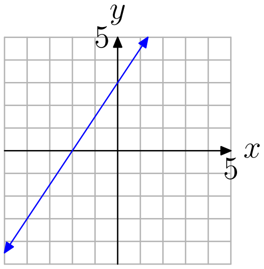

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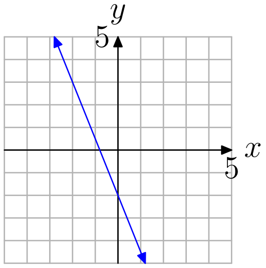
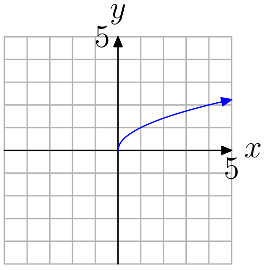
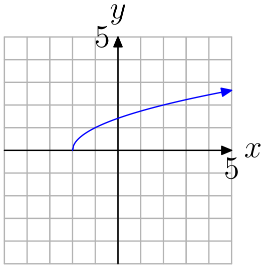


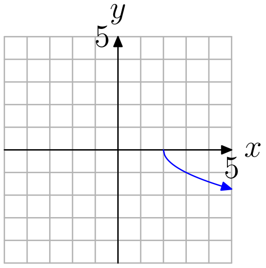
**33.**



**34.**



**35.**



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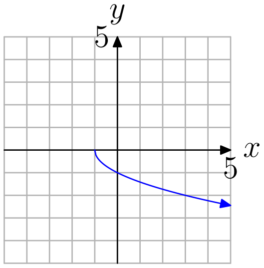


In **Exercises 37-68**, find the formula for the inverse function $f^{-1}(x)$.

**37.** $f(x) = 5x^3 - 5$

**38.** $f(x) = 4x^7 - 3$

**39.** $f(x) = -\dfrac{9x-3}{7x+6}$

**40.** $f(x) = 6x - 4$

**41.** $f(x) = 7x - 9$

**42.** $f(x) = 7x + 4$

**43.** $f(x) = 3x^5 - 9$

**44.** $f(x) = 6x + 7$

**45.** $f(x) = \dfrac{4x+2}{4x+3}$

**46.** $f(x) = 5x^7 + 4$

**47.** $f(x) = \dfrac{4x-1}{2x+2}$

**48.** $f(x) = \sqrt[7]{8x-3}$

**49.** $f(x) = \sqrt[3]{-6x-4}$

**50.** $f(x) = \dfrac{8x-7}{3x-6}$

**51.** $f(x) = \sqrt[7]{-3x-5}$

**52.** $f(x) = \sqrt[9]{8x+2}$

**53.** $f(x) = \sqrt[3]{6x+7}$

**54.** $f(x) = \dfrac{3x+7}{2x+8}$

**55.** $f(x) = -5x + 2$

**56.** $f(x) = 6x + 8$

**57.** $f(x) = 9x^9 + 5$

**58.** $f(x) = 4x^5 - 9$

**59.** $f(x) = \dfrac{9x-3}{9x+7}$

**60.** $f(x) = \sqrt[3]{9x-7}$

**61.** $f(x) = x^4,\ x \le 0$

**62.** $f(x) = x^4,\ x \ge 0$

**63.** $f(x) = x^2 - 1,\ x \le 0$

**64.** $f(x) = x^2 + 2,\ x \ge 0$

**65.** $f(x) = x^4 + 3,\ x \le 0$

**66.** $f(x) = x^4 - 5,\ x \ge 0$

**67.** $f(x) = (x-1)^2,\ x \le 1$

**68.** $f(x) = (x+2)^2,\ x \ge -2$

## 8.4 *Answers*

---

**1.** not one-to-one

**3.** not one-to-one

**5.** not one-to-one

**7.** one-to-one

**9.** one-to-one

**11.** one-to-one

**13.** $-\dfrac{9}{2x^2-5x+2}$

**15.** $2\sqrt{-x-3}$

**17.** $3\sqrt{4x+1}$

**19.** $-\dfrac{125}{x^2}+\dfrac{15}{x}-4$

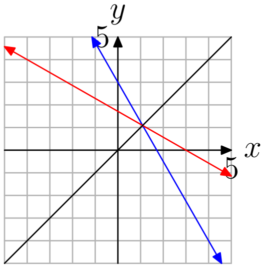
**21.** $6\sqrt{-4x+4}$

**23.** $3\sqrt{-2x+1}$

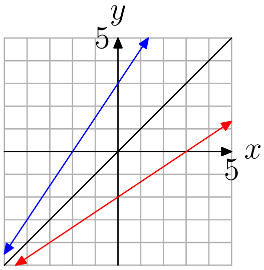
**25.** $-5/x+1$

**27.** $15x+11$

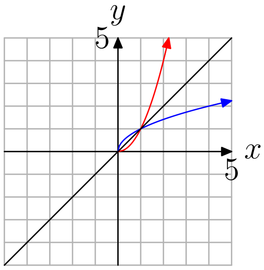
**29.**



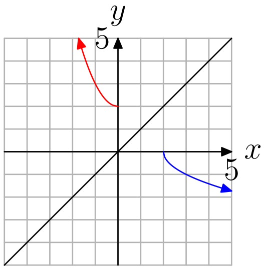
**31.**



**33.**



**35.**



**37.** $\sqrt[3]{\dfrac{x+5}{5}}$

**39.** $-\dfrac{6x-3}{7x+9}$

**41.** $\dfrac{x+9}{7}$

**43.** $\sqrt[5]{\dfrac{x+9}{3}}$

**45.** $-\dfrac{3x-2}{4x-4}$

**47.** $-\dfrac{2x+1}{2x-4}$

**49.** $-\dfrac{x^3+4}{6}$

**51.** $-\dfrac{x^7+5}{3}$

**53.** $\dfrac{x^3-7}{6}$

**55.** $-\dfrac{x-2}{5}$

**57.** $\sqrt[9]{\dfrac{x-5}{9}}$

**59.** $-\dfrac{7x+3}{9x-9}$

**61.** $-\sqrt[4]{x}$

**63.** $-\sqrt{x+1}$

**65.** $-\sqrt[4]{x-3}$

**67.** $-\sqrt{x}+1$

# 8.5 Logarithmic Functions

We can now apply the inverse function theory from the previous section to the exponential function. From Section 8.2, we know that the function $f(x) = b^x$ is either increasing (if $b > 1$) or decreasing (if $0 < b < 1$), and therefore is one-to-one. Consequently, $f$ has an inverse function $f^{-1}$.

As an example, let's consider the exponential function $f(x) = 2^x$. $f$ is increasing, has domain $D_f = (-\infty, \infty)$, and range $R_f = (0, \infty)$. Its graph is shown in **Figure 1**(a). The graph of the inverse function $f^{-1}$ is a reflection of the graph of $f$ across the line $y = x$, and is shown in **Figure 1**(b). Since domains and ranges are interchanged, the domain of the inverse function is $D_{f^{-1}} = (0, \infty)$ and the range is $R_{f^{-1}} = (-\infty, \infty)$.

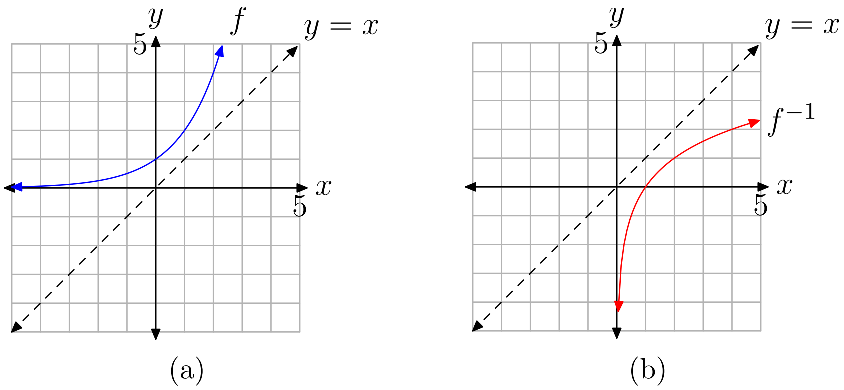


**Figure 1.** The graphs of $f(x) = 2^x$ and its inverse $f^{-1}(x)$ are reflections across the line $y = x$.

Unfortunately, when we try to use the procedure given in Section 8.4 to find a formula for $f^{-1}$, we run into a problem. Starting with $y = 2^x$, we then interchange $x$ and $y$ to obtain $x = 2^y$. But now we have no algebraic method for solving this last equation for $y$. It follows that the inverse of $f(x) = 2^x$ has no formula involving the usual arithmetic operations and functions that we're familiar with. Thus, the inverse function is a new function. The name of this new function is the *logarithm of x to base 2*, and it's denoted by $f^{-1}(x) = \log_2(x)$.

Recall that the defining relationship between a function and its inverse (Property 14 in Section 8.4) simply states that the inputs and outputs of the two functions are interchanged. Thus, the relationship between $2^x$ and its inverse $\log_2(x)$ takes the following form:

$$v = \log_2(u) \iff u = 2^v$$

More generally, for each exponential function $f(x) = b^x$ ($b > 0$, $b \neq 1$), the inverse function $f^{-1}(x)$ is called the *logarithm of $x$ to base $b$*, and is denoted by $\log_b(x)$. The defining relationship is given in the following definition.

---

[12] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

**Definition 1.** *If* $b > 0$ *and* $b \neq 1$*, then the* **logarithm of** $u$ **to base** $b$ *is defined by the relationship*

$$v = \log_b(u) \quad \Longleftrightarrow \quad u = b^v. \tag{2}$$

In order to understand the logarithm function better, let's work through a few simple examples.

▶ **Example 3.** *Compute* $\log_2(8)$.

Label the required value by $v$, so $v = \log_2(8)$. Then by (**2**), using $b = 2$ and $u = 8$, it follows that $2^v = 8$, and therefore $v = 3$ (solving by inspection).

◇

In the last example, note that $\log_2(8) = 3$ is the exponent $v$ such that $2^v = 8$. Thus, in general, one way to interpret the definition of the logarithm in (**2**) is that $\log_b(u)$ is the exponent $v$ such that $b^v = u$. In other words, the value of the logarithm is the exponent!

▶ **Example 4.** *Compute* $\log_{10}(10\,000)$.

Again, label the required value by $v$, so $v = \log_{10}(10\,000)$. By (**2**), it follows that $10^v = 10\,000$, and therefore $v = 4$. Note that here again we have found the exponent v=4 that is needed for base 10 in order to get $10^v = 10\,000$.

◇

▶ **Example 5.** *Compute* $\log_3\left(\frac{1}{9}\right)$.

$$\begin{aligned} & v = \log_3\left(\frac{1}{9}\right) \\ \Longrightarrow\ & 3^v = \frac{1}{9} \qquad \text{by } (\mathbf{2}) \\ \Longrightarrow\ & v = -2 \qquad \text{since } 3^{-2} = \frac{1}{9} \end{aligned}$$

◇

▶ **Example 6.** *Solve the equation* $\log_5(x) = 1$.

$$\begin{aligned} & \log_5(x) = 1 \\ \Longrightarrow\ & 5^1 = x \qquad \text{by } (\mathbf{2}) \\ \Longrightarrow\ & x = 5 \end{aligned}$$

◇

▶ **Example 7.** *Solve the equation* $\log_b(64) = 3$ *for* $b$.

$$\begin{aligned} &\log_b(64) = 3 \\ \Longrightarrow\ & b^3 = 64 \qquad \text{by } (\mathbf{2}) \\ \Longrightarrow\ & b = \sqrt[3]{64} = 4 \end{aligned}$$

◇

▶ **Example 8.** *Solve the equation* $\log_{1/2}(x) = -2$.

$$\begin{aligned} &\log_{1/2}(x) = -2 \\ \Longrightarrow\ & \left(\frac{1}{2}\right)^{-2} = x \qquad \text{by } (\mathbf{2}) \\ \Longrightarrow\ & x = \frac{1}{\left(\frac{1}{2}\right)^2} = \frac{1}{\frac{1}{4}} = 4 \end{aligned}$$

◇

The composition relationships in Property 15 of Section 8.4, applied to $b^x$ and $\log_b(x)$, become

**Property 9.**

$$\log_b(b^x) = x \tag{10}$$

and

$$b^{\log_b(x)} = x. \tag{11}$$

Both equations are important. Note that (**11**) again shows that the $\log_b(x)$ is the exponent $v$ such that $b^v = x$. Equation (**10**) will be used frequently in this and later sections to help us solve exponential equations.

Logarithmic functions are used in many areas of science and engineering. For example, they are used to define the Richter scale for the magnitudes of earthquakes, the decibel scale for the loudness of sound, and the astronomical scale for stellar brightness. They are also important tools for use in computation (as we will see in Section 8.8). Our main use of logarithms in this textbook will be to solve exponential equations, and thereby help us study physical phenomena that are described by exponential functions (as in Section 8.7).

## *Computing Logarithms*

In Examples **3**–**8** above, we were able to compute the logarithms by converting to exponential equations that could be solved by inspection. But it's easy to see that most of the time this won't work. For example, how would we compute the value of $\log_2(7)$?

Fortunately, mathematicians have found other methods for computing logarithms to high accuracy, and they can now be easily approximated using a calculator or computer.

Your calculator has built-in buttons for computing two different logarithms, $\log_{10}(x)$ and $\log_e(x)$. $\log_{10}(x)$ is called the *common logarithm*, and $\log_e(x)$ is called the *natural logarithm*.

**Common Logarithm**: The common logarithm $\log_{10}(x)$ is computed using the `LOG` button on your calculator. Notice also that its inverse function $10^x$, can be computed using the same button in conjunction with the `2ND` button. The common logarithm is usually the most convenient one to use for computations involving scientific notation (because we use a base 10 number system), and therefore is the logarithm most often used in the physical sciences. Because of that, it's often just abbreviated by $\log(x)$, and we'll do that as well in the remainder of the text.

**Common Logarithm**. $\log(x)$ and $\log_{10}(x)$ are equivalent notations. Thus, we have the defining relationship

$$v = \log(u) \quad \Longleftrightarrow \quad u = 10^v.$$

The composition properties for the common logarithm are

$$\log(10^x) = x \tag{12}$$

and

$$10^{\log(x)} = x.$$

**Natural Logarithm**: The natural logarithm $\log_e(x)$ is computed using the `LN` button on your calculator. Its inverse function, $e^x$, is computed using the same button in conjunction with the `2ND` button. The natural logarithm turns out to be the most convenient one to use in mathematics, because a lot of formulas, especially in calculus, are much simpler when the natural logarithm is used. The natural logarithm is abbreviated by $\ln(x)$.

**Natural Logarithm**. $\ln(x)$ and $\log_e(x)$ are equivalent notations. Thus, we have the defining relationship

$$v = \ln(u) \quad \Longleftrightarrow \quad u = e^v.$$

The composition properties for the common logarithm are

$$\ln(e^x) = x \tag{13}$$

and

$$e^{\ln(x)} = x.$$

Note that when using your calculator to compute $\log(x)$ and $\ln(x)$, you will usually only obtain approximate values, as these values frequently are irrational numbers.

What about other bases? You can also compute these on your calculator, but we'll first need to develop the *Change of Base Formula* in the next section. However, at this point, we can at least solve exponential equations involving bases 10 and e, as shown in the next two examples.

▶ **Example 14.** *Solve the equation* $704 = 2(10)^x$.

The first step is to isolate the exponential on the right side by dividing both sides by 2:

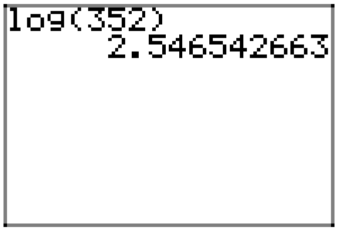
$$352 = 10^x$$

Then simply apply the $\log_{10}(x)$ function to both sides of the equation:

$$\log_{10}(352) = \log_{10}(10^x)$$

But (**10**) implies that $\log_{10}(10^x) = x$. Therefore, $x = \log_{10}(352) = \log(352)$ is the exact solution. The approximate value, using a calculator, is 2.546542663 (see **Figure 2**).

Alternatively, instead of taking the logarithm of both sides in the second step, you can apply (**2**) to the equation $352 = 10^x$ to get $x = \log_{10}(352)$. ◇



```
log(352)
        2.546542663
```

**Figure 2.** Approximation of $\log(352) = \log_{10}(352)$.

This last example shows how logarithms can be used for solving exponential equations. The basic strategy is to first isolate the exponential on one side of the equation, and then take appropriate logarithms of both sides. Here's one more example for now, and then we'll return to this process repeatedly in the remaining sections, especially when we work with application problems.

▶ **Example 15.** *Solve the equation* $30 = 20e^x$.

First isolate the exponential on the right side by dividing both sides by 20:

$$1.5 = e^x$$

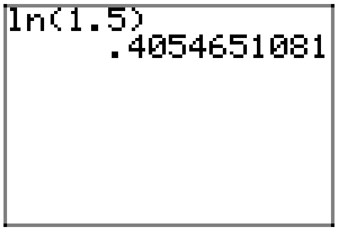
This time, since the base of the exponential function is $e$, apply the natural logarithm function to both sides:

$$\log_e(1.5) = \log_e(e^x)$$

Simplify the right side, since $\log_e(e^x) = x$ by (**10**):

$$\log_e(1.5) = x$$

Therefore, $x = \log_e(1.5) = \ln(1.5)$ is the exact solution. The approximate value, using a calculator, is 0.4054651081 (see **Figure 3**).



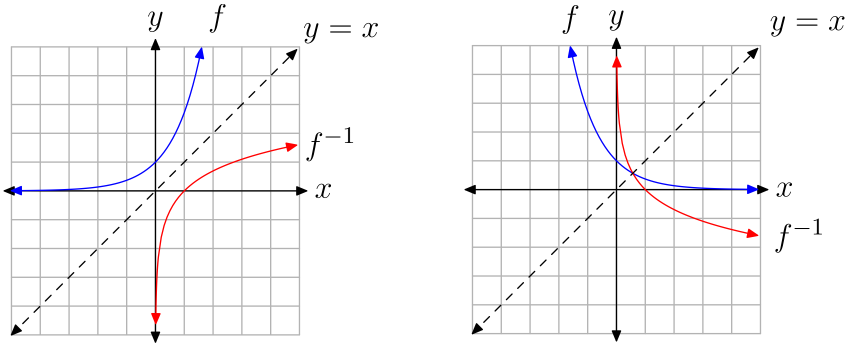
```
ln(1.5)
         .4054651081
```

**Figure 3.** Approximation of $\ln 1.5 = \log_e(1.5)$.

In the next section, we'll learn how to solve exponential equations involving other bases.

## *Graphs of Logarithmic Functions*

At the beginning of this section, we looked at the graphs of $f(x) = 2^x$ and its inverse function $f^{-1}(x) = \log_2(x)$. More generally, the graph of the exponential function $f(x) = b^x$ for $b > 1$ is shown in **Figure 4**(a), along with its inverse logarithmic function $f^{-1}(x) = \log_b(x)$. According to Section 8.4, the two graphs are reflections across the line $y = x$. Similarly, the graph for $0 < b < 1$ is shown in **Figure 4**(b).



(a) $b > 1$ (b) $0 < b < 1$

**Figure 4.** The graphs of $f(x) = b^x$ and $f^{-1}(x) = \log_b(x)$ are reflections across the line $y = x$.

Because domains and ranges of inverse functions are interchanged, it follows that

**Property 16.**

$$\text{Domain}(\log_b(x)) = (0, \infty)$$

and

$$\text{Range}(\log_b(x)) = (-\infty, \infty).$$

In particular, note that the logarithm of a negative number, as well as the logarithm of 0, are *not defined*.

Two particular points on the graph of the logarithm are noteworthy. Since $b^0 = 1$, it follows that $\log_b(1) = 0$, and therefore the $x$-intercept of the graph of $\log_b(x)$ is $(1, 0)$. Similarly, since $b^1 = b$, it follows that $\log_b(b) = 1$, and therefore $(b, 1)$ is on the graph.

**Property 17.**

$$\log_b(1) = 0 \qquad \textit{and} \qquad \log_b(b) = 1$$

Finally, since the graph of $b^x$ has a horizontal asymptote $y = 0$, the graph of $\log_b(x)$ must have a vertical asymptote $x = 0$. This behavior is a consequence of the fact that inputs and outputs of inverse functions are interchanged, and can be observed in **Figure 4**.

In the final example below, we'll apply a transformation to the logarithm and see how that affects the graph.

▶ **Example 18.** *Plot the graph of the function* $f(x) = \log_2(x + 1)$.

The graph of $f(x) = \log_2(x + 1)$ will be the same as the graph of $g(x) = \log_2(x)$ shifted one unit to the left. The graph of $g$ is shown in **Figure 1**(b). The $x$-intercept $(1, 0)$ on the graph of $g$ will be shifted one unit to the left to $(0, 0)$ on the graph of $f$. Likewise, the vertical asymptote $x = 0$ on the graph of $g$ will be shifted one unit to the left to the line $x = -1$ on the graph of $f$. The final graph of $f$ is shown in **Figure 5**.

◆

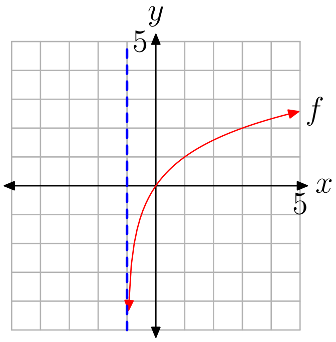


**Figure 5.** The graph of $f(x) = \log_2(x + 1)$.

## 8.5 Exercises

In **Exercises 1-18**, find the exact value of the function at the given value $b$.

**1.** $f(x) = \log_3(x)$; $b = \sqrt[5]{3}$.

**2.** $f(x) = \log_5(x)$; $b = 3125$.

**3.** $f(x) = \log_2(x)$; $b = \dfrac{1}{16}$.

**4.** $f(x) = \log_2(x)$; $b = 4$.

**5.** $f(x) = \log_5(x)$; $b = 5$.

**6.** $f(x) = \log_2(x)$; $b = 8$.

**7.** $f(x) = \log_2(x)$; $b = 32$.

**8.** $f(x) = \log_4(x)$; $b = \dfrac{1}{16}$.

**9.** $f(x) = \log_5(x)$; $b = \dfrac{1}{3125}$.

**10.** $f(x) = \log_5(x)$; $b = \dfrac{1}{25}$.

**11.** $f(x) = \log_5(x)$; $b = \sqrt[6]{5}$.

**12.** $f(x) = \log_3(x)$; $b = \sqrt[3]{3}$.

**13.** $f(x) = \log_6(x)$; $b = \sqrt[6]{6}$.

**14.** $f(x) = \log_5(x)$; $b = \sqrt[5]{5}$.

**15.** $f(x) = \log_2(x)$; $b = \sqrt[6]{2}$.

**16.** $f(x) = \log_4(x)$; $b = \dfrac{1}{4}$.

**17.** $f(x) = \log_3(x)$; $b = \dfrac{1}{9}$.

**18.** $f(x) = \log_4(x)$; $b = 64$.

In **Exercises 19-26**, use a calculator to evaluate the function at the given value $p$. Round your answer to the nearest hundredth.

**19.** $f(x) = \ln(x)$; $p = 10.06$.

**20.** $f(x) = \ln(x)$; $p = 9.87$.

**21.** $f(x) = \ln(x)$; $p = 2.40$.

**22.** $f(x) = \ln(x)$; $p = 9.30$.

**23.** $f(x) = \log(x)$; $p = 7.68$.

**24.** $f(x) = \log(x)$; $p = 652.22$.

**25.** $f(x) = \log(x)$; $p = 6.47$.

**26.** $f(x) = \log(x)$; $p = 86.19$.

In **Exercises 27-34**, solve the given equation, and round your answer to the nearest hundredth.

**27.** $13 = e^{8x}$

**28.** $2 = 8e^{x}$

**29.** $19 = 10^{8x}$

**30.** $17 = 10^{2x}$

**31.** $7 = 6(10)^{x}$

**32.** $7 = e^{9x}$

**33.** $13 = 8e^{x}$

**34.** $5 = 7(10)^{x}$

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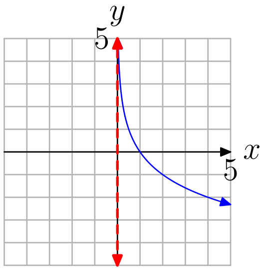
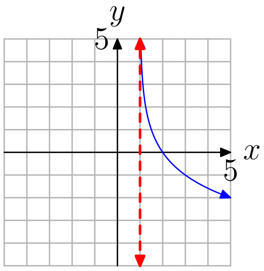
In **Exercises 35-42**, the graph of a logarithmic function of the form $f(x) = \log_b(x - a)$ is shown. The dashed red line is a vertical asymptote. Determine the domain of the function. Express your answer in interval notation.

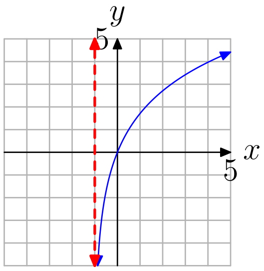
**35.**



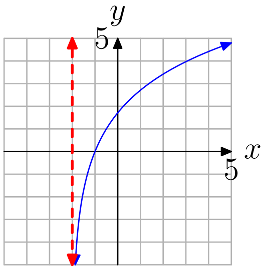
**36.**



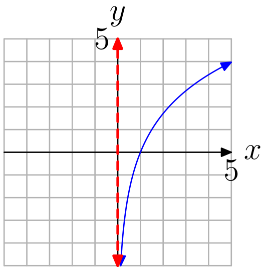
**37.**



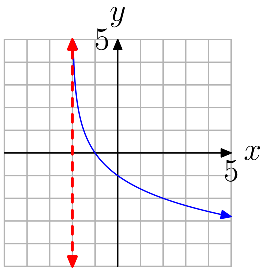
**38.**



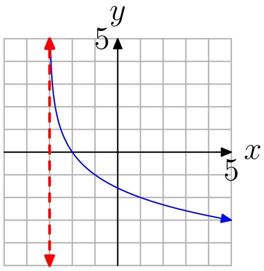
**39.**



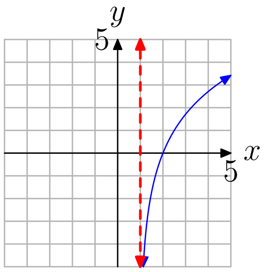
**40.**



**41.**



**42.**



## 8.5 *Answers*

---

**1.** $\frac{1}{5}$

**3.** $-4$

**5.** $1$

**7.** $5$

**9.** $-5$

**11.** $\frac{1}{6}$

**13.** $\frac{1}{6}$

**15.** $\frac{1}{6}$

**17.** $-2$

**19.** $2.31$

**21.** $0.88$

**23.** $0.89$

**25.** $0.81$

**27.** $0.32$

**29.** $0.16$

**31.** $0.07$

**33.** $0.49$

**35.** $(0, \infty)$

**37.** $(-1, \infty)$

**39.** $(0, \infty)$

**41.** $(-3, \infty)$

ïz£

# 8.6 Properties of Logarithms; Solving Exponential Equations

Logarithms were actually discovered and used in ancient times by both Indian and Islamic mathematicians. They were not used widely, though, until the 1600âĂŹs, when logarithms simplified the large amounts of hand computations needed in the scientific explorations of the times. In particular, after the invention of the telescope, calculations involving astronomical data became very important, and logarithms became an essential mathematical tool. Indeed, until the invention of the computer and electronic calculator in recent times, hand calculations using logarithms were a staple of every science student's curriculum.

The usefulness of logarithms in calculations is based on the following three important properties, known generally as the *properties of logarithms*.

**Properties of Logarithms**

a) $\log_b(MN) = \log_b(M) + \log_b(N)$

b) $\log_b\left(\dfrac{M}{N}\right) = \log_b(M) - \log_b(N)$

c) $\log_b(M^r) = r\log_b(M)$

provided that $M, N, b > 0$.

The first property says that the "log of a product is the sum of the logs." The second says that the "log of a quotient is the difference of the logs." And the third property is sometimes referred to as the "power rule". Loosely speaking, when taking the log of a power, you can just move the exponent out in front of the log.

We won't go into the details of the computation procedures using properties (a) and (b), since these procedures are no longer necessary after the invention of the calculator. But the idea is that a time-consuming product of two numbers, for example two 10-digit numbers, can be transformed by property (a) into a much simpler addition problem. Similarly, a large and difficult quotient can be transformed by property (b) into a much simpler subtraction problem. Properties (a) and (b) are also the basis for the slide rule, a mechanical computation device that preceded the electronic calculator (very fast and useful, but only accurate to about three digits).

Property (c), on the other hand, is still useful for difficult computations. If you try to compute a large power, say $2^{100}$, on a calculator or computer, you'll get an error message. That's because all calculators and computers can only handle numbers and exponents within a certain range. So to compute a large power, it's necessary to use

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property (c) to turn it into a multiplication problem. The details of this procedure are given in Section 8.8.

Even though properties (a) and (b) are no longer necessary for computation purposes, that does not mean they are not important. Logarithmic functions serve many purposes in mathematics and the sciences, and all of the logarithm properties are useful in various ways.

Where do the logarithm properties come from? Actually, they're all derived from the laws of exponents, using the fact that the exponential function is the inverse of the logarithm function. Since we'll only be using property (c) in this book, we'll show how that property is derived. Properties (a) and (b) are derived in a similar manner.

**Proof of (c):** Start on the right side of the equation, and label $\log_b(M)$ by $x$:

$$x = \log_b(M)$$

Use Definition 1 in Section 8.5 to rewrite the equation in exponential form:

$$b^x = M$$

Raise both sides to the $r$th power:

$$(b^x)^r = M^r$$

Apply one of the Laws of Exponents to the left side:

$$b^{rx} = M^r$$

Apply the base $b$ logarithmic function to both sides:

$$\log_b(b^{rx}) = \log_b(M^r)$$

Apply formula (10) in Section 8.5 to the left side:

$$rx = \log_b(M^r)$$

Substitute back for $x$ from the first line above:

$$r\log_b(M) = \log_b(M^r)$$

This is the formula in property (c).

◆

## *Change of Base Formula*

We can now prove a conversion formula that will enable us to compute the logarithm to any base.

**Change of Base Formula:**

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

**Proof:** Start on the left side of the equation, and label $\log_a(x)$ by $r$:

$$r = \log_a(x)$$

Use Definition 1 in Section 8.5 to rewrite the equation in exponential form:

$$a^r = x$$

Apply the base $b$ logarithmic function to both sides:

$$\log_b(a^r) = \log_b(x)$$

Apply property (c) to the left side:

$$r \log_b(a) = \log_b(x)$$

Divide by $\log_b(a)$:

$$r = \frac{\log_b(x)}{\log_b(a)}$$

Substitute back for $r$ from the first line above:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

This is the Change of Base Formula.

◆

▶ **Example 1.** *Compute* $\log_2(5)$.

Before applying the Change of Base Formula, let's see if we can estimate the value of $\log_2(5)$. First recall from Property 9 in Section 8.5 that $2^{\log_2(5)} = 5$. Now how large would the exponent on a base of 2 need to be for the power to equal 5? Since $2^2 = 4$ (too small) and $2^3 = 8$ (too large), we should expect $\log_2(5)$ to lie somewhere between 2 and 3. Indeed, applying the Change of Base Formula with the common logarithm yields

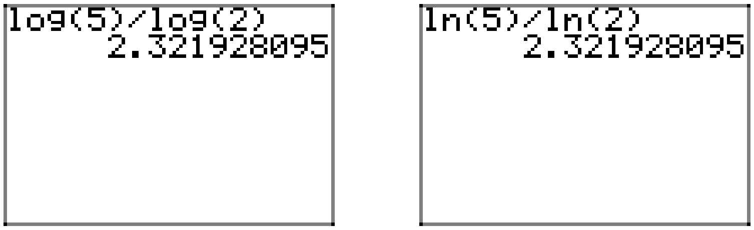
$$\log_2(5) = \frac{\log_{10}(5)}{\log_{10}(2)} = \frac{\log(5)}{\log(2)} \approx \frac{.6989700043}{.3010299957} \approx 2.321928095.$$

According to the formula, we could instead use the natural logarithm to obtain the same answer, as in

$$\log_2(5) = \frac{\log_e(5)}{\log_e(2)} = \frac{\ln(5)}{\ln(2)} \approx \frac{1.609437912}{.6931471806} \approx 2.321928095.$$

Calculator keystrokes are shown in **Figure 1**.

◆



```
log(5)/log(2)
      2.321928095
```

```
ln(5)/ln(2)
      2.321928095
```

**Figure 1.** Computing $\log_2(5)$ using the Change of Base Formula.

Another way to view the Change of Base Formula is that it says that *all logarithms are multiples of each other*, since

$$\log_a(x) = \left(\frac{1}{\log_b(a)}\right)\log_b(x).$$

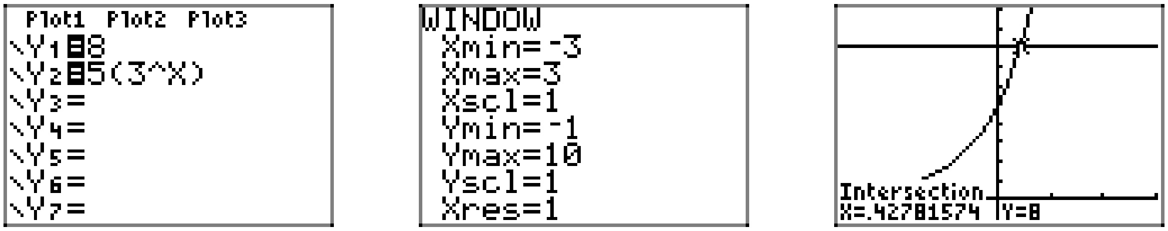
Thus, $\log_a(x)$ is a constant multiple of $\log_b(x)$, where the constant is $1/\log_b(a)$.

## Solving Exponential Equations

Property (c) ($\log_b(M^r) = r\log_b(M)$) is also used extensively to help solve exponential equations, and thus will be an important tool when we work with applications in the next section. In general terms, the main strategy for solving exponential equations is to (1) first isolate the exponential, then (2) apply a logarithmic function to both sides, and then (3) use property (c). We'll illustrate the strategy with several examples.

▶ **Example 2.** *Solve* $8 = 5(3^x)$.

Before trying the procedure outlined above, let's first approximate the solution using a graphical approach. Graph both sides of the equation in your calculator, and then find the intersection of the two curves to obtain $x \approx 0.42781574$ (see **Figure 2**).



```
 Plot1  Plot2  Plot3
\Y1=8
\Y2=5(3^X)
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
WINDOW
 Xmin=-3
 Xmax=3
 Xscl=1
 Ymin=-1
 Ymax=10
 Yscl=1
 Xres=1
```

```
Intersection
X=.42781574  Y=8
```

**Figure 2.** Approximating the solution of $8 = 5(3^x)$ graphically.

Now we'll solve the equation algebraically. First isolate the exponential function on one side of the equation by dividing both sides by 5:

$$1.6 = 3^x$$

Then take the logarithm of both sides. Use either the common or natural log:

$$\log(1.6) = \log(3^x)$$

Now use property (c) to move the exponent in front of the log on the right side:

$$\log(1.6) = x\log(3)$$

Finally, solve for $x$ by dividing both sides by $\log(3)$:

$$\frac{\log(1.6)}{\log(3)} = x$$

Thus, the exact value of $x$ is $\dfrac{\log(1.6)}{\log(3)}$, and the approximate value is 0.42781574. Note that this is the same as the graphical approximation found earlier.

◇

▶ **Example 3.** *Solve* $300 = 100(1.05^{5x})$.

$$\begin{aligned} & 300 = 100(1.05^{5x}) & & \\ \Longrightarrow\ & 3 = 1.05^{5x} & & \text{isolate the exponential} \\ \Longrightarrow\ & \log(3) = \log(1.05^{5x}) & & \text{apply the common log function} \\ \Longrightarrow\ & \log(3) = 5x\log(1.05) & & \text{use property (c)} \\ \Longrightarrow\ & \frac{\log(3)}{5\log(1.05)} = x & & \text{divide} \\ \Longrightarrow\ & x \approx 4.503417061 & & \end{aligned}$$

◇

If the base of the exponential is either 10 or $e$, the correct choice of logarithm leads to a faster solution:

▶ **Example 4.** *Solve* $3 = 4e^x$.

$$\begin{aligned} & 3 = 4e^x & & \\ \Longrightarrow\ & 0.75 = e^x & & \text{isolate the exponential} \\ \Longrightarrow\ & \ln(0.75) = \ln(e^x) & & \text{apply the natural log function} \\ \Longrightarrow\ & \ln(0.75) = x & & \text{since } \ln(e^x) = x \\ \Longrightarrow\ & x \approx -.2876820725 & & \end{aligned}$$

In this case, because the base of the exponential function is $e$, the use of the natural log function simplifies the solution.

◇

We can now turn our attention to solving more interesting application problems, such as the questions raised at the end of Section 8.3.

▶ **Example 5.** *If you deposit \$1000 in an account paying 6% interest compounded continuously, how long will it take for you to have \$1500 in your account?*

First, recall the continuous compound interest formula from Section 8.3:

$$P(t) = P_0 e^{rt} \tag{6}$$

In this case, $P_0 = 1000$ and $r = .06$. Inserting these values into the formula, we obtain

$$P(t) = 1000e^{0.06t}.$$

Now we want the future value $P(t)$ of the account at some time $t$ to equal \$1500. Therefore, we must solve the equation

$$1500 = 1000e^{0.06t}.$$

Following the steps in the previous example,

$$\begin{aligned} & 1500 = 1000e^{0.06t} & & \\ \implies & 1.5 = e^{0.06t} & & \text{isolate the exponential} \\ \implies & \ln(1.5) = \ln(e^{0.06t}) & & \text{apply the natural log function} \\ \implies & \ln(1.5) = 0.06t & & \text{since } \ln(e^x) = x \\ \implies & \frac{\ln(1.5)}{0.06} = t & & \text{divide} \\ \implies & t \approx 6.757751802. & & \end{aligned}$$

Thus, it would take about 6 years and 9 months.

◇

▶ **Example 7.** *If you deposit \$1000 in an account paying 5% interest compounded monthly, how long will it take for your money to double?*

First, recall the discrete compound interest formula from Section 8.3:

$$P(t) = P_0 \left(1 + \frac{r}{n}\right)^{nt} \tag{8}$$

In this case, $P_0 = 1000$, $r = .05$, and $n = 12$. Inserting these values into the formula, we obtain

$$P(t) = 1000\left(1 + \frac{.05}{12}\right)^{12t}.$$

Now we want the future value $P(t)$ of the account at some time $t$ to equal twice the initial amount. In other words, we want $P(t)$ to equal 2000. Therefore, we must solve the equation

$$2000 = 1000\left(1 + \frac{.05}{12}\right)^{12t}.$$

Following the steps in Examples **2** and **3**,

$$\begin{aligned}
& 2000 = 1000\left(1+\frac{.05}{12}\right)^{12t} && \\
\Longrightarrow\ & 2 = \left(1+\frac{.05}{12}\right)^{12t} && \text{isolate the exponential} \\
\Longrightarrow\ & \log(2) = \log\left(\left(1+\frac{.05}{12}\right)^{12t}\right) && \text{apply the common log function} \\
\Longrightarrow\ & \log(2) = 12t\log\left(1+\frac{.05}{12}\right) && \text{use property (c)} \\
\Longrightarrow\ & \frac{\log(2)}{12\log\left(1+\frac{.05}{12}\right)} = t && \text{divide} \\
\Longrightarrow\ & t \approx 13.89180473. &&
\end{aligned}$$

Thus, it would take about 13.9 years for your money to double.

◇

## 8.6 Exercises

In **Exercises 1-10**, use a calculator to evaluate the function at the given value $p$. Round your answer to the nearest hundredth.

**1.** $f(x) = \log_4(x)$; $p = 57.60$.

**2.** $f(x) = \log_4(x)$; $p = 11.22$.

**3.** $f(x) = \log_7(x)$; $p = 2.98$.

**4.** $f(x) = \log_3(x)$; $p = 2.27$.

**5.** $f(x) = \log_6(x)$; $p = 2.56$.

**6.** $f(x) = \log_8(x)$; $p = 289.27$.

**7.** $f(x) = \log_8(x)$; $p = 302.67$.

**8.** $f(x) = \log_5(x)$; $p = 15.70$.

**9.** $f(x) = \log_8(x)$; $p = 46.13$.

**10.** $f(x) = \log_4(x)$; $p = 15.59$.

In **Exercises 11-18**, perform each of the following tasks.

a) Approximate the solution of the given equation using your graphing calculator. Load each side of the equation into the `Y=` menu of your calculator. Adjust the `WINDOW` parameters so that the point of intersection of the graphs is visible in the viewing window. Use the `intersect` utility in the `CALC` menu of your calculator to determine the x-coordinate of the point of intersection. Then make an accurate copy of the image in your viewing window on your homework paper.

b) Solve the given equation algebraically, and round your answer to the nearest hundredth.

**11.** $20 = 3(1.2)^x$

**12.** $15 = 2(1.8)^x$

**13.** $14 = 1.4^{5x}$

**14.** $16 = 1.8^{4x}$

**15.** $-4 = 0.2^x - 9$

**16.** $12 = 2.9^x + 2$

**17.** $13 = 0.1^{x+1}$

**18.** $19 = 1.2^{x-6}$

In **Exercises 19-34**, solve the given equation algebraically, and round your answer to the nearest hundredth.

**19.** $20 = e^{x-3}$

**20.** $-4 = e^x - 9$

**21.** $23 = 0.9^x + 9$

**22.** $10 = e^x + 7$

**23.** $19 = e^x + 5$

**24.** $4 = 7(2.3)^x$

**25.** $18 = e^{x+4}$

**26.** $15 = e^{x+6}$

**27.** $8 = 2.7^{3x}$

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**28.** $7 = e^{x+1}$

**29.** $7 = 1.1^{8x}$

**30.** $6 = 0.2^{x-8}$

**31.** $-7 = 1.3^x - 9$

**32.** $11 = 3(0.7)^x$

**33.** $23 = e^x + 9$

**34.** $20 = 3.2^{x+1}$

---

**35.** Suppose that you invest \$17,000 at 6% interest compounded daily. How many years will it take for your investment to double? Round your answer to the nearest hundredth.

**36.** Suppose that you invest \$6,000 at 9% interest compounded continuously. How many years will it take for your investment to double? Round your answer to the nearest hundredth.

**37.** Suppose that you invest \$16,000 at 6% interest compounded daily. How many years will it take for your investment to reach \$26,000? Round your answer to the nearest hundredth.

**38.** Suppose that you invest \$15,000 at 5% interest compounded monthly. How many years will it take for your investment to double? Round your answer to the nearest hundredth.

**39.** Suppose that you invest \$18,000 at 3% interest compounded monthly. How many years will it take for your investment to double? Round your answer to the nearest hundredth.

**40.** Suppose that you invest \$7,000 at 5% interest compounded daily. How many years will it take for your investment to reach \$13,000? Round your answer to the nearest hundredth.

**41.** Suppose that you invest \$16,000 at 9% interest compounded continuously. How many years will it take for your investment to double? Round your answer to the nearest hundredth.

**42.** Suppose that you invest \$16,000 at 2% interest compounded continuously. How many years will it take for your investment to reach \$25,000? Round your answer to the nearest hundredth.

**43.** Suppose that you invest \$2,000 at 5% interest compounded continuously. How many years will it take for your investment to reach \$10,000? Round your answer to the nearest hundredth.

**44.** Suppose that you invest \$4,000 at 6% interest compounded continuously. How many years will it take for your investment to reach \$10,000? Round your answer to the nearest hundredth.

**45.** Suppose that you invest \$4,000 at 3% interest compounded daily. How many years will it take for your investment to reach \$14,000? Round your answer to the nearest hundredth.

**46.** Suppose that you invest \$13,000 at 2% interest compounded monthly. How many years will it take for your investment to reach \$20,000? Round your answer to the nearest hundredth.

**47.** Suppose that you invest \$20,000 at 7% interest compounded continuously. How many years will it take for your investment to reach \$30,000? Round your answer to the nearest hundredth.

**48.** Suppose that you invest \$16,000 at 4% interest compounded continuously. How many years will it take for your investment to double? Round your answer to the nearest hundredth.

**49.** Suppose that you invest \$8,000 at 8% interest compounded continuously. How many years will it take for your investment to double? Round your answer to the nearest hundredth.

**50.** Suppose that you invest \$3,000 at 3% interest compounded daily. How many years will it take for your investment to double? Round your answer to the nearest hundredth.

## 8.6 *Answers*

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**1.** 2.92

**3.** 0.56

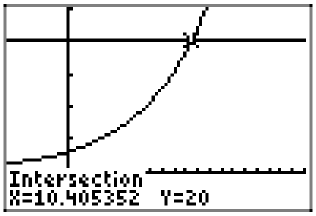
**5.** 0.52

**7.** 2.75

**9.** 1.84

**11.**

**a)**

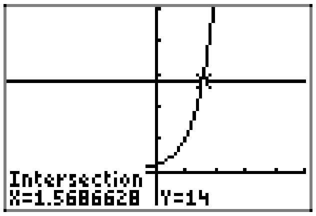


```
Intersection
X=10.405352  Y=20
```

**b)** 10.41

**13.**

**a)**

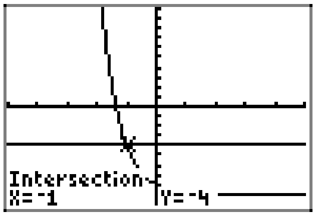


```
Intersection
X=1.5686628  Y=14
```

**b)** 1.57

**15.**

**a)**

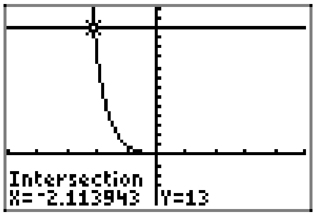


```
Intersection
X=-1         Y=-4
```

**b)** −1.00

**17.**

**a)**



```
Intersection
X=-2.113943  Y=13
```

**b)** −2.11

**19.** 6.00

**21.** −25.05

**23.** 2.64

**25.** −1.11

**27.** 0.70

**29.** 2.55

**31.** 2.64

**33.** 2.64

**35.** 11.55 years

**37.** 8.09 years

**39.** 23.13 years

**41.** 7.70 years

**43.** 32.19 years

**45.** 41.76 years

**47.** 5.79 years

**49.** 8.66 years

# 8.7 Exponential Growth and Decay

## Exponential Growth Models

Recalling the investigations in Section 8.3, we started by developing a formula for discrete compound interest. This led to another formula for continuous compound interest,

$$P(t) = P_0 e^{rt}, \tag{1}$$

where $P_0$ is the initial amount (principal) and $r$ is the annual interest rate in decimal form. If money in a bank account grows at an annual rate $r$ (via payment of interest), and if the growth is continually added in to the account (i.e., interest is continuously compounded), then the balance in the account at time $t$ years is $P(t)$, as given by formula (1).

But we can use the exact same analysis for quantities other than money. If $P(t)$ represents the amount of some quantity at time $t$ years, and if $P(t)$ grows at an annual rate $r$ with the growth continually added in, then we can conclude in the same manner that $P(t)$ must have the form

$$P(t) = P_0 e^{rt}, \tag{2}$$

where $P_0$ is the initial amount at time $t = 0$, namely $P(0)$.

A classic example is *uninhibited population growth*. If a population $P(t)$ of a certain species is placed in a good environment, with plenty of nutrients and space to grow, then it will grow according to formula (2). For example, the size of a bacterial culture in a petri dish will follow this formula very closely if it is provided with optimal living conditions. Many other species of animals and plants will also exhibit this behavior if placed in an environment in which they have no predators. For example, when the British imported rabbits into Australia in the late 18th century for hunting, the rabbit population exploded because conditions were good for living and reproducing, and there were no natural predators of the rabbits.

**Exponential Growth**

If a function $P(t)$ grows continually at a rate $r > 0$, then $P(t)$ has the form

$$P(t) = P_0 e^{rt}, \tag{3}$$

where $P_0$ is the initial amount $P(0)$. In this case, the quantity $P(t)$ is said to exhibit *exponential growth*, and $r$ is the *growth rate*.

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**Remarks 4.**

1. If a physical quantity (such as population) grows according to formula (**3**), we say that the quantity is *modeled* by the exponential growth function $P(t)$.
2. Some may argue that population growth of rabbits, or even bacteria, is not really continuous. After all, rabbits are born one at a time, so the population actually grows in discrete chunks. This is certainly true, but if the population is large, then the growth will appear to be continuous. For example, consider the world population of humans. There are so many people in the world that there are many new births and deaths each second. Thus, the time difference between each 1 unit change in the population is just a tiny fraction of a second, and consequently the discrete growth will act virtually the same as continuous growth. (This is analogous to the almost identical results for continuous compounding and discrete daily compounding that we found in Section 8.3; compounding each second or millisecond would be even closer.)
3. Likewise, using the continuous exponential growth formula (**3**) to model discrete quantities will sometimes result in fractional answers. In this case, the results will need to be rounded off in order to make sense. For example, an answer of 224.57 rabbits is not actually possible, so the answer should be rounded to 225.
4. In formula (**3**), if time is measured in years (as we have done so far in this chapter), then $r$ is the annual growth rate. However, time can instead be measured in any convenient units. The same formula applies, except that the growth rate $r$ is given in terms of the particular time units used. For example, if time $t$ is measured in hours, then $r$ is the hourly growth rate.

In Section 8.2, we showed that a function of the form $b^t$ with $b > 1$ is an exponential growth function. Likewise, if $A > 0$, then the more general exponential function $Ab^t$ also exhibits exponential growth, since the graph of $Ab^t$ is just a vertical scaling of the graph of $b^t$. However, the exponential growth function in formula (**3**) appears to be different. We will show below that the function $P_0e^{rt}$ can in fact be written in the form $Ab^t$ with $b > 1$.

Let's first look at a specific example. Suppose $P(t) = 4e^{0.8t}$. Using the Laws of Exponents, we can rewrite $P(t)$ as

$$P(t) = 4e^{0.8t} = 4(e^{0.8})^t. \tag{5}$$

Since $e^{0.8} \approx 2.22554$, it follows that

$$P(t) \approx 4(2.22554)^t.$$

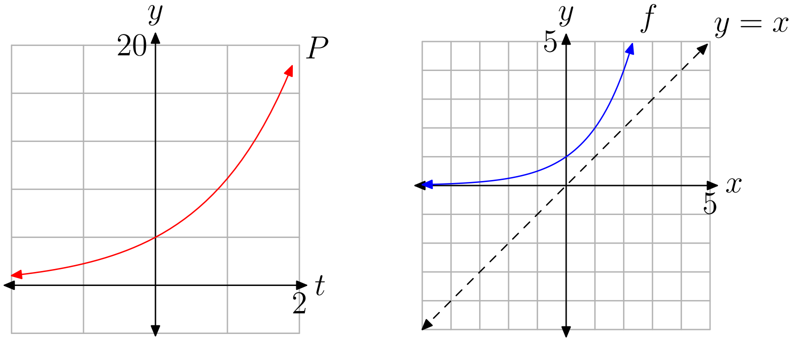
Because the base $\approx 2.22554$ is larger than 1, this shows that $P(t)$ is an exponential growth function, as seen in **Figure 1**(a)).

Now suppose that $P(t)$ is any function of the form $P_0e^{rt}$ with $r > 0$. As in (**5**) above, we can use the Laws of Exponents to rewrite $P(t)$ as

$$P(t) = P_0e^{rt} = P_0(e^r)^t = P_0b^t \quad \text{with} \quad b = e^r.$$

To prove that $b > 1$, consider the graph of $y = e^x$ shown in **Figure 1**(b). Recall that $e \approx 2.718$, so $e > 1$, and therefore $y = e^x$ is itself an exponential growth curve. Also, the $y$-intercept is $(0, 1)$ since $e^0 = 1$. It follows that $b = e^r > 1$ since $r > 0$ (see **Figure 1**(b)).

Therefore, functions of the form $P(t) = P_0 e^{rt}$ with $r > 0$ are exponential growth functions.



(a) $P(t) = 4e^{0.8t}$

(b) $b = e^r > 1$ since $r > 0$

**Figure 1.**

## Applications of Exponential Growth

We will now examine the role of exponential growth functions in some real-world applications. In the following examples, assume that the population is modeled by an exponential growth function as in formula (**3**).

▶ **Example 6.** *Suppose that the population of a certain country grows at an annual rate of 2%. If the current population is 3 million, what will the population be in 10 years?*

This is a future value problem. If we measure population in millions and time in years, then $P(t) = P_0 e^{rt}$ with $P_0 = 3$ and $r = 0.02$. Inserting these particular values into formula (**3**), we obtain

$$P(t) = 3e^{0.02t}.$$

The population in 10 years is $P(10) = 3e^{(0.02)(10)} \approx 3.664208$ million. —◇—

▶ **Example 7.** *In the same country as in* **Example 6**, *how long will it take the population to reach 5 million?*

As before,

$$P(t) = 3e^{0.02t}.$$

Now we want to know when the future value $P(t)$ of the population at some time $t$ will equal 5 million. Therefore, we need to solve the equation $P(t) = 5$ for time $t$, which leads to the exponential equation

$$5 = 3e^{0.02t}.$$

Using the procedure for solving exponential equations that was presented in Section 8.6,

$$\begin{aligned}
& 5 = 3e^{0.02t} & & \\
\Longrightarrow\ & \frac{5}{3} = e^{0.02t} & & \text{isolate the exponential} \\
\Longrightarrow\ & \ln\left(\frac{5}{3}\right) = \ln(e^{0.02t}) & & \text{apply the natural log function} \\
\Longrightarrow\ & \ln\left(\frac{5}{3}\right) = 0.02t & & \text{since } \ln(e^x) = x \\
\Longrightarrow\ & \frac{\ln\left(\frac{5}{3}\right)}{0.02} = t & & \text{division} \\
\Longrightarrow\ & t \approx 25.54128. & &
\end{aligned}$$

Thus, it would take about 25.54 years for the population to reach 5 million.

The population of bacteria is typically measured by weight, as in the next two examples.

▶ **Example 8.** *Suppose that a size of a bacterial culture is given by the function*

$$P(t) = 100e^{0.15t},$$

*where the size $P(t)$ is measured in grams and time $t$ is measured in hours. How long will it take for the culture to double in size?*

The initial size is $P_0 = 100$ grams, so we want to know when the future value $P(t)$ at some time $t$ will equal 200. Therefore, we need to solve the equation $P(t) = 200$ for time $t$, which leads to the exponential equation

$$200 = 100e^{0.15t}.$$

Using the same procedure as in the last example,

$$\begin{aligned}
& 200 = 100e^{0.15t} & & \\
\Longrightarrow\ & 2 = e^{0.15t} & & \text{isolate the exponential} \\
\Longrightarrow\ & \ln(2) = \ln(e^{0.15t}) & & \text{apply the natural log function} \\
\Longrightarrow\ & \ln(2) = 0.15t & & \text{since } \ln(e^x) = x \\
\Longrightarrow\ & \frac{\ln(2)}{0.15} = t & & \text{division} \\
\Longrightarrow\ & t \approx 4.620981. & &
\end{aligned}$$

Thus, it would take about 4.62 hours for the size to double. ◇

The last example deserves an additional comment. Suppose that we had started with 1000 grams instead of 100. Then to double in size would require a future value of 2000 grams. Therefore, in this case, we would have to solve the equation

$$2000 = 1000e^{0.15t}.$$

But the first step is to isolate the exponential by dividing both sides by 1000 to get

$$2 = e^{0.15t},$$

and this is the same as the second line of the solution in the last example, so the answer will be the same. Likewise, repeating this argument for any initial amount will lead to the same second line, and therefore the same answer. Thus, the *doubling time* depends only on $r$, not on the initial amount $P_0$.

## Exponential Decay Models

We've observed that if a quantity increases continually at a rate $r$, then it is modeled by a function of the form $P(t) = P_0e^{rt}$. But what if a quantity *decreases* instead? Although we won't present the details here, the analysis can be carried out in the same way as the derivation of the continuous compounding formula in Section 8.3. The only difference is that the growth rate $r$ in the formulas must be replaced by $-r$ since the quantity is decreasing. The conclusion is that the quantity is modeled by a function of the form $P(t) = P_0e^{-rt}$ instead of $P_0e^{rt}$.

**Exponential Decay**

If a function $P(t)$ decreases continually at a rate $r > 0$, then $P(t)$ has the form

$$P(t) = P_0e^{-rt}, \tag{9}$$

where $P_0$ is the initial amount $P(0)$. In this case, the quantity $P(t)$ is said to exhibit *exponential decay*, and $r$ is the *decay rate*.

In Section 8.2, we showed that a function of the form $b^t$ with $b < 1$ is an exponential decay function. Likewise, if $A > 0$, then the more general exponential function $Ab^t$ also exhibits exponential decay, since the graph of $Ab^t$ is just a vertical scaling of the graph of $b^t$. However, the exponential decay function in formula (9) appears to be different. We will show below that the function $P_0e^{-rt}$ can in fact be written in the form $Ab^t$ with $b < 1$.

Let's first look at a specific example. Suppose $P(t) = 4e^{-0.8t}$. Using the Laws of Exponents, we can rewrite $P(t)$ as

$$P(t) = 4e^{-0.8t} = 4(e^{-0.8})^t. \tag{10}$$

Since $e^{-0.8} \approx 0.44933$, it follows that

$$P(t) \approx 4(0.44933)^t.$$

Because the base $\approx 0.44933$ is less than 1, this shows that $P(t)$ is an exponential decay function, as seen in **Figure 2**(a)).

Now suppose that $P(t)$ is any function of the form $P_0e^{-rt}$ with $r > 0$. As in (**10**) above, we can use the Laws of Exponents to rewrite $P(t)$ as

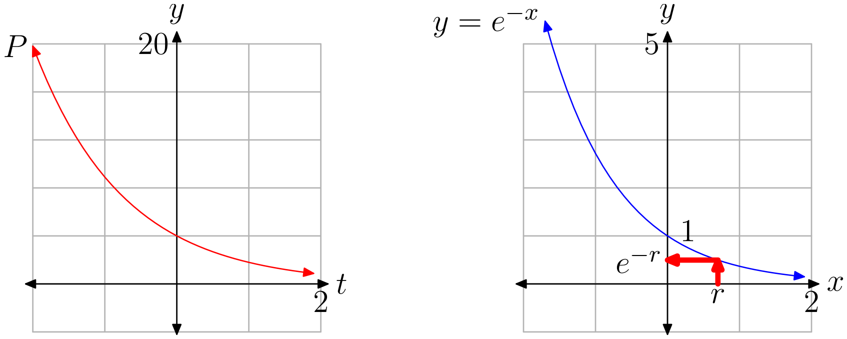
$$P(t) = P_0e^{-rt} = P_0(e^{-r})^t = P_0b^t \quad \text{with} \quad b = e^{-r}.$$

To prove that $b < 1$, consider the graph of $y = e^{-x}$ shown in **Figure 2**(b). Now

$$e^{-x} = (e^{-1})^x = \left(\frac{1}{e}\right)^x$$

and $1/e \approx 0.36788 < 1$, so $y = e^{-x}$ is itself an exponential decay curve. (Alternatively, you can observe that the graph of $y = e^{-x}$ is the reflection of the graph of $y = e^x$ across the $y$-axis.) Also, the $y$-intercept is $(0, 1)$ since $e^{-0} = 1$. It follows that $b = e^{-r} < 1$ since $r > 0$ (see **Figure 2**(b)).

Therefore, functions of the form $P(t) = P_0e^{-rt}$ with $r > 0$ are exponential decay functions.



(a) $P(t) = 4e^{-0.8t}$ (b) $b = e^{-r} < 1$ since $r > 0$

**Figure 2.**

## *Applications of Exponential Decay*

The main example of exponential decay is *radioactive decay*. Radioactive elements and isotopes spontaneously emit subatomic particles, and this process gradually changes the substance into a different isotope. For example, the radioactive isotope Uranium-238 eventually decays into the stable isotope Lead-206. This is a random process for individual atoms, but overall the mass of the substance decreases according to the exponential decay formula (**9**).

▶ **Example 11.** *Suppose that a certain radioactive element has an annual decay rate of 10%. Starting with a 200 gram sample of the element, how many grams will be left in 3 years?*

This is a future value problem. If we measuring size in grams and time in years, then $P(t) = P_0 e^{-rt}$ with $P_0 = 200$ and $r = 0.10$. Inserting these particular values into formula (**9**), we obtain

$$P(t) = 200e^{-0.10t}.$$

The amount in 3 years is $P(3) = 200e^{-(0.10)(3)} \approx 148.1636$ grams.

◆

▶ **Example 12.** *Using the same element as in* **Example 11**, *if a particular sample of the element decays to 50 grams after 5 years, how big was the original sample?*

This is a present value problem, where the unknown is the initial amount $P_0$. As before, $r = 0.10$, so

$$P(t) = P_0 e^{-0.10t}.$$

Since $P(5) = 50$, we have the equation

$$50 = P(5) = P_0 e^{-(0.10)(5)}.$$

This equation can be solved by division:

$$\frac{50}{e^{-(0.10)(5)}} = P_0$$

Finish by calculating the value of the left side to get $P_0 \approx 82.43606$ grams.

◆

▶ **Example 13.** *Suppose that a certain radioactive isotope has an annual decay rate of 5%. How many years will it take for a 100 gram sample to decay to 40 grams?*

Use $P(t) = P_0 e^{-rt}$ with $P_0 = 100$ and $r = 0.05$, so

$$P(t) = 100e^{-0.05t}.$$

Now we want to know when the future value $P(t)$ of the size of the sample at some time $t$ will equal 40. Therefore, we need to solve the equation $P(t) = 40$ for time $t$, which leads to the exponential equation

$$40 = 100e^{-0.05t}.$$

Using the procedure for solving exponential equations that was presented in Section 8.6,

$$\begin{aligned}
& 40 = 100e^{-0.05t} && \\
\Longrightarrow\ & 0.4 = e^{-0.05t} && \text{isolate the exponential} \\
\Longrightarrow\ & \ln(0.4) = \ln(e^{-0.05t}) && \text{apply the natural log function} \\
\Longrightarrow\ & \ln(0.4) = -0.05t && \text{since } \ln(e^x) = x \\
\Longrightarrow\ & \frac{\ln(0.4)}{-0.05} = t && \text{division} \\
\Longrightarrow\ & t \approx 18.32581. &&
\end{aligned}$$

Thus, it would take approximately 18.33 years for the sample to decay to 40 grams.

We saw earlier that exponential growth processes have a fixed doubling time. Similarly, exponential decay processes have a fixed *half-life*, the time in which one-half the original amount decays.

▶ **Example 14.** *Using the same element as in* **Example 13**, *what is the half-life of the element?*

As before, $r = 0.05$, so

$$P(t) = P_0 e^{-0.05t}.$$

The initial size is $P_0$ grams, so we want to know when the future value $P(t)$ at some time $t$ will equal one-half the initial amount, $P_0/2$. Therefore, we need to solve the equation $P(t) = P_0/2$ for time $t$, which leads to the exponential equation

$$\frac{P_0}{2} = P_0 e^{-0.05t}.$$

Using the same procedure as in the last example,

$$\begin{aligned}
& \frac{P_0}{2} = P_0 e^{-0.05t} & & \\
\Longrightarrow\ & \frac{1}{2} = e^{-0.05t} & & \text{isolate the exponential} \\
\Longrightarrow\ & \ln\left(\frac{1}{2}\right) = \ln(e^{-0.05t}) & & \text{apply the natural log function} \\
\Longrightarrow\ & \ln\left(\frac{1}{2}\right) = -0.05t & & \text{since } \ln(e^x) = x \\
\Longrightarrow\ & \frac{\ln\left(\frac{1}{2}\right)}{-0.05} = t & & \text{division} \\
\Longrightarrow\ & t \approx 13.86294. & &
\end{aligned}$$

Thus, the half-life is approximately 13.86 years.

The process of radioactive decay also forms the basis of the carbon-14 dating technique. The Earth's atmosphere contains a tiny amount of the radioactive isotope carbon-14, and therefore plants and animals also contain some carbon-14 due to their interaction with the atmosphere. However, this interaction ends when a plant or animal dies, so the carbon-14 begins to decay (the decay rate is 0.012%). By comparing the amount of carbon-14 in a bone, for example, with the normal amount in a living animal, scientists can compute the age of the bone.

▶ **Example 15.** *Suppose that only 1.5% of the normal amount of carbon-14 remains in a fragment of bone. How old is the bone?*

Use $P(t) = P_0 e^{-rt}$ with $r = 0.00012$, so

$$P(t) = P_0 e^{-0.00012t}.$$

The initial size is $P_0$ grams, so we want to know when the future value $P(t)$ at some time $t$ will equal 1.5% of the initial amount, $0.015P_0$. Therefore, we need to solve the equation $P(t) = 0.015P_0$ for time $t$, which leads to the exponential equation

$$0.015P_0 = P_0 e^{-0.00012t}.$$

Using the same procedure as in **Example 14**,

$$\begin{aligned}
& 0.015P_0 = P_0 e^{-0.00012t} & \\
\Longrightarrow\ & 0.015 = e^{-0.00012t} & \text{isolate the exponential} \\
\Longrightarrow\ & \ln(0.015) = \ln(e^{-0.00012t}) & \text{apply the natural log function} \\
\Longrightarrow\ & \ln(0.015) = -0.00012t & \text{since } \ln(e^x) = x \\
\Longrightarrow\ & \frac{\ln(0.015)}{-0.00012} = t & \text{division} \\
\Longrightarrow\ & t \approx 34998. &
\end{aligned}$$

Thus, the bone is approximately 34998 years old.

◇

While the carbon-14 technique only works on plants and animals, there are other similar dating techniques, using other radioactive isotopes, that are used to date rocks and other inorganic matter.

## 8.7 *Exercises*

**1.** Suppose that the population of a certain town grows at an annual rate of 6%. If the population is currently $5,000$, what will it be in 7 years? Round your answer to the nearest integer.

**2.** Suppose that the population of a certain town grows at an annual rate of 5%. If the population is currently 2,000, how many years will it take for it to double? Round your answer to the nearest hundredth.

**3.** Suppose that a certain radioactive isotope has an annual decay rate of 7.2%. How many years will it take for a 227 gram sample to decay to 93 grams? Round your answer to the nearest hundredth.

**4.** Suppose that a certain radioactive isotope has an annual decay rate of 6.8%. How many years will it take for a 399 gram sample to decay to 157 grams? Round your answer to the nearest hundredth.

**5.** Suppose that the population of a certain town grows at an annual rate of 8%. If the population is currently 4,000, how many years will it take for it to double? Round your answer to the nearest hundredth.

**6.** Suppose that a certain radioactive isotope has an annual decay rate of 19.2%. Starting with a 443 gram sample, how many grams will be left after 9 years? Round your answer to the nearest hundredth.

**7.** Suppose that a certain radioactive isotope has an annual decay rate of 17.4%. What is the half-life (in years) of the isotope? Round your answer to the nearest hundredth.

**8.** Suppose that the population of a certain town grows at an annual rate of 7%. If the population is currently 8,000, how many years will it take for it to reach 18,000? Round your answer to the nearest hundredth.

**9.** Suppose that a certain radioactive isotope has an annual decay rate of 17.3%. Starting with a 214 gram sample, how many grams will be left after 5 years? Round your answer to the nearest hundredth.

**10.** Suppose that the population of a certain town grows at an annual rate of 7%. If the population grows to $2,000$ in 7 years, what was the original population? Round your answer to the nearest integer.

**11.** Suppose that the population of a certain town grows at an annual rate of 3%. If the population is currently 3,000, how many years will it take for it to double? Round your answer to the nearest hundredth.

**12.** Suppose that a certain radioactive isotope has an annual decay rate of 12.5%. Starting with a 127 gram sample, how many grams will be left after 6 years? Round your answer to the nearest hundredth.

**13.** Suppose that a certain radioactive isotope has an annual decay rate of 13.1%.

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Starting with a 353 gram sample, how many grams will be left after 7 years? Round your answer to the nearest hundredth.

**14.** Suppose that the population of a certain town grows at an annual rate of 2%. If the population grows to $9,000$ in 4 years, what was the original population? Round your answer to the nearest integer.

**15.** Suppose that the population of a certain town grows at an annual rate of 2%. If the population is currently 7,000, how many years will it take for it to double? Round your answer to the nearest hundredth.

**16.** Suppose that a certain radioactive isotope has an annual decay rate of 5.3%. How many years will it take for a 217 gram sample to decay to 84 grams? Round your answer to the nearest hundredth.

**17.** Suppose that a certain radioactive isotope has an annual decay rate of 18.7%. How many years will it take for a 324 gram sample to decay to 163 grams? Round your answer to the nearest hundredth.

**18.** Suppose that the population of a certain town grows at an annual rate of 8%. If the population is currently 8,000, how many years will it take for it to reach 18,000? Round your answer to the nearest hundredth.

**19.** Suppose that a certain radioactive isotope has an annual decay rate of 2.3%. If a particular sample decays to 25 grams after 8 years, how big (in grams) was the original sample? Round your answer to the nearest hundredth.

**20.** Suppose that the population of a certain town grows at an annual rate of 4%. If the population is currently 7,000, how many years will it take for it to reach 17,000? Round your answer to the nearest hundredth.

**21.** Suppose that a certain radioactive isotope has an annual decay rate of 9.8%. If a particular sample decays to 11 grams after 6 years, how big (in grams) was the original sample? Round your answer to the nearest hundredth.

**22.** Suppose that the population of a certain town grows at an annual rate of 5%. If the population grows to $6,000$ in 3 years, what was the original population? Round your answer to the nearest integer.

**23.** Suppose that the population of a certain town grows at an annual rate of 8%. If the population is currently $6,000$, what will it be in 5 years? Round your answer to the nearest integer.

**24.** Suppose that a certain radioactive isotope has an annual decay rate of 15.8%. What is the half-life (in years) of the isotope? Round your answer to the nearest hundredth.

**25.** Suppose that the population of a certain town grows at an annual rate of 9%. If the population grows to $7,000$ in 5 years, what was the original population? Round your answer to the nearest integer.

**26.** Suppose that a certain radioactive isotope has an annual decay rate of 18.6%. If a particular sample decays to 41 grams after 3 years, how big (in grams) was the original sample? Round your answer to the nearest hundredth.

**27.** Suppose that a certain radioactive isotope has an annual decay rate of 5.2%. What is the half-life (in years) of the isotope? Round your answer to the nearest hundredth.

**28.** Suppose that a certain radioactive isotope has an annual decay rate of 6.5%. What is the half-life (in years) of the isotope? Round your answer to the nearest hundredth.

**29.** Suppose that the population of a certain town grows at an annual rate of 8%. If the population is currently 2,000, how many years will it take for it to reach 7,000? Round your answer to the nearest hundredth.

**30.** Suppose that a certain radioactive isotope has an annual decay rate of 3.7%. If a particular sample decays to 47 grams after 8 years, how big (in grams) was the original sample? Round your answer to the nearest hundredth.

**31.** Suppose that the population of a certain town grows at an annual rate of 6%. If the population is currently $7,000$, what will it be in 7 years? Round your answer to the nearest integer.

**32.** Suppose that the population of a certain town grows at an annual rate of 4%. If the population is currently $1,000$, what will it be in 3 years? Round your answer to the nearest integer.

In **Exercises 33-40**, use the fact that the decay rate of carbon-14 is 0.012%. Round your answer to the nearest year.

**33.** Suppose that only 8.6% of the normal amount of carbon-14 remains in a fragment of bone. How old is the bone?

**34.** Suppose that only 5.2% of the normal amount of carbon-14 remains in a fragment of bone. How old is the bone?

**35.** Suppose that 90.1% of the normal amount of carbon-14 remains in a piece of wood. How old is the wood?

**36.** Suppose that 83.6% of the normal amount of carbon-14 remains in a piece of cloth. How old is the cloth?

**37.** Suppose that only 6.2% of the normal amount of carbon-14 remains in a fragment of bone. How old is the bone?

**38.** Suppose that only 1.3% of the normal amount of carbon-14 remains in a fragment of bone. How old is the bone?

**39.** Suppose that 96.7% of the normal amount of carbon-14 remains in a piece of cloth. How old is the cloth?

**40.** Suppose that 84.9% of the normal amount of carbon-14 remains in a piece of wood. How old is the wood?

## 8.7 Answers

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**1.** 7610 people

**3.** 12.39 yrs

**5.** 8.66 yrs

**7.** 3.98 yrs

**9.** 90.11g

**11.** 23.10 yrs

**13.** 141.10g

**15.** 34.66 yrs

**17.** 3.67 yrs

**19.** 30.05g

**21.** 19.80g

**23.** $8,951$ people

**25.** $4,463$ people

**27.** 13.33 yrs

**29.** 15.66 yrs

**31.** $10,654$ people

**33.** 20445 years

**35.** 869 years

**37.** 23172 years

**39.** 280 years

# 8.8 Additional Topics

## *Computing Large Powers*

Logarithms were originally used to compute large products and powers. Prior to the age of calculators and computers, mathematics students spent many hours learning and practicing these procedures. In current times, most of these computations can be done easily on a calculator, so the original use of logarithms is usually not taught anymore.

However, calculators are still limited. They cannot compute large powers such as $253^{789}$ (try it!), and most computer programs can't either (all such tools have a limit on the size of the computations they can perform).

So how can we compute large powers such as these? The idea is to use our knowledge of the properties of logarithmic and exponential functions. Here is the procedure:

1. First, let $y = 253^{789}$, and take the log of both sides:

$$\begin{aligned} \log(y) &= \log(253^{789}) \\ &= 789\log(253) && \text{(property of logs)} \\ &\approx 1896.062091 && \text{(calculator approximation)} \end{aligned}$$

2. Now the idea is to exponentiate both sides, using the function $10^x$. However, your calculator still cannot compute $10^{1896.062091}$ (try it). So now we separate out the integer part, and our final answer will be in scientific notation:

$$\begin{aligned} y = 10^{\log(y)} &= 10^{1896.062091} = 10^{1896+0.062091} = 10^{1896} \cdot 10^{0.062091} \\ &\approx 10^{1896} \cdot 1.153694972 \qquad \text{(calculator approximation)} \end{aligned}$$

Thus, the final answer is approximately $1.153695 \cdot 10^{1896}$.

Here is one additional example:

▶ **Example 1.** *Compute the value* $2^{400}$*, and express your answer in scientific notation.*

1. Let $y = 2^{400}$, and take the log of both sides:

$$\begin{aligned} \log(y) &= \log(2^{400}) \\ &= 400\log(2) && \text{(property of logs)} \\ &\approx 120.4119983 && \text{(calculator approximation)} \end{aligned}$$

2. Exponentiate both sides, using the function $10^x$ and separating out the integer part of the exponent:

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$$\begin{aligned} y &= 10^{\log(y)} = 10^{120.4119983} = 10^{120+4119983} = 10^{120} \cdot 10^{0.4119983} \\ &\approx 10^{120} \cdot 2.582250083 \qquad \text{(calculator approximation)} \end{aligned}$$

The final answer is approximately $2.582250 \cdot 10^{120}$.

◆

# 8.8 Exercises

In **Exercises 1-10**, compute the value of the expression. Express your answer in scientific notation $c \cdot 10^n$.

**1.** $131^{808}$

**2.** $132^{759}$

**3.** $148^{524}$

**4.** $143^{697}$

**5.** $187^{642}$

**6.** $198^{693}$

**7.** $162^{803}$

**8.** $142^{569}$

**9.** $134^{550}$

**10.** $153^{827}$

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## 8.8 Answers

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**1.** $5.691 \cdot 10^{1710}$

**3.** $1.649 \cdot 10^{1137}$

**5.** $3.329 \cdot 10^{1458}$

**7.** $1.740 \cdot 10^{1774}$

**9.** $8.084 \cdot 10^{1169}$

## 8.9 Index

# 9 Radical Functions

In this chapter, we will study *radical functions* — in other words, functions that involve square, cubic, and other roots of algebraic expressions (for example, $\sqrt{x}$ or $\sqrt[3]{x+2}$). There are a number of subtleties and tricks to these functions, and it is important to learn how to manipulate them.

Radical functions are closely related to *power functions* (for example, $x^2$ or $(2-x)^5$). In fact, the graph of $\sqrt{x}$ is exactly what you would see if you reflected the graph of $x^2$ across the line $y = x$ and erased everything below the $x$-axis! It turns out that $\sqrt{x}$ is so closely related to $x^2$ that we say that those functions are *inverses* of each other; whatever one does, the other undoes.

Radical functions have many interesting applications, are studied extensively in many mathematics courses, and are used often in science and engineering. If you have ever wanted to calculate the shortest distance between two places, or predict how long a stairway is based upon the height it reaches, radical functions can help you with these calculations.

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# 9.1 The Square Root Function

In this section we turn our attention to the square root function, the function defined by the equation

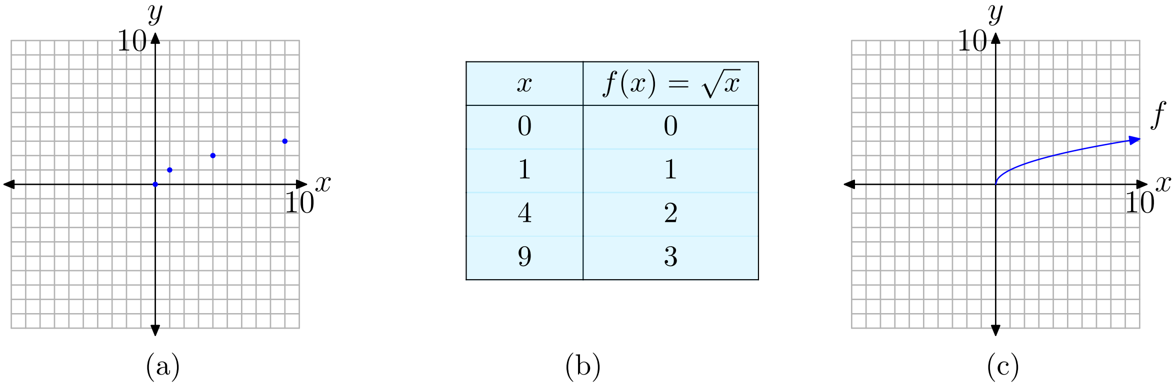
$$f(x) = \sqrt{x}. \tag{1}$$

We begin the section by drawing the graph of the function, then we address the domain and range. After that, we'll investigate a number of different transformations of the function.

## *The Graph of the Square Root Function*

Let's create a table of points that satisfy the equation of the function, then plot the points from the table on a Cartesian coordinate system on graph paper. We'll continue creating and plotting points until we are convinced of the eventual shape of the graph.

We know we cannot take the square root of a negative number. Therefore, we don't want to put any negative $x$-values in our table. To further simplify our computations, let's use numbers whose square root is easily calculated. This brings to mind perfect squares such as 0, 1, 4, 9, and so on. We've placed these numbers as $x$-values in the table in **Figure 1**(b), then calculated the square root of each. In **Figure 1**(a), you see each of the points from the table plotted as a solid dot. If we continue to add points to the table, plot them, the graph will eventually fill in and take the shape of the solid curve shown in **Figure 1**(c).



| $x$ | $f(x) = \sqrt{x}$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 4 | 2 |
| 9 | 3 |

(a) (b) (c)

**Figure 1.** Creating the graph of $f(x) = \sqrt{x}$.

The point plotting approach used to draw the graph of $f(x) = \sqrt{x}$ in **Figure 1** is a tested and familiar procedure. However, a more sophisticated approach involves the theory of inverses developed in the previous chapter.

In a sense, taking the square root is the "inverse" of squaring. Well, not quite, as the squaring function $f(x) = x^2$ in **Figure 2**(a) fails the horizontal line test and is not one-to-one. However, if we limit the domain of the squaring function, then the graph of $f(x) = x^2$ in **Figure 2**(b), where $x \geq 0$, does pass the horizontal line test and is

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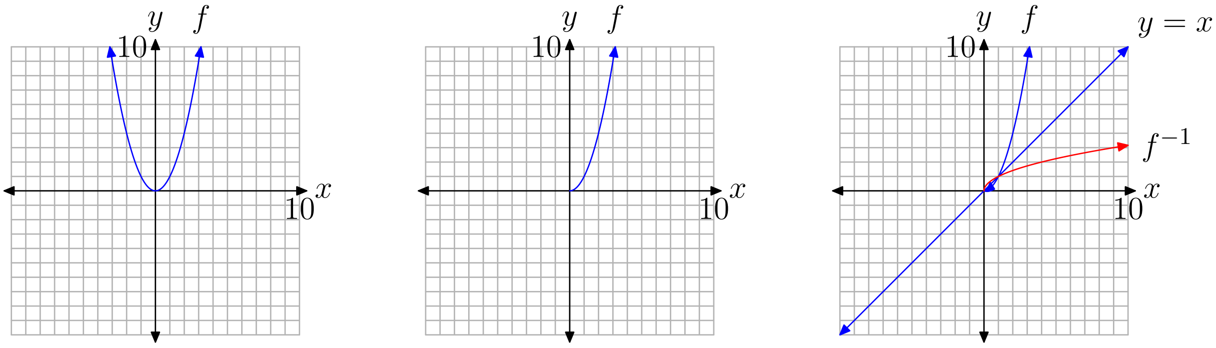
one-to-one. Therefore, the graph of $f(x) = x^2$, $x \geq 0$, has an inverse, and the graph of its inverse is found by reflecting the graph of $f(x) = x^2$, $x \geq 0$, across the line $y = x$ (see **Figure 2**(c)).



(a) $f(x) = x^2$.

(b) $f(x) = x^2$, $x \geq 0$.

(c) Reflecting the graph in (b) across the line $y = x$ produces the graph of $f^{-1}(x) = \sqrt{x}$.

**Figure 2.** Sketching the inverse of $f(x) = x^2$, $x \geq 0$.

To find the equation of the inverse, recall that the procedure requires that we switch the roles of $x$ and $y$, then solve the resulting equation for $y$. Thus, first write $f(x) = x^2$, $x \geq 0$, in the form

$$y = x^2, \quad x \geq 0.$$

Next, switch $x$ and $y$.

$$x = y^2, \quad y \geq 0 \tag{2}$$

When we solve this last equation for $y$, we get two solutions,

$$y = \pm\sqrt{x}. \tag{3}$$

However, in **equation (2)**, note that $y$ must be greater than or equal to zero. Hence, we must choose the nonnegative answer in **equation (3)**, so the inverse of $f(x) = x^2$, $x \geq 0$, has equation

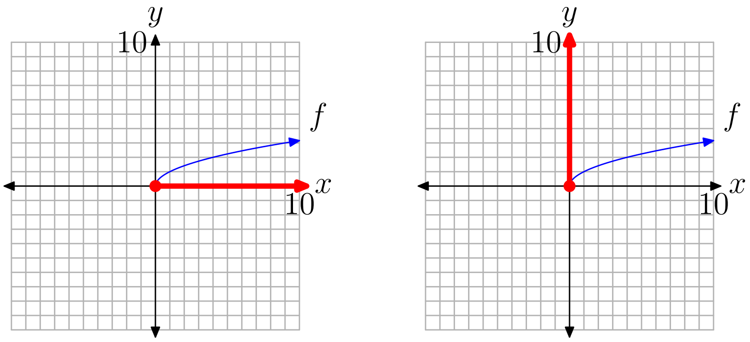
$$f^{-1}(x) = \sqrt{x}.$$

This is the equation of the reflection of the graph of $f(x) = x^2$, $x \geq 0$, that is pictured in **Figure 2**(c). Note the exact agreement with the graph of the square root function in **Figure 1**(c).

The sequence of graphs in **Figure 2** also help us identify the domain and range of the square root function.

- In **Figure 2**(a), the parabola opens outward indefinitely, both left and right. Consequently, the domain is $D_f = (-\infty, \infty)$, or all real numbers. Also, the graph has vertex at the origin and opens upward indefinitely, so the range is $R_f = [0, \infty)$.
- In **Figure 2**(b), we restricted the domain. Thus, the graph of $f(x) = x^2$, $x \geq 0$, now has domain $D_f = [0, \infty)$. The range is unchanged and is $R_f = [0, \infty)$.
- In **Figure 2**(c), we've reflected the graph of $f(x) = x^2$, $x \geq 0$, across the line $y = x$ to obtain the graph of $f^{-1}(x) = \sqrt{x}$. Because we've interchanged the role of $x$ and $y$, the domain of the square root function must equal the range of $f(x) = x^2$, $x \geq 0$. That is, $D_{f^{-1}} = [0, \infty)$. Similarly, the range of the square root function must equal the domain of $f(x) = x^2$, $x \geq 0$. Hence, $R_{f^{-1}} = [0, \infty)$.

Of course, we can also determine the domain and range of the square root function by projecting all points on the graph onto the $x$- and $y$-axes, as shown in **Figures 3**(a) and (b), respectively.



(a) Domain = $[0, \infty)$ (b) Range = $[0, \infty)$

**Figure 3.** Project onto the axes to find the domain and range.

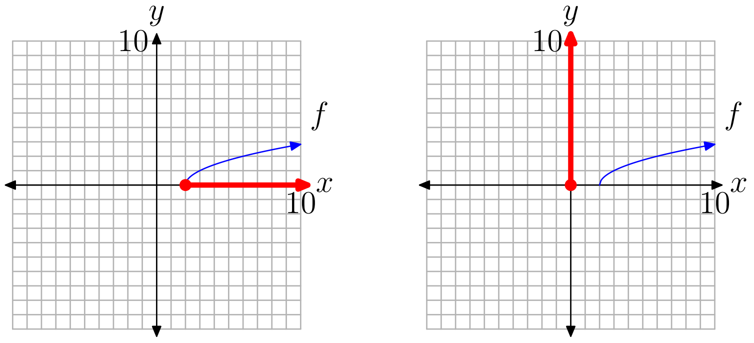
Some might object to the range, asking "How do we know that the graph of the square root function picture in **Figure 3**(b) rises indefinitely?" Again, the answer lies in the sequence of graphs in **Figure 2**. In **Figure 2**(c), note that the graph of $f(x) = x^2$, $x \geq 0$, opens indefinitely to the right as the graph rises to infinity. Hence, after reflecting this graph across the line $y = x$, the resulting graph must rise upward indefinitely as it moves to the right. Thus, the range of the square root function is $[0, \infty)$.

## *Translations*

If we shift the graph of $y = \sqrt{x}$ right and left, or up and down, the domain and/or range are affected.

▶ **Example 4.** *Sketch the graph of $f(x) = \sqrt{x - 2}$. Use your graph to determine the domain and range.*

We know that the basic equation $y = \sqrt{x}$ has the graph shown in **Figure 1**(c). If we replace $x$ with $x - 2$, the basic equation $y = \sqrt{x}$ becomes $y = \sqrt{x - 2}$. From our previous work with geometric transformations, we know that this will shift the graph two units to the right, as shown in **Figures 4**(a) and (b).



(a) Domain $= [2, \infty)$ (b) Range $= [0, \infty)$

**Figure 4.** To draw the graph of $f(x) = \sqrt{x-2}$, shift the graph of $y = \sqrt{x}$ two units to the right.

To find the domain, we project each point on the graph of $f$ onto the $x$-axis, as shown in **Figure 4**(a). Note that all points to the right of or including 2 are shaded on the $x$-axis. Consequently, the domain of $f$ is

$$\text{Domain} = [2, \infty) = \{x : x \geq 2\}.$$

As there has been no shift in the vertical direction, the range remains the same. To find the range, we project each point on the graph onto the $y$-axis, as shown in **Figure 4**(b). Note that all points at and above zero are shaded on the $y$-axis. Thus, the range of $f$ is

$$\text{Range} = [0, \infty) = \{y : y \geq 0\}.$$

We can find the domain of this function algebraically by examining its defining equation $f(x) = \sqrt{x-2}$. We understand that we cannot take the square root of a negative number. Therefore, the expression under the radical must be nonnegative (positive or zero). That is,

$$x - 2 \geq 0.$$

Solving this inequality for $x$,

$$x \geq 2.$$

Thus, the domain of $f$ is Domain $= [2, \infty)$, which matches the graphical solution above.

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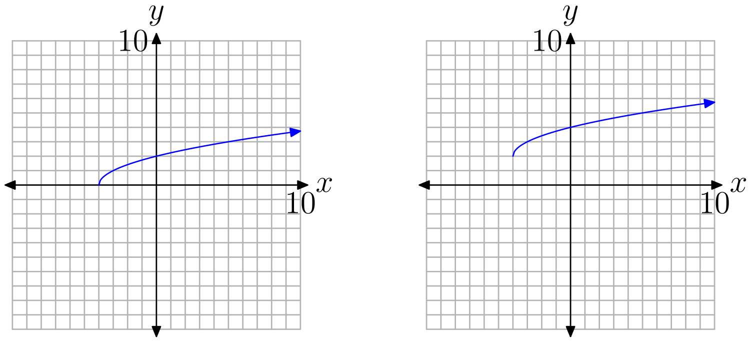
Let's look at another example.

▶ **Example 5.** *Sketch the graph of* $f(x) = \sqrt{x+4} + 2$. *Use your graph to determine the domain and range of* $f$.

Again, we know that the basic equation $y = \sqrt{x}$ has the graph shown in **Figure 1**(c). If we replace $x$ with $x + 4$, the basic equation $y = \sqrt{x}$ becomes $y = \sqrt{x+4}$. From our previous work with geometric transformations, we know that this will shift the graph of $y = \sqrt{x}$ four units to the left, as shown in **Figure 5**(a).

If we now add 2 to the equation $y = \sqrt{x+4}$ to produce the equation $y = \sqrt{x+4}+2$, this will shift the graph of $y = \sqrt{x+4}$ two units upward, as shown in **Figure 5**(b).



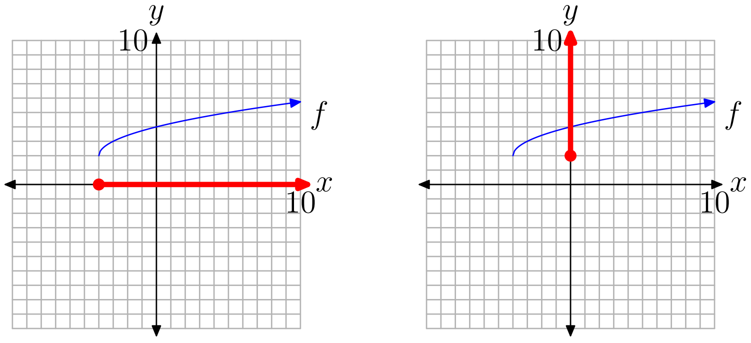
(a) To draw the graph of $y = \sqrt{x+4}$, shift the graph of $y = \sqrt{x}$ four units to the left.

(b) To draw the graph of $y = \sqrt{x+4}+2$, shift the graph of $y = \sqrt{x+4}$ two units upward.

**Figure 5.** Translating the original equation $y = \sqrt{x}$ to get the graph of $y = \sqrt{x+4}+2$.

To identify the domain of $f(x) = \sqrt{x+4}+2$, we project all points on the graph of $f$ onto the $x$-axis, as shown in **Figure 6**(a). Note that all points to the right of or including $-4$ are shaded on the $x$-axis. Thus, the domain of $f(x) = \sqrt{x+4}+2$ is

$$\text{Domain} = [-4, \infty) = \{x : x \geq -4\}.$$



(a) Shading the domain of $f$.

(b) Shading the range of $f$.

**Figure 6.** Project points of $f$ onto the axes to determine the domain and range.

Similarly, to find the range of $f$, project all points on the graph of $f$ onto the $y$-axis, as shown in **Figure 6**(b). Note that all points on the $y$-axis greater than or including 2 are shaded. Consequently, the range of $f$ is

$$\text{Range} = [2, \infty) = \{y : y \geq 2\}.$$

We can also find the domain of $f$ algebraically by examining the equation $f(x) = \sqrt{x+4} + 2$. We cannot take the square root of a negative number, so the expression under the radical must be nonnegative (zero or positive). Consequently,

$$x + 4 \geq 0.$$

Solving this inequality for $x$,

$$x \geq -4.$$

Thus, the domain of $f$ is Domain $= [-4, \infty)$, which matches the graphical solution presented above.

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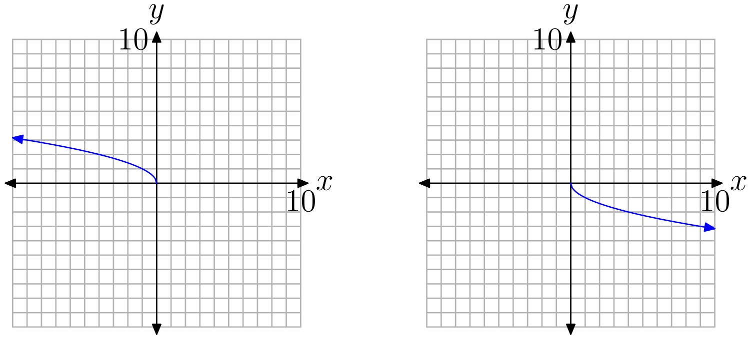
## Reflections

If we start with the basic equation $y = \sqrt{x}$, then replace $x$ with $-x$, then the graph of the resulting equation $y = \sqrt{-x}$ is captured by reflecting the graph of $y = \sqrt{x}$ (see **Figure 1**(c)) horizontally across the $y$-axis. The graph of $y = \sqrt{-x}$ is shown in **Figure 7**(a).

Similarly, the graph of $y = -\sqrt{x}$ would be a vertical reflection of the graph of $y = \sqrt{x}$ across the $x$-axis, as shown in **Figure 7**(b).



(a) To obtain the graph of $y = \sqrt{-x}$, reflect the graph of $y = \sqrt{x}$ across the $y$-axis.

(b) To obtain the graph of $y = -\sqrt{x}$, reflect the graph of $y = \sqrt{x}$ across the $x$-axis.

**Figure 7.** Reflecting the graph of $y = \sqrt{x}$ across the $x$- and $y$-axes.

More often than not, you will be asked to perform a reflection **and** a translation.

▶ **Example 6.** *Sketch the graph of* $f(x) = \sqrt{4-x}$. *Use the resulting graph to determine the domain and range of* $f$.

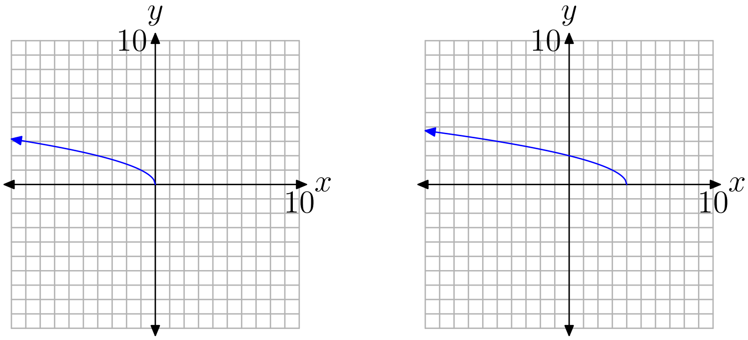
First, rewrite the equation $f(x) = \sqrt{4-x}$ as follows:

$$f(x) = \sqrt{-(x-4)}.$$

**Reflections First**. It is usually more intuitive to perform reflections before translations.

With this thought in mind, we first sketch the graph of $y = \sqrt{-x}$, which is a reflection of the graph of $y = \sqrt{x}$ across the $y$-axis. This is shown in **Figure 8**(a).

Now, in $y = \sqrt{-x}$, replace $x$ with $x - 4$ to obtain $y = \sqrt{-(x-4)}$. This shifts the graph of $y = \sqrt{-x}$ four units to the right, as pictured in **Figure 8**(b).



(a) The graph of $y = \sqrt{-x}$.

(b) The graph of $y = \sqrt{-(x-4)}$.

**Figure 8.** A reflection followed by a translation.

To find the domain of the function $f(x) = \sqrt{-(x-4)}$, or equivalently, $f(x) = \sqrt{4-x}$, project each point on the graph of $f$ onto the $x$-axis, as shown in **Figure 9**(a). Note that all real numbers less than or equal to 4 are shaded on the $x$-axis. Hence, the domain of $f$ is

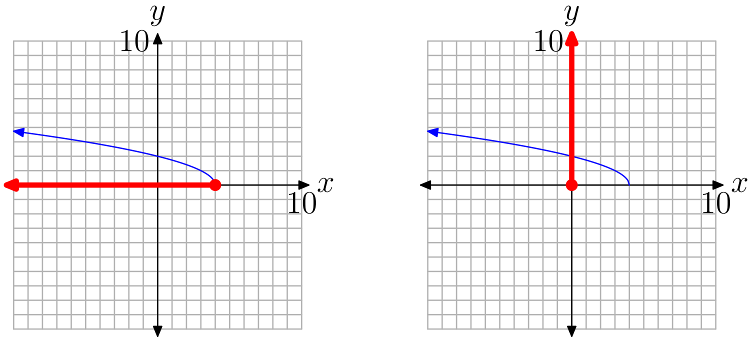
$$\text{Domain} = (-\infty, 4] = \{x : x \leq 4\}.$$

Similarly, to obtain the range of $f$, project each point on the graph of $f$ onto the $y$-axis, as shown in **Figure 9**(b). Note that all real numbers greater than or equal to zero are shaded on the $y$-axis. Hence, the range of $f$ is

$$\text{Range} = [0, \infty) = \{y : y \geq 0\}.$$

We can also find the domain of the function $f$ by examining the equation $f(x) = \sqrt{4-x}$. We cannot take the square root of a negative number, so the expression under the radical must be nonnegative (zero or positive). Consequently,

$$4 - x \geq 0.$$



(a) Project onto the $x$-axis to determine the domain.

(b) Project onto the $y$-axis to determine the range.

**Figure 9.** Determining the domain and range of $f(x) = \sqrt{4-x}$.

Solve this last inequality for $x$. First subtract 4 from both sides of the inequality, then multiply both sides of the resulting inequality by $-1$. Of course, multiplying by a negative number reverses the inequality symbol.

$$\begin{aligned} -x &\geq -4 \\ x &\leq 4 \end{aligned}$$

Thus, the domain of $f$ is $\{x : x \leq 4\}$. In interval notation, Domain $= (-\infty, 4]$. This agree nicely with the graphical result found above.

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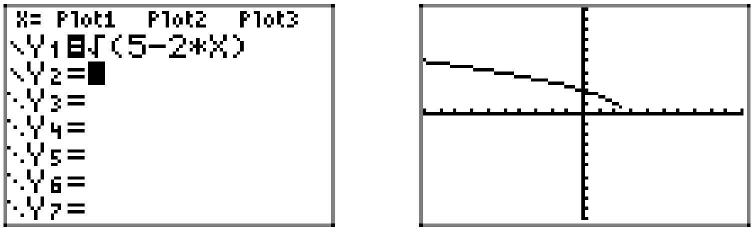
More often than not, it will take a combination of your graphing calculator and a little algebraic manipulation to determine the domain of a square root function.

▶ **Example 7.** *Sketch the graph of $f(x) = \sqrt{5-2x}$. Use the graph and an algebraic technique to determine the domain of the function.*

Load the function into `Y1` in the `Y=` menu of your calculator, as shown in **Figure 10**(a). Select `6:ZStandard` from the `ZOOM` menu to produce the graph shown in **Figure 10**(b).



```
Plot1  Plot2  Plot3
\Y1=√(5-2*X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

(a)

(b)

**Figure 10.** Drawing the graph of $f(x) = \sqrt{5-2x}$ on the graphing calculator.

Look carefully at the graph in **Figure 10**(b) and note that it's difficult to tell if the graph comes all the way down to "touch" the $x$-axis near $x \approx 2.5$. However, our previous experience with the square root function makes us believe that this is just an artifact of insufficient resolution on the calculator that is preventing the graph from "touching" the $x$-axis at $x \approx 2.5$.

An algebraic approach will settle the issue. We can determine the domain of $f$ by examining the equation $f(x) = \sqrt{5 - 2x}$. We cannot take the square root of a negative number, so the expression under the radical must be nonnegative (zero or positive). Consequently,

$$5 - 2x \geq 0.$$

Solve this last inequality for $x$. First, subtract 5 from both sides of the inequality.

$$-2x \geq -5$$

Next, divide both sides of this last inequality by $-2$. Remember that we must reverse the inequality the moment we divide by a negative number.

$$\begin{aligned} \frac{-2x}{-2} &\leq \frac{-5}{-2} \\ x &\leq \frac{5}{2} \end{aligned}$$

Thus, the domain of $f$ is $\{x : x \leq 5/2\}$. In interval notation, Domain $= (-\infty, 5/2]$.

Further introspection reveals that this argument also settles the issue of whether or not the graph "touches" the $x$-axis at $x = 5/2$. If you remain unconvinced, then substitute $x = 5/2$ in $f(x) = \sqrt{5 - 2x}$ to see

$$f(5/2) = \sqrt{5 - 2(5/2)} = \sqrt{0} = 0.$$

Thus, the graph of $f$ "touches" the $x$-axis at the point $(5/2, 0)$.

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## 9.1 Exercises

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In **Exercises 1-10**, complete each of the following tasks.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis.
ii. Complete the table of points for the given function. Plot each of the points on your coordinate system, then use them to help draw the graph of the given function.
iii. Use different colored pencils to project all points onto the $x$- and $y$-axes to determine the domain and range. Use interval notation to describe the domain of the given function.

**1.** $f(x) = -\sqrt{x}$

| $x$ | 0 | 1 | 4 | 9 |
|---|---|---|---|---|
| $f(x)$ | | | | |

**2.** $f(x) = \sqrt{-x}$

| $x$ | 0 | $-1$ | $-4$ | $-9$ |
|---|---|---|---|---|
| $f(x)$ | | | | |

**3.** $f(x) = \sqrt{x+2}$

| $x$ | $-2$ | $-1$ | 2 | 7 |
|---|---|---|---|---|
| $f(x)$ | | | | |

**4.** $f(x) = \sqrt{5-x}$

| $x$ | $-4$ | 1 | 4 | 5 |
|---|---|---|---|---|
| $f(x)$ | | | | |

**5.** $f(x) = \sqrt{x} + 2$

| $x$ | 0 | 1 | 4 | 9 |
|---|---|---|---|---|
| $f(x)$ | | | | |

**6.** $f(x) = \sqrt{x} - 1$

| $x$ | 0 | 1 | 4 | 9 |
|---|---|---|---|---|
| $f(x)$ | | | | |

**7.** $f(x) = \sqrt{x+3} + 2$

| $x$ | $-3$ | $-2$ | 1 | 6 |
|---|---|---|---|---|
| $f(x)$ | | | | |

**8.** $f(x) = \sqrt{x-1} + 3$

| $x$ | 1 | 2 | 5 | 10 |
|---|---|---|---|---|
| $f(x)$ | | | | |

**9.** $f(x) = \sqrt{3-x}$

| $x$ | $-6$ | $-1$ | 2 | 3 |
|---|---|---|---|---|
| $f(x)$ | | | | |

**10.** $f(x) = -\sqrt{x+3}$

| $x$ | $-3$ | $-2$ | 1 | 6 |
|---|---|---|---|---|
| $f(x)$ | | | | |

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In **Exercises 11-20**, perform each of the following tasks.

i. Set up a coordinate system on a sheet of graph paper. Label and scale each axis. *Remember to draw all lines with a ruler.*
ii. Use geometric transformations to draw the graph of the given function on your coordinate system without the use of a graphing calculator. *Note: You may **check** your solution with your calculator, but you should be able to produce the graph without the use of your calculator.*
iii. Use different colored pencils to project the points on the graph of the function onto the $x$- and $y$-axes. Use interval notation to describe the domain and range of the function.

**11.** $f(x) = \sqrt{x} + 3$

**12.** $f(x) = \sqrt{x+3}$

**13.** $f(x) = \sqrt{x-2}$

**14.** $f(x) = \sqrt{x} - 2$

**15.** $f(x) = \sqrt{x+5} + 1$

**16.** $f(x) = \sqrt{x-2} - 1$

**17.** $f(x) = -\sqrt{x+4}$

**18.** $f(x) = -\sqrt{x} + 4$

**19.** $f(x) = -\sqrt{x} + 3$

**20.** $f(x) = -\sqrt{x+3}$

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**21.** To draw the graph of the function $f(x) = \sqrt{3-x}$, perform each of the following steps in sequence without the aid of a calculator.

i. Set up a coordinate system and sketch the graph of $y = \sqrt{x}$. Label the graph with its equation.
ii. Set up a second coordinate system and sketch the graph of $y = \sqrt{-x}$. Label the graph with its equation.
iii. Set up a third coordinate system and sketch the graph of $y = \sqrt{-(x-3)}$. Label the graph with its equation. This is the graph of $f(x) = \sqrt{3-x}$. Use interval notation to state the domain and range of this function.

**22.** To draw the graph of the function $f(x) = \sqrt{-x-3}$, perform each of the following steps in sequence.

i. Set up a coordinate system and sketch the graph of $y = \sqrt{x}$. Label the graph with its equation.
ii. Set up a second coordinate system and sketch the graph of $y = \sqrt{-x}$. Label the graph with its equation.
iii. Set up a third coordinate system and sketch the graph of $y = \sqrt{-(x+3)}$. Label the graph with its equation. This is the graph of $f(x) = \sqrt{-x-3}$. Use interval notation to state the domain and range of this function.

**23.** To draw the graph of the function $f(x) = \sqrt{-x-1}$, perform each of the following steps in sequence without the aid of a calculator.

i. Set up a coordinate system and sketch the graph of $y = \sqrt{x}$. Label the graph with its equation.
ii. Set up a second coordinate system and sketch the graph of $y = \sqrt{-x}$. Label the graph with its equation.
iii. Set up a third coordinate system and sketch the graph of $y = \sqrt{-(x+1)}$. Label the graph with its equation. This is the graph of $f(x) = \sqrt{-x-1}$. Use interval notation to state the domain and range of this function.

**24.** To draw the graph of the function $f(x) = \sqrt{1-x}$, perform each of the following steps in sequence.

i. Set up a coordinate system and sketch the graph of $y = \sqrt{x}$. Label the graph with its equation.
ii. Set up a second coordinate system and sketch the graph of $y = \sqrt{-x}$. Label the graph with its equation.
iii. Set up a third coordinate system and sketch the graph of $y = \sqrt{-(x-1)}$. Label the graph with its equation. This is the graph of $f(x) = \sqrt{1-x}$. Use interval notation to state the domain and range of this function.

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In **Exercises 25-28**, perform each of the following tasks.

i. Draw the graph of the given function with your graphing calculator. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label your graph with its equation. Use the graph to determine the domain of the function and describe the domain with interval notation.
ii. Use a purely algebraic approach to determine the domain of the given function. Use interval notation to describe your result. Does it agree with the graphical result from part (i)?

**25.** $f(x) = \sqrt{2x+7}$

**26.** $f(x) = \sqrt{7-2x}$

**27.** $f(x) = \sqrt{12-4x}$

**28.** $f(x) = \sqrt{12+2x}$

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In **Exercises 29-40**, find the domain of the given function algebraically.

**29.** $f(x) = \sqrt{2x+9}$

**30.** $f(x) = \sqrt{-3x+3}$

**31.** $f(x) = \sqrt{-8x-3}$

**32.** $f(x) = \sqrt{-3x+6}$

**33.** $f(x) = \sqrt{-6x-8}$

**34.** $f(x) = \sqrt{8x-6}$

**35.** $f(x) = \sqrt{-7x+2}$

**36.** $f(x) = \sqrt{8x-3}$

**37.** $f(x) = \sqrt{6x+3}$

**38.** $f(x) = \sqrt{x-5}$

**39.** $f(x) = \sqrt{-7x-8}$

**40.** $f(x) = \sqrt{7x+8}$

## 9.1 Answers

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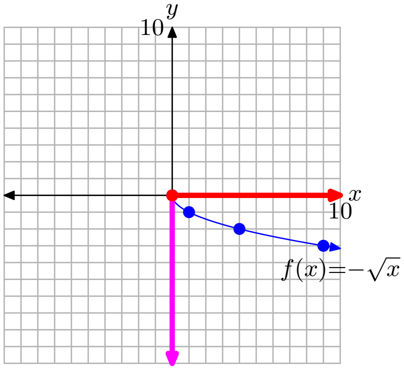
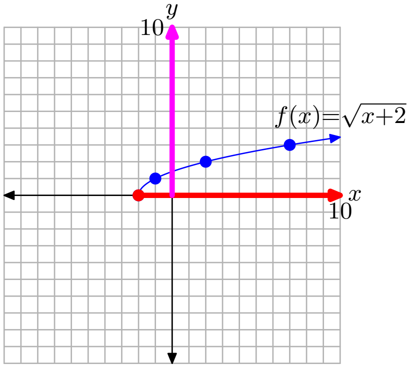
**1.** Domain = $[0, \infty)$, Range = $(-\infty, 0]$.

| $x$ | 0 | 1 | 4 | 9 |
|---|---|---|---|---|
| $f(x)$ | 0 | $-1$ | $-2$ | $-3$ |



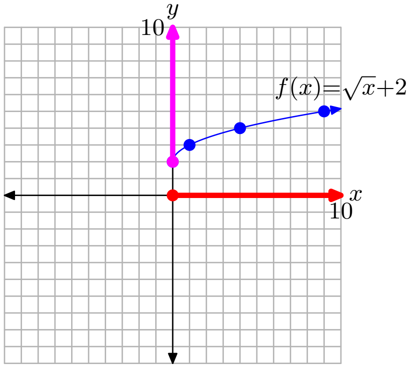
**3.** Domain = $[-2, \infty)$, Range = $[0, \infty)$.

| $x$ | $-2$ | $-1$ | 2 | 7 |
|---|---|---|---|---|
| $f(x)$ | 0 | 1 | 2 | 3 |



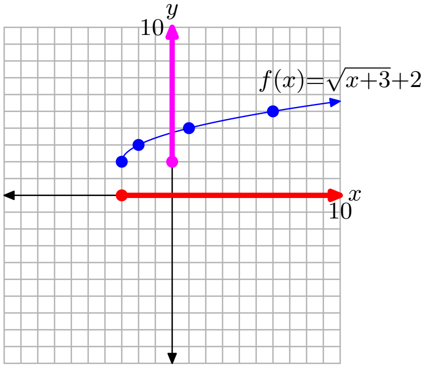
**5.** Domain = $[0, \infty)$, Range = $[2, \infty)$.

| $x$ | 0 | 1 | 4 | 9 |
|---|---|---|---|---|
| $f(x)$ | 2 | 3 | 4 | 5 |



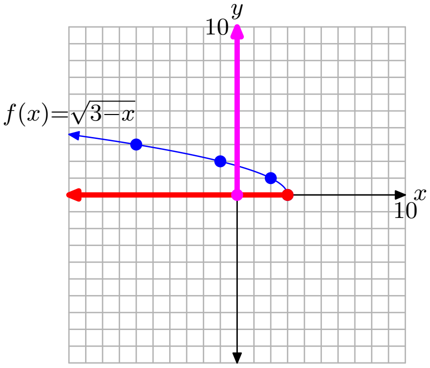
**7.** Domain = $[-3, \infty)$, Range = $[2, \infty)$.

| $x$ | $-3$ | $-2$ | 1 | 6 |
|---|---|---|---|---|
| $f(x)$ | 2 | 3 | 4 | 5 |

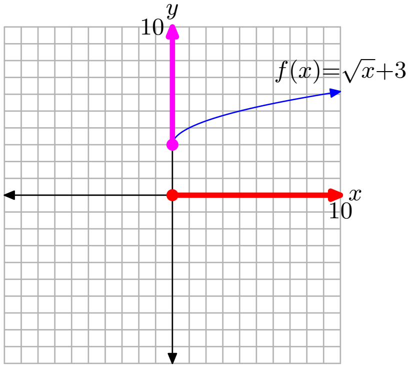


**9.** Domain = $(-\infty, 3]$, Range = $[0, \infty)$.

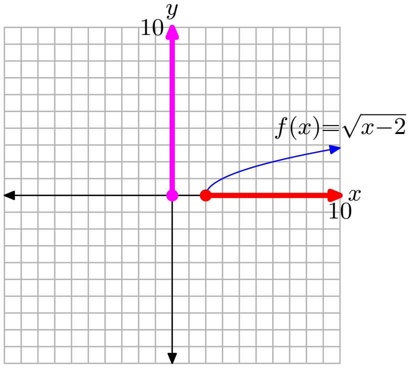
| $x$ | $-6$ | $-1$ | $2$ | $3$ |
|---|---|---|---|---|
| $f(x)$ | 3 | 2 | 1 | 0 |



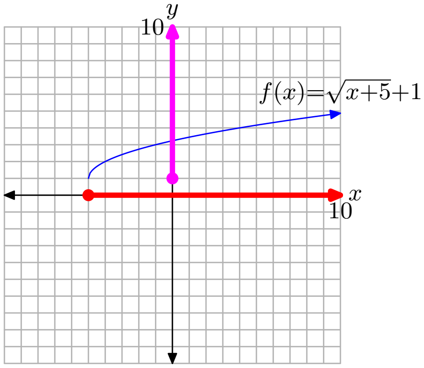
**11.** Domain = $[0, \infty)$, Range = $[3, \infty)$.



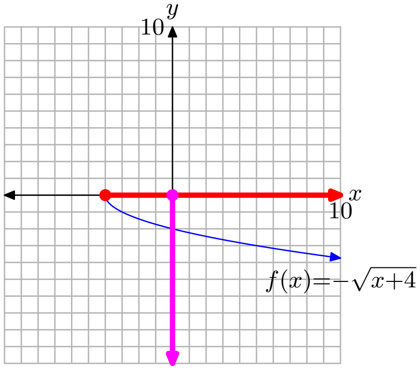
**13.** Domain = $[2, \infty)$, Range = $[0, \infty)$.



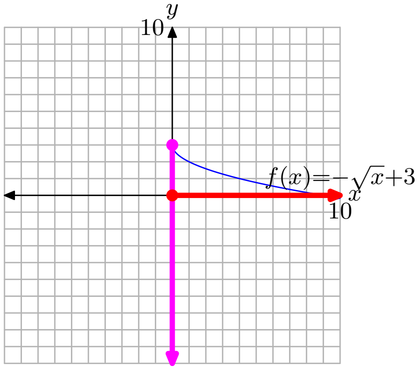
**15.** Domain = $[-5, \infty)$, Range = $[1, \infty)$.



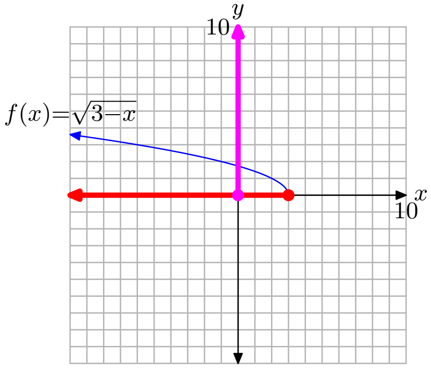
**17.** Domain = $[-4, \infty)$, Range = $(-\infty, 0]$.



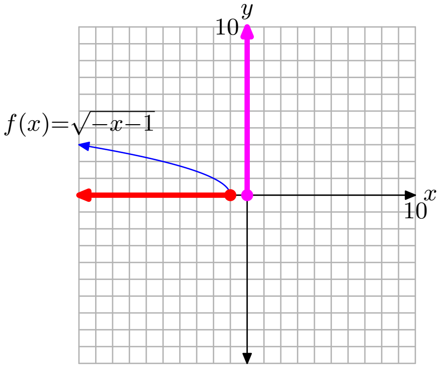
**19.** Domain = $[0, \infty)$, Range = $(-\infty, 3]$.



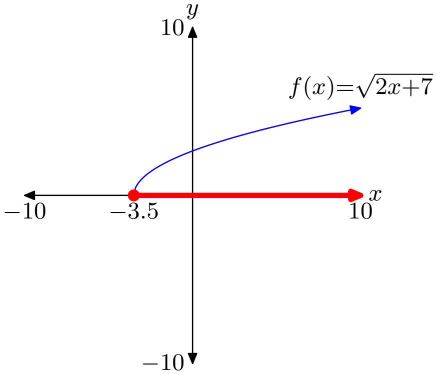
**21.** Domain $= (-\infty, 3]$, Range $= [0, \infty)$.



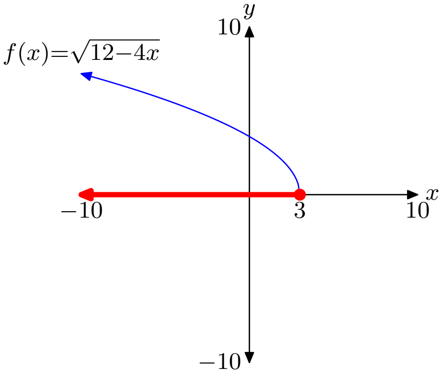
**23.** Domain $= (-\infty, -1]$, Range $= [0, \infty)$.



**25.** Domain $= [-7/2, \infty)$



**27.** Domain $= (-\infty, 3]$



**29.** $\left[-\frac{9}{2}, \infty\right)$

**31.** $\left(-\infty, -\frac{3}{8}\right]$

**33.** $\left(-\infty, -\frac{4}{3}\right]$

**35.** $\left(-\infty, \frac{2}{7}\right]$

**37.** $\left[-\frac{1}{2}, \infty\right)$

**39.** $\left(-\infty, -\frac{8}{7}\right]$

# 9.2 Multiplication Properties of Radicals

Recall that the equation $x^2 = a$, where $a$ is a positive real number, has two solutions, as indicated in **Figure 1**.

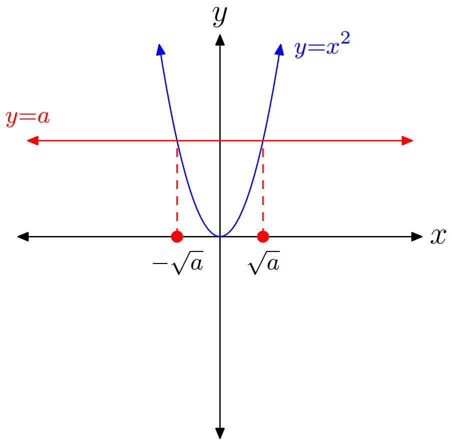


**Figure 1.** The equation $x^2 = a$, where $a$ is a positive real number, has two solutions.

Here are the key facts.

**Solutions of $x^2 = a$.** If $a$ is a positive real number, then:

1. The equation $x^2 = a$ has two real solutions.
2. The notation $\sqrt{a}$ denotes the **unique positive** real solution.
3. The notation $-\sqrt{a}$ denotes the **unique negative** real solution.

Note the use of the word **unique**. When we say that $\sqrt{a}$ is the unique positive real solution,[4] we mean that it is the only one. There are no other positive real numbers that are solutions of $x^2 = a$. A similar statement holds for the unique negative solution.

Thus, the equations $x^2 = a$ and $x^2 = b$ have unique positive solutions $x = \sqrt{a}$ and $x = \sqrt{b}$, respectively, provided that $a$ and $b$ are positive real numbers. Furthermore, because they are solutions, they can be substituted into the equations $x^2 = a$ and $x^2 = b$ to produce the results

$$\left(\sqrt{a}\right)^2 = a \qquad \text{and} \qquad \left(\sqrt{b}\right)^2 = b,$$

respectively. Again, these results are dependent upon the fact that $a$ and $b$ are positive real numbers.

Similarly, the equation

---

[3] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

[4] Technically, the notation $\sqrt{\ }$ calls for a **nonnegative** real square root, so as to include the possibility $\sqrt{0}$.

$$x^2 = ab$$

has unique positive solution $x = \sqrt{ab}$, provided $a$ and $b$ are positive numbers. However, note that

$$\left(\sqrt{a}\sqrt{b}\right)^2 = \left(\sqrt{a}\right)^2 \left(\sqrt{b}\right)^2 = ab,$$

making $\sqrt{a}\sqrt{b}$ a **second** positive solution of $x^2 = ab$. However, because $\sqrt{ab}$ is the *unique* positive solution of $x^2 = ab$, this forces

$$\sqrt{ab} = \sqrt{a}\sqrt{b}.$$

This discussion leads to the following property of radicals.

**Property 1.** Let $a$ and $b$ be positive real numbers. Then,

$$\sqrt{ab} = \sqrt{a}\sqrt{b}. \qquad (2)$$

This result can be used in two distinctly different ways.

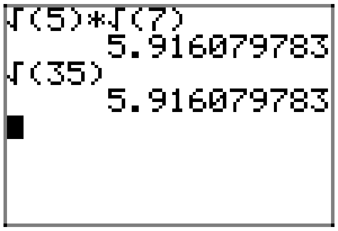
- You can use the result to multiply two square roots, as in

$$\sqrt{7}\sqrt{5} = \sqrt{35}.$$

- You can also use the result to factor, as in

$$\sqrt{35} = \sqrt{5}\sqrt{7}.$$

It is interesting to check this result on the calculator, as shown in **Figure 2**.

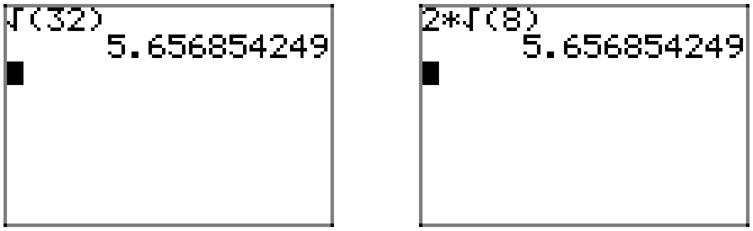


```
√(5)*√(7)
         5.916079783
√(35)
         5.916079783
█
```

**Figure 2.** Checking the result $\sqrt{5}\sqrt{7} = \sqrt{35}$.

## *Simple Radical Form*

In this section we introduce the concept of *simple radical form*, but let's first start with a little story. Martha and David are studying together, working a homework problem from their textbook. Martha arrives at an answer of $\sqrt{32}$, while David gets the result $2\sqrt{8}$. At first, David and Martha believe that their solutions are different numbers, but they've been mistaken before so they decide to compare decimal approximations of their results on their calculators. Martha's result is shown in **Figure 3**(a), while David's is shown in **Figure 3**(b).



```
√(32)
      5.656854249
```

```
2*√(8)
      5.656854249
```

(a) Martha's result. (b) David's result.

**Figure 3.** Comparing $\sqrt{32}$ with $2\sqrt{8}$.

Martha finds that $\sqrt{32} \approx 5.656854249$ and David finds that his solution $2\sqrt{8} \approx 5.656854249$. David and Martha conclude that their solutions match, but they want to know why the two very different looking radical expressions are identical.

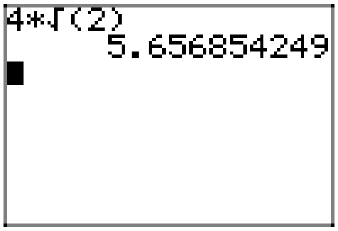
The following calculation, using **Property 1**, shows why David's result is identical to Martha's.

$$\sqrt{32} = \sqrt{4}\sqrt{8} = 2\sqrt{8}$$

Indeed, there is even a third possibility, one that is much different from the results found by David and Martha. Consider the following calculation, which again uses **Property 1**.

$$\sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$$

In **Figure 4**, note that the decimal approximation of $4\sqrt{2}$ is identical to the decimal approximations for $\sqrt{32}$ (Martha's result in **Figure 3**(a)) and $2\sqrt{8}$ (David's result in **Figure 3**(b)).



```
4*√(2)
      5.656854249
```

**Figure 4.** Approximating $4\sqrt{2}$.

While all three of these radical expressions ($\sqrt{32}$, $2\sqrt{8}$, and $4\sqrt{2}$) are identical, it is somewhat frustrating to have so many different forms, particularly when we want to compare solutions. Therefore, we offer a set of guidelines for a special form of the answer which we will call *simple radical form*.

**The First Guideline for Simple Radical Form.** When possible, factor out a perfect square.

Thus, $\sqrt{32}$ is not in simple radical form, as it is possible to factor out a perfect square, as in

$$\sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}.$$

Similarly, David's result ($2\sqrt{8}$) is not in simple radical form, because he too can factor out a perfect square as follows.

$$2\sqrt{8} = 2(\sqrt{4}\sqrt{2}) = 2(2\sqrt{2}) = (2 \cdot 2)\sqrt{2} = 4\sqrt{2}.$$

If both Martha and David follow the "first guideline for simple radical form," their answers will look identical (both equal $4\sqrt{2}$). This is one of the primary advantages of simple radical form: the ability to compare solutions.

In the examples that follow (and in the exercises), it is helpful if you know the squares of the first 25 positive integers. We've listed them in the margin for you in **Table 1** for future reference.

| $x$ | $x^2$ |
|---|---|
| 2 | 4 |
| 3 | 9 |
| 4 | 16 |
| 5 | 25 |
| 6 | 36 |
| 7 | 49 |
| 8 | 64 |
| 9 | 81 |
| 10 | 100 |
| 11 | 121 |
| 12 | 144 |
| 13 | 169 |
| 14 | 196 |
| 15 | 225 |
| 16 | 256 |
| 17 | 289 |
| 18 | 324 |
| 19 | 361 |
| 20 | 400 |
| 21 | 441 |
| 22 | 484 |
| 23 | 529 |
| 24 | 576 |
| 25 | 625 |

**Table 1.** Squares.

Let's place a few more radical expressions in simple radical form.

▶ **Example 3.** *Place $\sqrt{50}$ in simple radical form.*

In **Table 1**, 25 is a square. Because $50 = 25 \cdot 2$, we can use **Property 1** to write

$$\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}.$$

◇

▶ **Example 4.** *Place $\sqrt{98}$ in simple radical form.*

In **Table 1**, 49 is a square. Because $98 = 49 \cdot 2$, we can again use **Property 1** and write

$$\sqrt{98} = \sqrt{49}\sqrt{2} = 7\sqrt{2}.$$

◇

▶ **Example 5.** *Place $\sqrt{288}$ in simple radical form.*

Some students seem able to pluck the optimal "perfect square" out of thin air. If you consult **Table 1**, you'll note that 144 is a square. Because $288 = 144 \cdot 2$, we can write

$$\sqrt{288} = \sqrt{144}\sqrt{2} = 12\sqrt{2}.$$

However, what if you miss that higher perfect square, think $288 = 4 \cdot 72$, and write

$$\sqrt{288} = \sqrt{4}\sqrt{72} = 2\sqrt{72}.$$

This approach is not incorrect, provided you realize that you're not finished. You can still factor a perfect square out of 72. Because $72 = 36 \cdot 2$, you can continue and write

$$2\sqrt{72} = 2(\sqrt{36}\sqrt{2}) = 2(6\sqrt{2}) = (2 \cdot 6)\sqrt{2} = 12\sqrt{2}.$$

Note that we arrived at the same simple radical form, namely $12\sqrt{2}$. It just took us a little longer. As long as we realize that we must continue until we can no longer factor out a perfect square, we'll arrive at the same simple radical form as the student who seems to magically pull the higher square out of thin air.

Indeed, here is another approach that is equally valid.

$$\sqrt{288} = \sqrt{4}\sqrt{72} = 2(\sqrt{4}\sqrt{18}) = 2(2\sqrt{18}) = (2 \cdot 2)\sqrt{18} = 4\sqrt{18}$$

We need to recognize that we are still not finished because we can extract another perfect square as follows.

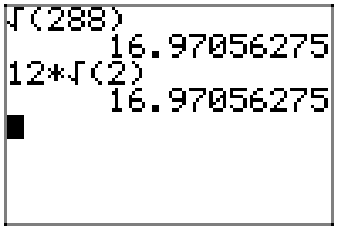
$$4\sqrt{18} = 4(\sqrt{9}\sqrt{2}) = 4(3\sqrt{2}) = (4 \cdot 3)\sqrt{2} = 12\sqrt{2}$$

Once again, same result. However, note that it behooves us to extract the largest square possible, as it minimizes the number of steps required to attain simple radical form.

**Checking Results with the Graphing Calculator.** Once you've placed a radical expression in simple radical form, you can use your graphing calculator to check your result. In this example, we found that

$$\sqrt{288} = 12\sqrt{2}. \qquad (6)$$

Enter the left- and right-hand sides of this result as shown in **Figure 5**. Note that each side produces the same decimal approximation, verifying the result in **equation (6)**.



```
√(288)
       16.97056275
12*√(2)
       16.97056275
█
```

**Figure 5.** Comparing $\sqrt{288}$ with its simple radical form $12\sqrt{2}$.

## *Helpful Hints*

Recall that raising a power of a base to another power requires that we multiply exponents.

**Raising a Power of a Base to another Power.**

$$(a^m)^n = a^{mn}$$

In particular, when you square a power of a base, you must multiply the exponent by 2. For example,

$$(2^5)^2 = 2^{10}.$$

Conversely, because taking a square root is the "inverse" of squaring,[5] when taking a square root we must *divide* the existing exponent by 2, as in

$$\sqrt{2^{10}} = 2^5.$$

Note that squaring $2^5$ gives $2^{10}$, so taking the square root of $2^{10}$ must return you to $2^5$. When you square, you double the exponent. Therefore, when you take the square root, you must halve the exponent.

Similarly,

- $(2^6)^2 = 2^{12}$ so $\sqrt{2^{12}} = 2^6$.
- $(2^7)^2 = 2^{14}$ so $\sqrt{2^{14}} = 2^7$.
- $(2^8)^2 = 2^{16}$ so $\sqrt{2^{16}} = 2^8$.

This leads to the following result.

**Taking the Square Root of an Even Power.** When taking a square root of $x^n$, when $x$ is a positive real number and $n$ is an even natural number, divide the exponent by two. In symbols,

$$\sqrt{x^n} = x^{n/2}.$$

Note that this agrees with the definition of rational exponents presented in Chapter 8, as in

$$\sqrt{x^n} = (x^n)^{1/2} = x^{n/2}.$$

On another note, recall that raising a product to a power requires that we raise each factor to that power.

**Raising a Product to a Power.**

$$(ab)^n = a^n b^n.$$

In particular, if you square a product, you must square each factor. For example,

$$(5^3 7^4)^2 = (5^3)^2 (7^4)^2 = 5^6 7^8.$$

Note that we multiplied each existing exponent in this product by 2.

---

[5] Well, not always. Consider $(-2)^2 = 4$, but $\sqrt{4} = 2$ does not return to $-2$. However, when you start with a postive number and square, then taking the positive square root is the inverse operation and returns you to the original positive number. Return to Chapter 8 (the section on inverse functions) if you want to reread a full discussion of this trickiness.

**Property 1** is similar, in that when we take the square root of a product, we take the square root of each factor. Because taking a square root is the inverse of squaring, we must *divide* each existing exponent by 2, as in

$$\sqrt{5^6 7^8} = \sqrt{5^6}\sqrt{7^8} = 5^3 7^4.$$

Let's look at some examples that employ this technique.

▶ **Example 7.** *Simplify* $\sqrt{2^4 3^6 5^{10}}$.

When taking the square root of a product of exponential factors, divide each exponent by 2.

$$\sqrt{2^4 3^6 5^{10}} = 2^2 3^3 5^5$$

If needed, you can expand the exponential factors and multiply to provide a single numerical answer.

$$2^2 3^3 5^5 = 4 \cdot 27 \cdot 3125 = 337\,500$$

A calculator was used to obtain the final solution.

◇

▶ **Example 8.** *Simplify* $\sqrt{2^5 3^3}$.

In this example, the difficulty is the fact that the exponents are not divisible by 2. However, if possible, the "first guideline of simple radical form" requires that we factor out a perfect square. So, extract each factor raised to the highest possible power that is divisible by 2, as in

$$\sqrt{2^5 3^3} = \sqrt{2^4\ 3^2}\sqrt{2 \cdot 3}$$

Now, divide each exponent by 2.

$$\sqrt{2^4\ 3^2}\sqrt{2 \cdot 3} = 2^2 3^1 \sqrt{2 \cdot 3}$$

Finally, simplify by expanding each exponential factor and multiplying.

$$2^2 3^1 \sqrt{2 \cdot 3} = 4 \cdot 3\sqrt{2 \cdot 3} = 12\sqrt{6}$$

◇

▶ **Example 9.** *Simplify* $\sqrt{3^7 5^2 7^5}$.

Extract each factor to the highest possible power that is divisible by 2.

$$\sqrt{3^7 5^2 7^5} = \sqrt{3^6 5^2 7^4}\sqrt{3 \cdot 7}$$

Divide each exponent by 2.

$$\sqrt{3^6 5^2 7^4}\sqrt{3 \cdot 7} = 3^3 5^1 7^2 \sqrt{3 \cdot 7}$$

Expand each exponential factor and multiply.

$$3^3 5^1 7^2 \sqrt{3 \cdot 7} = 27 \cdot 5 \cdot 49 \sqrt{3 \cdot 7} = 6\,615\sqrt{21}$$

◆

▶ **Example 10.** *Place $\sqrt{216}$ in simple radical form.*

If we prime factor 216, we can attack this problem with the same technique used in the previous examples. Before we prime factor 216, here are a few divisibility tests that you might find useful.

**Divisibility Tests.**

- If a number ends in 0, 2, 4, 6, or 8, it is an **even** number and is divisible by 2.
- If the last two digits of a number form a number that is divisible by 4, then the entire number is divisible by 4.
- If a number ends in 0 or 5, it is divisible by 5.
- If the sum of the digits of a number is divisible by 3, then the entire number is divisible by 3.
- If the sum of the digits of a number is divisible by 9, then the entire number is divisible by 9.

For example, in order:

- The number 226 ends in a 6, so it is even and divisible by 2. Indeed, $226 = 2 \cdot 113$.
- The last two digits of 224 are 24, which is divisible by 4, so the entire number is divisible by 4. Indeed, $224 = 4 \cdot 56$.
- The last digit of 225 is a 5. Therefore 225 is divisible by 5. Indeed, $225 = 5 \cdot 45$.
- The sum of the digits of 222 is $2+2+2 = 6$, which is divisible by 3. Therefore, 222 is divisible by 3. Indeed, $222 = 3 \cdot 74$.
- The sum of the digits of 684 is $6 + 8 + 4 = 18$, which is divisible by 9. Therefore, 684 is divisible by 9. Indeed, $684 = 9 \cdot 76$.

Now, let's prime factor 216. Note that $2+1+6 = 9$, so 216 is divisible by 9. Indeed, $216 = 9 \cdot 24$. In **Figure 6**, we use a "factor tree" to continue factoring until all of the "leaves" are prime numbers.

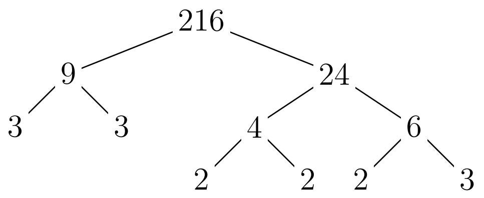


**Figure 6.** Using a factor tree to prime factor 216.

Thus,

$$216 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3,$$

or in exponential form,

$$216 = 2^3 \cdot 3^3.$$

Thus,

$$\sqrt{216} = \sqrt{2^3\,3^3} = \sqrt{2^2\,3^2}\,\sqrt{2 \cdot 3} = 2 \cdot 3\sqrt{2 \cdot 3} = 6\sqrt{6}.$$

◆

Prime factorization is an unbelievably useful tool!

Let's look at another example.

▶ **Example 11.** *Place* $\sqrt{2592}$ *in simple radical form.*

If we find the prime factorization for 2592, we can attack this example using the same technique we used in the previous example. We note that the sum of the digits of 2592 is $2 + 5 + 9 + 2 = 18$, which is divisible by 9. Therefore, 2592 is also divisible by 9.

$$2592 = 9 \cdot 288$$

The sum of the digits of 288 is $2 + 8 + 8 = 18$, which is divisible by 9, so 288 is also divisible by 9.

$$2592 = 9 \cdot (9 \cdot 32)$$

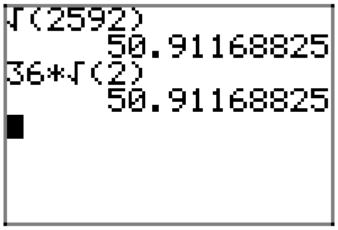
Continue in this manner until the leaves of your "factor tree" are all primes. Then, you should get

$$2592 = 2^5\,3^4.$$

Thus,

$$\sqrt{2592} = \sqrt{2^5 3^4} = \sqrt{2^4 3^4}\sqrt{2} = 2^2 3^2\sqrt{2} = 4 \cdot 9\sqrt{2} = 36\sqrt{2}.$$

Let's use the graphing calculator to check this result. Enter each side of $\sqrt{2592} = 36\sqrt{2}$ separately and compare approximations, as shown in **Figure 7**.



```
√(2592)
      50.91168825
36*√(2)
      50.91168825
█
```

**Figure 7.** Comparing $\sqrt{2592}$ with its simple radical form $36\sqrt{2}$.

◆

## An Important Property of Square Roots

One of the most common mistakes in algebra occurs when practitioners are asked to simplify the expression $\sqrt{x^2}$, where $x$ is any arbitrary real number. Let's examine two of the most common errors.

- Some will claim that the following statement is true for any arbitrary real number $x$.

  $$\sqrt{x^2} = \pm x.$$

  This is easily seen to be incorrect. Simply substitute any real number for $x$ to check this claim. We will choose $x = 3$ and substitute it into each side of the proposed statement.

  $$\sqrt{3^2} = \pm 3$$

  If we simplify the left-hand side, we produce the following result.

  $$\begin{aligned} \sqrt{3^2} &= \pm 3 \\ 3 &= \pm 3 \end{aligned}$$

  It is not correct to state that 3 and ±3 are equal.
- A second error is to claim that

  $$\sqrt{x^2} = x$$

  for any arbitrary real number $x$. Although this is certainly true if you substitute nonnegative numbers for $x$, look what happens when you substitute $-3$ for $x$.

  $$\sqrt{(-3)^2} = -3$$

  If we simplify the left-hand side, we produce the following result.

  $$\begin{aligned} \sqrt{9} &= -3 \\ 3 &= -3 \end{aligned}$$

  Clearly, 3 and $-3$ are not equal.

In both cases, what has been forgotten is the fact that $\sqrt{\ }$ calls for a positive (nonnegative if you want to include the case $\sqrt{0}$) square root. In both of the errors above, namely $\sqrt{x^2} = \pm x$ and $\sqrt{x^2} = x$, the left-hand side is calling for a nonnegative response, but nothing has been done to insure that the right-hand side is also nonnegative. Does anything come to mind?

Sure, if we wrap the right-hand side in absolute values, as in

$$\sqrt{x^2} = |x|,$$

then both sides are calling for a nonnegative response. Indeed, note that

$$\sqrt{(-3)^2} = |-3|, \qquad \sqrt{0^2} = |0|, \qquad \text{and} \qquad \sqrt{3^2} = |3|$$

are all valid statements.

This discussion leads to the following result.

**The Positive Square Root of the Square of $x$.** If $x$ is any real number, then

$$\sqrt{x^2} = |x|.$$

The next task is to use this new property to produce a extremely useful property of absolute value.

## *A Multiplication Property of Absolute Value*

If we combine the law of exponents for squaring a product with our property for taking the square root of a product, we can write

$$\sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2}.$$

However, $\sqrt{(ab)^2} = |ab|$, while $\sqrt{a^2}\sqrt{b^2} = |a||b|$. This discussion leads to the following result.

**Product Rule for Absolute Value.** If $a$ and $b$ are any real numbers,

$$|ab| = |a||b|. \tag{12}$$

In words, the absolute value of a product is equal to the product of the absolute values.

We saw this property previously in the chapter on the absolute value function, where we provided a different approach to the proof of the property. It's interesting that we can prove this property in a completely new way using the properties of square root. We'll see we have need for the Product Rule for Absolute Value in the examples that follow.

For example, using the product rule, if $x$ is any real number, we could write

$$|3x| = |3||x| = 3|x|$$

However, there is no way we can remove the absolute value bars that surround $x$ unless we know the sign of $x$. If $x \geq 0$, then $|x| = x$ and the expression becomes

$$3|x| = 3x.$$

On the other hand, if $x < 0$, then $|x| = -x$ and the expression becomes

$$3|x| = 3(-x) = -3x.$$

Let's look at another example. Using the product rule, if $x$ is any real number, the expression $|-4x^3|$ can be manipulated as follows.

$$|-4x^3| = |-4||x^2||x|$$

However, $|-4| = 4$ and since $x^2 \geq 0$ for any value of $x$, $|x^2| = x^2$. Thus,

$$|-4||x^2||x| = 4x^2|x|.$$

Again, there is no way we can remove the absolute value bars around $x$ unless we know the sign of $x$. If $x \geq 0$, then $|x| = x$ and

$$4x^2|x| = 4x^2(x) = 4x^3.$$

On the other hand, if $x < 0$, then $|x| = -x$ and

$$4x^2|x| = 4x^2(-x) = -4x^3.$$

Let's use these ideas to simplify some radical expressions that contain variables.

## Variable Expressions

▶ **Example 13.** *Given that the $x$ represents any real numbers, place the radical expression*

$$\sqrt{48x^6}$$

*in simple radical form.*

Simple radical form demands that we factor out a perfect square, if possible. In this case, $48 = 16 \cdot 3$ and we factor out the highest power of $x$ that is divisible by 2.

$$\sqrt{48x^6} = \sqrt{16x^6}\sqrt{3}$$

We can now use **Property 1** to take the square root of each factor.

$$\sqrt{16x^6}\sqrt{3} = \sqrt{16}\sqrt{x^6}\sqrt{3}$$

Now, remember that the notation $\sqrt{\phantom{x}}$ calls for a **nonnegative** square root, so we must insure that each response in the equation above is nonnegative. Thus,

$$\sqrt{16}\sqrt{x^6}\sqrt{3} = 4|x^3|\sqrt{3}.$$

Some comments are in order.

- The nonnegative square root of 16 is 4. That is, $\sqrt{16} = 4$.
- The nonnegative square root of $x^6$ is trickier. It is incorrect to say $\sqrt{x^6} = x^3$, because $x^3$ could be negative (if $x$ is negative). To insure a nonnegative square root, in this case we need to wrap our answer in absolute value bars. That is, $\sqrt{x^6} = |x^3|$.

We can use the Product Rule for Absolute Value to write $|x^3| = |x^2||x|$. Because $x^2$ is nonnegative, absolute value bars are redundant and not needed. That is, $|x^2||x| = x^2|x|$. Thus, we can simplify our solution a bit further and write

$$4|x^3|\sqrt{3} = 4x^2|x|\sqrt{3}.$$

Thus,

$$\sqrt{48x^6} = 4x^2|x|\sqrt{3}. \tag{14}$$

**Alternate Solution.** There is a variety of ways that we can place a radical expression in simple radical form. Here is another approach. Starting at the step above, where we first factored out a perfect square,

$$\sqrt{48x^6} = \sqrt{16x^6}\sqrt{3},$$

we could write

$$\sqrt{16x^6}\sqrt{3} = \sqrt{(4x^3)^2}\sqrt{3}.$$

Now, remember that the nonnegative square root of the square of an expression is the absolute value of that expression (we have to guarantee a nonnegative answer), so

$$\sqrt{(4x^3)^2}\sqrt{3} = |4x^3|\sqrt{3}.$$

However, $|4x^3| = |4||x^3|$ by our product rule and $|4||x^3| = 4|x^3|$. Thus,

$$|4x^3|\sqrt{3} = 4|x^3|\sqrt{3}.$$

Finally, $|x^3| = |x^2||x| = x^2|x|$ because $x^2 \geq 0$, so we can write

$$4|x^3|\sqrt{3} = 4x^2|x|\sqrt{3}. \tag{15}$$

We cannot remove the absolute value bar that surrounds $x$ unless we know the sign of $x$.

Note that the simple radical form (**15**) in the alternate solution is identical to the simple radical form (**14**) found with the previous solution technique.

◆

Let's look at another example.

▶ **Example 16.** *Given that $x < 0$, place $\sqrt{24x^6}$ in simple radical form.*

First, factor out a perfect square and write

$$\sqrt{24x^6} = \sqrt{4x^6}\sqrt{6}.$$

Now, use **Property 1** and take the square root of each factor.

$$\sqrt{4x^6}\sqrt{6} = \sqrt{4}\sqrt{x^6}\sqrt{6}$$

To insure a nonnegative response to $\sqrt{x^6}$, wrap your response in absolute values.

$$\sqrt{4}\sqrt{x^6}\sqrt{6} = 2|x^3|\sqrt{6}$$

However, as in the previous problem, $|x^3| = |x^2||x| = x^2|x|$, since $x^2 \geq 0$. Thus,

$$2|x^3|\sqrt{6} = 2x^2|x|\sqrt{6}.$$

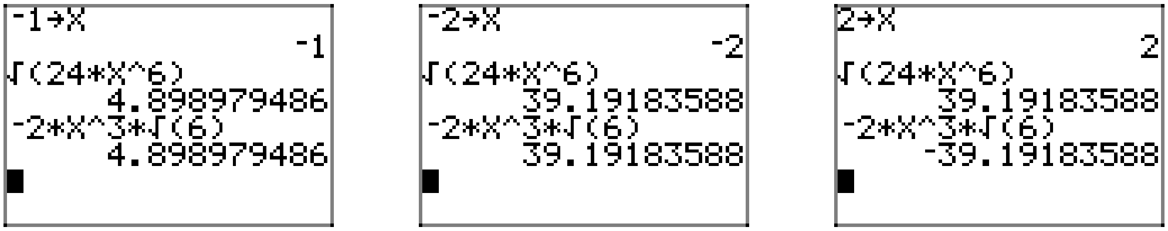
In this example, we were given the extra fact that $x < 0$, so $|x| = -x$ and we can write

$$2x^2|x|\sqrt{6} = 2x^2(-x)\sqrt{6} = -2x^3\sqrt{6}.$$

It is instructive to test the validity of the answer

$$\sqrt{24x^6} = -2x^3\sqrt{6}, \quad x < 0,$$

using a calculator. So, set $x = -1$ with the command `-1 STO▶X`. That is, enter $-1$, then push the `STO▶` button, followed by `X`, then press the `ENTER` key. The result is shown in **Figure 8**(a). Next, enter `√(24*X^6)` and press `ENTER` to capture the second result shown in **Figure 8**(a). Finally, enter `-2*X^3√(6)` and press `ENTER`. Note that the two expressions $\sqrt{24x^6}$ and $-2x^3\sqrt{6}$ agree at $x = -1$, as seen in **Figure 8**(a). We've also checked the validity of the result at $x = -2$ in **Figure 8**(b). However, note that our result is not valid at $x = 2$ in **Figure 8**(c). This occurs because $\sqrt{24x^6} = -2x^3\sqrt{6}$ only if $x$ is negative.



```
-1→X
                -1
√(24*X^6)
       4.898979486
-2*X^3*√(6)
       4.898979486
```

(a) Check with $x = -1$.

```
-2→X
                -2
√(24*X^6)
       39.19183588
-2*X^3*√(6)
       39.19183588
```

(b) Check with $x = -2$.

```
2→X
                 2
√(24*X^6)
       39.19183588
-2*X^3*√(6)
      -39.19183588
```

(c) Check with $x = 2$.

**Figure 8.** Spot-checking the validity of $\sqrt{24x^6} = -2x^3\sqrt{6}$.

It is somewhat counterintuitive that the result

$$\sqrt{24x^6} = -2x^3\sqrt{6}, \quad x < 0,$$

contains a negative sign. After all, the expression $\sqrt{24x^6}$ calls for a nonnegative result, but we have a negative sign. However, on closer inspection, if $x < 0$, then $x$ is a negative number and the right-hand side $-2x^3\sqrt{6}$ is a positive number ($-2$ is negative, $x^3$ is negative because $x$ is negative, and the product of two negatives is a positive).

◆

Let's look at another example.

▶ **Example 17.** *If $x < 3$, simplify $\sqrt{x^2 - 6x + 9}$.*

The expression under the radical is a perfect square trinomial and factors.

$$\sqrt{x^2 - 6x + 9} = \sqrt{(x-3)^2}$$

However, the nonnegative square root of the square of an expression is the absolute value of that expression, so

$$\sqrt{(x-3)^2} = |x-3|.$$

Finally, because we are told that $x < 3$, this makes $x - 3$ a negative number, so

$$|x-3| = -(x-3). \tag{18}$$

Again, the result $\sqrt{x^2 - 6x + 9} = -(x-3)$, provided $x < 3$, is somewhat counterintuitive as we are expecting a positive result. However, if $x < 3$, the result $-(x-3)$ is positive. You can test this by substituting several values of $x$ that are less than 3 into the expression $-(x-3)$ and noting that the result is positive. For example, if $x = 2$, then $x$ is less than 3 and

$$-(x-3) = -(2-3) = -(-1) = 1,$$

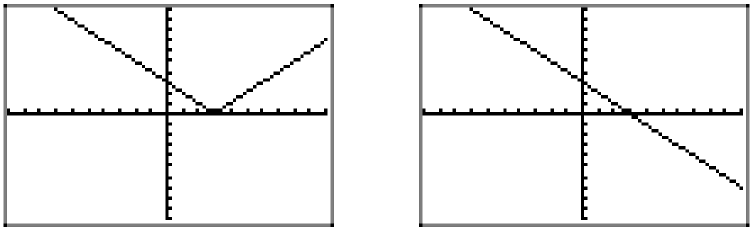
which, of course, is a positive result.

It is even more informative to note that our result is equivalent to

$$\sqrt{x^2 - 6x + 9} = -x + 3, \quad x < 3.$$

This is easily seen by distributing the minus sign in the result (**18**).

We've drawn the graph of $y = \sqrt{x^2 - 6x + 9}$ on our calculator in **Figure 9**(a). In **Figure 9**(b), we've drawn the graph of $y = -x + 3$. Note that the graphs agree when $x < 3$. Indeed, when you consider the left-hand branch of the "V" in **Figure 9**(a), you can see that the slope of this branch is $-1$ and the $y$-intercept is 3. The equation of this branch is $y = -x + 3$, so it agrees with the graph of $y = -x + 3$ in **Figure 9**(b) when $x$ is less than 3.



(a) The graph of $y = \sqrt{x^2 - 6x + 9}$.

(b) The graph of $y = -x + 3$.

**Figure 9.** Verifying graphically that $\sqrt{x^2 - 6x + 9} = -x + 3$ when $x < 3$.

◆

## 9.2 Exercises

**1.** Use a calculator to first approximate $\sqrt{5}\sqrt{2}$. On the same screen, approximate $\sqrt{10}$. Report the results on your homework paper.

**2.** Use a calculator to first approximate $\sqrt{7}\sqrt{10}$. On the same screen, approximate $\sqrt{70}$. Report the results on your homework paper.

**3.** Use a calculator to first approximate $\sqrt{3}\sqrt{11}$. On the same screen, approximate $\sqrt{33}$. Report the results on your homework paper.

**4.** Use a calculator to first approximate $\sqrt{5}\sqrt{13}$. On the same screen, approximate $\sqrt{65}$. Report the results on your homework paper.

In **Exercises 5-20**, place each of the radical expressions in simple radical form. As in Example 3 in the narrative, check your result with your calculator.

**5.** $\sqrt{18}$

**6.** $\sqrt{80}$

**7.** $\sqrt{112}$

**8.** $\sqrt{72}$

**9.** $\sqrt{108}$

**10.** $\sqrt{54}$

**11.** $\sqrt{50}$

**12.** $\sqrt{48}$

**13.** $\sqrt{245}$

**14.** $\sqrt{150}$

**15.** $\sqrt{98}$

**16.** $\sqrt{252}$

**17.** $\sqrt{45}$

**18.** $\sqrt{294}$

**19.** $\sqrt{24}$

**20.** $\sqrt{32}$

In **Exercises 21-26**, use prime factorization (as in Examples 10 and 11 in the narrative) to assist you in placing the given radical expression in simple radical form. Check your result with your calculator.

**21.** $\sqrt{2016}$

**22.** $\sqrt{2700}$

**23.** $\sqrt{14175}$

**24.** $\sqrt{44000}$

**25.** $\sqrt{20250}$

**26.** $\sqrt{3564}$

In **Exercises 27-46**, place each of the given radical expressions in simple radical form. Make no assumptions about the sign of the variables. Variables can either represent positive or negative numbers.

**27.** $\sqrt{(6x-11)^4}$

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**28.** $\sqrt{16h^8}$

**29.** $\sqrt{25f^2}$

**30.** $\sqrt{25j^8}$

**31.** $\sqrt{16m^2}$

**32.** $\sqrt{25a^2}$

**33.** $\sqrt{(7x+5)^{12}}$

**34.** $\sqrt{9w^{10}}$

**35.** $\sqrt{25x^2-50x+25}$

**36.** $\sqrt{49x^2-42x+9}$

**37.** $\sqrt{25x^2+90x+81}$

**38.** $\sqrt{25f^{14}}$

**39.** $\sqrt{(3x+6)^{12}}$

**40.** $\sqrt{(9x-8)^{12}}$

**41.** $\sqrt{36x^2+36x+9}$

**42.** $\sqrt{4e^2}$

**43.** $\sqrt{4p^{10}}$

**44.** $\sqrt{25x^{12}}$

**45.** $\sqrt{25q^6}$

**46.** $\sqrt{16h^{12}}$

---

**47.** Given that $x < 0$, place the radical expression $\sqrt{32x^6}$ in simple radical form. Check your solution on your calculator for $x = -2$.

**48.** Given that $x < 0$, place the radical expression $\sqrt{54x^8}$ in simple radical form. Check your solution on your calculator for $x = -2$.

**49.** Given that $x < 0$, place the radical expression $\sqrt{27x^{12}}$ in simple radical form. Check your solution on your calculator for $x = -2$.

**50.** Given that $x < 0$, place the radical expression $\sqrt{44x^{10}}$ in simple radical form. Check your solution on your calculator for $x = -2$.

---

In **Exercises 51-54**, follow the lead of Example 17 in the narrative to simplify the given radical expression and check your result with your graphing calculator.

**51.** Given that $x < 4$, place the radical expression $\sqrt{x^2-8x+16}$ in simple radical form. Use a graphing calculator to show that the graphs of the original expression and your simple radical form agree for all values of $x$ such that $x < 4$.

**52.** Given that $x \geq -2$, place the radical expression $\sqrt{x^2+4x+4}$ in simple radical form. Use a graphing calculator to show that the graphs of the original expression and your simple radical form agree for all values of $x$ such that $x \geq -2$.

**53.** Given that $x \geq 5$, place the radical expression $\sqrt{x^2-10x+25}$ in simple radical form. Use a graphing calculator to show that the graphs of the original expression and your simple radical form agree for all values of $x$ such that $x \geq 5$.

**54.** Given that $x < -1$, place the radical expression $\sqrt{x^2+2x+1}$ in simple radical form. Use a graphing calculator to show that the graphs of the original expression and your simple radical form agree for all values of $x$ such that $x < -1$.

In **Exercises 55-72**, place each radical expression in simple radical form. Assume that all variables represent positive numbers.

**55.** $\sqrt{9d^{13}}$

**56.** $\sqrt{4k^2}$

**57.** $\sqrt{25x^2+40x+16}$

**58.** $\sqrt{9x^2-30x+25}$

**59.** $\sqrt{4j^{11}}$

**60.** $\sqrt{16j^6}$

**61.** $\sqrt{25m^2}$

**62.** $\sqrt{9e^9}$

**63.** $\sqrt{4c^5}$

**64.** $\sqrt{25z^2}$

**65.** $\sqrt{25h^{10}}$

**66.** $\sqrt{25b^2}$

**67.** $\sqrt{9s^7}$

**68.** $\sqrt{9e^7}$

**69.** $\sqrt{4p^8}$

**70.** $\sqrt{9d^{15}}$

**71.** $\sqrt{9q^{10}}$

**72.** $\sqrt{4w^7}$

In **Exercises 73-80**, place each given radical expression in simple radical form. Assume that all variables represent positive numbers.

**73.** $\sqrt{2f^5}\sqrt{8f^3}$

**74.** $\sqrt{3s^3}\sqrt{243s^3}$

**75.** $\sqrt{2k^7}\sqrt{32k^3}$

**76.** $\sqrt{2n^9}\sqrt{8n^3}$

**77.** $\sqrt{2e^9}\sqrt{8e^3}$

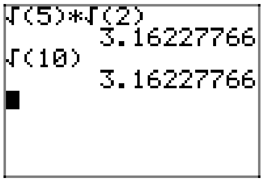
**78.** $\sqrt{5n^9}\sqrt{125n^3}$

**79.** $\sqrt{3z^5}\sqrt{27z^3}$

**80.** $\sqrt{3t^7}\sqrt{27t^3}$

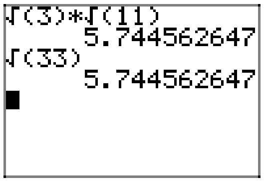
## 9.2 *Answers*

**1.**



```
√(5)*√(2)
           3.16227766
√(10)
           3.16227766
```

**3.**



```
√(3)*√(11)
          5.744562647
√(33)
          5.744562647
```

**5.** $3\sqrt{2}$

**7.** $4\sqrt{7}$

**9.** $6\sqrt{3}$

**11.** $5\sqrt{2}$

**13.** $7\sqrt{5}$

**15.** $7\sqrt{2}$

**17.** $3\sqrt{5}$

**19.** $2\sqrt{6}$

**21.** $12\sqrt{14}$

**23.** $45\sqrt{7}$

**25.** $45\sqrt{10}$

**27.** $(6x-11)^2$

**29.** $5|f|$

**31.** $4|m|$

**33.** $(7x+5)^6$

**35.** $|5x-5|$

**37.** $|5x+9|$

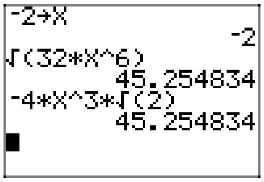
**39.** $(3x+6)^6$

**41.** $|6x+3|$

**43.** $2p^4|p|$

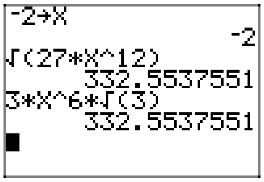
**45.** $5q^2|q|$

**47.** $-4x^3\sqrt{2}$



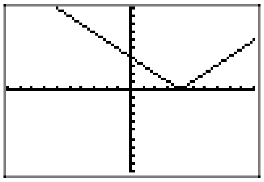
```
-2→X
                   -2
√(32*X^6)
            45.254834
-4*X^3*√(2)
            45.254834
```

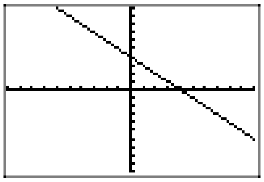
**49.** $3x^6\sqrt{3}$



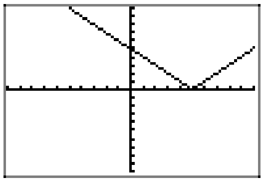
```
-2→X
                   -2
√(27*X^12)
          332.5537551
3*X^6*√(3)
          332.5537551
```

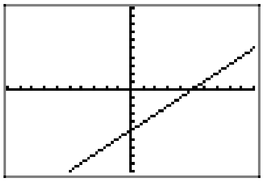
**51.** $-x+4$. The graphs of $y=-x+4$ and $y=\sqrt{x^2-8x+16}$ follow. Note that they agree for $x<4$.





**53.** $x-5$. The graphs of $y=x-5$ and $y=\sqrt{x^2-10x+25}$ follow. Note that they agree for $x\geq 5$.





**55.** $3d^6\sqrt{d}$

**57.** $5x+4$

**59.** $2j^5\sqrt{j}$

**61.** $5m$

**63.** $2c^2\sqrt{c}$

**65.** $5h^5$

**67.** $3s^3\sqrt{s}$

**69.** $2p^4$

**71.** $3q^5$

**73.** $4f^4$

**75.** $8k^5$

**77.** $4e^6$

**79.** $9z^4$

# 9.3 Division Properties of Radicals

Each of the equations $x^2 = a$ and $x^2 = b$ has a unique positive solution, $x = \sqrt{a}$ and $x = \sqrt{b}$, respectively, provided $a$ and $b$ are positive real numbers. Further, because they are solutions, they can be substituted into the equations $x^2 = a$ and $x^2 = b$ to produce the results

$$(\sqrt{a})^2 = a \qquad \text{and} \qquad (\sqrt{b})^2 = b,$$

respectively. These results are dependent upon the fact that $a$ and $b$ are positive real numbers.

Similarly, the equation

$$x^2 = \frac{a}{b}$$

has the unique positive solution

$$x = \sqrt{\frac{a}{b}},$$

provided $a$ and $b$ are positive real numbers. However, note that

$$\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{(\sqrt{a})^2}{(\sqrt{b})^2} = \frac{a}{b},$$

making $\sqrt{a}/\sqrt{b}$ a **second** positive solution of $x^2 = a/b$. However, because $\sqrt{a/b}$ is the *unique* positive solution of $x^2 = a/b$, this forces

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

This discussion leads us to the following property of radicals.

**Property 1.** Let $a$ and $b$ be positive real numbers. Then,

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

This result can be used in two distinctly different ways.

- You can use the result to divide two square roots, as in

$$\frac{\sqrt{13}}{\sqrt{7}} = \sqrt{\frac{13}{7}}.$$

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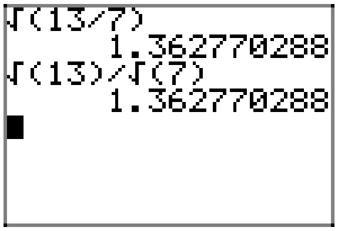
- You can also use the result to take the square root of a fraction. Simply take the square root of both numerator and denominator, as in

$$\sqrt{\frac{13}{7}}=\frac{\sqrt{13}}{\sqrt{7}}.$$

It is interesting to check these results on a calculator, as shown in **Figure 1**.

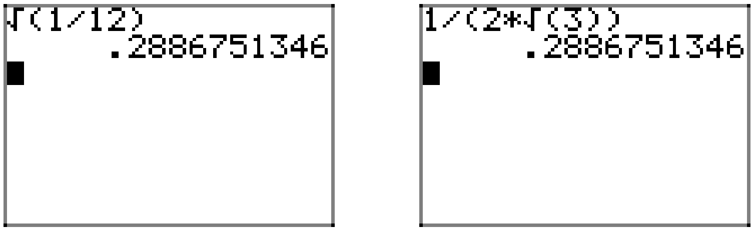


```
√(13/7)
      1.362770288
√(13)/√(7)
      1.362770288
```

**Figure 1.** Checking that $\sqrt{13/7}=\sqrt{13}/\sqrt{7}$.

## *Simple Radical Form Continued*

David and Martha are again working on a homework problem. Martha obtains the solution $\sqrt{1/12}$, but David's solution $1/(2\sqrt{3})$ is seemingly different. Having learned their lesson in an earlier assignment, they use their calculators to find decimal approximations of their solutions. Martha's approximation is shown in **Figure 2**(a) and David's approximation is shown in **Figure 2**(b).



```
√(1/12)
      .2886751346
```

```
1/(2*√(3))
      .2886751346
```

(a) Approximating Martha's $\sqrt{1/12}$.

(b) Approximating David's $1/(2\sqrt{3})$.

**Figure 2.** Comparing Martha's $\sqrt{1/12}$ with David's $1/(2\sqrt{3})$.

Martha finds that $\sqrt{1/12}\approx 0.2886751346$ and David finds that $1/(2\sqrt{3})\approx 0.2886751346$. They conclude that their answers match, but they want to know why such different looking answers are identical.

The following calculation shows why Martha's result is identical to David's. First, use the division property of radicals (**Property 1**) to take the square root of both numerator and denominator.

$$\sqrt{\frac{1}{12}}=\frac{\sqrt{1}}{\sqrt{12}}=\frac{1}{\sqrt{12}}$$

Next, use the “first guideline for simple radical form” and factor a perfect square from the denominator.

$$\frac{1}{\sqrt{12}}=\frac{1}{\sqrt{4}\sqrt{3}}=\frac{1}{2\sqrt{3}}$$

This clearly demonstrates that David and Martha’s solutions are identical.

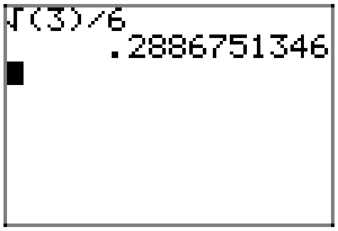
Indeed, there are other possible forms for the solution of David and Martha’s homework exercise. Start with Martha’s solution, then multiply both numerator and denominator of the fraction under the radical by 3.

$$\sqrt{\frac{1}{12}}=\sqrt{\frac{1}{12}\cdot\frac{3}{3}}=\sqrt{\frac{3}{36}}$$

Now, use the division property of radicals (**Property 1**), taking the square root of both numerator and denominator.

$$\sqrt{\frac{3}{36}}=\frac{\sqrt{3}}{\sqrt{36}}=\frac{\sqrt{3}}{6}$$

Note that the approximation of $\sqrt{3}/6$ in **Figure 3** is identical to Martha’s and David’s approximations in **Figures 2**(a) and (b).



```
√(3)/6
            .2886751346
```

**Figure 3.** Finding an approximation of $\sqrt{3}/6$.

While all three of the solution forms ($\sqrt{1/12}$, $1/(2\sqrt{3})$, and $\sqrt{3}/6$) are identical, it is very frustrating to have so many forms, particularly when we want to compare solutions. So, we are led to establish two more guidelines for simple radical form.

**The Second Guideline for Simple Radical Form.** Don’t leave fractions under a radical.

Thus, Martha’s $\sqrt{1/12}$ is not in simple radical form, because it contains a fraction under the radical.

**The Third Guideline for Simple Radical Form.** Don’t leave radicals in the denominator of a fraction.

| $x$ | $x^2$ |
|---|---|
| 2 | 4 |
| 3 | 9 |
| 4 | 16 |
| 5 | 25 |
| 6 | 36 |
| 7 | 49 |
| 8 | 64 |
| 9 | 81 |
| 10 | 100 |
| 11 | 121 |
| 12 | 144 |
| 13 | 169 |
| 14 | 196 |
| 15 | 225 |
| 16 | 256 |
| 17 | 289 |
| 18 | 324 |
| 19 | 361 |
| 20 | 400 |
| 21 | 441 |
| 22 | 484 |
| 23 | 529 |
| 24 | 576 |
| 25 | 625 |

**Table 1.** Squares.

Thus, David's $1/(2\sqrt{3})$ is not in simple radical form, because the denominator of his fraction contains a radical.

Only the equivalent form $\sqrt{3}/6$ obeys all three rules of simple radical form.

1. It is not possible to factor a perfect square from any radical in the expression $\sqrt{3}/6$.
2. There are no fractions under a radical in the expression $\sqrt{3}/6$.
3. The denominator in the expression $\sqrt{3}/6$ contains no radicals.

In this text and in this course, we will always follow the three guidelines for simple radical form.[8]

**Simple Radical Form.** When your answer is a radical expression:

1. If possible, factor out a perfect square.
2. Don't leave fractions under a radical.
3. Don't leave radicals in the denominator of a fraction.

In the examples that follow (and in the exercises), it is helpful if you know the squares of the first 25 positive integers. We've listed them in the margin for you in **Table 1** for future reference.

Let's place a few radical expressions in simple radical form. We'll start with some radical expressions that contain fractions under a radical.

► **Example 2.** *Place the expression $\sqrt{1/8}$ in simple radical form.*

The expression $\sqrt{1/8}$ contains a fraction under a radical. We could take the square root of both numerator and denominator, but that would produce $\sqrt{1}/\sqrt{8}$, which puts a radical in the denominator.

The better strategy is to change the form of 1/8 so that we have a perfect square in the denominator before taking the square root of the numerator and denominator. We note that if we multiply 8 by 2, the result is 16, a perfect square. This is hopeful, so we begin the simplification by multiplying both numerator and denominator of 1/8 by 2.

$$\sqrt{\frac{1}{8}} = \sqrt{\frac{1}{8} \cdot \frac{2}{2}} = \sqrt{\frac{2}{16}}$$

We now take the square root of both numerator and denominator. Because the denominator is now a perfect square, the result will not have a radical in the denominator.

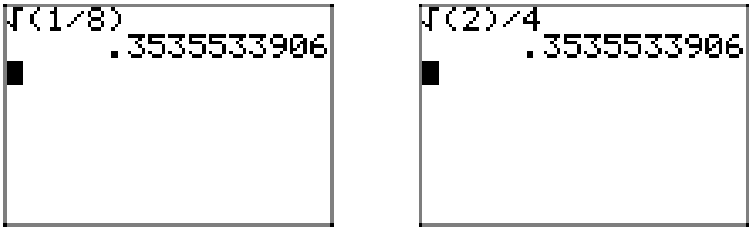
$$\sqrt{\frac{2}{16}} = \frac{\sqrt{2}}{\sqrt{16}} = \frac{\sqrt{2}}{4}$$

---

[8] In some courses, such as trigonometry and calculus, your instructor may relax these guidelines a bit. In some cases, it is easier to work with $1/\sqrt{2}$, for example, than it is to work with $\sqrt{2}/2$, even though they are equivalent.

This last result, $\sqrt{2}/4$ is in simple radical form. It is not possible to factor a perfect square from any radical, there are no fractions under any radical, and the denominator is free of radicals.

You can easily check your solution by using your calculator to compare the original expression with your simple radical form. In **Figure 4**(a), we've approximated the original expression, $\sqrt{1/8}$. In **Figure 4**(b), we've approximated our simple radical form, $\sqrt{2}/4$. Note that they yield identical decimal approximations.



```
√(1/8)
          .3535533906
```

```
√(2)/4
          .3535533906
```

(a) Approximating $\sqrt{1/8}$. (b) Approximating $\sqrt{2}/4$.

**Figure 4.** Comparing $\sqrt{1/8}$ and $\sqrt{2}/4$.

Let's look at another example.

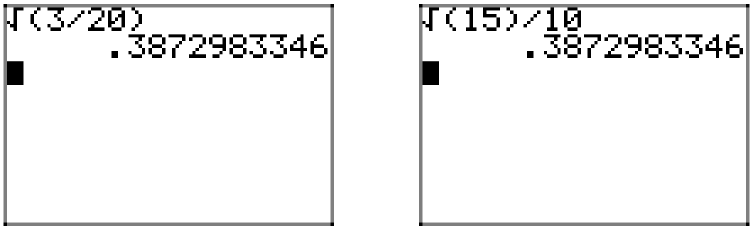
◆

▶ **Example 3.** *Place $\sqrt{3/20}$ in simple radical form.*

Following the lead from **Example 2**, we note that $5 \cdot 20 = 100$, a perfect square. So, we multiply both numerator and denominator by 5, then take the square root of both numerator and denominator once we have a perfect square in the denominator.

$$\sqrt{\frac{3}{20}} = \sqrt{\frac{3}{20} \cdot \frac{5}{5}} = \sqrt{\frac{15}{100}} = \frac{\sqrt{15}}{\sqrt{100}} = \frac{\sqrt{15}}{10}$$

Note that the decimal approximation of the simple radical form $\sqrt{15}/10$ in **Figure 5**(b) matches the decimal approximation of the original expression $\sqrt{3/20}$ in **Figure 5**(a).



```
√(3/20)
          .3872983346
```

```
√(15)/10
          .3872983346
```

(a) Approximating $\sqrt{3/20}$. (b) Approximating $\sqrt{15}/10$.

**Figure 5.** Comparing the original $\sqrt{3/20}$ with the simple radical form $\sqrt{15}/10$.

◆

We will now show how to deal with an expression having a radical in its denominator, but first we pause to explain a new piece of terminology.

**Rationalizing the Denominator.** The process of eliminating radicals from the denominator is called **rationalizing the denominator** because it results in a fraction where the denominator is free of radicals and is a rational number.

▶ **Example 4.** *Place the expression* $5/\sqrt{18}$ *in simple radical form.*

In the previous examples, making the denominator a perfect square seemed a good tactic. We apply the same tactic in this example, noting that $2 \cdot 18 = 36$ is a perfect square. However, the strategy is slightly different, as we begin the solution by multiplying both numerator and denominator by $\sqrt{2}$.

$$\frac{5}{\sqrt{18}} = \frac{5}{\sqrt{18}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

We now multiply numerators and denominators. In the denominator, the multiplication property of radicals is used, $\sqrt{18}\sqrt{2} = \sqrt{36}$.

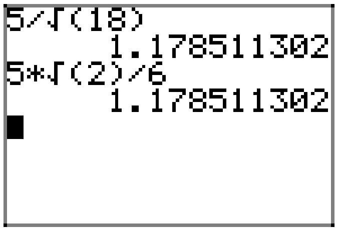
$$\frac{5}{\sqrt{18}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{\sqrt{36}}$$

The strategy should now be clear. Because the denominator is a perfect square, $\sqrt{36} = 6$, clearing all radicals from the denominator of our result.

$$\frac{5\sqrt{2}}{\sqrt{36}} = \frac{5\sqrt{2}}{6}$$

The last result is in simple radical form. It is not possible to extract a perfect square root from any radical, there are no fractions under any radical, and the denominator is free of radicals.

In **Figure 6**, we compare the approximation for our original expression $5/\sqrt{18}$ with our simple radical form $5\sqrt{2}/6$.



```
5/√(18)
      1.178511302
5*√(2)/6
      1.178511302
```

**Figure 6.** Comparing $5/\sqrt{18}$ with $5\sqrt{2}/6$.

◆

Let's look at another example.

▶ **Example 5.** *Place the expression* $18/\sqrt{27}$ *in simple radical form.*

Note that $3 \cdot 27 = 81$ is a perfect square. We begin by multiplying both numerator and denominator of our expression by $\sqrt{3}$.

$$\frac{18}{\sqrt{27}} = \frac{18}{\sqrt{27}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

Multiply numerators and denominators. In the denominator, $\sqrt{27}\sqrt{3} = \sqrt{81}$.

$$\frac{18}{\sqrt{27}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{18\sqrt{3}}{\sqrt{81}}$$

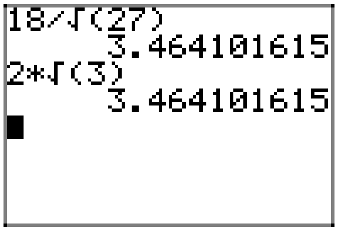
Of course, $\sqrt{81} = 9$, so

$$\frac{18\sqrt{3}}{\sqrt{81}} = \frac{18\sqrt{3}}{9}$$

We can now reduce to lowest terms, dividing numerator and denominator by 9.

$$\frac{18\sqrt{3}}{9} = 2\sqrt{3}$$

In **Figure 7**, we compare approximations of the original expression $18/\sqrt{27}$ and its simple radical form $2\sqrt{3}$.



```
18/√(27)
         3.464101615
2*√(3)
         3.464101615
█
```

**Figure 7.** Comparing $18/\sqrt{27}$ with its simple radical form $2\sqrt{3}$.

◆

## Helpful Hints

In the previous section, we learned that if you square a product of exponential expressions, you multiply each of the exponents by 2.

$$(2^3 3^4 5^5)^2 = 2^6 3^8 5^{10}$$

Because taking the square root is the "inverse" of squaring,[9] we divide each of the exponents by 2.

---

[9] As we have pointed out in previous sections, taking the positive square root is the inverse of squaring, only if we restrict the domain of the squaring function to nonnegative real numbers, which we do here.

$$\sqrt{2^6 3^8 5^{10}} = 2^3 3^4 5^5$$

We also learned that prime factorization is an extremely powerful tool that is quite useful when placing radical expressions in simple radical form. We'll see that this is even more true in this section.

Let's look at an example.

▶ **Example 6.** *Place the expression* $\sqrt{1/98}$ *in simple radical form.*

Sometimes it is not easy to figure out how to scale the denominator to get a perfect square, even when provided with a table of perfect squares. This is when prime factorization can come to the rescue and provide a hint. So, first express the denominator as a product of primes in exponential form: $98 = 2 \cdot 49 = 2 \cdot 7^2$.

$$\sqrt{\frac{1}{98}} = \sqrt{\frac{1}{2 \cdot 7^2}}$$

We can now easily see what is preventing the denominator from being a perfect square. The problem is the fact that not all of the exponents in the denominator are divisible by 2. We can remedy this by multiplying both numerator and denominator by 2.

$$\sqrt{\frac{1}{2 \cdot 7^2}} = \sqrt{\frac{1}{2 \cdot 7^2} \cdot \frac{2}{2}} = \sqrt{\frac{2}{2^2 7^2}}$$

Note that each prime in the denominator now has an exponent that is divisible by 2. We can now take the square root of both numerator and denominator.

$$\sqrt{\frac{2}{2^2 7^2}} = \frac{\sqrt{2}}{\sqrt{2^2 7^2}}$$

Take the square root of the denominator by dividing each exponent by 2.

$$\frac{\sqrt{2}}{\sqrt{2^2 7^2}} = \frac{\sqrt{2}}{2^1 \cdot 7^1}$$

Then, of course, $2 \cdot 7 = 14$.

$$\frac{\sqrt{2}}{2 \cdot 7} = \frac{\sqrt{2}}{14}$$

In **Figure 8**, note how the decimal approximations of the original expression $\sqrt{1/98}$ and its simple radical form $\sqrt{2}/14$ match, strong evidence that we've found the correct simple radical form. That is, we cannot take a perfect square out of any radical, there are no fractions under any radical, and the denominators are clear of all radicals.

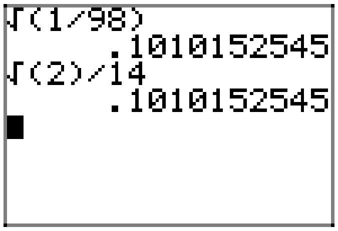
◆

Let's look at another example.

▶ **Example 7.** *Place the expression* $12/\sqrt{54}$ *in simple radical form.*

Prime factor the denominator: $54 = 2 \cdot 27 = 2 \cdot 3^3$.

$$\frac{12}{\sqrt{54}} = \frac{12}{\sqrt{2 \cdot 3^3}}$$



```
√(1/98)
          .1010152545
√(2)/14
          .1010152545
```

**Figure 8.** Comparing the original $\sqrt{1/98}$ with its simple radical form $\sqrt{2}/14$.

Neither prime in the denominator has an exponent divisible by 2. If we had another 2 and one more 3, then the exponents would be divisible by 2. This encourages us to multiply both numerator and denominator by $\sqrt{2 \cdot 3}$.

$$\frac{12}{\sqrt{2 \cdot 3^3}} = \frac{12}{\sqrt{2 \cdot 3^3}} \cdot \frac{\sqrt{2 \cdot 3}}{\sqrt{2 \cdot 3}} = \frac{12\sqrt{2 \cdot 3}}{\sqrt{2^2 3^4}}$$

Divide each of the exponents in the denominator by 2.

$$\frac{12\sqrt{2 \cdot 3}}{\sqrt{2^2 3^4}} = \frac{12\sqrt{2 \cdot 3}}{2^1 \cdot 3^2}$$

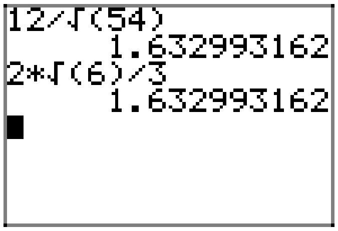
Then, in the numerator, $2 \cdot 3 = 6$, and in the denominator, $2 \cdot 3^2 = 18$.

$$\frac{12\sqrt{2 \cdot 3}}{2 \cdot 3^2} = \frac{12\sqrt{6}}{18}$$

Finally, reduce to lowest terms by dividing both numerator and denominator by 6.

$$\frac{12\sqrt{6}}{18} = \frac{2\sqrt{6}}{3}$$

In **Figure 9**, the approximation for the original expression $12/\sqrt{54}$ matches that of its simple radical form $2\sqrt{6}/3$.



```
12/√(54)
          1.632993162
2*√(6)/3
          1.632993162
```

**Figure 9.** Comparing approximations of the original expression $12/\sqrt{54}$ with its simple radical form $2\sqrt{6}/3$.

◆

## Variable Expressions

If $x$ is any real number, recall again that

$$\sqrt{x^2} = |x|.$$

If we combine the law of exponents for squaring a quotient with our property for taking the square root of a quotient, we can write

$$\sqrt{\left(\frac{a}{b}\right)^2} = \sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}}$$

However, $\sqrt{(a/b)^2} = |a/b|$, while $\sqrt{a^2}/\sqrt{b^2} = |a|/|b|$. This discussion leads to the following key result.

**Quotient Rule for Absolute Value.** If $a$ and $b$ are any real numbers, then

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|},$$

provided $b \neq 0$. In words, the absolute value of a quotient is the quotient of the absolute values.

We saw this property previously in the chapter on the absolute value function, where we provided a different approach to the proof of the property. It's interesting that we can prove this property in a completely new way using the properties of square root. We'll see we have need for the Quotient Rule for Absolute Value in the examples that follow.

For example, if $x$ is any real number except zero, using the quotient rule for absolute value we could write

$$\left|\frac{3}{x}\right| = \frac{|3|}{|x|} = \frac{3}{|x|}.$$

However, there is no way to remove the absolute value bars that surround $x$ unless we know the sign of $x$. If $x > 0$ (remember, no zeros in the denominator), then $|x| = x$ and the expression becomes

$$\frac{3}{|x|} = \frac{3}{x}.$$

On the other hand, if $x < 0$, then $|x| = -x$ and the expression becomes

$$\frac{3}{|x|} = \frac{3}{-x} = -\frac{3}{x}.$$

Let's look at another example.

▶ **Example 8.** *Place the expression* $\sqrt{18/x^6}$ *in simple radical form. Discuss the domain.*

Note that $x$ cannot equal zero, otherwise the denominator of $\sqrt{18/x^6}$ would be zero, which is not allowed. However, whether $x$ is positive or negative, $x^6$ will be a positive number (raising a nonzero number to an even power always produces a positive real number), and $\sqrt{18/x^6}$ is well-defined.

Keeping in mind that $x$ is nonzero, but could either be positive or negative, we proceed by first invoking **Property 1**, taking the positive square root of both numerator and denominator of our radical expression.

$$\sqrt{\frac{18}{x^6}} = \frac{\sqrt{18}}{\sqrt{x^6}}$$

From the numerator, we factor a perfect square. In the denominator, we use absolute value bars to insure a positive square root.

$$\frac{\sqrt{18}}{\sqrt{x^6}} = \frac{\sqrt{9}\sqrt{2}}{|x^3|} = \frac{3\sqrt{2}}{|x^3|}$$

We can use the Product Rule for Absolute Value to write $|x^3| = |x^2||x| = x^2|x|$. Note that we do not need to wrap $x^2$ in absolute value bars because $x^2$ is already positive.

$$\frac{3\sqrt{2}}{|x^3|} = \frac{3\sqrt{2}}{x^2|x|}$$

Because $x$ could be positive or negative, we cannot remove the absolute value bars around $x$. We are done.

◇

Let's look at another example.

▶ **Example 9.** *Place the expression* $\sqrt{12/x^5}$ *in simple radical form. Discuss the domain.*

Note that $x$ cannot equal zero, otherwise the denominator of $\sqrt{12/x^5}$ would be zero, which is not allowed. Further, if $x$ is a negative number, then $x^5$ will also be a negative number (raising a negative number to an odd power produces a negative number). If $x$ were negative, then $12/x^5$ would also be negative and $\sqrt{12/x^5}$ would be undefined (you cannot take the square root of a negative number). Thus, $x$ must be a positive real number or the expression $\sqrt{12/x^5}$ is undefined.

We proceed, keeping in mind that $x$ is a positive real number. One possible approach is to first note that another factor of $x$ is needed to make the denominator a perfect square. This motivates us to multiply both numerator and denominator inside the radical by $x$.

$$\sqrt{\frac{12}{x^5}} = \sqrt{\frac{12}{x^5} \cdot \frac{x}{x}} = \sqrt{\frac{12x}{x^6}}.$$

We can now use **Property 1** to take the square root of both numerator and denominator.

$$\sqrt{\frac{12x}{x^6}} = \frac{\sqrt{12x}}{\sqrt{x^6}}$$

In the numerator, we factor out a perfect square. In the denominator, absolute value bars would insure a positive square root. However, we've stated that $x$ must be a positive number, so $x^3$ is already positive and absolute value bars are not needed.

$$\frac{\sqrt{12x}}{\sqrt{x^6}} = \frac{\sqrt{4}\sqrt{3x}}{x^3} = \frac{2\sqrt{3x}}{x^3}$$

◇

Let's look at another example.

▶ **Example 10.** *Given that $x < 0$, place $\sqrt{27/x^{10}}$ in simple radical form.*

One possible approach would be to factor out a perfect square and write

$$\sqrt{\frac{27}{x^{10}}} = \sqrt{\frac{9}{x^{10}}}\sqrt{3} = \sqrt{\left(\frac{3}{x^5}\right)^2}\sqrt{3} = \left|\frac{3}{x^5}\right|\sqrt{3}.$$

Now, $|3/x^5| = |3|/(|x^4||x|) = 3/(x^4|x|)$, since $x^4 > 0$. Thus,

$$\left|\frac{3}{x^5}\right|\sqrt{3} = \frac{3}{x^4|x|}\sqrt{3}.$$

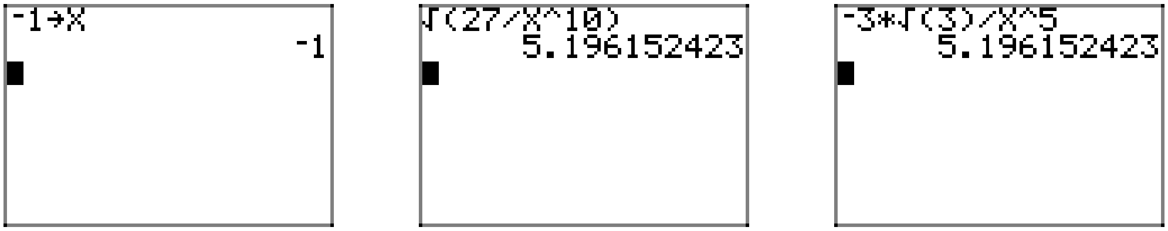
However, we are given that $x < 0$, so $|x| = -x$ and we can write

$$\frac{3}{x^4|x|}\sqrt{3} = \frac{3}{(x^4)(-x)}\sqrt{3} = -\frac{3}{x^5}\sqrt{3}.$$

We can move $\sqrt{3}$ into the numerator and write

$$-\frac{3}{x^5}\sqrt{3} = -\frac{3\sqrt{3}}{x^5}. \qquad (11)$$

Again, it's instructive to test the validity of this result using your graphing calculator. Supposedly, the result is true for all values of $x < 0$. So, store $-1$ in $x$, then enter the original expression and its simple radical form, then compare the approximations, as shown in **Figures 10**(a), (b), and (c).



```
-1→X
                -1
```

```
√(27/X^10)
       5.196152423
```

```
-3*√(3)/X^5
       5.196152423
```

(a) Store $-1$ in $x$.

(b) Approximate $\sqrt{27/x^{10}}$.

(c) Approximate $-3\sqrt{3}/x^5$.

**Figure 10.** Comparing the original expression and its simple radical form at $x = -1$.

**Alternative approach.** A slightly different approach would again begin by taking the square root of both numerator and denominator.

$$\sqrt{\frac{27}{x^{10}}}=\frac{\sqrt{27}}{\sqrt{x^{10}}}$$

Now, $\sqrt{27}=\sqrt{9}\sqrt{3}=3\sqrt{3}$ and we insure that $\sqrt{x^{10}}$ produces a positive number by using absolute value bars. That is, $\sqrt{x^{10}}=|x^5|$ and

$$\frac{\sqrt{27}}{\sqrt{x^{10}}}=\frac{3\sqrt{3}}{|x^5|}.$$

However, using the product rule for absolute value and the fact that $x^4>0$, $|x^5|=|x^4||x|=x^4|x|$ and

$$\frac{3\sqrt{3}}{|x^5|}=\frac{3\sqrt{3}}{x^4|x|}.$$

Finally, we are given that $x<0$, so $|x|=-x$ and we can write

$$\frac{3\sqrt{3}}{x^4|x|}=\frac{3\sqrt{3}}{(x^4)(-x)}=-\frac{3\sqrt{3}}{x^5}. \tag{12}$$

Note that the simple radical form (**12**) of our alternative approach matches perfectly the simple radical form (**11**) of our first approach.

◆

## 9.3 Exercises

**1.** Use a calculator to first approximate $\sqrt{5}/\sqrt{2}$. On the same screen, approximate $\sqrt{5/2}$. Report the results on your homework paper.

**2.** Use a calculator to first approximate $\sqrt{7}/\sqrt{5}$. On the same screen, approximate $\sqrt{7/5}$. Report the results on your homework paper.

**3.** Use a calculator to first approximate $\sqrt{12}/\sqrt{2}$. On the same screen, approximate $\sqrt{6}$. Report the results on your homework paper.

**4.** Use a calculator to first approximate $\sqrt{15}/\sqrt{5}$. On the same screen, approximate $\sqrt{3}$. Report the results on your homework paper.

In **Exercises 5-16**, place each radical expression in simple radical form. As in Example 2 in the narrative, check your result with your calculator.

**5.** $\sqrt{\dfrac{3}{8}}$

**6.** $\sqrt{\dfrac{5}{12}}$

**7.** $\sqrt{\dfrac{11}{20}}$

**8.** $\sqrt{\dfrac{3}{2}}$

**9.** $\sqrt{\dfrac{11}{18}}$

**10.** $\sqrt{\dfrac{7}{5}}$

**11.** $\sqrt{\dfrac{4}{3}}$

**12.** $\sqrt{\dfrac{16}{5}}$

**13.** $\sqrt{\dfrac{49}{12}}$

**14.** $\sqrt{\dfrac{81}{20}}$

**15.** $\sqrt{\dfrac{100}{7}}$

**16.** $\sqrt{\dfrac{36}{5}}$

In **Exercises 17-28**, place each radical expression in simple radical form. As in Example 4 in the narrative, check your result with your calculator.

**17.** $\dfrac{1}{\sqrt{12}}$

**18.** $\dfrac{1}{\sqrt{8}}$

**19.** $\dfrac{1}{\sqrt{20}}$

**20.** $\dfrac{1}{\sqrt{27}}$

**21.** $\dfrac{6}{\sqrt{8}}$

**22.** $\dfrac{4}{\sqrt{12}}$

---

[10] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

**23.** $\dfrac{5}{\sqrt{20}}$

**24.** $\dfrac{9}{\sqrt{27}}$

**25.** $\dfrac{6}{2\sqrt{3}}$

**26.** $\dfrac{10}{3\sqrt{5}}$

**27.** $\dfrac{15}{2\sqrt{20}}$

**28.** $\dfrac{3}{2\sqrt{18}}$

---

In **Exercises 29-36**, place the given radical expression in simple form. Use prime factorization as in Example 8 in the narrative to help you with the calculations. As in Example 6, check your result with your calculator.

**29.** $\dfrac{1}{\sqrt{96}}$

**30.** $\dfrac{1}{\sqrt{432}}$

**31.** $\dfrac{1}{\sqrt{250}}$

**32.** $\dfrac{1}{\sqrt{108}}$

**33.** $\sqrt{\dfrac{5}{96}}$

**34.** $\sqrt{\dfrac{2}{135}}$

**35.** $\sqrt{\dfrac{2}{1485}}$

**36.** $\sqrt{\dfrac{3}{280}}$

---

In **Exercises 37-44**, place each of the given radical expressions in simple radical form. Make no assumptions about the sign of any variable. Variables can represent either positive or negative numbers.

**37.** $\sqrt{\dfrac{8}{x^4}}$

**38.** $\sqrt{\dfrac{12}{x^6}}$

**39.** $\sqrt{\dfrac{20}{x^2}}$

**40.** $\sqrt{\dfrac{32}{x^{14}}}$

**41.** $\dfrac{2}{\sqrt{8x^8}}$

**42.** $\dfrac{3}{\sqrt{12x^6}}$

**43.** $\dfrac{10}{\sqrt{20x^{10}}}$

**44.** $\dfrac{12}{\sqrt{6x^4}}$

---

In **Exercises 45-48**, follow the lead of Example 8 in the narrative to craft a solution.

**45.** Given that $x < 0$, place the radical expression $6/\sqrt{2x^6}$ in simple radical form. Check your solution on your calculator for $x = -1$.

**46.** Given that $x > 0$, place the radical expression $4/\sqrt{12x^3}$ in simple radical form. Check your solution on your calculator for $x = 1$.

**47.** Given that $x > 0$, place the radical expression $8/\sqrt{8x^5}$ in simple radical form. Check your solution on your calculator for $x = 1$.

**48.** Given that $x < 0$, place the radical expression $15/\sqrt{20x^6}$ in simple radical form. Check your solution on your calculator for $x = -1$.

---

In **Exercises 49-56**, place each of the radical expressions in simple form. Assume that all variables represent positive numbers.

**49.** $\sqrt{\dfrac{12}{x}}$

**50.** $\sqrt{\dfrac{18}{x}}$

**51.** $\sqrt{\dfrac{50}{x^3}}$

**52.** $\sqrt{\dfrac{72}{x^5}}$

**53.** $\dfrac{1}{\sqrt{50x}}$

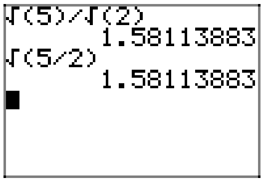
**54.** $\dfrac{2}{\sqrt{18x}}$

**55.** $\dfrac{3}{\sqrt{27x^3}}$

**56.** $\dfrac{5}{\sqrt{10x^5}}$

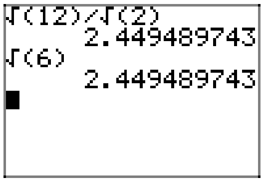
## 9.3 Answers

**1.**



```
√(5)/√(2)
          1.58113883
√(5/2)
          1.58113883
```

**3.**



```
√(12)/√(2)
         2.449489743
√(6)
         2.449489743
```

**5.** $\sqrt{6}/4$

**7.** $\sqrt{55}/10$

**9.** $\sqrt{22}/6$

**11.** $2\sqrt{3}/3$

**13.** $7\sqrt{3}/6$

**15.** $10\sqrt{7}/7$

**17.** $\sqrt{3}/6$

**19.** $\sqrt{5}/10$

**21.** $3\sqrt{2}/2$

**23.** $\sqrt{5}/2$

**25.** $\sqrt{3}$

**27.** $3\sqrt{5}/4$

**29.** $\sqrt{6}/24$

**31.** $\sqrt{10}/50$

**33.** $\sqrt{30}/24$

**35.** $\sqrt{330}/495$

**37.** $2\sqrt{2}/x^2$

**39.** $2\sqrt{5}/|x|$

**41.** $\sqrt{2}/(2x^4)$

**43.** $\sqrt{5}/(x^4|x|)$

**45.** $-3\sqrt{2}/x^3$

**47.** $2\sqrt{2x}/x^3$

**49.** $2\sqrt{3x}/x$

**51.** $5\sqrt{2x}/x^2$

**53.** $\sqrt{2x}/(10x)$

**55.** $\sqrt{3x}/(3x^2)$

# 9.4 Radical Expressions

In the previous two sections, we learned how to multiply and divide square roots. Specifically, we are now armed with the following two properties.

**Property 1.** Let $a$ and $b$ be any two real nonnegative numbers. Then,

$$\sqrt{a}\sqrt{b} = \sqrt{ab},$$

and, provided $b \neq 0$,

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}.$$

In this section, we will simplify a number of more extensive expressions containing square roots, particularly those that are fundamental to your work in future mathematics courses.

Let's begin by building some fundamental skills.

## *The Associative Property*

We recall the associative property of multiplication.

**Associative Property of Multiplication.** Let $a$, $b$, and $c$ be any real numbers. The *associative property of multiplication* states that

$$(ab)c = a(bc). \tag{2}$$

Note that the order of the numbers on each side of **equation (2)** has not changed. The numbers on each side of the equation are in the order $a$, $b$, and then $c$.

However, the grouping has changed. On the left, the parentheses around the product of $a$ and $b$ instruct us to perform that product first, then multiply the result by $c$. On the right, the grouping is different; the parentheses around $b$ and $c$ instruct us to perform that product first, then multiply by $a$. The key point to understand is the fact that the different groupings make no difference. We get the same answer in either case.

For example, consider the product $2 \cdot 3 \cdot 4$. If we multiply 2 and 3 first, then multiply the result by 4, we get

$$(2 \cdot 3) \cdot 4 = 6 \cdot 4 = 24.$$

On the other hand, if we multiply 3 and 4 first, then multiply the result by 2, we get

$$2 \cdot (3 \cdot 4) = 2 \cdot 12 = 24.$$

---

[11] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

Note that we get the same result in either case. That is,

$$(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4).$$

The associative property, seemingly trivial, takes on an extra level of sophistication if we apply it to expressions containing radicals. Let's look at an example.

► **Example 3.** *Simplify the expression* $3(2\sqrt{5})$*. Place your answer in simple radical form.*

Currently, the parentheses around 2 and $\sqrt{5}$ require that we multiply those two numbers first. However, the associative property of multiplication allows us to regroup, placing the parentheses around 3 and 2, multiplying those two numbers first, then multiplying the result by $\sqrt{5}$. We arrange the work as follows.

$$3(2\sqrt{5}) = (3 \cdot 2)\sqrt{5} = 6\sqrt{5}.$$

Readers should note the similarity to a very familiar manipulation.

$$3(2x) = (3 \cdot 2)x = 6x$$

◆

In practice, when we became confident with this regrouping, we began to skip the intermediate step and simply state that $3(2x) = 6x$. In a similar vein, once you become confident with regrouping, you should simply state that $3(2\sqrt{5}) = 6\sqrt{5}$. If called upon to explain your answer, you must be ready to explain how you regrouped according to the associative property of multiplication. Similarly,

$$-4(5\sqrt{7}) = -20\sqrt{7}, \quad 12(5\sqrt{11}) = 60\sqrt{11}, \quad \text{and} \quad -5(-3\sqrt{3}) = 15\sqrt{3}.$$

## The Commutative Property of Multiplication

We recall the commutative property of multiplication.

**Commutative Property of Multiplication.** Let $a$ and $b$ be any real numbers. The *commutative property of multiplication* states that

$$ab = ba. \tag{4}$$

The commutative property states that the order of multiplication is irrelevant. For example, $2 \cdot 3$ is the same as $3 \cdot 2$; they both equal 6. This seemingly trivial property, coupled with the associative property of multiplication, allows us to change the order of multiplication and regroup as we please.

▶ **Example 5.** *Simplify the expression* $\sqrt{5}(2\sqrt{3})$. *Place your answer in simple radical form.*

What we'd really like to do is first multiply $\sqrt{5}$ and $\sqrt{3}$. In order to do this, we must first regroup, then switch the order of multiplication as follows.

$$\sqrt{5}(2\sqrt{3}) = (\sqrt{5} \cdot 2)\sqrt{3} = (2\sqrt{5})\sqrt{3}$$

This is allowed by the associative and commutative properties of multiplication. Now, we regroup again and multiply.

$$(2\sqrt{5})\sqrt{3} = 2(\sqrt{5}\sqrt{3}) = 2\sqrt{15}$$

◆

In practice, this is far too much work for such a simple calculation. Once we understand the associative and commutative properties of multiplication, the expression $a \cdot b \cdot c$ is unambiguous. Parentheses are not needed. We know that we can change the order of multiplication and regroup as we please. Therefore, when presented with the product of three numbers, simply multiply two of your choice together, then multiply the result by the third remaining number.

In the case of $\sqrt{5}(2\sqrt{3})$, we choose to first multiply $\sqrt{5}$ and $\sqrt{3}$, which is $\sqrt{15}$, then multiply this result by 2 to get $2\sqrt{15}$. Similarly,

$$\sqrt{5}(2\sqrt{7}) = 2\sqrt{35} \qquad \text{and} \qquad \sqrt{x}(3\sqrt{5}) = 3\sqrt{5x}.$$

▶ **Example 6.** *Simplify the expression* $\sqrt{6}(4\sqrt{8})$. *Place your answer in simple radical form.*

We start by multiplying $\sqrt{6}$ and $\sqrt{8}$, then the result by 4.

$$\sqrt{6}(4\sqrt{8}) = 4\sqrt{48}$$

Now, $48 = 16 \cdot 3$, so we can extract a perfect square.

$$4\sqrt{48} = 4(\sqrt{16}\sqrt{3}) = 4(4\sqrt{3})$$

Again, we choose to multiply the fours, then the result by the square root of three. That is,

$$4(4\sqrt{3}) = 16\sqrt{3}.$$

◆

By induction, we can argue that the associative and commutative properties will allow us to group and arrange the product of more than three numbers in any order that we please.

▶ **Example 7.** *Simplify the expression* $(2\sqrt{12})(3\sqrt{3})$. *Place your answer in simple radical form.*

We'll first take the product of 2 and 3, then the product of $\sqrt{12}$ and $\sqrt{3}$, then multiply these results together.

$$(2\sqrt{12})(3\sqrt{3}) = (2 \cdot 3)(\sqrt{12}\sqrt{3}) = 6\sqrt{36}$$

Of course, $\sqrt{36} = 6$, so we can simplify further.

$$6\sqrt{36} = 6 \cdot 6 = 36$$

◇

## The Distributive Property

Recall the distributive property for real numbers.

**Distributive Property.** Let $a$, $b$, and $c$ be any real numbers. Then,

$$a(b + c) = ab + ac. \tag{8}$$

You might recall the following operation, where you "distribute the 2," multiplying each term in the parentheses by 2.

$$2(3 + x) = 6 + 2x$$

You can do precisely the same thing with radical expressions.

$$2(3 + \sqrt{5}) = 6 + 2\sqrt{5}$$

Like the familiar example above, we "distributed the 2," multiplying each term in the parentheses by 2.

Let's look at more examples.

▶ **Example 9.** *Use the distributive property to expand the expression* $\sqrt{12}(3+\sqrt{3})$, *placing your final answer in simple radical form.*

First, distribute the $\sqrt{12}$, multiplying each term in the parentheses by $\sqrt{12}$. Note that $\sqrt{12}\sqrt{3} = \sqrt{36}$.

$$\sqrt{12}(3 + \sqrt{3}) = 3\sqrt{12} + \sqrt{36} = 3\sqrt{12} + 6$$

However, this last expression is not in simple radical form, as we can factor out a perfect square ($12 = 4 \cdot 3$).

$$\begin{aligned} 3\sqrt{12} + 6 &= 3(\sqrt{4}\sqrt{3}) + 6 \\ &= 3(2\sqrt{3}) + 6 \\ &= 6\sqrt{3} + 6 \end{aligned}$$

◆

It doesn't matter whether the monomial factor is in the front or rear of the sum, you still distribute the monomial times each term in the parentheses.

▶ **Example 10.** *Use the distributive property to expand* $(\sqrt{3}+2\sqrt{2})\sqrt{6}$*. Place your answer in simple radical form.*

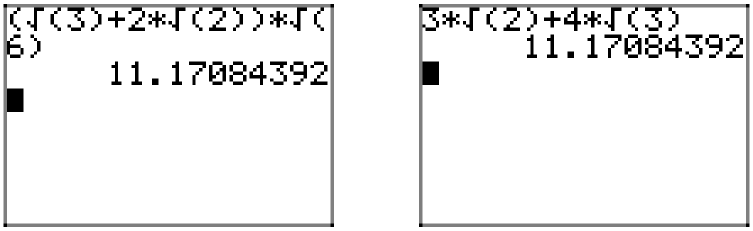
First, multiply each term in the parentheses by $\sqrt{6}$.

$$(\sqrt{3} + 2\sqrt{2})\sqrt{6} = \sqrt{18} + 2\sqrt{12}$$

To obtain the second term of this result, we chose to first multiply $\sqrt{2}$ and $\sqrt{6}$, which is $\sqrt{12}$, then we multiplied this result by 2. Now, we can factor perfect squares from both 18 and 12.

$$\begin{aligned} \sqrt{18} + 2\sqrt{12} &= \sqrt{9}\sqrt{2} + 2(\sqrt{4}\sqrt{3}) \\ &= 3\sqrt{2} + 2(2\sqrt{3}) \\ &= 3\sqrt{2} + 4\sqrt{3} \end{aligned}$$

Remember, you can check your results with your calculator. In **Figure 1**(a), we've found a decimal approximation for the original expression $(\sqrt{3} + 2\sqrt{2})\sqrt{6}$, and in **Figure 1**(b) we have a decimal approximation for our solution $3\sqrt{2} + 4\sqrt{3}$. Note that they are the same, providing evidence that our solution is correct.



```
(√(3)+2*√(2))*√(
6)
       11.17084392
```

```
3*√(2)+4*√(3)
       11.17084392
```

(a) Approximating $(\sqrt{3} + 2\sqrt{2})\sqrt{6}$.

(b) Approximating $3\sqrt{2} + 4\sqrt{3}$.

**Figure 1.** Comparing the original expression with its simple radical form.

◆

The distributive property is also responsible in helping us combine "like terms." For example, you might remember that $3x + 5x = 8x$, a seemingly simple calculation, but it is the distributive property that actually provides this solution. Note how we use the distributive property to factor $x$ from each term.

$$3x + 5x = (3+5)x$$

Hence, $3x + 5x = 8x$. You can do the same thing with radical expressions.

$$3\sqrt{2} + 5\sqrt{2} = (3+5)\sqrt{2}$$

Hence, $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$, and the structure of this result is identical to that shown in $3x + 5x = 8x$. There is no difference in the way we combine these "like terms." We repeat the common factor and add coefficients. For example,

$$2\sqrt{3} + 9\sqrt{3} = 11\sqrt{3}, \quad -4\sqrt{2} + 2\sqrt{2} = -2\sqrt{2}, \quad \text{and} \quad -3x\sqrt{x} + 5x\sqrt{x} = 2x\sqrt{x}.$$

In each case above, we're adding "like terms," by repeating the common factor and adding coefficients.

In the case that we don't have like terms, as in $3x + 5y$, there is nothing to be done. In like manner, each of the following expressions have no like terms that you can combine. They are as simplified as they are going to get.

$$3\sqrt{2} + 5\sqrt{3}, \quad 2\sqrt{11} - 8\sqrt{10}, \quad \text{and} \quad 2\sqrt{x} + 5\sqrt{y}$$

However, there are times when it can look as if you don't have like terms, but when you place everything in simple radical form, you discover that you do have like terms that can be combined by adding coefficients.

► **Example 11.** *Simplify the expression $5\sqrt{27} + 8\sqrt{3}$, placing the final expression in simple radical form.*

We can extract a perfect square ($27 = 9 \cdot 3$).

$$\begin{aligned} 5\sqrt{27} + 8\sqrt{3} &= 5(\sqrt{9}\sqrt{3}) + 8\sqrt{3} \\ &= 5(3\sqrt{3}) + 8\sqrt{3} \\ &= 15\sqrt{3} + 8\sqrt{3} \end{aligned}$$

Note that we now have "like terms" that can be combined by adding coefficients.

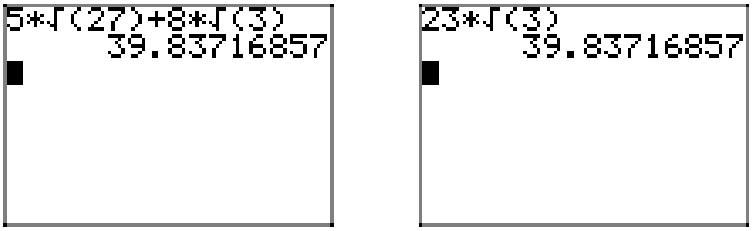
$$15\sqrt{3} + 8\sqrt{3} = 23\sqrt{3}$$

A comparison of the original expression and its simplified form is shown in **Figures 2**(a) and (b).

◆

► **Example 12.** *Simplify the expression $2\sqrt{20} + \sqrt{8} + 3\sqrt{5} + 4\sqrt{2}$, placing the result in simple radical form.*

We can extract perfect squares ($20 = 4 \cdot 5$ and $8 = 4 \cdot 2$).



```
5*√(27)+8*√(3)
      39.83716857
```

```
23*√(3)
      39.83716857
```

(a) Approximating $5\sqrt{27}+8\sqrt{3}$.

(b) Approximating $23\sqrt{3}$.

**Figure 2.** Comparing the original expression with its simplified form.

$$\begin{aligned}2\sqrt{20}+\sqrt{8}+3\sqrt{5}+4\sqrt{2} &= 2(\sqrt{4}\sqrt{5})+\sqrt{4}\sqrt{2}+3\sqrt{5}+4\sqrt{2}\\ &= 2(2\sqrt{5})+2\sqrt{2}+3\sqrt{5}+4\sqrt{2}\\ &= 4\sqrt{5}+2\sqrt{2}+3\sqrt{5}+4\sqrt{2}\end{aligned}$$

Now we can combine like terms by adding coefficients.

$$4\sqrt{5}+2\sqrt{2}+3\sqrt{5}+4\sqrt{2}=7\sqrt{5}+6\sqrt{2}$$

◆

Fractions can be a little tricky.

▶ **Example 13.** *Simplify* $\sqrt{27}+1/\sqrt{12}$, *placing the result in simple radical form.*

We can extract a perfect square root $(27 = 9 \cdot 3)$. The denominator in the second term is $12 = 2^2 \cdot 3$, so one more 3 is needed in the denominator to make a perfect square.

$$\begin{aligned}\sqrt{27}+\frac{1}{\sqrt{12}} &= \sqrt{9}\sqrt{3}+\frac{1}{\sqrt{12}}\cdot\frac{\sqrt{3}}{\sqrt{3}}\\ &= 3\sqrt{3}+\frac{\sqrt{3}}{\sqrt{36}}\\ &= 3\sqrt{3}+\frac{\sqrt{3}}{6}\end{aligned}$$

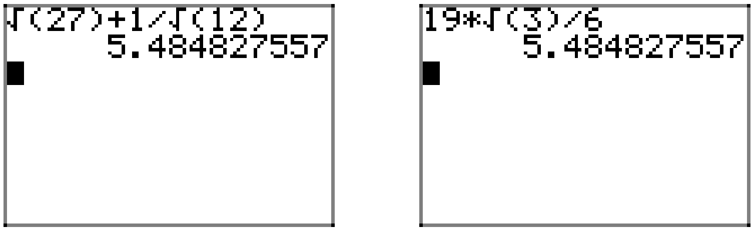
To add these fractions, we need a common denominator of 6.

$$\begin{aligned}3\sqrt{3}+\frac{\sqrt{3}}{6} &= \frac{3\sqrt{3}}{1}\cdot\frac{6}{6}+\frac{\sqrt{3}}{6}\\ &= \frac{18\sqrt{3}}{6}+\frac{\sqrt{3}}{6}\end{aligned}$$

We can now combine numerators by adding coefficients.

$$\frac{18\sqrt{3}}{6}+\frac{\sqrt{3}}{6}=\frac{19\sqrt{3}}{6}$$

Decimal approximations of the original expression and its simplified form are shown in **Figures 3**(a) and (b).



```
√(27)+1/√(12)
      5.484827557
```

```
19*√(3)/6
      5.484827557
```

(a) Approximating $\sqrt{27}+1/\sqrt{12}$.

(b) Approximating $19\sqrt{3}/6$.

**Figure 3.** Comparing the original expression and its simple radical form.

◆

At first glance, the lack of a monomial in the product $(x+1)(x+3)$ makes one think that the distributive property will not help us find the product. However, if we think of the second factor as a single unit, we can distribute it times each term in the first factor.

$$(x+1)(x+3) = x(x+3) + 1(x+3)$$

Apply the distributive property a second time, then combine like terms.

$$\begin{aligned} x(x+3)+1(x+3) &= x^2+3x+x+3 \\ &= x^2+4x+3 \end{aligned}$$

We can handle products with radical expressions in the same manner.

▶ **Example 14.** *Simplify* $(2+\sqrt{2})(3+5\sqrt{2})$. *Place your result in simple radical form.*

Think of the second factor as a single unit and distribute it times each term in the first factor.

$$(2+\sqrt{2})(3+5\sqrt{2}) = 2(3+5\sqrt{2}) + \sqrt{2}(3+5\sqrt{2})$$

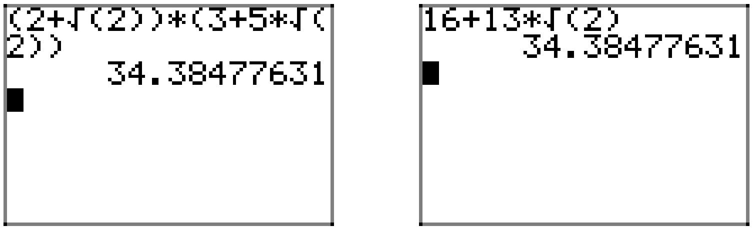
Now, use the distributive property again.

$$2(3+5\sqrt{2}) + \sqrt{2}(3+5\sqrt{2}) = 6+10\sqrt{2}+3\sqrt{2}+5\sqrt{4}$$

Note that in finding the last term, $\sqrt{2}\sqrt{2}=\sqrt{4}$. Now, $\sqrt{4}=2$, then we can combine like terms.

$$\begin{aligned} 6+10\sqrt{2}+3\sqrt{2}+5\sqrt{4} &= 6+10\sqrt{2}+3\sqrt{2}+5(2) \\ &= 6+10\sqrt{2}+3\sqrt{2}+10 \\ &= 16+13\sqrt{2} \end{aligned}$$

Decimal approximations of the original expression and its simple radical form are shown in **Figures 4**(a) and (b).



```
(2+√(2))*(3+5*√(
2))
      34.38477631
```

```
16+13*√(2)
      34.38477631
```

(a) Approximating $(2+\sqrt{2})(3+5\sqrt{2})$.

(b) Approximating $16+13\sqrt{2}$.

**Figure 4.** Comparing the original expression with its simple radical form.

◆

## Special Products

There are two special products that have important applications involving radical expressions, perhaps one more than the other. The first is the well-known difference of two squares pattern.

**Difference of Squares.** Let $a$ and $b$ be any numbers. Then,

$$(a+b)(a-b)=a^2-b^2.$$

This pattern involves two binomial factors having identical first and second terms, the terms in one factor separated by a plus sign, the terms in the other factor separated by a minus sign. When we see this pattern of multiplication, we should square the first term of either factor, square the second term, then subtract the results. For example,

$$(2x+3)(2x-3)=4x^2-9.$$

This special product applies equally well when the first and/or second terms involve radical expressions.

► **Example 15.** *Use the difference of squares pattern to multiply* $(2+\sqrt{11})(2-\sqrt{11})$.

Note that this multiplication has the form $(a+b)(a-b)$, so we apply the difference of squares pattern to get

$$(2+\sqrt{11})(2-\sqrt{11})=(2)^2-(\sqrt{11})^2.$$

Of course, $2^2=4$ and $(\sqrt{11})^2=11$, so we can finish as follows.

$$(2)^2-(\sqrt{11})^2=4-11=-7$$

◆

▶ **Example 16.** *Use the difference of squares pattern to multiply* $(2\sqrt{5}+3\sqrt{7})(2\sqrt{5}-3\sqrt{7})$.

Again, this product has the special form $(a+b)(a-b)$, so we apply the difference of squares pattern to get

$$(2\sqrt{5}+3\sqrt{7})(2\sqrt{5}-3\sqrt{7}) = (2\sqrt{5})^2 - (3\sqrt{7})^2.$$

Next, we square a product of two factors according to the rule $(ab)^2 = a^2b^2$. Thus,

$$(2\sqrt{5})^2 = (2)^2(\sqrt{5})^2 = 4 \cdot 5 = 20$$

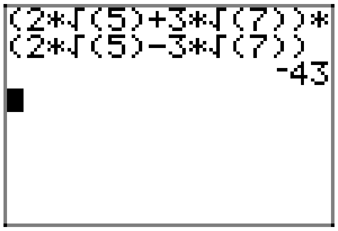
and

$$(3\sqrt{7})^2 = (3)^2(\sqrt{7})^2 = 9 \cdot 7 = 63.$$

Thus, we can complete the multiplication $(2\sqrt{5}+3\sqrt{7})(2\sqrt{5}-3\sqrt{7})$ with

$$(2\sqrt{5})^2 - (3\sqrt{7})^2 = 20 - 63 = -43.$$

This result is easily verified with a calculator, as shown in **Figure 5**.



```
(2*√(5)+3*√(7))*
(2*√(5)-3*√(7))
             -43
```

**Figure 5.** Approximating $(2\sqrt{5}+3\sqrt{7})(2\sqrt{5}-3\sqrt{7})$.

◇

The second pattern of interest is the shortcut for squaring a binomial.

**Squaring a Binomial.** Let $a$ and $b$ be numbers. Then,

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Here we square the first and second terms of the binomial, then produce the middle term of the result by multiplying the first and second terms and doubling that result. For example,

$$(2x+9)^2 = (2x)^2 + 2(2x)(9) + (9)^2 = 4x^2 + 36x + 81.$$

This pattern can also be applied to binomials containing radical expressions.

▶ **Example 17.** *Use the squaring a binomial pattern to expand* $(2\sqrt{x}+\sqrt{5})^2$. *Place your result in simple radical form. Assume that* $x$ *is a positive real number* ($x > 0$).

Applying the squaring a binomial pattern, we get

$$(2\sqrt{x}+\sqrt{5})^2 = (2\sqrt{x})^2 + 2(2\sqrt{x})(\sqrt{5}) + (\sqrt{5})^2.$$

As before, $(2\sqrt{x})^2 = (2)^2(\sqrt{x})^2 = 4x$ and $(\sqrt{5})^2 = 5$. In the case of $2(2\sqrt{x})(\sqrt{5})$, note that we are multiplying four numbers together. The associative and commutative properties state that we can multiply these four numbers in any order that we please. So, the product of 2 and 2 is 4, the product of $\sqrt{x}$ and $\sqrt{5}$ is $\sqrt{5x}$, then we multiply these results to produce the result $4\sqrt{5x}$. Thus,

$$(2\sqrt{x})^2 + 2(2\sqrt{x})(\sqrt{5}) + (\sqrt{5})^2 = 4x + 4\sqrt{5x} + 5.$$

◆

## Rationalizing Denominators

As we saw in the previous section, the instruction “rationalize the denominator” is a request to remove all radical expressions from the denominator. Of course, this is the “third guideline of simple radical form,” but there are times, particularly in calculus, when the instruction changes to “rationalize the numerator.” Of course, this is a request to remove all radicals from the numerator.

You can’t have both worlds. You can either remove radical expressions from the denominator or from the numerator, but not both. If no instruction is given, assume that the “third guideline of simple radical form” is in play and remove all radical expressions from the denominator. We’ve already done a little of this in previous sections, but here we address a slightly more complicated type of expression.

▶ **Example 18.** *In the expression*

$$\frac{3}{2+\sqrt{2}},$$

*rationalize the denominator.*

The secret lies in the difference of squares pattern, $(a+b)(a-b) = a^2 - b^2$. For example,

$$(2+\sqrt{2})(2-\sqrt{2}) = (2)^2 - (\sqrt{2})^2 = 4 - 2 = 2.$$

This provides a terrific hint at how to proceed with rationalizing the denominator of the expression $3/(2+\sqrt{2})$. Multiply both numerator and denominator by $2-\sqrt{2}$.

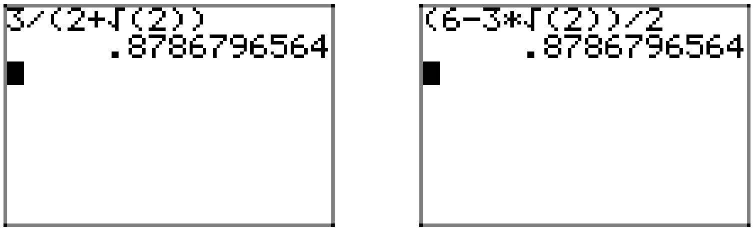
$$\frac{3}{2+\sqrt{2}} = \frac{3}{2+\sqrt{2}} \cdot \frac{2-\sqrt{2}}{2-\sqrt{2}}$$

Multiply numerators and denominators.

$$\begin{aligned}\frac{3}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2}}{2-\sqrt{2}} &= \frac{3(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})}\\ &= \frac{6-3\sqrt{2}}{(2)^2-(\sqrt{2})^2}\\ &= \frac{6-3\sqrt{2}}{4-2}\\ &= \frac{6-3\sqrt{2}}{2}\end{aligned}$$

Note that it is tempting to cancel the 2 in the denominator into the 6 in the numerator, but you are not allowed to cancel terms that are separated by a minus sign. This is a common error, so don't fall prey to this mistake.

In **Figures 6**(a) and (b), we compare decimal approximations of the original expression and its simple radical form.



```
3/(2+√(2))
        .8786796564
```

```
(6-3*√(2))/2
        .8786796564
```

(a) Approximating $3/(2+\sqrt{2})$.

(b) Approximating $(6-3\sqrt{2})/2$.

**Figure 6.** Comparing the original expression with its simple radical form.

◆

▶ **Example 19.** *In the expression*

$$\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}},$$

*rationalize the denominator.*

Multiply numerator and denominator by $\sqrt{3}+\sqrt{2}$.

$$\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$

Multiply numerators and denominators.

$$\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{(\sqrt{3}+\sqrt{2})^2}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}$$

In the denominator, we have the difference of two squares. Thus,

$$(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2}) = (\sqrt{3})^2 - (\sqrt{2})^2 = 3 - 2 = 1.$$

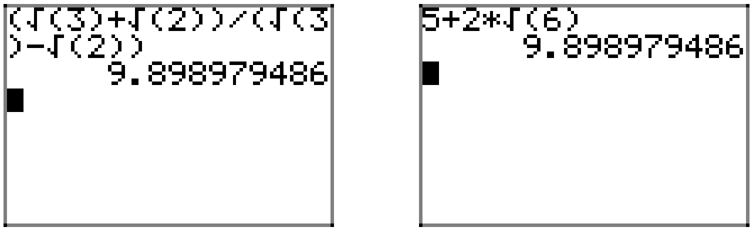
Note that this clears the denominator of radicals. This is the reason we multiply numerator and denominator by $\sqrt{3}+\sqrt{2}$. In the numerator, we can use the squaring a binomial shortcut to multiply.

$$\begin{aligned}(\sqrt{3}+\sqrt{2})^2 &= (\sqrt{3})^2 + 2(\sqrt{3})(\sqrt{2}) + (\sqrt{2})^2\\ &= 3 + 2\sqrt{6} + 2\\ &= 5 + 2\sqrt{6}\end{aligned}$$

Thus, we can complete the simplification started above.

$$\frac{(\sqrt{3}+\sqrt{2})^2}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})} = \frac{5+2\sqrt{6}}{1} = 5 + 2\sqrt{6}$$

In **Figures 7**(a) and (b), we compare the decimal approximations of the original expression with its simple radical form.



```
(√(3)+√(2))/(√(3
)-√(2))
         9.898979486
```

```
5+2*√(6)
         9.898979486
```

(a) Approximating $(\sqrt{3}+\sqrt{2})/(\sqrt{3}-\sqrt{2})$.

(b) Approximating $5+2\sqrt{6}$.

**Figure 7.** Comparing the original expression with its simple radical form.

## *Revisiting the Quadratic Formula*

We can use what we've learned to place solutions provided by the quadratic formula in simple form. Let's look at an example.

▶ **Example 20.** *Solve the equation $x^2 = 2x + 2$ for $x$. Place your solution in simple radical form.*

The equation is nonlinear, so make one side zero.

$$x^2 - 2x - 2 = 0$$

Compare this result with the general form $ax^2 + bx + c = 0$ and note that $a = 1$, $b = -2$ and $c = -2$. Write down the quadratic formula, make the substitutions, then simplify.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} = \frac{2 \pm \sqrt{12}}{2}$$

Note that we can factor a perfect square from the radical in the numerator.

$$x = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm \sqrt{4}\sqrt{3}}{2} = \frac{2 \pm 2\sqrt{3}}{2}$$

At this point, note that both numerator and denominator are divisible by 2. There are several ways that we can proceed with the reduction.

- Some people prefer to factor, then cancel.

$$x = \frac{2 \pm 2\sqrt{3}}{2} = \frac{2(1 \pm \sqrt{3})}{2} = \frac{\cancel{2}(1 \pm \sqrt{3})}{\cancel{2}} = 1 \pm \sqrt{3}$$

- Some prefer to use the distributive property.

$$x = \frac{2 \pm 2\sqrt{3}}{2} = \frac{2}{2} \pm \frac{2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

In each case, the final form of the answer is in simple radical form and it is reduced to lowest terms.

◆

**Warning 21.** *When working with the quadratic formula, one of the most common algebra mistakes is to cancel addends instead of factors, as in*

$$\frac{2 \pm 2\sqrt{3}}{2} = \frac{\cancel{2} \pm 2\sqrt{3}}{\cancel{2}} = \pm 2\sqrt{3}.$$

*Please try to avoid making this mistake.*

Let's look at another example.

▶ **Example 22.** *Solve the equation* $3x^2 - 2x = 6$ *for* $x$*. Place your solution in simple radical form.*

This equation is nonlinear. Move every term to one side of the equation, making the other side of the equation equal to zero.

$$3x^2 - 2x - 6 = 0$$

Compare with the general form $ax^2 + bx + c = 0$ and note that $a = 3$, $b = -2$, and $c = -6$. Write down the quadratic formula and substitute.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)} = \frac{2 \pm \sqrt{76}}{6}$$

Factor a perfect square from the radical in the numerator.

$$x = \frac{2 \pm \sqrt{76}}{6} = \frac{2 \pm \sqrt{4}\sqrt{19}}{6} = \frac{2 \pm 2\sqrt{19}}{6}$$

We choose to factor and cancel.

$$x = \frac{2 \pm 2\sqrt{19}}{6} = \frac{2(1 \pm \sqrt{19})}{2 \cdot 3} = \frac{\cancel{2}(1 \pm \sqrt{19})}{\cancel{2} \cdot 3} = \frac{1 \pm \sqrt{19}}{3}$$

◆

## *9.4 Exercises*

In **Exercises 1-14**, place each of the radical expressions in simple radical form. Check your answer with your calculator.

**1.** $2(5\sqrt{7})$

**2.** $-3(2\sqrt{3})$

**3.** $-\sqrt{3}(2\sqrt{5})$

**4.** $\sqrt{2}(3\sqrt{7})$

**5.** $\sqrt{3}(5\sqrt{6})$

**6.** $\sqrt{2}(-3\sqrt{10})$

**7.** $(2\sqrt{5})(-3\sqrt{3})$

**8.** $(-5\sqrt{2})(-2\sqrt{7})$

**9.** $(-4\sqrt{3})(2\sqrt{6})$

**10.** $(2\sqrt{5})(-3\sqrt{10})$

**11.** $(2\sqrt{3})^2$

**12.** $(-3\sqrt{5})^2$

**13.** $(-5\sqrt{2})^2$

**14.** $(7\sqrt{11})^2$

In **Exercises 15-22**, use the distributive property to multiply. Place your final answer in simple radical form. Check your result with your calculator.

**15.** $2(3+\sqrt{5})$

**16.** $-3(4-\sqrt{7})$

**17.** $2(-5+4\sqrt{2})$

**18.** $-3(4-3\sqrt{2})$

**19.** $\sqrt{2}(2+\sqrt{2})$

**20.** $\sqrt{3}(4-\sqrt{6})$

**21.** $\sqrt{2}(\sqrt{10}+\sqrt{14})$

**22.** $\sqrt{3}(\sqrt{15}-\sqrt{33})$

In **Exercises 23-30**, combine like terms. Place your final answer in simple radical form. Check your solution with your calculator.

**23.** $-5\sqrt{2}+7\sqrt{2}$

**24.** $2\sqrt{3}+3\sqrt{3}$

**25.** $2\sqrt{6}-8\sqrt{6}$

**26.** $\sqrt{7}-3\sqrt{7}$

**27.** $2\sqrt{3}-4\sqrt{2}+3\sqrt{3}$

**28.** $7\sqrt{5}+2\sqrt{7}-3\sqrt{5}$

**29.** $2\sqrt{3}+5\sqrt{2}-7\sqrt{3}+2\sqrt{2}$

**30.** $3\sqrt{11}-2\sqrt{7}-2\sqrt{11}+4\sqrt{7}$

In **Exercises 31-40**, combine like terms where possible. Place your final answer in simple radical form. Use your calculator to check your result.

**31.** $\sqrt{45}+\sqrt{20}$

**32.** $-4\sqrt{45}-4\sqrt{20}$

**33.** $2\sqrt{18}-\sqrt{8}$

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**34.** $-\sqrt{20}+4\sqrt{45}$

**35.** $-5\sqrt{27}+5\sqrt{12}$

**36.** $3\sqrt{12}-2\sqrt{27}$

**37.** $4\sqrt{20}+4\sqrt{45}$

**38.** $-2\sqrt{18}-5\sqrt{8}$

**39.** $2\sqrt{45}+5\sqrt{20}$

**40.** $3\sqrt{27}-4\sqrt{12}$

---

In **Exercises 41-48**, simplify each of the given rational expressions. Place your final answer in simple radical form. Check your result with your calculator.

**41.** $\sqrt{2}-\dfrac{1}{\sqrt{2}}$

**42.** $3\sqrt{3}-\dfrac{3}{\sqrt{3}}$

**43.** $2\sqrt{2}-\dfrac{2}{\sqrt{2}}$

**44.** $4\sqrt{5}-\dfrac{5}{\sqrt{5}}$

**45.** $5\sqrt{2}+\dfrac{3}{\sqrt{2}}$

**46.** $6\sqrt{3}+\dfrac{2}{\sqrt{3}}$

**47.** $\sqrt{8}-\dfrac{12}{\sqrt{2}}-3\sqrt{2}$

**48.** $\sqrt{27}-\dfrac{6}{\sqrt{3}}-5\sqrt{3}$

---

In **Exercises 49-60**, multiply to expand each of the given radical expressions. Place your final answer in simple radical form. Use your calculator to check your result.

**49.** $(2+\sqrt{3})(3-\sqrt{3})$

**50.** $(5+\sqrt{2})(2-\sqrt{2})$

**51.** $(4+3\sqrt{2})(2-5\sqrt{2})$

**52.** $(3+5\sqrt{3})(1-2\sqrt{3})$

**53.** $(2+3\sqrt{2})(2-3\sqrt{2})$

**54.** $(3+2\sqrt{5})(3-2\sqrt{5})$

**55.** $(2\sqrt{3}+3\sqrt{2})(2\sqrt{3}-3\sqrt{2})$

**56.** $(8\sqrt{2}+\sqrt{5})(8\sqrt{2}-\sqrt{5})$

**57.** $(2+\sqrt{5})^2$

**58.** $(3-\sqrt{2})^2$

**59.** $(\sqrt{3}-2\sqrt{5})^2$

**60.** $(2\sqrt{3}+3\sqrt{2})^2$

---

In **Exercises 61-68**, place each of the given rational expressions in simple radical form by "rationalizing the denominator." Check your result with your calculator.

**61.** $\dfrac{1}{\sqrt{5}+\sqrt{3}}$

**62.** $\dfrac{1}{2\sqrt{3}-\sqrt{2}}$

**63.** $\dfrac{6}{2\sqrt{5}-\sqrt{2}}$

**64.** $\dfrac{9}{3\sqrt{3}-\sqrt{6}}$

**65.** $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$

**66.** $\dfrac{3-\sqrt{5}}{3+\sqrt{5}}$

**67.** $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

**68.** $\dfrac{2\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

---

In **Exercises 69-76**, use the quadratic formula to find the solutions of the given equation. Place your solutions in simple radical form and reduce your solutions to lowest terms.

**69.** $3x^2 - 8x = 5$

**70.** $5x^2 - 2x = 1$

**71.** $5x^2 = 2x + 1$

**72.** $3x^2 - 2x = 11$

**73.** $7x^2 = 6x + 2$

**74.** $11x^2 + 6x = 4$

**75.** $x^2 = 2x + 19$

**76.** $100x^2 = 40x - 1$

---

In **Exercises 77-80**, we will suspend the usual rule that you should rationalize the denominator. Instead, just this one time, rationalize the numerator of the resulting expression.

**77.** Given $f(x) = \sqrt{x}$, evaluate the expression

$$\frac{f(x)-f(2)}{x-2},$$

and then "rationalize the numerator."

**78.** Given $f(x) = \sqrt{x+2}$, evaluate the expression

$$\frac{f(x)-f(3)}{x-3},$$

and then "rationalize the numerator."

**79.** Given $f(x) = \sqrt{x}$, evaluate the expression

$$\frac{f(x+h)-f(x)}{h},$$

and then "rationalize the numerator."

**80.** Given $f(x) = \sqrt{x-3}$, evaluate the expression

$$\frac{f(x+h)-f(x)}{h},$$

and then "rationalize the numerator."

## 9.4 Answers

---

**1.** $10\sqrt{7}$

**3.** $-2\sqrt{15}$

**5.** $15\sqrt{2}$

**7.** $-6\sqrt{15}$

**9.** $-24\sqrt{2}$

**11.** 12

**13.** 50

**15.** $6+2\sqrt{5}$

**17.** $-10+8\sqrt{2}$

**19.** $2\sqrt{2}+2$

**21.** $2\sqrt{5}+2\sqrt{7}$

**23.** $2\sqrt{2}$

**25.** $-6\sqrt{6}$

**27.** $5\sqrt{3}-4\sqrt{2}$

**29.** $7\sqrt{2}-5\sqrt{3}$

**31.** $5\sqrt{5}$

**33.** $4\sqrt{2}$

**35.** $-5\sqrt{3}$

**37.** $20\sqrt{5}$

**39.** $16\sqrt{5}$

**41.** $\sqrt{2}/2$

**43.** $\sqrt{2}$

**45.** $13\sqrt{2}/2$

**47.** $-7\sqrt{2}$

**49.** $3+\sqrt{3}$

**51.** $-22-14\sqrt{2}$

**53.** $-14$

**55.** $-6$

**57.** $9+4\sqrt{5}$

**59.** $23-4\sqrt{15}$

**61.** $\dfrac{\sqrt{5}-\sqrt{3}}{2}$

**63.** $\dfrac{2\sqrt{5}+\sqrt{2}}{3}$

**65.** $7+4\sqrt{3}$

**67.** $5+2\sqrt{6}$

**69.** $(4\pm\sqrt{31})/3$

**71.** $(1\pm\sqrt{6})/5$

**73.** $(3\pm\sqrt{23})/7$

**75.** $1\pm2\sqrt{5}$

**77.** $\dfrac{1}{\sqrt{x}+\sqrt{2}}$

**79.** $\dfrac{1}{\sqrt{x+h}+\sqrt{x}}$

# 9.5 Radical Equations

In this section we are going to solve equations that contain one or more radical expressions. In the case where we can *isolate the radical expression on one side of the equation*, we can simply raise both sides of the equation to a power that will eliminate the radical expression. For example, if

$$\sqrt{x-1}=2, \tag{1}$$

then we can square both sides of the equation, eliminating the radical.

$$\begin{aligned}\left(\sqrt{x-1}\right)^2&=(2)^2\\ x-1&=4\end{aligned}$$

Now that the radical is eliminated, we can appeal to well understood techniques to solve the equation that remains. In this case, we need only add 1 to both sides of the equation to obtain

$$x=5.$$

This solution is easily checked. Substitute $x=5$ in the original **equation (1)**.

$$\begin{aligned}\sqrt{x-1}&=2\\ \sqrt{5-1}&=2\\ \sqrt{4}&=2\end{aligned}$$

The last line is valid because the "positive square root of 4" is indeed 2.

This seems pretty straight forward, but there are some subtleties. Let's look at another example, one with an equation quite similar to **equation (1)**.

► **Example 2.** *Solve the equation* $\sqrt{x-1}=-2$ *for* $x$.

If you carefully study the equation

$$\sqrt{x-1}=-2, \tag{3}$$

you might immediately detect a difficulty. The left-hand side of the equation calls for a "positive square root," but the right-hand side of the equation is negative. Intuitively, there can be no solutions.

A look at the graphs of each side of the equation also reveals the problem. The graphs of $y=\sqrt{x-1}$ and $y=-2$ are shown in **Figure 1**. Note that the graphs do not intersect, so the equation $\sqrt{x-1}=-2$ has no solution.

However, note what happens when we square both sides of **equation (3)**.

$$\begin{aligned}(\sqrt{x-1})^2&=(-2)^2\\ x-1&=4\end{aligned} \tag{4}$$

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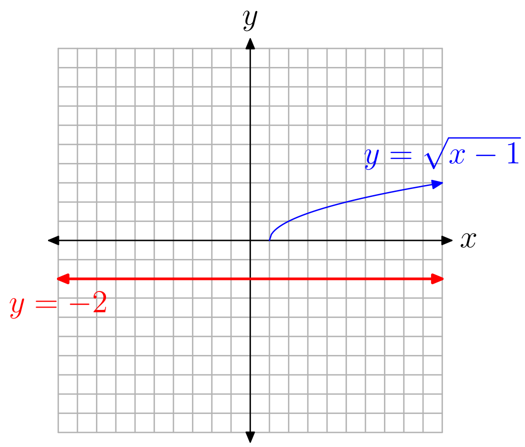


**Figure 1.** The graphs of $y = \sqrt{x-1}$ and $y = -2$ do not intersect.

This result is identical to the result we got when we squared both sides of the equation $\sqrt{x-1} = 2$ above. If we continue, adding 1 to both sides of the equation, we get

$$x = 5.$$

But this cannot be correct, as both intuition and the graphs in **Figure 1** have shown that the equation $\sqrt{x-1} = -2$ has no solutions.

Let's check the solution $x = 5$ in the original **equation (3)**.

$$\begin{aligned}\sqrt{x-1} &= -2\\ \sqrt{5-1} &= -2\\ \sqrt{4} &= -2\end{aligned}$$

Because the "positive square root of 4" does not equal $-2$, this last line is incorrect and the "solution" $x = 5$ does not check in the equation $\sqrt{x-1} = -2$. Because the only solution we found does not check, the equation has no solutions.

◆

The discussion in **Example 2** dictates caution.

**Warning 5.** *Whenever you square both sides of an equation, there is a possibility that you can introduce* extraneous *solutions, "extra" solutions that will not check in the original problem.*

There is only one way to avoid this dilemma of extraneous equations.

**Checking Solutions.** Whenever you square both sides of an equation, you **must** check each of your solutions in the **original** equation. This is the only way you can be sure you have a valid solution.

## *Squaring a Binomial*

As we've seen time and time again, the squaring a binomial pattern is of utmost importance.

**Squaring a Binomial.** If $a$ and $b$ are any real numbers, then

$$(a+b)^2 = a^2 + 2ab + b^2.$$

The squaring a binomial pattern will play a major role in the rest of the examples in this section.

Let's look at some examples of its use.

▶ **Example 6.** *Expand and simplify* $(1+\sqrt{x})^2$ *by using the squaring a binomial pattern. Assume that* $x \geq 0$.

The assumption that $x \geq 0$ is required, otherwise the expression $\sqrt{x}$ involves the square root of a negative number, which is not a real number.

The squaring a binomial pattern tells us to square the first and second terms. However, there is also a middle term, which is found by taking the product of the first and second terms, then multiplying the result by 2.

$$\begin{aligned}(1+\sqrt{x})^2 &= (1)^2 + 2(1)(\sqrt{x}) + (\sqrt{x})^2 \\ &= 1 + 2\sqrt{x} + x\end{aligned}$$

◆

Let's look at another example.

▶ **Example 7.** *Expand and simplify* $(\sqrt{x+1}-\sqrt{x})^2$ *by using the squaring a binomial pattern. Comment on the domain of this expression.*

In order for this expression to make sense, we must avoid taking the square root of a negative number. Hence, both expressions under the square roots must be nonnegative (positive or zero). That is,

$$x+1 \geq 0 \qquad \text{and} \qquad x \geq 0$$

Solving each of these inequalities independently, we get the fact that

$$x \geq -1 \qquad \text{and} \qquad x \geq 0.$$

Because of the word "and," the requested domain is the set of all numbers that satisfy *both* inequalities, namely, the set of all real numbers that are greater than or equal to zero. That is, the domain of the expression is $\{x : x \geq 0\}$.

We will now expand the expression $(\sqrt{x+1}-\sqrt{x})^2$ using the squaring a binomial pattern.

$$\begin{aligned}(\sqrt{x+1}-\sqrt{x})^2 &= (\sqrt{x+1})^2 - 2(\sqrt{x+1})(\sqrt{x}) + (\sqrt{x})^2 \\ &= x+1+2\sqrt{(x+1)x}+x \\ &= 2x+1+2\sqrt{x^2+x}\end{aligned}$$

## Isolate the Radical

Our mantra will be the strategy phrase “Isolate the radical.”

> **Isolate the Radical.** When you solve equations containing one radical, isolate the radical by itself on one side of the equation.

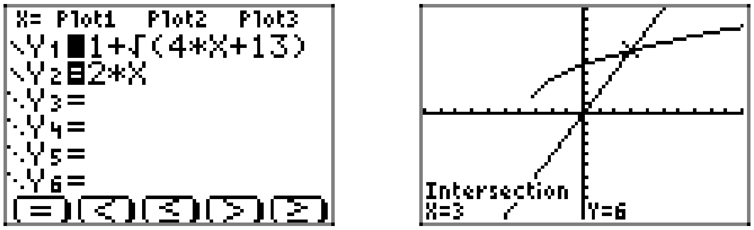
Although this is not always possible (some equations might contain more than one radical exprssion), it is possible in our next example.

▶ **Example 8.** *Solve the equation*

$$1+\sqrt{4x+13}=2x \tag{9}$$

*for $x$.*

Let’s look at a graphing calculator solution. We’ve loaded the left- and right-hand sides of $1+\sqrt{4x+13}=2x$ into `Y1` and `Y2`, respectively, as shown in **Figure 2**(a). We then use `6:ZStandard` and the `intersect` utility on the `CALC` menu to determine the coordinates of the point of intersection of $y=1+\sqrt{4x+13}$ and $y=2x$, as shown in **Figure 2**(b).



```
X= Plot1  Plot2  Plot3
\Y1 ■1+√(4*X+13)
\Y2 ■2*X
\Y3=
\Y4=
\Y5=
\Y6=
```

(a) Loading $y=1+\sqrt{4x+13}$ and $y=2x$ into the `Y=` menu.

(b) The solution is $x\approx 3$.

**Figure 2.** Solving $1+\sqrt{4x+13}=2x$ on the graphing calculator. Note that there is only one point of intersection.

We will now present an algebraic solution, but note that we are forewarned that there is only one solution and we believe that the solution is $x\approx 3$. Of course, this is only an approximation, as is always the case when we pick up our calculator (our approximating machine).

Chant the strategy phrase "isolate the radical," then isolate the radical on one side of the equation. We will accomplish this directive by subtracting 1 from both sides of the equation.

$$\begin{aligned} 1+\sqrt{4x+13} &= 2x \\ \sqrt{4x+13} &= 2x-1 \end{aligned}$$

Next, square both sides of the equation.

$$(\sqrt{4x+13})^2 = (2x-1)^2$$

Squaring eliminates the radical on the left, but we must use the squaring a binomial pattern to square the binomial on the right-side of the equation.

$$\begin{aligned} 4x+13 &= (2x)^2 - 2(2x)(1) + (1)^2 \\ 4x+13 &= 4x^2 - 4x + 1 \end{aligned}$$

We've succeed in clearing all square roots from the equation with our "isolate the radical" strategy. The equation that remains is nonlinear (there is a power of $x$ higher than 1), so we want to make one side of the equation equal to zero. We will do this by subtracting $4x$ and 13 from both sides of the equation.

$$\begin{aligned} 0 &= 4x^2 - 4x + 1 - 4x - 13 \\ 0 &= 4x^2 - 8x - 12 \end{aligned}$$

At this point, note that each term on the right-hand side of the equation is divisible by 4. Divide both sides of the equation by 4, then use the *ac*-test to factor the result.

$$\begin{aligned} 0 &= x^2 - 2x - 3 \\ 0 &= (x-3)(x+1) \end{aligned}$$

Set each factor on the right-hand side of this last equation to obtain the solutions $x = 3$ and $x = -1$.

Note that $x = 3$ matches the solution found by graphing in **Figure 2**(b). However, an "extra" solution $x = -1$ has appeared. Remember that we squared both sides of the original equation, so it is possible that extraneous solutions have been introduced. We need to check each of our solutions by substituting them into the **original** equation.

Our graph in **Figure 2**(b) adds credence to the analytical solution $x = 3$, so let's check that one first. Substitute $x = 3$ in the original equation.

$$\begin{aligned} 1+\sqrt{4x+13} &= 2x \\ 1+\sqrt{4(3)+13} &= 2(3) \\ 1+\sqrt{25} &= 6 \\ 1+5 &= 6 \end{aligned}$$

Clearly, $x = 3$ checks and is a valid solution.

Next, let's check the "suspect" solution $x = -1$ by substituting it into the original equation.

$$\begin{aligned} 1 + \sqrt{4x+13} &= 2x \\ 1 + \sqrt{4(-1)+13} &= 2(-1) \\ 1 + \sqrt{9} &= -2 \\ 1 + 3 &= -2 \end{aligned}$$

Clearly, $x = -1$ does not check and is not a solution.

Thus, the only solution of $1 + \sqrt{4x+13} = 2x$ is $x = 3$. Readers should take note how that graphical solution and the analytic solution complement one another.

◆

Before looking at another example, let's look at one of the most common mistakes made in the algebraic solution of **equation (9)**.

## A Common Algebraic Mistake

In this section we discuss one of the most common algebraic mistakes encountered when solving equations that contain radical expressions.

**Warning 10.** *Many of the computations in this section are* **incorrect***. They are examples of common algebra mistakes made when solving equations containing radicals. Keep this in mind and read the material in this section* **very** *carefully.*

When presented with the equation

$$1 + \sqrt{4x+13} = 2x, \tag{11}$$

some will square both sides of the equation in the following manner.

$$(1)^2 + (\sqrt{4x+13})^2 = (2x)^2, \tag{12}$$

arriving at

$$1 + 4x + 13 = 4x^2.$$

Make one side zero, then divide both sides of the resulting equation by 2.

$$\begin{aligned} 0 &= 4x^2 - 4x - 14 \\ 0 &= 2x^2 - 2x - 7 \end{aligned}$$

The careful reader will already realize that we've traveled the wrong path, as this result is quite different from that at a similar point in the solution of **Example 8**. However, we can continue with the solution by using the quadratic formula to solve the last equation for $x$. When we compare $2x^2 - 2x - 7$ with $ax^2 + bx + c$, note that $a = 2$, $b = -2$, and $c = -7$. Thus,

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-7)}}{2(2)} \\ &= \frac{2 \pm \sqrt{60}}{4}. \end{aligned}$$

However, neither of these "solutions" represent the correct solution found in **Example 8**, namely, $x = 3$. So, what have we done wrong?

The mistake occurred in the very first step when we squared both sides of the **equation (11)**. Indeed, to get **equation (12)**, we did not actually square both sides of **equation (11)**. Rather, we squared each of the individual terms on each side of the equation.

This is a serious mistake. In essence, we started with an equation having the form

$$a + b = c, \tag{13}$$

then squared "both sides" in the following manner.

$$a^2 + b^2 = c^2. \tag{14}$$

This is not valid. For example, start with

$$2 + 3 = 5,$$

a completely valid equation as the sum of 2 and 3 is 5. Now "square" as we did in **equation (14)** to get

$$2^2 + 3^2 = 5^2.$$

However, note that this simplifies as

$$4 + 9 = 25,$$

so we no longer have a valid equation.

The mistake made here is that we squared each of the individual terms on each side of the equation instead of squaring "each side" of the equation. If we had done that, we would have been all right, as is seen in this calculation.

$$\begin{aligned} 2 + 3 &= 5 \\ (2 + 3)^2 &= 5^2 \\ 2^2 + 2(2)(3) + 3^2 &= 5^2 \\ 4 + 12 + 9 &= 25 \end{aligned}$$

Just remember, $a + b = c$ does not imply $a^2 + b^2 = c^2$.

**Warning 15.** *We will now return to correct computations.*

## *More than One Radical*

Let's look at an equation that contains more than one radical.

▶ **Example 16.** *Solve the equation*

$$\sqrt{2x} + \sqrt{2x + 3} = 3 \tag{17}$$

*for $x$.*

We'll start with a graphical solution of the equation. First, load the equations $y = \sqrt{2x} + \sqrt{2x + 3}$ and $y = 3$ into the `Y=` menu, as shown in **Figure 3**(a).

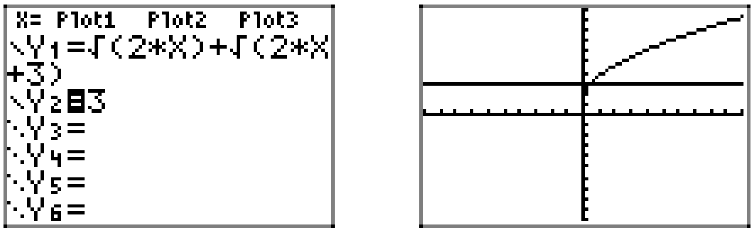
We cannot take the square root of a negative number, so when we consider the function defined by the equation $y = \sqrt{2x} + \sqrt{2x + 3}$, both expressions under the radicals must be nonnegative. That is,

$$2x \geq 0 \qquad \text{and} \qquad 2x + 3 \geq 0.$$

Solving each of these independently,

$$x \geq 0 \qquad \text{and} \qquad x \geq -\frac{3}{2}.$$

The numbers that are greater than or equal to zero *and* greater than or equal to $-3/2$ are the numbers greater than or equal to zero. Hence, the domain of the function defined by the equation $y = \sqrt{2x} + \sqrt{2x + 3}$ is $\{x : x \geq 0\}$. Thus, it should not come as a shock when the graph of $y = \sqrt{2x} + \sqrt{2x + 3}$ lies entirely to the right of zero, as shown in **Figure 3**(b).



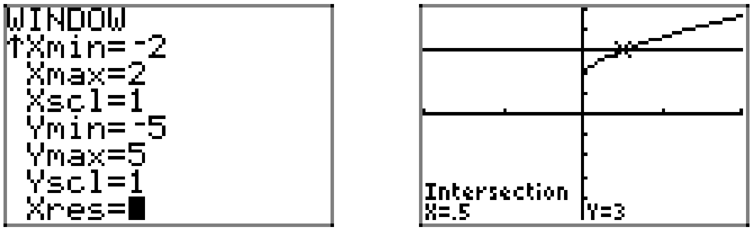
```
Y1=√(2*X)+√(2*X
+3)
Y2=3
Y3=
Y4=
Y5=
Y6=
```

(a) Load each side of **equation (17)** into `Y1` and `Y2`.

(b) The graph of $y = \sqrt{2x} + \sqrt{2x + 3}$ lies completely to the right of zero.

**Figure 3.** Drawing the graphs of $y = \sqrt{2x} + \sqrt{2x + 3}$ and $y = 3$.

It's a bit difficult to see the point of intersection in **Figure 3**(b), so let's adjust the `WINDOW` settings as shown in **Figure 4**(a). As you can see **Figure 4**(b), this highlights the point of intersection a bit more clearly and the `5:intersect` utility in the `CALC` menu finds the point of intersection shown in **Figure 4**(b).



```
WINDOW
↑Xmin=-2
 Xmax=2
 Xscl=1
 Ymin=-5
 Ymax=5
 Yscl=1
 Xres=█
```

```
Intersection
X=.5          Y=3
```

(a) Adjust the view.

(b) Use `5:intersect` to find the point of intersection.

**Figure 4.** Solving $\sqrt{2x} + \sqrt{2x+3} = 3$ graphically.

The graphing calculator reports one solution (there's only one point of intersection), and the $x$-value of the point of intersection is approximately $x \approx 0.5$.

Now, let's look at an algebraic solution. Since are two radical expressions in this equation, we will isolate one of them on one side of the equation. We choose to isolate the more complex of the two radical expressions on the left-hand side of the equation, then square both sides of the resulting equation.

$$\begin{aligned} \sqrt{2x} + \sqrt{2x+3} &= 3 \\ \sqrt{2x+3} &= 3 - \sqrt{2x} \\ (\sqrt{2x+3})^2 &= (3 - \sqrt{2x})^2 \end{aligned}$$

On the left, squaring eliminates the radical. To square the binomial on the right, we use the squaring a binomial pattern to obtain

$$\begin{aligned} 2x + 3 &= (3)^2 - 2(3)(\sqrt{2x}) + (\sqrt{2x})^2 \\ 2x + 3 &= 9 - 6\sqrt{2x} + 2x. \end{aligned}$$

We still have one radical expression left on the right-hand side of this equation, so we'll follow the mantra "isolate the radical." First, subtract $2x$ from both sides of the equation to obtain

$$3 = 9 - 6\sqrt{2x},$$

then subtract 9 from both sides of the equation.

$$-6 = -6\sqrt{2x}$$

We've succeeded in isolating the radical term on one side of the equation. Now, divide both sides of the equation by $-6$, then square both sides of the resulting equation.

$$\begin{aligned}1 &= \sqrt{2x}\\(1)^2 &= (\sqrt{2x})^2\\1 &= 2x\end{aligned}$$

Divide both sides of the last result by 2.

$$x = \frac{1}{2}$$

Note that this agrees nicely with our graphical solution ($x \approx 0.5$), but let's check our solution by substituting $x = 1/2$ into the **original** equation.

$$\begin{aligned}\sqrt{2x} + \sqrt{2x+3} &= 3\\\sqrt{2(1/2)} + \sqrt{2(1/2)+3} &= 3\\\sqrt{1} + \sqrt{4} &= 3\\1 + 2 &= 3\end{aligned}$$

This last statement is true, so the solution $x = 1/2$ checks.

◆

## 9.5 Exercises

For the rational functions in **Exercises 1-6**, perform each of the following tasks.

i. Load the function $f$ and the line $y = k$ into your graphing calculator. Adjust the viewing window so that all point(s) of intersection of the two graphs are visible in your viewing window.
ii. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graphs with their equations. *Remember to draw all lines with a ruler.*
iii. Use the `intersect` utility to determine the coordinates of the point(s) of intersection. Plot the point of intersection on your homework paper and label it with its coordinates.
iv. Solve the equation $f(x) = k$ algebraically. Place your work and solution next to your graph. Do the solutions agree?

**1.** $f(x) = \sqrt{x+3}$, $k = 2$

**2.** $f(x) = \sqrt{4-x}$, $k = 3$

**3.** $f(x) = \sqrt{7-2x}$, $k = 4$

**4.** $f(x) = \sqrt{3x+5}$, $k = 5$

**5.** $f(x) = \sqrt{5+x}$, $k = 4$

**6.** $f(x) = \sqrt{4-x}$, $k = 5$

In **Exercises 7-12**, use an algebraic technique to solve the given equation. Check your solutions.

**7.** $\sqrt{-5x+5} = 2$

**8.** $\sqrt{4x+6} = 7$

**9.** $\sqrt{6x-8} = 8$

**10.** $\sqrt{2x+4} = 2$

**11.** $\sqrt{-3x+1} = 3$

**12.** $\sqrt{4x+7} = 3$

For the rational functions in **Exercises 13-16**, perform each of the following tasks.

i. Load the function $f$ and the line $y = k$ into your graphing calculator. Adjust the viewing window so that all point(s) of intersection of the two graphs are visible in your viewing window.
ii. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graphs with their equations. *Remember to draw all lines with a ruler.*
iii. Use the `intersect` utility to determine the coordinates of the point(s) of intersection. Plot the point of intersection on your homework paper and label it with its coordinates.
iv. Solve the equation $f(x) = k$ algebraically. Place your work and solution next to your graph. Do the solutions agree?

**13.** $f(x) = \sqrt{x+3} + x$, $k = 9$

**14.** $f(x) = \sqrt{x+6} - x$, $k = 4$

**15.** $f(x) = \sqrt{x-5} - x$, $k = -7$

**16.** $f(x) = \sqrt{x+5} + x$, $k = 7$

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In **Exercises 17-24**, use an algebraic technique to solve the given equation. Check your solutions.

**17.** $\sqrt{x+1}+x=5$

**18.** $\sqrt{x+8}-x=8$

**19.** $\sqrt{x+4}+x=8$

**20.** $\sqrt{x+8}-x=2$

**21.** $\sqrt{x+5}-x=3$

**22.** $\sqrt{x+5}+x=7$

**23.** $\sqrt{x+9}-x=9$

**24.** $\sqrt{x+7}+x=5$

---

For the rational functions in **Exercises 25-28**, perform each of the following tasks.

i. Load the function $f$ and the line $y = k$ into your graphing calculator. Adjust the viewing window so that all point(s) of intersection of the two graphs are visible in your viewing window.
ii. Copy the image in your viewing window onto your homework paper. Label and scale each axis with xmin, xmax, ymin, and ymax. Label the graphs with their equations. *Remember to draw all lines with a ruler.*
iii. Use the `intersect` utility to determine the coordinates of the point(s) of intersection. Plot the point of intersection on your homework paper and label it with its coordinates.
iv. Solve the equation $f(x) = k$ algebraically. Place your work and solution next to your graph. Do the solutions agree?

**25.** $f(x)=\sqrt{x-1}+\sqrt{x+6},\ k=7$

**26.** $f(x)=\sqrt{x+2}+\sqrt{x+9},\ k=7$

**27.** $f(x)=\sqrt{x+2}+\sqrt{3x+4},\ k=2$

**28.** $f(x)=\sqrt{6x+7}+\sqrt{3x+3},\ k=1$

---

In **Exercises 29-40**, use an algebraic technique to solve the given equation. Check your solutions.

**29.** $\sqrt{x+46}-\sqrt{x-35}=1$

**30.** $\sqrt{x-16}+\sqrt{x+16}=8$

**31.** $\sqrt{x-19}+\sqrt{x-6}=13$

**32.** $\sqrt{x+31}-\sqrt{x+12}=1$

**33.** $\sqrt{x-2}-\sqrt{x-49}=1$

**34.** $\sqrt{x+13}+\sqrt{x+8}=5$

**35.** $\sqrt{x+27}-\sqrt{x-22}=1$

**36.** $\sqrt{x+10}+\sqrt{x+13}=3$

**37.** $\sqrt{x+30}-\sqrt{x-38}=2$

**38.** $\sqrt{x+36}-\sqrt{x+11}=1$

**39.** $\sqrt{x-17}+\sqrt{x+3}=10$

**40.** $\sqrt{x+18}+\sqrt{x+13}=5$

## 9.5 *Answers*

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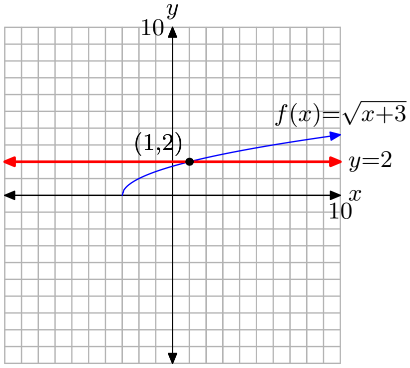
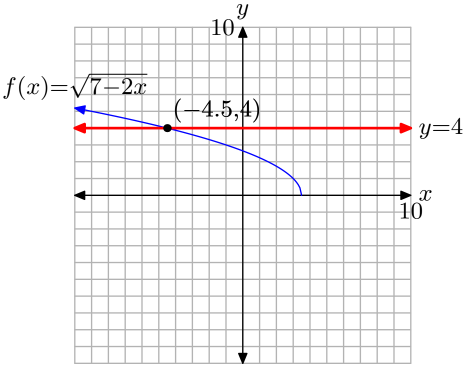
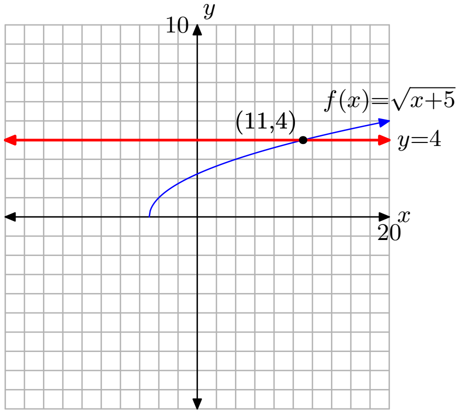
**1.** $x = 1$



**3.** $x = -9/2$



**5.** $x = 11$

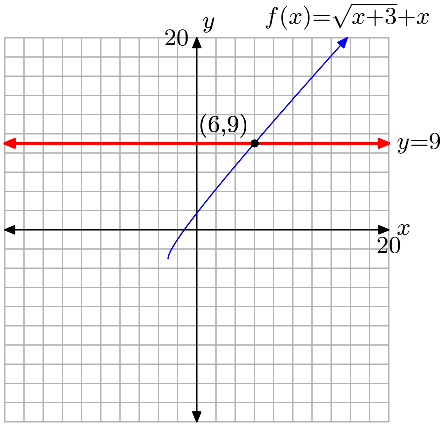


**7.** $\frac{1}{5}$

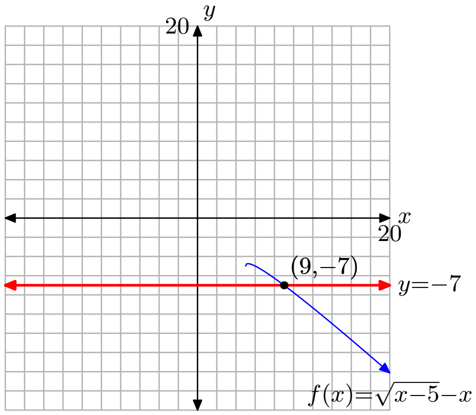
**9.** 12

**11.** $-\frac{8}{3}$

**13.** $x = 6$



**15.** $x = 9$



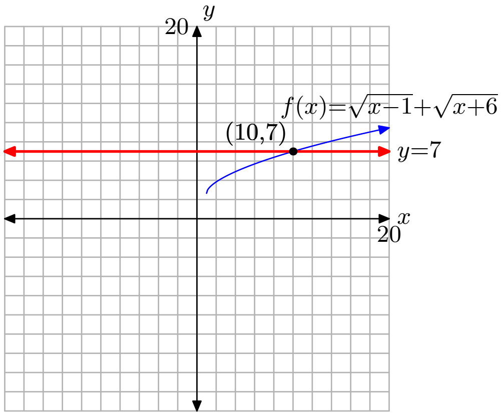
**17.** 3

**19.** 5

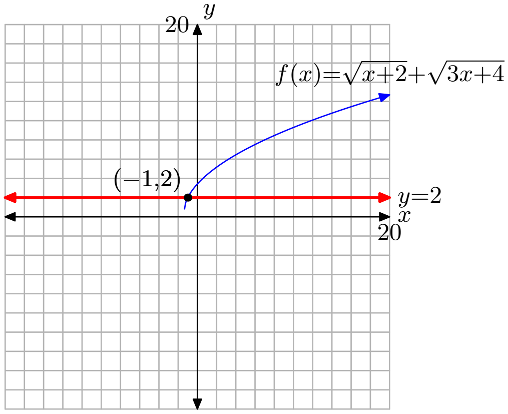
**21.** $-1$

**23.** $-8, -9$

**25.** $x = 10$



**27.** $x = -1$



**29.** 1635

**31.** 55

**33.** 578

**35.** 598

**37.** 294

**39.** 33

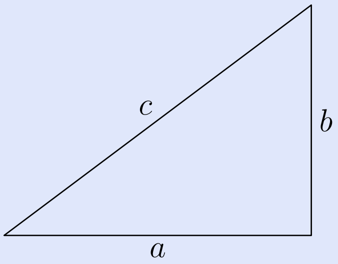
## 9.6 The Pythagorean Theorem

Pythagoras was a Greek mathematician and philosopher, born on the island of Samos (ca. 582 BC). He founded a number of schools, one in particular in a town in southern Italy called Crotone, whose members eventually became known as the Pythagoreans. The inner circle at the school, the *Mathematikoi*, lived at the school, rid themselves of all personal possessions, were vegetarians, and observed a strict vow of silence. They studied mathematics, philosophy, and music, and held the belief that numbers constitute the true nature of things, giving numbers a mystical or even spiritual quality.

Pythagoras.

Today, nothing is known of Pythagoras's writings, perhaps due to the secrecy and silence of the Pythagorean society. However, one of the most famous theorems in all of mathematics does bear his name, the *Pythagorean Theorem*.

**Pythagorean Theorem.** Let $c$ represent the length of the **hypotenuse**, the side of a right triangle directly opposite the right angle (a right angle measures 90°) of the triangle. The remaining sides of the right triangle are called the **legs** of the right triangle, whose lengths are designated by the letters $a$ and $b$.



The relationship involving the legs and hypotenuse of the right triangle, given by

$$a^2 + b^2 = c^2, \tag{1}$$

is called the **Pythagorean Theorem**.

Note that the Pythagorean Theorem can only be applied to right triangles.

Let's look at a simple application of the Pythagorean Theorem (1).

▶ **Example 2.** *Given that the length of one leg of a right triangle is 4 centimeters and the hypotenuse has length 8 centimeters, find the length of the second leg.*

Let's begin by sketching and labeling a right triangle with the given information. We will let $x$ represent the length of the missing leg.

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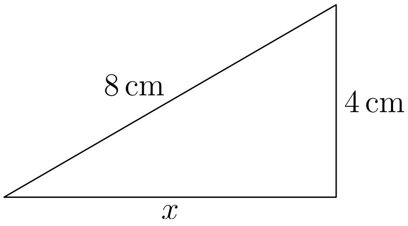


**Figure 1.** A sketch makes things a bit easier.

Here is an important piece of advice.

**Tip 3.** *The hypotenuse is the longest side of the right triangle. It is located directly opposite the right angle of the triangle. Most importantly, it is the quantity that is* **isolated** *by itself in the Pythagorean Theorem.*

$$a^2 + b^2 = c^2$$

*Always isolate the quantity representing the hypotenuse on one side of the equation. The legs go on the other side of the equation.*

So, taking the tip to heart, and noting the lengths of the legs and hypotenuse in **Figure 1**, we write

$$4^2 + x^2 = 8^2.$$

Square, then isolate $x$ on one side of the equation.

$$\begin{aligned} 16 + x^2 &= 64 \\ x^2 &= 48 \end{aligned}$$

Normally, we would take plus or minus the square root in solving this equation, but $x$ represents the length of a leg, which must be a positive number. Hence, we take just the positive square root of 48.

$$x = \sqrt{48}$$

Of course, place your answer in simple radical form.

$$\begin{aligned} x &= \sqrt{16}\sqrt{3} \\ x &= 4\sqrt{3} \end{aligned}$$

If need be, you can use your graphing calculator to approximate this length. To the nearest hundredth of a centimeter, $x \approx 6.93$ centimeters.

◆

## *Proof of the Pythagorean Theorem*

It is not known whether Pythagoras was the first to provide a proof of the Pythagorean Theorem. Many mathematical historians think not. Indeed, it is not even known if Pythagoras crafted a proof of the theorem that bears his name, let alone was the first to provide a proof.

There is evidence that the ancient Babylonians were aware of the Pythagorean Theorem over a 1000 years before the time of Pythagoras. A clay tablet, now referred to as Plimpton 322 (see **Figure 2**), contains examples of *Pythagorean Triples*, sets of three numbers that satisfy the Pythagorean Theorem (such as 3, 4, 5).

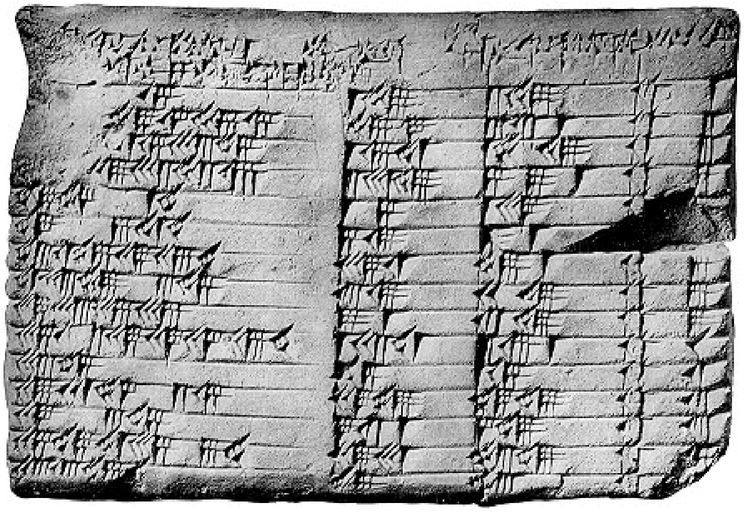


**Figure 2.** A photograph of Plimpton 322.

One of the earliest recorded proofs of the Pythagorean Theorem dates from the Han dynasty (206 BC to AD 220), and is recorded in the *Chou Pei Suan Ching* (see **Figure 3**). You can see that this figure specifically addresses the case of the 3, 4, 5 right triangle. Mathematical historians are divided as to whether or not the image was meant to be part of a general proof or was just devised to address this specific case. There is also disagreement over whether the proof was provided by a more modern commentator or dates back further in time.

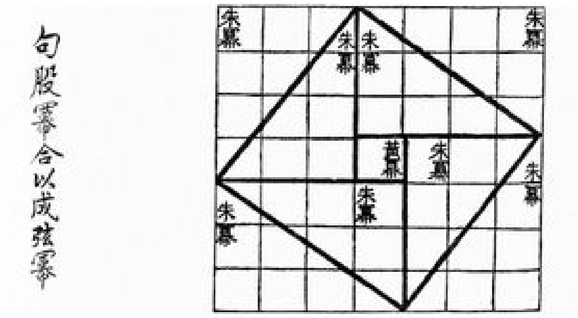
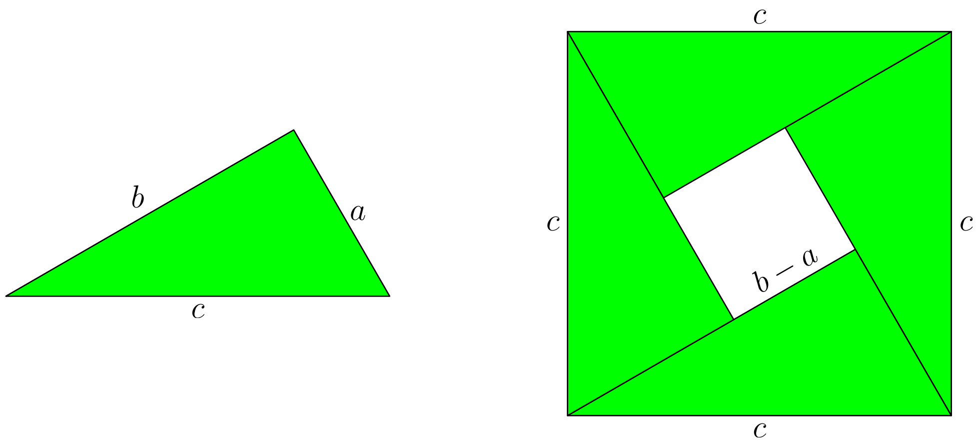


**Figure 3.** A figure from the Chou Pei Suan Ching.

However, **Figure 3** does suggest a path we might take on the road to a proof of the Pythagorean Theorem. Start with an arbitrary right triangle having legs of lengths $a$ and $b$, and hypotenuse having length $c$, as shown in **Figure 4**(a).

Next, make four copies of the triangle shown in **Figure 4**(a), then rotate and translate them into place as shown in **Figure 4**(b). Note that this forms a big square that is $c$ units on a side.

Further, the position of the triangles in **Figure 4**(b) allows for the formation of a smaller, unshaded square in the middle of the larger square. It is not hard to calculate the length of the side of this smaller square. Simply subtract the length of the smaller leg from the larger leg of the original triangle. Thus, the side of the smaller square has length $b - a$.



(a) (b)

**Figure 4.** Proof of the Pythagorean Theorem.

Now, we will calculate the area of the large square in **Figure 4**(b) in two separate ways.

- First, the large square in **Figure 4**(b) has a side of length $c$. Therefore, the area of the large square is

$$\text{Area} = c^2.$$

- Secondly, the large square in **Figure 4**(b) is made up of 4 triangles of the same size and one smaller square having a side of length $b - a$. We can calculate the area of the large square by summing the area of the 4 triangles and the smaller square.

1. The area of the smaller square is $(b-a)^2$.
2. The area of each triangle is $ab/2$. Hence, the area of four triangles of equal size is four times this number; i.e., $4(ab/2)$.

   Thus, the area of the large square is

$$\begin{aligned}\text{Area} &= \text{Area of small square} + 4 \cdot \text{Area of triangle}\\ &= (b-a)^2 + 4\left(\frac{ab}{2}\right).\end{aligned}$$

We calculated the area of the larger square twice. The first time we got $c^2$; the second time we got $(b-a)^2 + 4(ab/2)$. Therefore, these two quantities must be equal.

$$c^2 = (b-a)^2 + 4\left(\frac{ab}{2}\right)$$

Expand the binomial and simplify.

$$\begin{aligned}c^2 &= b^2 - 2ab + a^2 + 2ab\\ c^2 &= b^2 + a^2\end{aligned}$$

That is,

$$a^2 + b^2 = c^2,$$

and the Pythagorean Theorem is proven.

## Applications of the Pythagorean Theorem

In this section we will look at a few applications of the Pythagorean Theorem, one of the most applied theorems in all of mathematics. Just ask your local carpenter.

The ancient Egyptians would take a rope with 12 equally spaced knots like that shown in **Figure 5**, and use it to square corners of their buildings. The tool was instrumental in the construction of the pyramids.

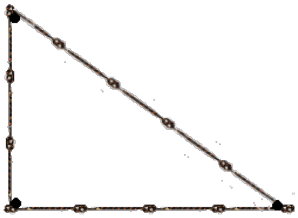


**Figure 5.** A basic 3-4-5 right triangle for squaring corners.

The Pythagorean theorem is also useful in surveying, cartography, and navigation, to name a few possibilities.

Let's look at a few examples of the Pythagorean Theorem in action.

► **Example 4.** *One leg of a right triangle is 7 meters longer than the other leg. The length of the hypotenuse is 13 meters. Find the lengths of all sides of the right triangle.*

Let $x$ represent the length of one leg of the right triangle. Because the second leg is 7 meters longer than the first leg, the length of the second leg can be represented by the expression $x+7$, as shown in **Figure 6**, where we've also labeled the length of the hypotenuse (13 meters).

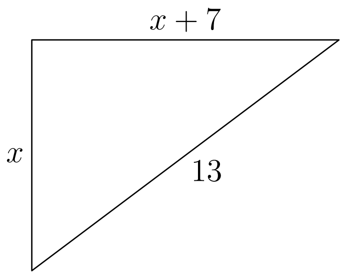


**Figure 6.** The second leg is 7 meters longer than the first.

Remember to isolate the length of the hypotenuse on one side of the equation representing the Pythagorean Theorem. That is,

$$x^2 + (x+7)^2 = 13^2.$$

Note that the legs go on one side of the equation, the hypotenuse on the other. Square and simplify. Remember to use the squaring a binomial pattern.

$$\begin{aligned} x^2 + x^2 + 14x + 49 &= 169 \\ 2x^2 + 14x + 49 &= 169 \end{aligned}$$

This equation is nonlinear, so make one side zero by subtracting 169 from both sides of the equation.

$$\begin{aligned} 2x^2 + 14x + 49 - 169 &= 0 \\ 2x^2 + 14x - 120 &= 0 \end{aligned}$$

Note that each term on the left-hand side of the equation is divisible by 2. Divide both sides of the equation by 2.

$$x^2 + 7x - 60 = 0$$

Let's use the quadratic formula with $a = 1$, $b = 7$, and $c = -60$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-7 \pm \sqrt{7^2 - 4(1)(-60)}}{2(1)}$$

Simplify.

$$x = \frac{-7 \pm \sqrt{289}}{2}$$

Note that 289 is a perfect square ($17^2 = 289$). Thus,

$$x = \frac{-7 \pm 17}{2}.$$

Thus, we have two solutions,

$$x = 5 \quad \text{or} \quad x = -12.$$

Because length must be a positive number, we eliminate $-12$ from consideration. Thus, the length of the first leg is $x = 5$ meters. The length of the second leg is $x + 7$, or 12 meters.

**Check.** Checking is an easy matter. The legs are 5 and 12 meters, respectively, and the hypotenuse is 13 meters. Note that the second leg is 7 meters longer than the first. Also,

$$5^2 + 12^2 = 25 + 144 = 169,$$

which is the square of 13.

The integral sides of the triangle in the previous example, 5, 12, and 13, are an example of a *Pythagorean Triple*.

**Pythagorean Triple.** A set of positive integers $a$, $b$, and $c$, is called a Pythagorean Triple if they satisfy the Pythagorean Theorem; that is, if

$$a^2 + b^2 = c^2.$$

If the greatest common factor of $a$, $b$, and $c$ is 1, then the triple $(a, b, c)$ is called a **primitive** Pythagorean Triple.

Thus, for example, the Pythagorean Triple $(5, 12, 13)$ is primitive.

Let's look at another example.

► **Example 5.** *If $(a, b, c)$ is a Pythagorean Triple, show that any positive integral multiple is also a Pythagorean Triple.*

Thus, if the positive integers $(a, b, c)$ is a Pythagorean Triple, we must show that $(ka, kb, kc)$, where $k$ is a positive integer, is also a Pythagorean Triple.

However, we know that

$$a^2 + b^2 = c^2.$$

Multiply both sides of this equation by $k^2$.

$$k^2a^2 + k^2b^2 = k^2c^2$$

This last result can be written

$$(ka)^2 + (kb)^2 = (kc)^2.$$

Hence, $(ka, kb, kc)$ is a Pythagorean Triple.

Hence, because $(3, 4, 5)$ is a Pythagorean Triple, you can double everything to get another triple $(6, 8, 10)$. Note that $6^2 + 8^2 = 10^2$ is easily checked. Similarly, tripling gives another triple $(9, 12, 15)$, and so on.

In **Example 5**, we showed that $(5, 12, 13)$ was a triple, so we can take multiples to generate other Pythagorean Triples, such as $(10, 24, 26)$ or $(15, 36, 39)$, and so on.

Formulae for generating Pythagorean Triples have been know since antiquity.

▶ **Example 6.** *The following formula for generating Pythagorean Triples was published in Euclid's (325–265 BC)* Elements, *one of the most successful textbooks in the history of mathematics. If $m$ and $n$ are positive integers with $m > n$, show*

$$\begin{aligned} a &= m^2 - n^2, \\ b &= 2mn, \\ c &= m^2 + n^2, \end{aligned} \tag{7}$$

*generates Pythagorean Triples.*

We need only show that the formulae for $a$, $b$, and $c$ satisfy the Pythagorean Theorem. With that is mind, let's first compute $a^2 + b^2$.

$$\begin{aligned} a^2 + b^2 &= (m^2 - n^2)^2 + (2mn)^2 \\ &= m^4 - 2m^2n^2 + n^4 + 4m^2n^2 \\ &= m^4 + 2m^2n^2 + n^4 \end{aligned}$$

On the other hand,

$$\begin{aligned} c^2 &= (m^2 + n^2)^2 \\ &= m^4 + 2m^2n^2 + n^4. \end{aligned}$$

Hence, $a^2 + b^2 = c^2$, and the expressions for $a$, $b$, and $c$ form a Pythagorean Triple.

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It is both interesting and fun to generate Pythagorean Triples with the formulae from **Example 6**. Choose $m = 4$ and $n = 2$, then

$$\begin{aligned} a &= m^2 - n^2 = (4)^2 - (2)^2 = 12, \\ b &= 2mn = 2(4)(2) = 16, \\ c &= m^2 + n^2 = (4)^2 + (2)^2 = 20. \end{aligned}$$

It is easy to check that the triple $(12, 16, 20)$ will satisfy $12^2 + 16^2 = 20^2$. Indeed, note that this triple is a multiple of the basic $(3, 4, 5)$ triple, so it must also be a Pythagorean Triple.

It can also be shown that if $m$ and $n$ are relatively prime, and are not both odd or both even, then the formulae in **Example 6** will generate a **primitive** Pythagorean Triple. For example, choose $m = 5$ and $n = 2$. Note that the greatest common divisor of $m = 5$ and $n = 2$ is one, so $m$ and $n$ are relatively prime. Moreover, $m$ is odd while $n$ is even. These values of $m$ and $n$ generate

$$\begin{aligned} a &= m^2 - n^2 = (5)^2 - (2)^2 = 21, \\ b &= 2mn = 2(5)(2) = 20, \\ c &= m^2 + n^2 = (5)^2 + (2)^2 = 29. \end{aligned}$$

Note that

$$\begin{aligned} 21^2 + 20^2 &= 441 + 400 \\ &= 841 \\ &= 29^2. \end{aligned}$$

Hence, $(21, 20, 29)$ is a Pythagorean Triple. Moreover, the greatest common divisor of 21, 20, and 29 is one, so $(21, 20, 29)$ is primitive.

The practical applications of the Pythagorean Theorem are numerous.

► **Example 8.** *A painter leans a 20 foot ladder against the wall of a house. The base of the ladder is on level ground 5 feet from the wall of the house. How high up the wall of the house will the ladder reach?*

Consider the triangle in **Figure 7**. The hypotenuse of the triangle represents the ladder and has length 20 feet. The base of the triangle represents the distance of the base of the ladder from the wall of the house and is 5 feet in length. The vertical leg of the triangle is the distance the ladder reaches up the wall and the quantity we wish to determine.

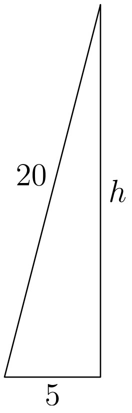


**Figure 7.** A ladder leans against the wall of a house.

Applying the Pythagorean Theorem,

$$5^2 + h^2 = 20^2.$$

Again, note that the square of the length of the hypotenuse is the quantity that is isolated on one side of the equation.

Next, square, then isolate the term containing $h$ on one side of the equation by subtracting 25 from both sides of the resulting equation.

$$\begin{aligned} 25 + h^2 &= 400 \\ h^2 &= 375 \end{aligned}$$

We need only extract the positive square root.

$$h = \sqrt{375}$$

We could place the solution in simple form, that is, $h = 5\sqrt{15}$, but the nature of the problem warrants a decimal approximation. Using a calculator and rounding to the nearest tenth of a foot,

$$h \approx 19.4.$$

Thus, the ladder reaches about 19.4 feet up the wall.

◇

## *The Distance Formula*

We often need to calculate the distance between two points $P$ and $Q$ in the plane. Indeed, this is such a frequently recurring need, we'd like to develop a formula that will quickly calculate the distance between the given points $P$ and $Q$. Such a formula is the goal of this last section.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two arbitrary points in the plane, as shown in **Figure 8**(a) and let $d$ represent the distance between the two points.

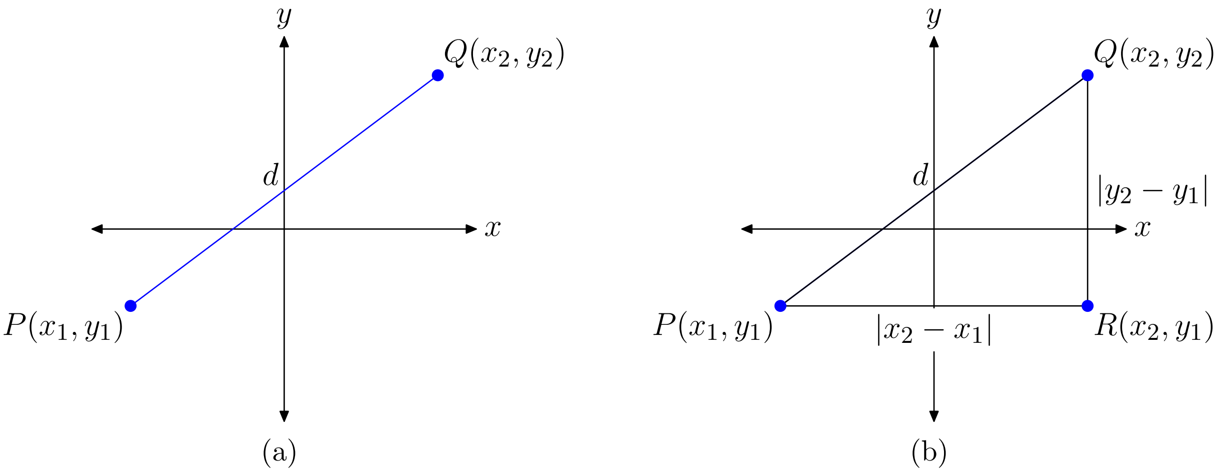


**Figure 8.** Finding the distance between the points $P$ and $Q$.

To find the distance $d$, first draw the right triangle $\triangle PQR$, with legs parallel to the axes, as shown in **Figure 8**(b). Next, we need to find the lengths of the legs of the right triangle $\triangle PQR$.

- The distance between $P$ and $R$ is found by subtracting the $x$ coordinate of $P$ from the $x$-coordinate of $R$ and taking the absolute value of the result. That is, the distance between $P$ and $R$ is $|x_2 - x_1|$.
- The distance between $R$ and $Q$ is found by subtracting the $y$-coordinate of $R$ from the $y$-coordinate of $Q$ and taking the absolute value of the result. That is, the distance between $R$ and $Q$ is $|y_2 - y_1|$.

We can now use the Pythagorean Theorem to calculate $d$. Thus,

$$d^2 = (|x_2 - x_1|)^2 + (|y_2 - y_1|)^2.$$

However, for any real number $a$,

$$(|a|)^2 = |a| \cdot |a| = |a^2| = a^2,$$

because $a^2$ is nonnegative. Hence, $(|x_2 - x_1|)^2 = (x_2 - x_1)^2$ and $(|y_2 - y_1|)^2 = (y_2 - y_1)^2$ and we can write

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Taking the positive square root leads to the *Distance Formula*.

**The Distance Formula.** Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two arbitrary points in the plane. The distance $d$ between the points $P$ and $Q$ is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{9}$$

The direction of subtraction is unimportant. Because you square the result of the subtraction, you get the same response regardless of the direction of subtraction (e.g. $(5-2)^2 = (2-5)^2$). Thus, it doesn't matter which point you designate as the point $P$, nor does it matter which point you designate as the point $Q$. Simply subtract $x$-coordinates and square, subtract $y$-coordinates and square, add, then take the square root.

Let's look at an example.

▶ **Example 10.** *Find the distance between the points* $P(-4, -2)$ *and* $Q(4, 4)$.

It helps the intuition if we draw a picture, as we have in **Figure 9**. One can now take a compass and open it to the distance between points $P$ and $Q$. Then you can place your compass on the horizontal axis (or any horizontal gridline) to estimate the distance between the points $P$ and $Q$. We did that on our graph paper and estimate the distance $d \approx 10$.

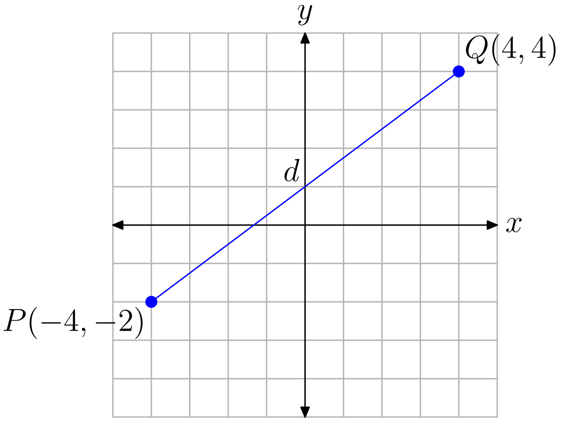


**Figure 9.** Gauging the distance between $P(-4, -2)$ and $Q(4, 4)$.

Let's now use the distance formula to obtain an exact value for the distance $d$. With $(x_1, y_1) = P(-4, -2)$ and $(x_2, y_2) = Q(4, 4)$,

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4 - (-4))^2 + (4 - (-2))^2} \\ &= \sqrt{8^2 + 6^2} \\ &= \sqrt{64 + 36} \\ &= \sqrt{100} \\ &= 10. \end{aligned}$$

It's not often that your exact result agrees with your approximation, so never worry if you're off by just a little bit.

## 9.6 Exercises

In **Exercises 1-8**, state whether or not the given triple is a Pythagorean Triple. Give a reason for your answer.

**1.** $(8, 15, 17)$

**2.** $(7, 24, 25)$

**3.** $(8, 9, 17)$

**4.** $(4, 9, 13)$

**5.** $(12, 35, 37)$

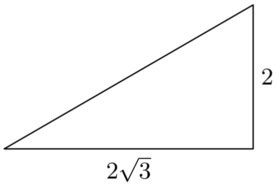
**6.** $(12, 17, 29)$

**7.** $(11, 17, 28)$

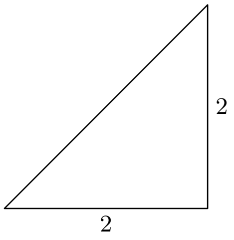
**8.** $(11, 60, 61)$

In **Exercises 9-16**, set up an equation to model the problem constraints and solve. Use your answer to find the missing side of the given right triangle. Include a sketch with your solution and check your result.

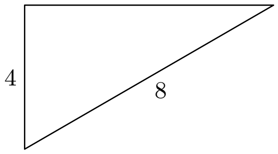
**9.**



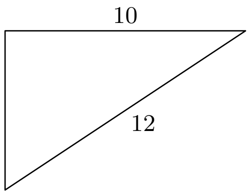
**10.**



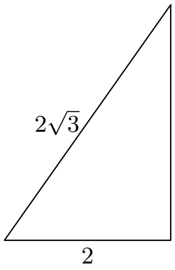
**11.**



**12.**



**13.**



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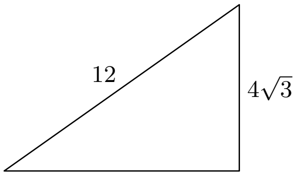
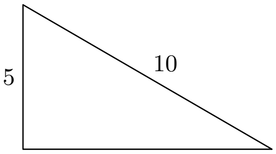
[16] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

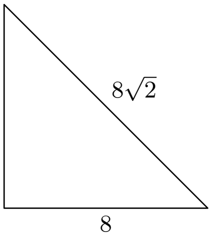
**14.**



**15.**



**16.**



In **Exercises 17-20**, set up an equation that models the problem constraints. Solve the equation and use the result to answer the question. Look back and check your result.

**17.** The legs of a right triangle are consecutive positive integers. The hypotenuse has length 5. What are the lengths of the legs?

**18.** The legs of a right triangle are consecutive even integers. The hypotenuse has length 10. What are the lengths of the legs?

**19.** One leg of a right triangle is 1 centimeter less than twice the length of the first leg. If the length of the hypotenuse is 17 centimeters, find the lengths of the legs.

**20.** One leg of a right triangle is 3 feet longer than 3 times the length of the first leg. The length of the hypotenuse is 25 feet. Find the lengths of the legs.

**21.** Pythagoras is credited with the following formulae that can be used to generate Pythagorean Triples.

$$\begin{aligned} a &= m \\ b &= \frac{m^2-1}{2}, \\ c &= \frac{m^2+1}{2} \end{aligned}$$

Use the technique of Example 6 to demonstrate that the formulae given above will generate Pythagorean Triples, provided that $m$ is an *odd* positive integer larger than one. Secondly, generate at least 3 instances of Pythagorean Triples with Pythagoras's formula.

**22.** Plato (380 BC) is credited with the following formulae that can be used to generate Pythagorean Triples.

$$\begin{aligned} a &= 2m \\ b &= m^2-1, \\ c &= m^2+1 \end{aligned}$$

Use the technique of Example 6 to demonstrate that the formulae given above will generate Pythagorean Triples, provided that $m$ is a positive integer larger than 1. Secondly, generate at least 3 instances of Pythagorean Triples with Plato's formula.

In **Exercises 23-28**, set up an equation that models the problem constraints. Solve the equation and use the result to answer the question. Look back and check your result.

**23.** Fritz and Greta are planting a 12-foot by 18-foot rectangular garden, and are laying it out using string. They would like to know the length of a diagonal to make sure that right angles are formed. Find the length of a diagonal. Approximate your answer to within 0.1 feet.

**24.** Angelina and Markos are planting a 20-foot by 28-foot rectangular garden, and are laying it out using string. They would like to know the length of a diagonal to make sure that right angles are formed. Find the length of a diagonal. Approximate your answer to within 0.1 feet.

**25.** The base of a 36-foot long guy wire is located 16 feet from the base of the telephone pole that it is anchoring. How high up the pole does the guy wire reach? Approximate your answer to within 0.1 feet.

**26.** The base of a 35-foot long guy wire is located 10 feet from the base of the telephone pole that it is anchoring. How high up the pole does the guy wire reach? Approximate your answer to within 0.1 feet.

**27.** A stereo receiver is in a corner of a 13-foot by 16-foot rectangular room. Speaker wire will run under a rug, diagonally, to a speaker in the far corner. If 3 feet of slack is required on each end, how long a piece of wire should be purchased? Approximate your answer to within 0.1 feet.

**28.** A stereo receiver is in a corner of a 10-foot by 15-foot rectangular room. Speaker wire will run under a rug, diagonally, to a speaker in the far corner. If 4 feet of slack is required on each end, how long a piece of wire should be purchased? Approximate your answer to within 0.1 feet.

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In **Exercises 29-38**, use the distance formula to find the exact distance between the given points.

**29.** $(-8,-9)$ and $(6,-6)$

**30.** $(1,0)$ and $(-9,-2)$

**31.** $(-9,1)$ and $(-8,7)$

**32.** $(0,9)$ and $(3,1)$

**33.** $(6,-5)$ and $(-9,-2)$

**34.** $(-9,6)$ and $(1,4)$

**35.** $(-7,7)$ and $(-3,6)$

**36.** $(-7,-6)$ and $(-2,-4)$

**37.** $(4,-3)$ and $(-9,6)$

**38.** $(-7,-1)$ and $(4,-5)$

---

In **Exercises 39-42**, set up an equation that models the problem constraints. Solve the equation and use the result to answer the question. Look back and check your result.

**39.** Find $k$ so that the point $(4,k)$ is $2\sqrt{2}$ units away from the point $(2,1)$.

**40.** Find $k$ so hat the point $(k,1)$ is $2\sqrt{2}$ units away from the point $(0,-1)$.

**41.** Find $k$ so that the point $(k, 1)$ is $\sqrt{17}$ units away from the point $(2, -3)$.

**42.** Find $k$ so that the point $(-1, k)$ is $\sqrt{13}$ units away from the point $(-4, -3)$.

**43.** Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Plot the points $P(0, 5)$ and $Q(4, -3)$ on your coordinate system.

**a)** Plot several points that are equidistant from the points $P$ and $Q$ on your coordinate system. What graph do you get if you plot **all** points that are equidistant from the points $P$ and $Q$? Determine the equation of the graph by examining the resulting image on your coordinate system.

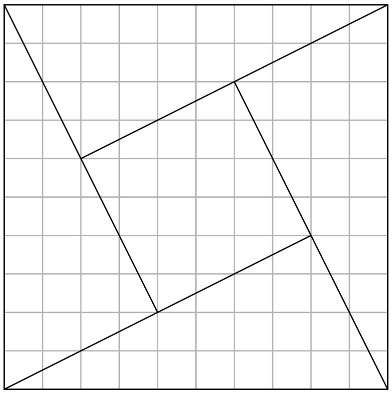
**b)** Use the distance formula to find the equation of the graph of all points that are equidistant from the points $P$ and $Q$. *Hint: Let $(x, y)$ represent an arbitrary point on the graph of all points equidistant from points $P$ and $Q$. Calculate the distances from the point $(x, y)$ to the points $P$ and $Q$ separately, then set them equal and simplify the resulting equation. Note that this analytical approach should provide an equation that matches that found by the graphical approach in part (a).*

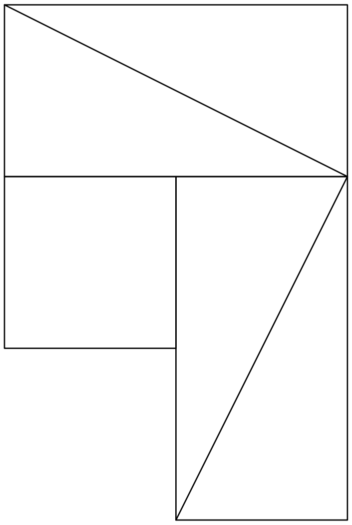
**44.** Set up a coordinate system on a sheet of graph paper. Label and scale each axis. Plot the point $P(0, 2)$ and label it with its coordinates. Draw the line $y = -2$ and label it with its equation.

**a)** Plot several points that are equidistant from the point $P$ and the line $y = -2$ on your coordinate system. What graph do you get if you plot **all** points that are equidistant from the points $P$ and the line $y = -2$.

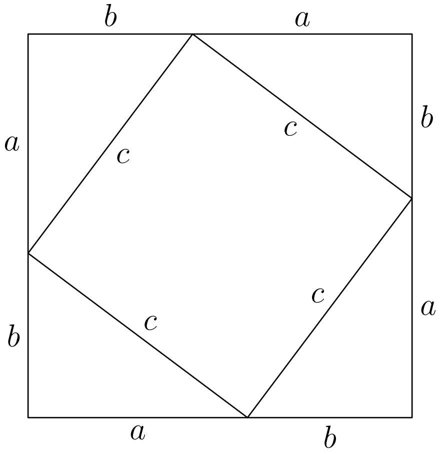
**b)** Use the distance formula to find the equation of the graph of all points that are equidistant from the points $P$ and the line $y = -2$. *Hint: Let $(x, y)$ represent an arbitrary point on the graph of all points equidistant from points $P$ and the line $y = -2$. Calculate the distances from the point $(x, y)$ to the points $P$ and the line $y = -2$ separately, then set them equal and simplify the resulting equation.*

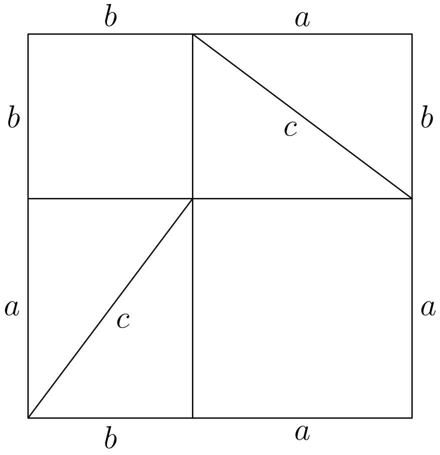
**45.** Copy the following figure onto a sheet of graph paper. Cut the pieces of the first figure out with a pair of scissors, then rearrange them to form the second figure. Explain how this proves the Pythagorean Theorem.





**46.** Compare this image to the one that follows and explain how this proves the Pythagorean Theorem.





## 9.6 *Answers*

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**1.** Yes, because $8^2 + 15^2 = 17^2$

**3.** No, because $8^2 + 9^2 \neq 17^2$

**5.** Yes, because $12^2 + 35^2 = 37^2$

**7.** No, because $11^2 + 17^2 \neq 28^2$

**9.** 4

**11.** $4\sqrt{3}$

**13.** $2\sqrt{2}$

**15.** $5\sqrt{3}$

**17.** The legs have lengths 3 and 4.

**19.** The legs have lengths 8 and 15 centimeters.

**21.** $(3, 4, 5)$, $(5, 12, 13)$, and $(7, 24, 25)$, with $m = 3$, 5, and 7, respectively.

**23.** 21.63 ft

**25.** 32.25 ft

**27.** 26.62 ft

**29.** $\sqrt{205}$

**31.** $\sqrt{37}$

**33.** $\sqrt{234} = 3\sqrt{26}$

**35.** $\sqrt{17}$

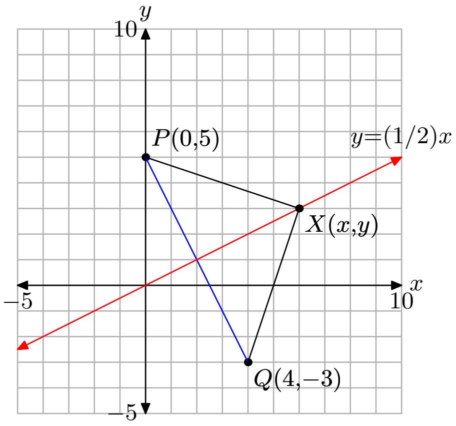
**37.** $\sqrt{250} = 5\sqrt{10}$

**39.** $k = 3, -1$.

**41.** $k = 1, 3$.

**43.**

**a)** In the figure that follows, $XP = XQ$.



**b)** $y = (1/2)x$

## 9.7 Index